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WEAK-STAR CONVERGENCE IN THE DUAL OF THE CONTINUOUS FUNCTIONS ON THE n-CUBE, $1 \leqslant n \leqslant \infty$

BY

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ABSTRACT. Let n be a positive integer and let $J = \times_{j=1}^{n}[0,1]_j$ denote the n-cube. Let $\mathbf{C} = \mathbf{C}(J)$ denote the (sup norm) space of continuous (real-valued) functions defined on J, and let $\mathfrak{M}$ denote the (variation norm) space of (real-valued) signed Borel measures defined on the Borel subsets of J. Let $\langle \mu_l \rangle$ be a sequence of elements of $\mathfrak{M}$. Necessary and sufficient conditions are given in order that $\lim_l \int f\,d\mu_l$ exists for every $f \in \mathbf{C}$. After considering a finite dimensional case, the infinite dimensional case is entertained.

I. Introduction to the finite dimensional case. We begin with a brief unchronological orientation. One of the Riesz representation theorems establishes an isometric isomorphism between $\mathfrak{M}$ and the dual, $\mathbf{C}^*$, of $\mathbf{C}$: $\mu \in \mathfrak{M}$ which corresponds to $L \in \mathbf{C}^*$ via the equation $L(f) = \int_J f\,d\mu = \int f\,d\mu$, $f \in \mathbf{C}$.

For $x = (x^1,\ldots,x^n)$, $y \in \mathbf{R}^n$, $x < y$ means $x^j < y^j$, $i \leqslant j \leqslant n$. For $\bar{0} = (0,\ldots,0) \leqslant x_1 \leqslant x_2 \leqslant \bar{1} = (1,\ldots,1)$, the closed subinterval $[x_1, x_2] = \{x;\ x_1 \leqslant x \leqslant x_2\}$. The distribution function Γ of $\mu \in \mathfrak{M}$ is defined on J by $\Gamma(x) = \mu([\bar{0}, x])$ for $x > 0$ and $\Gamma(x) = 0$ otherwise. The variation norm, $\|\Gamma\|$, of Γ is equal to $\|\mu\|$ and $\int f\,d\mu = \int f\,d\Gamma$, where the latter integral is a Riemann-Stieltjes integral. Let $\mathbf{\Gamma}$ denote the space of distribution functions.

Let $\langle \mu_l \rangle$ be a sequence in $\mathfrak{M}$ and suppose that $\lim_l \int_J f\,d\mu_l$ exists, $f \in \mathbf{C}$. Let $L(f)$ denote this limit. Then $L \in \mathbf{C}^*$ and $\mu_l \xrightarrow{w} \mu$, i.e., $\lim_l \int f\,d(\mu_l - \mu) = 0$, $f \in \mathbf{C}$. Thus, it suffices to consider weak-star convergence to zero.

Let S denote the set of all proper subsets of $\{1,2,\ldots,n\}$, and for $\theta \in S$, let $J_\theta = \{x \in J;\ x^j = 1,\ j \in \theta\}$.

Let ν denote Lebesgue measure on J, and let ν_θ denote m-dimensional Lebesgue measure on J_θ, where $\theta \in S$, $|\theta|$ is the number of elements in θ and $m = m_\theta = n - |\theta|$.

Conditions for weak-star convergence to zero follow.

THEOREM 1. *Let $\langle \mu_l \rangle$ be a sequence in $\mathfrak{M}$ and let $\langle \Gamma_l \rangle$ be the corresponding sequence of distribution functions. Then, $\mu_l \xrightarrow{w} 0$ if and only if the following three conditions are met*:

(i) $\|\mu_l\| \leqslant M$ *for some M and all l*;

(ii) $\mu_l(J) \to 0$, *as $l \to \infty$*;

(iii) $\forall \theta \in S$: $\int_{J_\theta} |\Gamma_l|\,d\nu_\theta \to 0$ *as $l \to \infty$*.

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For instance, when $n = 2$, $S = \{\phi, \{1\}, \{2\}\}$ and (iii) is the union of three statements:

$$\int_0^1\int_0^1 |\Gamma_l(x, y)|\,dx\,dy \to 0, \quad \int_0^1 |\Gamma_l(1, y)|\,dy \to 0 \quad \text{and} \quad \int_0^1 |\Gamma_l(x, 1)|\,dx \to 0.$$

Notice that the necessity of (i) is a consequence of the uniform boundedness theorem and (ii) merely says that $\mu_l(J) = \int_J 1\,d\mu_l \to 0$, so (iii) is the crucial condition. G. Högnäs [4] considered the case $n = 1$; however, our approach is quite different. Our proof of Theorem 1 is based on two Helly theorems and a technical result (Theorem 2). Before establishing Theorem 1, we will go back to the one-dimensional case and recall several facts to motivate our strategy. Theorem 2 is a multidimensional version of some of these facts, so the one-dimensional version of Theorem 1 turns out to be a consequence of two Helly theorems and the Lebesgue Dominated Convergence Theorem. In this section we give a proof of Theorem 1. §II contains a discussion of n-dimensional Riemann-Stieltjes integration, including proofs of the Helly theorems, for the interested reader. We give a proof of Theorem 2 in §III. §IV is a brief discussion of the infinite dimensional case.

A subinterval $[x_1, x_2]$ of J has 2^n corners, namely

$$\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_n}^n\right); \qquad i_j = 1, 2 \text{ for each } j = 1, 2, \ldots, n.$$

We define

$$\gamma_{i_1, i_2, \ldots, i_n} = \operatorname{sign}\left[\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_n}^n\right)\right] = \begin{cases} + & \text{if } \sum_{j=1}^n i_j \text{ is even}, \\ - & \text{if } \sum_{j=1}^n i_j \text{ is odd}. \end{cases}$$

For a function g we define

$$\Delta g(I) = \sum_{i_1, \ldots, i_n} \gamma_{i_1, \ldots, i_n} g\left(x_{i_1}^1, \ldots, x_{i_n}^n\right).$$

In accordance with our usage of superscripts, a partition σ_j of $[0, 1]_j$, $1 \leqslant j \leqslant n$, will be given by

$$\sigma_j\colon 0 = x_0^j < x_1^j < \cdots < x_{m_j}^j = 1.$$

By a partition $\sigma = \times_{j=1}^n \sigma_j$ of J, we mean the set of subintervals

$$I_{i_1, \ldots, i_n} = \mathop{\times}_{j=1}^{n} \left[x_{i_j - 1}^j, x_{i_j}^j\right]; \qquad 1 \leqslant i_j \leqslant m_j,\ j = 1, 2, \ldots, n.$$

In case all the partitions σ_j are disjoint, i.e.,

$$\sigma_j = \left\{\left[0, x_1^j\right], \left(x_1^j, x_2^j\right], \ldots, \left(x_{m_j - 1}^j, x_{m_j}^j\right]\right\},$$

σ is referred to as a disjoint partition.

A function g is said to be of bounded variation (on J) if and only if

$$\|g\| = \sup_{\sigma} \sum_{I \in \sigma} |\Delta g(I)| < \infty,$$

where σ ranges over all partitions of J; $\|g\|$ is called the total variation of g (on J).

