

ORTHOGONAL GEODESIC AND MINIMAL DISTRIBUTIONS

BY

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ABSTRACT. Let \mathfrak{F} be a smooth distribution on a Riemannian manifold M with \mathfrak{G} the orthogonal distribution. We say that \mathfrak{F} is geodesic provided \mathfrak{F} is integrable with leaves which are totally geodesic submanifolds of M . The notion of minimality of a submanifold of M may be defined in terms of a criterion involving any orthonormal frame field tangent to the given submanifold. If this criterion is satisfied by any orthonormal frame field tangent to \mathfrak{G} then we say \mathfrak{G} is minimal. Suppose that \mathfrak{F} and \mathfrak{G} are orthogonal geodesic and minimal distributions on a submanifold of Euclidean space. Then each leaf of \mathfrak{F} is also a submanifold of Euclidean space with mean curvature normal vector field η . We show that the integral of $|\eta|^2$ over M is bounded below by an intrinsic constant and give necessary and sufficient conditions for equality to hold.

We study the relationships between the geometry of M and the integrability of \mathfrak{G} . For example, if \mathfrak{F} and \mathfrak{G} are orthogonal geodesic and minimal distributions on a space of nonnegative sectional curvature then \mathfrak{G} is integrable iff \mathfrak{F} and \mathfrak{G} are parallel distributions. Similarly if M^n has constant negative sectional curvature and $\dim \mathfrak{G} = 2 < n$ then \mathfrak{G} is not integrable. If \mathfrak{F} is geodesic and \mathfrak{G} is integrable then we characterize the local structure of the Riemannian metric in the case that the leaves of \mathfrak{G} are flat submanifolds of M with parallel second fundamental form.

1. Introduction. Let $(M, \langle \cdot, \cdot \rangle)$ denote an n -dimensional Riemannian manifold with ∇ the Riemannian connection on the full tensor algebra of M . If S is a tensor of type (k, k) on M then we say $S \in \Gamma[\text{End } \Lambda^k(TM)]$ provided S is alternating in the first k and in the last k indices. At each point $x \in M$, S_x may be identified with an element of $\text{End } \Lambda^k T_x M$. Given $S \in \Gamma[\text{End } \Lambda^k(TM)]$ and $T \in \Gamma[\text{End } \Lambda^j(TM)]$ we may define $S * T \in \Gamma[\text{End } \Lambda^{k+j}(TM)]$ by wedging the respective covariant and contravariant components of S and T . The multiplication, $*$, is associative and commutative.

DEFINITION 1.1. $S \in \Gamma[\text{End } \Lambda^k(TM)]$ is a Codazzi tensor of type (k, k) provided

$$(1) \quad 0 = \sum (-1)^{j+1} (\nabla_{X_j} S) (X_1 \wedge X_2 \wedge \cdots \wedge X_{j-1} \wedge X_{j+1} \wedge \cdots \wedge X_{k+1})$$

for all smooth vector fields X_1, X_2, \dots, X_{k+1} on M .

It is routine to show that if S and T are Codazzi tensors of types (k, k) and (j, j) respectively then $S * T$ is a Codazzi tensor of type $(k + j, k + j)$.

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Let \mathfrak{F} denote a smooth codimension k distribution on M . The distribution orthogonal to \mathfrak{F} will be denoted by \mathfrak{G} and we let $P: TM \rightarrow \mathfrak{G}$ denote orthogonal projection onto \mathfrak{G} . If $\{e_1, e_2, \dots, e_k\}$ is an arbitrary local orthonormal frame field for \mathfrak{F} then we say \mathfrak{F} is minimal provided $P[\sum \nabla_{e_i} e_i] = \sum \nabla_{e_i} e_i$. In particular, if \mathfrak{F} is integrable then each leaf of \mathfrak{F} is a minimal submanifold of M . The distribution \mathfrak{F} is said to be parallel provided \mathfrak{F} is invariant under parallel translation along any path in M . Clearly if \mathfrak{F} is parallel then so is \mathfrak{G} . Equivalently \mathfrak{F} is parallel iff both \mathfrak{F} and \mathfrak{G} are integrable with totally geodesic leaves. In fact, if \mathfrak{F} is parallel then M splits locally as a Riemannian product $L_1 \times L_2$ with $\mathfrak{F} = TL_1$ and $\mathfrak{G} = TL_2$.