A point $x \in J$ is said to lie on coordinate planes if $x^j = 0$ for at least one $j = 1, 2, \ldots, n$. Let $\mathbf{B}$ denote the set of all functions of bounded variation vanishing at all the points lying on the coordinate planes.

Using the linearity property of the integral, to every $g \in \mathbf{B}$ we can associate an $L \in \mathbf{C}^*$ by defining $L(f) = \int_J f\,dg$, $f \in \mathbf{C}$, and hence a $\mu = \mu_g \in \mathfrak{M}$ such that

$$\text{(1-1)} \qquad \int f\,dg = L(f) = \int f\,d\mu = \int f\,d\Gamma_\mu, \qquad f \in \mathbf{C}.$$

The relations $\|L\| = \|\mu\| = \|\Gamma_\mu\| \leqslant \|g\|$ obtain and the equivalence classes $\{h \in \mathbf{B};\ \int f\,dh = \int f\,dg,\ f \in \mathbf{C}\}$ comprise a partition of $\mathbf{B}$. With this in mind, given a sequence $\langle \mu_l \rangle$ in $\mathfrak{M}$, by a corresponding sequence $\langle g_l \rangle$ in $\mathbf{B}$ we mean any sequence such that (1-1) holds for each pair μ_l, g_l, for $l = 1, 2, \ldots$. Let $\langle g_l \rangle$ be a sequence in $\mathbf{B}$ and $g \in \mathbf{B}$. Pointwise convergence on J is denoted $g_l \to g$, and weak-star convergence by $g_l \overset{w}{\to} g$, where the latter is defined as usual:

$$\int_J f\,dg_l \to \int_J f\,dg, \qquad f \in \mathbf{C}.$$

Put $n = 1$ and recall the following facts: (i) a function g in $\mathbf{B}$ is the difference of two nondecreasing functions p and q in $\mathbf{B}$; moreover, p and q can be so chosen that $\|g\| = p(1) + q(1)$; (ii) a distribution function is right continuous on $(0, 1)$; (iii) if a uniformly bounded sequence of distribution functions converges pointwise to a function g, then $g \in \mathbf{B}$, but g need not be in $\mathbf{\Gamma}$; (iv) a function of bounded variation has only a countable number of points of discontinuity; (v) a function of bounded variation has left side limits on $(0, 1]$ and right side limits on $[0, 1)$; (vi) a function in $\mathbf{B}$ corresponds to the zero functional $\Leftrightarrow$ it is zero at one and is zero a.e.; (vii) a function g in $\mathbf{B}$ is a distribution function $\Leftrightarrow$ it is right continuous on $(0, 1)$.

When $n > 1$, things are more complicated; however, basic facts tend to be quite similar. The case $n = 2$ is a nice case to consider in order to see what is happening in the sequel: the resulting spaces are flexible enough to display the types of things that can occur, it is easy to draw pictures and there is a rather complete treatment of two-dimensional Riemann-Stieltjes integration in [5]. To illustrate, suppose that $n = 2$ and $L \in \mathbf{C}^*$ is defined by $L(f) = f(\frac{1}{2}, \frac{1}{2})$. Then μ corresponds to a unit mass at the point $(\frac{1}{2}, \frac{1}{2})$, but Γ is discontinuous on $\{(\frac{1}{2}, y);\ \frac{1}{2} \leqslant y \leqslant 1\} \cup \{(x, \frac{1}{2});\ \frac{1}{2} \leqslant x \leqslant 1\}$; however, if we think of a point as a hyperplane in $\mathbf{R}^1$, then (iv) says that the discontinuities of elements of $\mathbf{B}$ lie on a countable set of hyperplanes and this is a valid statement for $1 \leqslant n < \infty$. Theorem 2 following the Helly theorems below is an n-dimensional version of (vi).

$[\mathrm{H}_1]$ Let $\langle g_l \rangle$ be a sequence in $\mathbf{B}$ with $\|g_l\| \leqslant M$, $l = 1, 2, \ldots$. Then there exists a subsequence $\langle g_{l_k} \rangle$ such that $g_{l_k} \to g$ and $\|g\| \leqslant M$.

$[\mathrm{H}_2]$ Let $\langle g_l \rangle$ be a sequence in $\mathbf{B}$ with $\|g_l\| \leqslant M$, $l = 1, 2, \ldots$. If $g_l \to g$, then $g \in \mathbf{B}$ and $g_n \overset{w}{\to} g$.

For $\theta \in S$ and h a function on J, h_θ denotes the restriction of h to J_θ.

THEOREM 2. *Let $g \in \mathbf{B}$. Then the following statements are equivalent*:

(i) $\int f\,dg = 0$, $f \in \mathbf{C}$.

(ii) $g(\bar{1}) = 0$ *and* $\forall \theta \in S$, g_θ *vanishes at all of its points of continuity.*

(iii) $g(\bar{1}) = 0$ *and* $\forall \theta \in S$, $g_\theta = 0$, *a.e. on* J_θ.

PROOF OF THEOREM 1. *Necessity*. (i) By the Principle of Uniform Boundedness there exists an M such that $\|\Gamma_l\| = \|\mu_l\| \leqslant M$.

(ii) Let $f \equiv 1$ on J. Then

$$\Gamma_l(\bar{1}) = \int d\Gamma_l = \int d\mu_l \to 0 \quad \text{as } l \to \infty.$$

(iii) Since $\|\Gamma_l\| \leqslant M$, by [H$_1$] there exists a subsequence $\langle \Gamma_{l_k} \rangle$ such that $\Gamma_{l_k} \to g \in \mathbf{B}$ with $\|g\| \leqslant M$. By [H$_2$] then $\Gamma_{l_k} \xrightarrow{w} g$. Since $\Gamma_{l_k} \xrightarrow{w} 0$ this implies $\int f\,dg = 0$, $f \in \mathbf{C}$, so that by Theorem 2, for all $\theta \in S$, g_θ vanishes at all of its continuity points, which implies

$$\int_{J_\theta} |g_\theta|\, d\nu_\theta = 0, \qquad \theta \in S.$$

Clearly, for every $\theta \in S$, $|(\Gamma_{l_k})_\theta| \to |g_\theta|$ and for $k = 1, 2, \ldots,$ we have $\|(\Gamma_{l_k})_\theta\|_\infty \leqslant M$, $\|g_\theta\|_\infty \leqslant M$. By Lebesgue's Dominated Convergence Theorem we find $\int_{J_\theta} |(\Gamma_{l_k})_\theta| \to 0$, $\theta \in S$.

If, for some $\theta \in S$, we do not have $\int_{J_\theta} |(\Gamma_l)_\theta| \to 0$, then there exists a subsequence $\langle \Gamma_m \rangle$ of $\langle \Gamma_l \rangle$, and some $\varepsilon > 0$ such that

$$\int_{J_\theta} |(\Gamma_m)_\theta| \geqslant \varepsilon, \qquad m = 1, 2, \ldots.$$

However, since $\Gamma_m \xrightarrow{w} 0$ we can extract a subsequence $\langle \Gamma_{m_k} \rangle$ as above such that $\int_{J_\theta} |(\Gamma_{m_k})_\theta| \to 0$, which is a contradiction.

Sufficiency. By [H$_1$] there exists a subsequence $\langle \Gamma_{l_k} \rangle$ such that $\Gamma_{l_k} \to g \in \mathbf{B}$. Therefore, g(1) = 0 and $(\Gamma_{l_k})_\theta \to g_\theta$, $\theta \in S$. By Lebesgue's Dominated Convergence Theorem, for every $\theta \in S$,

$$\int_{J_\theta} |(\Gamma_{l_k})_\theta| \to \int_{J_\theta} |g_\theta|,$$

from which it follows that for all $\theta \in S$, $g_\theta = 0$, a.e. on J_θ. By Theorem 2, we must have $\int f\,dg = 0$, $f \in \mathbf{C}$. By [H$_2$], $\Gamma_{l_k} \xrightarrow{w} g$, and hence $\Gamma_{l_k} \xrightarrow{w} 0$, so that $\mu_{l_k} \xrightarrow{w} 0$.

If $\mu_l \xrightarrow{w} 0$ is not true, then for some subsequence $\langle \mu_m \rangle$ of $\langle \mu_l \rangle$, some $f \in \mathbf{C}$ and some $\varepsilon > 0$, we must have

$$\left| \int f\,d\mu_m \right| \geqslant \varepsilon, \qquad m = 1, 2, \ldots.$$

However, we can extract a subsequence $\langle \mu_{m_k} \rangle$, as above, such that $\mu_{m_k} \xrightarrow{w} 0$. This is a contradiction and the proof is complete.

II. A discussion of n-dimensional Riemann-Stieltjes integration. For $\bar{0} < t \in J$, denote the closed subinterval $\times_{n=1}^{n} [0, t^j]$ by I_t.

Let $g \in \mathbf{B}$ and $t > 0$. Then, $|g(t)| = |\Delta g(I_t)| \leqslant \|g\|$, so that

$$\|g\|_\infty \leqslant \|g\|, \qquad g \in \mathbf{B}.$$

Now let g, $h \in \mathbf{B}$ and let σ be any partition of J. We have

$$\sum_{I\in\sigma} |\Delta(g \pm h)(I)| = \sum_{I\in\sigma} |\Delta g(I) \pm \Delta h(I)|$$
$$\leqslant \sum_{I\in\sigma} |\Delta g(I)| + \sum_{I\in\sigma} |\Delta h(I)| \leqslant \|g\| + \|h\|,$$

from which it follows that

$$\|g \pm h\| \leqslant \|g\| + \|h\|. \tag{2-1}$$

Hence, $\mathbf{B}$ is a normed linear space under $\| \ \|$.

A function p on J is said to be positively monotone increasing if and only if for all subintervals I of J we have $\Delta p(I) \geqslant 0$. Such functions are called positively monotonely monotone in [5]. When p is positively monotone increasing and vanishes at all the points lying on coordinate planes we find $p \geqslant 0$ on J since for any $t > 0$ we have $p(t) = \Delta p(I_t)$. Moreover, in this case, we also have $\|p\| = p(\bar{1})$, so that $p \in \mathbf{B}$.

Given a subinterval K of J and a partition τ of K, we may extend τ to a partition σ of all of J such that for all subintervals $I \in \tau$, we have $I \in \sigma$. Let $g \in \mathbf{B}$ and consider the restriction $g \mid K$. Then,

$$\sum_{I\in\tau} |\Delta g(I)| \leqslant \sum_{I\in\sigma} |\Delta g(I)| \leqslant \|g\|$$

so that $g \mid K$ is of bounded variation on K and if we denote its total variation on K by $\|g\|_K$, then

$$\|g\|_K \leqslant \|g\|, \qquad K \text{ a subinterval of } J. \tag{2-2}$$

DEFINITION (2-1). Let $g \in \mathbf{B}$. We define the variation function of g, denoted Π_g or Π, as follows: $\Pi(t) = \|g\|_{I_t}$ for $t > 0$ and $\Pi(t) = 0$ otherwise.

We have $\Pi(\bar{1}) = \|g\|$, and that for every subinterval I,

$$\Delta\Pi(I) \geqslant |\Delta g(I)|, \tag{2-3}$$

so that Π is positively monotone increasing.

By (2-1), the difference of two positively monotone increasing functions in $\mathbf{B}$ is again in $\mathbf{B}$. For the converse we introduce

DEFINITION (2-2). Let $g \in \mathbf{B}$. The positive variation of g, denoted ψ, and the negative variation of g, denoted ϕ, are defined as $\psi = \frac{1}{2}(\Pi + g)$, $\phi = \frac{1}{2}(\Pi - g)$; on J.

It follows from (2-3) that ψ, ϕ are positively monotone increasing lying in $\mathbf{B}$. Moreover, by their definition

$$g = \psi - \phi, \qquad g \in \mathbf{B},$$

to which we shall refer as the Jordan decomposition of g.

Let $p, q \in \mathbf{B}$ be positively monotone increasing such that $g = p - q$ on J. By (2-1), for all $t > 0$ in J we have

$$\Pi(t) = \|g\|_{I_t} \leqslant \|p\|_{I_t} + \|q\|_{I_t} = p(t) + q(t).$$

However, $\Pi = \psi + \phi$ and so $p \geqslant \psi$, $q \geqslant \phi$ on J.

When $n = 1$, g is continuous at $x \Leftrightarrow \Pi_q$ is continuous at $x \Leftrightarrow$ each of ψ and ϕ is continuous at x. When $n > 1$, this is no longer the case. For examples, put $n = 2$, $u_j = \frac{1}{2} + 4^{-j}$, $x_0 = (\frac{1}{2}, \frac{1}{2})$, $x_j = (u_j, u_j)$, $y_j = (\frac{1}{2}, u_j)$ and consider two sequences of functionals, $K_j(f) = f(x_0) - f(x_j)$ and $L_j = f(x_0) - f(y_j)$. Compute their μ's and Γ's, and notice that both sequences are weak-star convergent to zero.

Let P_θ denote the orthogonal projection of J on J_θ, $\theta \in S$.

For x and $y \in J_\theta$, $x <_\theta y$ means $x^j < y^j$, $j \notin \theta$. We simply write $x < y$ whenever it is clear that we mean $x <_\theta y$. Let

$$J(m) = \bigtimes_{j=1}^{m} [0,1]_j,$$

so that $J_{(n)} = J$. Then J_θ and $J_{(m)}$ are naturally isomorphic ($m = n - |\theta|$) and $<$ is preserved under the natural isomorphism. Moreover, the natural isomorphism of J_θ and $J_{(m)}$ induces an isometry between $\mathbf{C}_\theta$ and $\mathbf{C}_{(m)}$, the space of continuous functions on J_θ and $J_{(m)}$, respectively. We mention $J_{(m)}$ to clarify statements made for J_θ.

Given a subinterval I of J such that $I \cap J_\theta = I_\theta \neq \varnothing$, then I_θ naturally corresponds to some subinterval in $J_{(m)}$. In the same vein, if σ is a (disjoint) partition of J and σ_θ is defined as

$$\sigma_\theta = \{I_\theta \mid I \in \sigma, I \cap J_\theta \neq \varnothing\},$$

then σ_θ is a (disjoint) partition of J_θ.