Suppose M is a compact oriented Riemannian manifold without boundary isometrically immersed into \mathbf{R}^N and \mathfrak{F} is a parallel distribution on M . Then each leaf of \mathfrak{F} is a submanifold of \mathbf{R}^N and given $p \in M$ we let $\eta(p)$ denote the mean curvature normal vector at p for the corresponding immersed leaf of \mathfrak{F} . In [1] we showed that

$$(2) \quad \int_M |\eta|^2 dV \geq \frac{\lambda_1 \text{vol } M}{n}$$

where λ_1 denotes the first eigenvalue of the Laplacian on M . Equality holds iff both M and the leaves of \mathfrak{F} are minimally immersed into a sphere of radius $R = (n/\lambda_1)^{1/2}$. This result was originally proved by R. Reilly [9] for the case $\mathfrak{F} = TM$. Its extension to arbitrary parallel distributions \mathfrak{F} uses the fact that the leaves of \mathfrak{F} are minimal submanifolds of M and that $S = P^k = P * P * \dots * P$ (k times) is a Codazzi tensor of type (k, k) . For arbitrary \mathfrak{F} and \mathfrak{G} we have the following result (see §2 for proof):

PROPOSITION 2.1. *$S = P^k$ is a Codazzi tensor of type (k, k) iff \mathfrak{F} is integrable with totally geodesic leaves and \mathfrak{G} is minimal.*

We will say \mathfrak{F} is geodesic provided \mathfrak{F} is integrable with totally geodesic leaves. Since each leaf of a geodesic distribution is a minimal submanifold of M we may extend inequality (2) to geodesic \mathfrak{F} provided \mathfrak{G} is minimal. For M and η as above, we have

THEOREM 2.2. *Suppose $\mathfrak{F} \neq 0$ is geodesic and \mathfrak{G} is minimal. Then*

$$\int_M |\eta|^2 dV \geq \frac{\lambda_1 \text{vol } M}{n}.$$

Equality holds iff both M and the leaves of \mathfrak{F} are minimally immersed into a sphere of radius $R = (n/\lambda_1)^{1/2}$.

The above results motivate the consideration of orthogonal geodesic and minimal distributions \mathfrak{F} and \mathfrak{G} . If, in addition, \mathfrak{G} is integrable then we will refer to \mathfrak{F} and \mathfrak{G} as orthogonal geodesic and minimal foliations of M . Parallel distributions provide examples of orthogonal geodesic and minimal foliations which are in some sense “trivial”. We will be interested in geometrical conditions which imply that this is the only type that can occur. Since the square norm of the second fundamental form of

the leaves of \mathfrak{F} may be written as a negative sum of sectional curvatures (Lemma 3.1), we have

THEOREM 3.2. *Let \mathfrak{F} and \mathfrak{G} be orthogonal geodesic and minimal distributions on a space M of nonnegative sectional curvature. Then \mathfrak{G} is integrable iff \mathfrak{F} and \mathfrak{G} are parallel.*

Other examples of orthogonal geodesic and minimal distributions may be constructed using a Riemannian submersion. Let M and N be Riemannian manifolds with metrics $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_N$ respectively. Suppose $\pi: M \rightarrow N$ is a submersion and let \mathfrak{F} denote the tangent space to the fibres of π with \mathfrak{G} the orthogonal distribution. The mapping π is said to be a Riemannian submersion provided that $\langle X, Y \rangle_M = \langle \pi_* X, \pi_* Y \rangle_N$ for all $X, Y \in \mathfrak{F}$. If X is a vector field tangent to \mathfrak{G} then $\nabla_X X$ is also tangent to \mathfrak{G} [8]. In particular, if the fibres of π are totally geodesic it follows that \mathfrak{F} and \mathfrak{G} are orthogonal geodesic and minimal distributions.

EXAMPLES. (a) With the standard metrics, $\pi: S^{2n+1} \rightarrow \mathbb{CP}^n$ is a Riemannian submersion, the fibres of which are geodesics of S^{2n+1} [8]. Since S^{2n+1} does not split as a Riemannian product it follows from Theorem 3.2 that \mathfrak{G} is not integrable. Note also that since $\lambda_1(S^n) = n$, we have equality in Theorem 2.2.

(b) Let G be a Lie group with a bi-invariant metric and let K be a closed subgroup of G . Then the natural metric on G/K makes the projection $\pi: G \rightarrow G/K$ into a Riemannian submersion with totally geodesic fibres [8].

(c) Let $\pi: P \rightarrow M$ be a principal G bundle with $\langle \cdot, \cdot \rangle_G$ an inner product on the Lie algebra of G and ω a connection 1-form on P . Let \mathfrak{F} denote the tangent space to the fibres of π with $\mathfrak{G} = \text{Ker } \omega$. Define a metric $\langle \cdot, \cdot \rangle$ on P by the conditions that \mathfrak{F} and \mathfrak{G} be orthogonal with $\langle X, Y \rangle = \langle \omega(X), \omega(Y) \rangle_G$ for $X, Y \in \mathfrak{F}$ and $\langle X, Y \rangle = \langle \pi_* X, \pi_* Y \rangle_M$ for $X, Y \in \mathfrak{G}$. Then π is a Riemannian submersion with totally geodesic fibres [4, 6].