DEFINITION (2-3). A subinterval in J_θ is given by $I_\theta = \bigtimes_{j=1}^{n}[x_1^j, x_2^j]$ where for all $j \in \theta$ we have $x_1^j = x_2^j = 1$ so that $[x_1^j, x_2^j] = \{1\}$. We may extend I_θ to a subinterval I of J by replacing each $\{1\}$ by the linear subinterval $[0, 1]$: $I = P_\theta^{-1}(I_\theta)$. We refer to I as the standard extension of I_θ.

Next, let h be any function on J_θ. We may extend h to a function f on J by defining $f(x) = h(P_\theta(x))$, $x \in J$. Then $f_\theta = f \mid J_\theta = h$, and we call f the standard extension of h.

Let I_θ be a subinterval of J_θ and let I be its standard extension. Then $I_\theta = I \cap J_\theta$ and the corners of I lie either in J_θ or else in coordinate planes.

Notice that if $g \in \mathbf{B}, \theta \in S$, then $\| g_\theta \| \leqslant \| g \|$. Also, observe that if $p \in \mathbf{B}$ is positively monotone increasing on J, then so is p_θ on J_θ; moreover, $\| p_\theta \| = \| p \| = p(\bar{1})$.

Let $g \in \mathbf{B}$ and let $I = \bigtimes_{j=1}^{n}[x_1^j, x_2^j]$ be a subinterval of J, $n > 1$. Fix some j and consider the sets

$$I_1 = \bigtimes_{k=1}^{n} [x_1^k, x_2^k] \quad \text{with } [x_1^j, x_2^j] = \{x_1^j\},$$

$$I_2 = \bigtimes_{k=1}^{n} [x_1^k, x_2^k] \quad \text{with } [x_1^j, x_2^j] = \{x_2^j\}.$$

Then I_1 is a subinterval in a hyperplane (a subset of J obtained by fixing a single coordinate) say H_1 and I_2 is a subinterval in a hyperplane H_2. Let $g_1 = g \mid H_1$, $g_2 = g \mid H_2$. Then

$$\Delta g(I) = \Delta g_2(I_2) - \Delta g_1(I_1). \tag{2-4}$$

We observe that in computing $\gamma_{i_1,\ldots,i_{n-1}}$ for vertices of I_1, we ignore the coordinate x_1^k, thinking of H_1 as $J_{(n-1)}$. When vertices of I_1 are thought of as vertices of I, then x_1^k has to be considered as a coordinate and in this case $\gamma_{i_1,\ldots,i_n}$ will have opposite sign to $\gamma_{i_1,\ldots,i_{n-1}}$ since in the sum $\Sigma_{j=1}^n i_j$ we have $i_k = 1$.

For a set $A \subset J_\theta$, we denote its interior by A^0 and its closure by $\bar{A}$, both with respect to relative topology on $\mathbf{R}_\theta^n$.

Given $t \in J^0$, there are 2^n subintervals having a corner at t such that their union is all of J. By a quadrant with respect to t we mean any one of these subintervals, containing only that portion of their boundary which is common to the boundary of J. Two quadrants of t will be given special name and symbol. The quadrant which has a corner at 0 will be called the left quadrant, denoted I_t^-, and the one with a corner at 1 will be referred to as the right quadrant, denoted I_t^+.

For a subinterval $I = [x_1, x_2]$ put

$$\|I\| = \max\{x_2^j - x_1^j \mid j = 1, 2, \ldots, n\}.$$

Definition (2-4). Let f be a function and $x \in J$. We shall say that the left limit of f at x, denoted $f(x-0)$, exists if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every subinterval $K \subset \bar{I}_x^-$ with $\|K\| < \delta$ and having a corner at x, we have

$$\forall y \in K^0: |f(y) - f(x-0)| < \varepsilon.$$

The 2^n quadrantal limits at a point $x \in J^0$ are defined similarly and the right limit is denoted $f(x+0)$.

If in the preceding definition we replace K^0 by K, the closed subinterval, then we talk about the strong quadrantal limit.

Definition (2-5). A function f is said to be left continuous at $x \in J^0$ if and only if the strong left limit of f at x exists and equals $f(x)$. The function is said to be continuous at x if and only if all the strong quadrantal limits at x exist and coincide with $f(x)$.

Definition (2-6). Let f be a function.

(a) The oscillation of f on a subinterval I, denoted $O(f; I)$, is defined as

$$O(f; I) = \sup\{|f(x) - f(y)| : x, y \in I\}.$$

(b) The oscillation of f at a point $x \in J^0$, denoted $O(f; x)$, is defined as

$$O(f; x) = \inf\{O(f; I) : x \in I^0\}.$$

Remark (2-1). Let f be a function and $x \in J^0$. Suppose $f(x-0)$ exists. Then, for every $\varepsilon > 0$, we can find a subinterval $K \subset \bar{I}_x^-$ with a corner at x such that for every subinterval $I \subset K^0$, we have $O(f; I) < \varepsilon$.

We note that, in the usual way, f will have a limit at a point x if and only if all the strong quadrantal limits of f at x exist and coincide. In this language, f is continuous at x if and only if its limit at x exists and coincides with $f(x)$. Furthermore, f will be continuous at $x \in J^0$ if and only if for every $\varepsilon > 0$ there exists a subinterval K centered at x such that $|f(y) - f(x)| < \varepsilon$, $y \in K^0$.

Finally, a function f is continuous at $x \in J^0$ if and only if $O(f; x) = 0$.

Let p be positively monotone increasing and $K \subset I$ subintervals of J. It is clear that K may be extended to a partition τ of I with $K \in \tau$. If we denote the subintervals in τ by H, then

$$\Delta p(I) = \Delta p(K) + \sum_{H \in \tau} \Delta p(H \neq K)$$

so that $\Delta p(K) \leqslant \Delta p(I)$.

PROPOSITION (2-1). *Let p be positively monotone increasing and $x \in J$. Then all the quadrantal limits of p at x exist.*

PROOF. We take $x \in J^0$ and prove the existence of $p(x-0)$. Clearly, $y < x$ implies $y \in I_x^-$. Let

$$\alpha \inf_{y<x} \Delta p([y, x]).$$

Consider any sequence $y_l \to x$, $y_l < x$ for $l = 1, 2, \ldots$. Given $\varepsilon > 0$ choose $K \subset \bar{I}_x^-$ such that K has a corner at x and $\Delta p(K) - \alpha < \varepsilon$. Since K is fixed and $y_l \to x$, for some l_0 we must have $l > l_0 \Rightarrow [y_l, x] \subset K$, and therefore $\Delta p([y, x]) \leqslant \Delta p(K)$. Hence

$$l > l_0 \Rightarrow |\Delta p([y_l, x]) - \alpha| < \varepsilon,$$

which means $\lim_{l\to\infty} \Delta p([y_l, x]) = \alpha$. For the sequence $\langle y_l \rangle$, fix all the coordinates of each of its terms except the jth coordinate. Then, as $l \to \infty$ we have $y_l^j \to x^j$ and $y_l^j < x^j$, $l = 1, 2, \ldots$. We know that

$$\lim_{l\to\infty} p(y_l^1, y_l^2, \ldots, y_l^j, \ldots, y_l^n)$$

exists for each $j = 1, 2, \ldots, n$, and is independent of the way y_l^j approaches x^j (the function is simply monotone increasing in the jth coordinate). It follows from the definition of Δp that $\lim_{y_l \to x} p(y_l)$ exists and is independent of the way y_l approaches x. But this is equivalent to the existence of $p(x-0)$.

PROPOSITION (2-2). *If p is positively monotone increasing then the points of discontinuity of p lie on a countable number of hyperplanes.*

PROOF. For each $x \in J^0$, all the quadrantal limits of p at x exist. Given $\varepsilon > 0$, we can choose 2^n subintervals according to Remark (2-1), one in each quadrant and with a corner at x. We then take K centered at x and contained in the union of the above 2^n subintervals. Then, for any $y \in K$, y lying in a quadrant, we have

$$O(p; y) < \varepsilon. \tag{2-5}$$

We now cover J with a finite number of such subintervals, say K_r, $r = 1, 2, \ldots, m$, centered at $x_r \in J$ (with obvious interpretation of K_r being centered at a boundary point x_r). It follows that (2-5) is satisfied by all the points which do not lie on the hyperplanes passing through x_r. Hence, the set of points $z \in J$ such that $O(p; z) \geqslant 1/l$, $l = 1, 2, \ldots$, is contained in the union of a finite number of hyperplanes. This means that the set of points in J at which the oscillation of p exceeds zero is contained in the union of a countable number of hyperplanes. See Remark (2-1).

PROPOSITION (2-3). *Let $\langle p_l \rangle$ be a sequence of positively monotone increasing functions in* **B**. *If the sequence is pointwise bounded on J, then there exists a subsequence of it which converges to a positively monotone increasing function, pointwise on J.*

PROOF. The proof is by induction on n, the dimension. For $n = 1$, this is Lemma 2 on p. 221 of [7]. Let $1 < n < \infty$ and suppose that the proposition is true for $1, 2, \ldots, n-1$. Now we proceed as follows.

Let E be the countable dense subset of J consisting of points all of whose coordinates are rational. Extract a subsequence $\langle p_{l_k} \rangle$ so that it converges on E and put $\lim p_{l_k} = p$ on E. Clearly, p is positively monotone increasing on E. For any point $t \in J - E$ define

$$p(t) = \sup p(x), \qquad x \in I_t \cap E.$$

We assert that p is positively monotone increasing. Let $I = \times_{j=1}^{n}[c^j, d^j]$ be any subinterval, $c < d$ points in J. Choose 2^n sequences in E, each converging to a corner of I and lying in the left quadrant of the corner to which they converge. Consider the corner c of I and let $\langle x_l \rangle$ be the sequence in E that converges to c. Given $\varepsilon > 0$, take $y \in E \cap I_c^-$ such that

$$p(c) - p(y) < \varepsilon \tag{2-6}$$

and choose l_0 such that $l > l_0 \Rightarrow y < x_l < c$. This means, for $l > l_0$ we have $I_y \subset I_{x_l}$ and hence ($p(c)$ is supremum),

$$l > l_0 \Rightarrow p(y) \leqslant p(x_l) \leqslant p(c),$$

which together with (2-6) implies $\lim_{l \to \infty} p(x_l) = p(c)$. Clearly, a similar result holds for all the corners of I. Let I_l be the subinterval having as its corners the points of the 2^n sequences (converging to the corners of I) for $l = 1, 2, \ldots$. It follows that

$$\Delta p(I) = \lim_{l \to \infty} \Delta p(I_l) \geqslant 0$$

and proves the assertion.

Let x_0 be a point of continuity of p. We assert that $p_{l_k}(x_0) \to p(x_0)$. Given $\eta > 0$, choose K centered at x_0 such that

$$\forall I \subset K^0 \colon O(p; I) < \eta/2. \tag{2-7}$$

Let $I = [x, y]$ be centered at x_0, $I \subset K^0$, and $x, y \in E$. Since $x < x_0 < y$, we find

$$p_{l_k}(x) \leqslant p_{l_k}(x_0) \leqslant p_{l_k}(y), \qquad k = 1, 2, \ldots. \tag{2-8}$$

(This follows from the fact that $I_x \subset I_{x_0} \subset I_y$.) Choose k_0 such that

$$k > k_0 \Rightarrow |p_{l_k}(x) - p(x)| < \eta/2 \quad \text{and} \quad |p_{l_k}(y) - p(y)| < \eta/2.$$

From (2-7) we have $|p(x) - p(x_0)| < \eta/2$ and $|p(y) - p(x_0)| < \eta/2$, which combined with preceding inequalities yields

$$k > k_0 \Rightarrow |p_{l_k}(x) - p(x_0)| < \eta \quad \text{and} \quad |p_{l_k}(y) - p(y_0)| < \eta.$$

The first inequality above and (2-8) give

$$k > k_0 \Rightarrow p_{l_k}(x_0) - p(x_0) \geqslant p_{l_k}(x) - p(x_0) > -\eta,$$

and similarly the second inequality gives

$$k > k_0 \Rightarrow p_{l_k}(x_0) - p(x_0) \leqslant p_{l_k}(y) - p(x_0) < \eta,$$

and the assertion follows.

Finally, the points of discontinuity of p lie on a countable number of hyperplanes $H_1, H_2, \ldots$. Since $p_{l_k} | H_1$ are positively monotone increasing for $k = 1, 2, \ldots$, we invoke the induction hypothesis to extract a subsequence of $\langle p_{l_k} \rangle$ say $\langle p_{m_1} \rangle$ which converges pointwise on H_1. Then, we extract a subsequence $\langle p_{m_2} \rangle$ of the sequence $\langle p_{m_1} \rangle$ which converges pointwise on H_2, and continue the process. The diagonal subsequence will then converge pointwise on J to a positively monotone increasing function.

One can replace the class of positively monotone increasing functions by functions in **B** in these propositions via an application of the Jordan decomposition. The consequent modification of Proposition (2-3) is [H_1].

When $n = 1$ and σ: $0 = x_0 < x_1 < \cdots < x_m = 1$ is a partition of J we define $\|\sigma\| = \max\{x_{k-1} - x_k\colon\ k = 1, 2, \ldots, m\}$, and for $1 \leqslant n < \infty$, we let $\|\sigma\| = \max\{\|\sigma_j\|\colon j = 1, 2, \ldots, n\}$. We say the partition σ is finer than τ, denoted $\tau \leqslant \sigma$, if and only if $\tau_j \leqslant \sigma_j$ for $j = 1, 2, \ldots, n$.

Let f, g be two functions and σ a partition. We define the sum of f with respect to g for σ by

$$S(f, g, \sigma) = \sum_{I \in \sigma} f(t_I) \Delta g(I),$$

where t_I is a point in I. The (Stieltjes) integral of f with respect to g, over J, is denoted

$$\int_J f\, dg; \tag{2-9}$$

we have not written J when the integral was understood to be over all of J. The following two definitions of integral will be considered.

THE REFINEMENT DEFINITION. We shall say that the refinement integral of f with respect to g exists if there exists a (real) number denoted by (2-9) such that for every $\varepsilon > 0$, there exists a partition σ with the property that for all $\tau \leqslant \sigma$, and independent of the choice of the t_I, we have

$$\left| S(f, g, \tau) - \int f\, dg \right| < \varepsilon. \tag{2-10}$$

THE NORM DEFINITION. We shall say that the norm integral of f with respect to g exists if there exists a number, also denoted by (2-9), such that for every $\varepsilon > 0$, we can find a $\delta > 0$, with the property that for all τ, with $\|\tau\| < \delta$, (2-10) is satisfied independent of the choice of t_I.

In (2-9), f is called the integrand and g the integrator. In consideration of integral, the integrand will always lie in **C** and the integrator will always lie in **B** in which case the integral exists in both senses defined above and the values coincide. This is a consequence of Theorem 6.8, p. 108 of **[10]** for $n = 1$, and Theorem 9.3, p. 129 of **[5]** for $n \geqslant 2$.

The integral is linear with respect to both the integrand and the integrator. The proof given for $n = 1$ in Theorems 9-2 and 9-3, p. 193 of [1], is valid for $1 \leqslant n < \infty$.

We have $|\int f\,dg| \leqslant \|f\|_\infty \cdot \|g\|$ since for any partition σ,

$$|S(f, g, \sigma)| = \left|\sum_{I \in \sigma} f(t_I) \cdot \Delta g(I)\right| \leqslant \|f\|_\infty \cdot \sum_{I \in \sigma} |\Delta g(I)| \leqslant \|f\|_\infty \cdot \|g\|.$$

From (2-2) it follows that if $f \in \mathbf{C}$ and $g \in \mathbf{B}$ and σ is a partition, then $\forall I \in \sigma$: $\int_I f\,dg$ exists. In fact, by Theorem 8.4, p. 126 of [5], we have

$$\int f\,dg = \sum_{I \in \sigma} \int_I f\,dg.$$

REMARK (2-2). Let $\theta \in S$, let f_θ be a continuous function on J_θ and let f be its standard extension. Then, for every $g \in \mathbf{B}$, $\int f\,dg = \int_{J_\theta} f_\theta\,dg_\theta$.

PROOF OF $[\mathrm{H}_2]$. $[\mathrm{H}_1]$ implies that $\|g\| \leqslant M$, so that $g \in \mathbf{B}$. Let f be any function in $\mathbf{C}$. By the uniform continuity of f on J, Theorem 7.3 on p. 180 of [6], for any $\varepsilon > 0$ we can find a $\delta > 0$ such that for every partition σ with $\|\sigma\| < \delta$, we have

$$\forall I \in \sigma, \forall t', t'' \in I: |f(t') - f(t'')| < \frac{\varepsilon}{4M}.$$

Let t_I be any point in $I \in \sigma$. We have

$$\begin{aligned}\int f\,dg &= \sum_{I \in \sigma} \int_I f\,dg = \sum_{I \in \sigma} \int_I [f - f(t_I) + g(t_I)]\,dg \\ &= \sum_{I \in \sigma} \int_I [f - f(t_I)]\,dg + \sum_{I \in \sigma} f(t_I) \int_I dg \\ &= \sum_{I \in \sigma} \int_I [f - f(t_I)]\,dg + \sum_{I \in \sigma} f(t_I) \cdot \Delta g(I).\end{aligned}$$

Keeping the partition σ fixed, a similar computation gives

$$\int f\,dg_l = \sum_{I \in \sigma} \int_I [f - f(t_I)]\,dg_l + \sum_{I \in \sigma} f(t_I) \cdot \Delta g_l(I), \qquad l = 1, 2, \ldots.$$

Let each σ_j have m_j linear subintervals, $j = 1, 2, \ldots, n$. Then σ has $m = m_1 \cdot m_2 \cdot \ldots \cdot m_n$ subintervals. From $g_l \to g$ it follows that $\Delta g_l(I) \to \Delta g(I)$ for each $I \in \sigma$. So, we can choose l_0 such that

$$l > l_0 \Rightarrow \forall I \in \sigma: |\Delta g_l(I) - \Delta g(I)| < \frac{\varepsilon}{2\|f\|_\infty \cdot m}.$$

Hence, for $l > l_0$ we find

$$\left|\sum_{I \in \sigma} f(t_I)\Delta g_l(I) - \sum_{I \in \sigma} f(t_I)\Delta g(I)\right| < \|f\|_\infty \cdot \sum_{I \in \sigma} |\Delta g_l(I) - \Delta g(I)| < \frac{\varepsilon}{2}.$$

Next, we take care of the sum involving integral. Thus,

$$\begin{aligned}\left|\sum_{I \in \sigma} \int_I [f - f(t_I)]\,dg_l - \sum_{I \in \sigma} \int_I [f - f(t_I)]\,dg\right| &= \left|\sum_{I \in \sigma} \int_I [f - f(t_I)]\,d(g_l - g)\right| \\ &\leqslant \frac{\varepsilon}{4M} \cdot \|g_n - g\| \leqslant \frac{\varepsilon}{2}.\end{aligned}$$

Therefore, by the triangle inequality

$$l > l_0 \Rightarrow \left| \int f\, dg_l - \int f\, dg \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $f \in \mathbf{C}$ was arbitrary, we have shown $g_l \xrightarrow{w} g$.

PROPOSITION (2-4). *Let Γ be the distribution function of $\mu \in \mathfrak{M}$. Then Γ_θ is right continuous on J_θ^0 for all $\theta \in S$.*

PROOF. It suffices to prove the assertion for a positive measure $\mu \in \mathfrak{M}$. Furthermore, for notational convenience we assume θ is the empty set, but the argument will work for all $\theta \in S$.

Consider any sequence of points $\langle x_l \rangle$ in I_x^+ such that $x_l \to x$ and $x_{l+1} < x_l$, $l = 1, 2, \ldots$. Clearly,

$$I_{x_{l+1}} \subset I_{x_l}, \qquad l = 1, 2, \ldots,$$

and

$$I_x = \bigcup_{l=1}^{\infty} I_{x_l}. \tag{2-11}$$

Proposition 14, p. 61 of [**8**] and (2-11) give $\mu(I_x) = \lim_{l \to \infty} \mu(I_{x_l})$, so that $\Gamma(x) = \lim_{l \to \infty} \Gamma(x_l)$. If Γ is not right continuous, we must have $\Gamma(x_l) \geqslant \Gamma(x + 0) > \Gamma(x)$ which contradicts the fact just established.

III. A proof of Theorem 2.

LEMMA (3-1). *Let $E \subset J^0$ with $\nu(E) = 0$. Then, for every $\delta > 0$, there exists a partition σ of J such that $\|\sigma\| < \delta$ and no points of σ (i.e., corners of subintervals in σ) are in E.*

PROOF. Let $n = 1$, τ: $0 = x_0 < \cdots < x_l = 1$ a partition of $J = [0, 1]$ such that every subinterval in τ has length equal to $\delta_1 < \delta$. Suppose some $x_i \in E$. Choose the points $x_i' < x_i < x_i''$ such that

$$x_i', x_i'' \notin E \quad \text{and} \quad x_i - x_i' < \frac{\delta_1}{2}, \quad x_i'' - x_i < \frac{\delta_1}{2}.$$

Then replace x_i by x_i', $x_i''i$. Continuing in this way we end up with a partition σ as desired.

Let the statement be true for $n - 1$, $n > 1$ and let J be n-dimensional. Let H_{x^n} be a hyperplane obtained by fixing the nth coordinate x^n and put $E_{x^n} = H_{x^n} \cap E$, i.e., E_{x^n} is a cross-section of E. Then E_{x^n} are measurable a.e. on $0 < x^n < 1$ and the function $\xi(x^n) = \nu_{(n-1)}(E_{x^n})$ is nonnegative on its domain with

$$\int_0^1 \xi \, d\nu_{(1)} = \nu(E) = 0.$$

It follows that $\xi = 0$, a.e. and hence $\nu_{(n-1)}(E_{x^n}) = 0$, a.e. on $0 < x^n < 1$. Choose a partition σ_n: $0 = x_0^n < \cdots < x_{m_n}^n = 1$ so that $\|\sigma_n\| < \delta$ and $\nu_{(n-1)}(E_{x_i^n}) = 0$ for $i = 1, 2, \ldots, m_n - 1$. Set

$$F = \bigcup_{i=1}^{m_n - 1} E_{x_i^n}$$

and we have $\nu_{(n-1)}(F) = 0$. Let $\theta = \{n\} \in S$. By the induction hypothesis there exists a partition τ of J_θ with norm less than δ which contains no points of $P_\theta(F)$; the intersection of P_θ^{-1} (points in τ) with hyperplanes $H_{x_i^n}$, $i = 0, 1, \ldots, m_n$, generates the points of a partition with required properties.

PROOF OF THEOREM 2. (i) $\Rightarrow$ (ii). Let $f \equiv 1$ on J. Then

$$g(\bar{1}) = \int dg = 0.$$

Recalling Remark (2-2), it suffices to consider the case where θ is the empty set.

Given $x \in J^0$, let $\langle x_l \rangle$ be a sequence in I_x^+, such that $x_l \to x$ and $x_{l+1} < x_l < 1$, $l = 1, 2, \ldots$. For each l, consider the closed sets

$$A_l^k = \bigtimes_{j=1}^{n} [\alpha_j, 1], \qquad k = 1, 2, \ldots, n,$$

with $\alpha_k = x_l^k$ (kth coordinate of x_l), and $\alpha_j = 0$ for $j \neq k$. Let

$$A_l = \bigcup_{k=1}^{n} A_l^k, \qquad l = 1, 2, \ldots.$$

For each l then we have two closed sets, I_x and A_l. By Urysohn Lemma, p. 207 of [**6**] for each l there exists an $f_l \in \mathbf{C}$ such that

$$f_l(I_x) \equiv 1, \quad f_l(A_l) \equiv 0, \quad \| f_l \|_\infty \leqslant 1, \qquad l = 1, 2, \ldots.$$

Now let $p \in \mathbf{B}$ be any positively monotone increasing functiion which is continuous at the point x. Let B_l be the closure of $J - (I_x \cup A_l)$ for $l = 1, 2, \ldots$. Then

$$\int f_l\, dp = \int_{I_x} f_l\, dp + \int_{B_l} f_l\, dp + \int_{A_l} f_l\, dp = p(x) + \int_{B_l} f_l\, dp, \qquad l = 1, 2, \ldots.$$

Since

$$\int_{B_l} f_l\, dp \leqslant \| p \|_{B_l} = p(x_l) - p(x), \qquad l = 1, 2, \ldots,$$

we obtain

$$p(x) \leqslant \int f_l\, dp \leqslant p(x_l), \qquad l = 1, 2, \ldots.$$

Since p is continuous at x, we find

$$\int f_l\, dp \to p(x) \quad \text{as } l \to \infty.$$

Now we write $g = \psi - \phi$, the Jordan decomposition. Let $x \in J$ be a point such that ψ, ϕ are both continuous at x. Construct the sequence $\langle f_l \rangle \subset \mathbf{C}$ as above. Then

$$0 = \int f_l\, dg = \int f_l\, d\psi - \int f_l\, d\phi \to \psi(x) - \phi(x) = g(x),$$

so that $g(x) = 0$. Suppose there is a point $y \in J^0$ with g continuous at y but ψ and ϕ both discontinuous there. Then, for every $\varepsilon > 0$, we can find a subinterval K, centered at y, such that

$$| g(z') - g(z'') | < \varepsilon, \qquad z', z'' \in K.$$

By Proposition (2-2), for some $z \in K$, ψ and ϕ are both continuous at z so that $g(z) = 0$. It follows that, for every $\varepsilon > 0$, $|g(z) - g(y)| = |g(y)| < \varepsilon$, and hence $g(y) = 0$. This completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). This is an immediate consequence of Proposition (2-2).

(iii) $\Rightarrow$ (i). Let $n = 1$. Given $\varepsilon > 0$ for every $f \in \mathbf{C}$ there is a $\delta > 0$ such that for all partitions σ with $\|\sigma\| < \delta$, we have $|S(f, g, \sigma) - \int f\,dg| < \varepsilon$. By Lemma (3-1) we may choose σ such that g vanishes at all the points of σ. Hence $S(f, g, \sigma) = 0$ and it follows that $|\int f\,dg| < \varepsilon$. Assume now the statement holds for all $k < n$, $n > 1$. Let $\theta_j\{j\}$, $1 \leq j \leq n$. Let h_1 denote the standard extension of $f_{\theta_1} = f | J_{\theta_1}$. By the induction hypothesis and Remark (2-2), we have

$$\int h_1\,dg = \int_{J_{\theta_1}} (h_1)_{\theta_1}\,dg_{\theta_1} = 0.$$

Let $f_1 = f - h_1$; then $(f_1)_{\theta_1} \equiv 0$ and $\int f_1\,dg = \int f\,dg$. Iterating this process, let h_2 denote the standard extension of $(f_1)_{\theta_2}$ and set $f_2 = f_1 - h_2$, so that $(f_2)_{\theta_j} \equiv 0$, $j = 1, 2$, and $\int f_2\,dg = \int f\,dg$. After n iterations we end up with $f_n \in \mathbf{C}$ such that $(f_n)_{\theta_j} \equiv 0$, $1 \leq j \leq n$, and $\int f_n\,dg = \int f\,dg$. The proof will be complete when we show the left side of the preceding equation vanishes. Let $\varepsilon > 0$ be given. Since f_n is uniformly continuous, we can find a $\delta > 0$ such that for every subinterval I with $\|I\| < \delta$, we have

$$x_1, x_2 \in I \Rightarrow |f_n(x_1) - f_n(x_2)| < \frac{\varepsilon}{n \cdot \|g\|}.$$

By Lemma (3-1) we can choose a partition σ with $\|\sigma\| < \delta$ and such that g vanishes at all the points of σ in J^0. Consider any one of the linear partitions comprising σ, say σ_1: $0 = x_1^1 < x_2^1 < \cdots < x_{m_1}^1 = 1$. Let $I_1 = \times_{j=1}^n [z_1^j, z_2^j]$ where $[z_1^1, z_2^1] = [x_{m_1-1}^1, 1]$ and $[z_1^j, z_2^j] = [0, 1]$ for $2 \leq j \leq n$. Then, for every $y \in I_1$ we have $|f_n(y)| < \varepsilon/n \cdot \|g\|$. Choose the subintervals $I_2, \ldots, I_n$ similar to I_1. Then

$$\left|\int f_n\,dg\right| \leq \sum_{j=1}^{n} \left|\int_{I_j} f_n\,dg\right| < n \cdot \frac{\varepsilon}{n \cdot \|g\|} \cdot \|g\| = \varepsilon.$$

COROLLARY (3-1). *The integral of every $f \in \mathbf{C}$ with respect to some $g \in \mathbf{B}$ vanishes if and only if $g(1) = 0$ and for every $\theta \in S$ all the quadrantal limits of g_θ vanish.*

PROOF. Clearly, the same argument works for all $\theta \in S$ and for convenience we take θ to be the empty set.

Suppose $\int f\,dg = 0$, $f \in \mathbf{C}$. Then $g(1) = 0$. Without loss of generality, let $0 < x$ be any point in J and we shall only show $g(x - 0) = 0$. Given $\varepsilon > 0$, choose $K \subset I_x^-$ such that K has a corner at x and

$$|g(y) - g(x - 0)| < \varepsilon, \qquad y \in K^0.$$

Choose $z \in K^0$ with g continuous there, so that $g(z) = 0$. This gives

$$\forall \varepsilon > 0: |g(x - 0)| < \varepsilon \Rightarrow g(x - 0) = 0.$$

The converse is obvious.

Let $\tilde{\Gamma} = \{g \in \mathbf{B} \mid g_\theta$ is right continuous on $J_\theta^0, \theta \in S\}$. By Proposition (2-4), $\tilde{\Gamma}$ contains all the distribution functions of elements in $\mathfrak{M}$. Coversely, every element of $\tilde{\Gamma}$ is a distribution function because by the preceding corollary we have

COROLLARY (3-2). *Let* $\Gamma_1, \Gamma_2 \in \tilde{\Gamma}$. *Then, for every* $f \in \mathbf{C}$,

$$\int f\,d\Gamma_1 = \int f\,d\Gamma_2 \Leftrightarrow \Gamma_1 \equiv \Gamma_2 \quad on\ J.$$

So, given a sequence $\langle \mu_l \rangle$ in $\mathfrak{M}$ the corresponding sequence $\langle \Gamma_l \rangle$ in Γ is unique, but infinitely many sequences $\langle g_l \rangle$ in $\mathbf{B}$ correspond to $\langle \mu_l \rangle$ such that $\mu_l \overset{w}{\to} 0$ if and only if $g_l \overset{w}{\to} 0$ where we assume each μ_l corresponds to g_l, $l = 1, 2, \ldots$. However, $\|\mu_l\| < M$ does not even imply that the sequence $\langle g_l \rangle$ is bounded. For instance, let $J = [0, 1]$, and for the sequence $\|\mu_l\| = 0$ choose $\langle g_l \rangle$ as follows:

$$g_l(x) = 0, \quad x \neq \tfrac{1}{2}, \qquad g_l(\tfrac{1}{2}) = l.$$

We have the following generalization of Theorem 1.

THEOREM 3. *Let* $\|\mu_l\| \leqslant M$. *Then* $\mu_l \overset{w}{\to} 0$ *if and only if for every corresponding sequence* $\langle g_l \rangle$ *in* $\mathbf{B}$ *we have*

(i) $g_l(\bar{1}) \to 0$, *as* $l \to \infty$;

(ii) $\forall \theta \in S$: $\int_{J_\theta} |g_l| \, d\nu_\theta \to 0$, *as* $l \to \infty$.

PROOF. Let $\langle \Gamma_l \rangle$ be the corresponding sequence of distribution functions. Then

$$\int f\,d(g_l - \Gamma_l) = 0, \qquad f \in \mathbf{C}, \tag{3-1}$$

so that by Theorem 2, $\forall \theta \in S$: $g_l = \Gamma_l$, a.e. on J_θ, $l = 1, 2, \ldots$. Hence,

$$\forall \theta \in S: \int_{J_\theta} |g_l| = \int_{J_\theta} |\Gamma_l|, \qquad l = 1, 2, \ldots.$$

Moreover, by letting $f \equiv 1$ on J in (3-1), we find $g_l(\bar{1}) \to 0$ if and only if $\Gamma_l(\bar{1}) \to 0$. Therefore, the theorem follows by Theorem 1.

The preceding theorem may be rephrased as: a sequence $\langle g_l \rangle$ in $\mathbf{B}$ converges weak-star to zero if and only if $\langle g_l \rangle$ satisfies conditions (i) and (ii) in the theorem and $\|L_l\| \leqslant M$ where $\langle L_l \rangle$ is the corresponding sequence in $\mathbf{C}^*$.

IV. The infinite dimensional case. Let Λ be an infinite index set and let $J = \times_{\alpha \in \Lambda} [0, 1]_\alpha$. Put the product topology on J and obtain a compact Hausdorff space. Let $\mathfrak{F}$ denote the set of finite subsets of Λ. For $F \in \mathfrak{F}$ and $x = \{x_\alpha\} \in J$, let $P_F(x) = \{y_\alpha\}$, where $y_\alpha = x_\alpha$, $\alpha \in F$, and $y_\alpha = 1$, $\alpha \notin F$. For $F \in \mathfrak{F}$, let C_F denote the subspace of $C(J)$ comprised of the functions $f \in C(J)$ with the property that $f(x) = f(y)$ whenever $P_F(x) = P_F(y)$. Let $C_{\mathfrak{F}} = \cup_{F \in \mathfrak{F}} C_F$. Then $C_{\mathfrak{F}}$ is a subalgebra of $C(J)$ that separates points and contains the constant functions, so $C_{\mathfrak{F}}$ is dense in $C(J)$. The Riesz representation theorem tells us that the dual of $C(J)$ is isomorphic and isometric to the space $\mathfrak{B}(J)$ of real-valued, regular Borel measures on J. A bounded sequence, $\langle \mu_n \rangle$, in $\mathfrak{B}(J)$ converges weakly to zero if and only if $\int f\,d\mu_n \to 0$ for each $f \in C_{\mathfrak{F}}$. For $\phi \neq F \in \mathfrak{F}$, let $J_F = \{(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n});\ F = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}\}$.

For $x \in J_F$, let $\Pi_F(x) = \{y_\alpha\}$, where $y_\alpha = x_\alpha$, $\alpha \in F$, and $y_\alpha = 1$, $\alpha \notin F$. For $\mu \in \mathfrak{B}(J)$, define μ^F on the Borel subsets E of J_F by $\mu^F(E) = \mu(P_F^{-1}(\Pi_F(E)))$. Notice that J_F is isomorphic to the card(F)-cube. Define the finite dimensional distribution Γ^F on J^F by $\Gamma^F(x) = \mu^F(\{y \in J_F;\ y \leqslant x\})$, $x > 0$, $\Gamma^F(x) = 0$, $x \not> 0$. Since $\int_J f\,d\mu = \int_{J_F} f_F\,d\mu^F$ when $f \in C_F$ and $f_F(x) = f(\Pi_F(x))$, we have the following characterization of weak-star convergence to zero.

THEOREM 4. *Let $\langle \mu_n \rangle$ be a bounded sequence of regular Borel measures on J. Then $\mu_n \overset{w}{\to} 0$ if and only if*

(i) *$\mu_n(J) \to 0$ and*

(ii) *for $\phi \neq F \in \mathfrak{F}$, $\int_{J^F} |(\Gamma_n)^F|\, dm_{\mathrm{card}(F)} \to 0$, where $m_{\mathrm{card}(F)}$ denotes* card(F) *dimensional Lebesgue measure on J_F.*

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