We may also construct orthogonal geodesic and minimal foliations by means of a Riemannian submersion. Again let $\pi: M \rightarrow N$ be a Riemannian submersion, but now with \mathfrak{F} the tangent space to the fibres of π and \mathfrak{G} the distribution orthogonal to \mathfrak{F} . Then \mathfrak{F} is geodesic and \mathfrak{G} is minimal iff the fibres of π are minimal submanifolds of M , and \mathfrak{F} is integrable. In fact, this construction characterizes locally orthogonal geodesic and minimal foliations of M . To see this, let \mathfrak{F} and \mathfrak{G} be orthogonal foliations of M with $(x^1, x^2, \dots, x^n): U \rightarrow \mathbb{R}^n$ a coordinate system which is adapted to \mathfrak{F} and \mathfrak{G} in the sense that $\{\partial/\partial x_1, \dots, \partial/\partial x_k\}$ span \mathfrak{G} and $\{\partial/\partial x_{k+1}, \dots, \partial/\partial x_n\}$ span \mathfrak{F} . With respect to this coordinate system the metric may be written

$$ds^2 = \sum g_{ij}(x^1, \dots, x^n) dx^i \otimes dx^j + \sum g_{\alpha\beta}(x^1, \dots, x^n) dx^\alpha \otimes dx^\beta$$

where Roman indices run from 1 to k and Greek indices run from $k+1$ to n . A routine argument shows that \mathfrak{F} is geodesic iff each $g_{\alpha\beta}$ is a function of x^{k+1}, \dots, x^n alone. Assume then that \mathfrak{F} is geodesic. If we identify U with its image in \mathbb{R}^n then the leaves of \mathfrak{G} are given as the fibres of $\pi: U \rightarrow \mathbb{R}^{n-k}$ where $\pi(x^1, x^2, \dots, x^n) = (x^{k+1}, \dots, x^n)$. Define a Riemannian metric dt^2 on the image of π by the formula $dt^2 = \sum g_{\alpha\beta}(x^{k+1}, \dots, x^n) dx^\alpha \otimes dx^\beta$. With this metric, π becomes a Riemannian

submersion onto its image. The fibres of π are minimal submanifolds of M iff $\det(g_{ij})$ is independent of x^{k+1}, \dots, x^n . Two extreme cases may occur. One possibility is that each g_{ij} is independent of x^{k+1}, \dots, x^n . In this case \mathfrak{F} and \mathfrak{G} are parallel and ds^2 splits as a product metric. The other possibility is that each g_{ij} is a function of x^{k+1}, \dots, x^n alone and $\det(g_{ij})$ is constant.

THEOREM 3.3 *Suppose \mathfrak{F} is geodesic and \mathfrak{G} is integrable. Then about any point of M there exists an adapted coordinate system such that*

$$ds^2 = \sum g_{ij}(x^{k+1}, \dots, x^n) dx^i \otimes dx^j + \sum g_{\alpha\beta}(x^{k+1}, \dots, x^n) dx^\alpha \otimes dx^\beta$$

iff each leaf of \mathfrak{G} is a flat submanifold of M with parallel second fundamental form.

Using this result we consider orthogonal geodesic and minimal foliations with $\dim \mathfrak{G} = 2$.

PROPOSITION 3.4. *Let \mathfrak{F} and \mathfrak{G} be orthogonal geodesic and minimal foliations of M with $\dim \mathfrak{G} = 2$. If each leaf of \mathfrak{G} has parallel second fundamental form, then in a neighborhood of any point of M there exists an adapted coordinate system such that either*

$$(i) \, ds^2 = \sum g_{ij}(x^1, x^2) dx^i \otimes dx^j + \sum g_{\alpha\beta}(x^3, \dots, x^n) dx^\alpha \otimes dx^\beta \text{ or}$$

$$(ii) \, ds^2 = \sum g_{ij}(x^3, \dots, x^n) dx^i \otimes dx^j + \sum g_{\alpha\beta}(x^3, \dots, x^n) dx^\alpha \otimes dx^\beta \text{ with } \det(g_{ij}) \text{ constant.}$$

Furthermore, if M is compact and connected, the choice may be made unambiguously between (i) and (ii) for all of M .

Assume \mathfrak{G} is integrable and $\dim \mathfrak{G} = 2$. In order for the leaves of \mathfrak{G} to have parallel second fundamental form it is necessary that $\tilde{R}(X, Y)Z \in \mathfrak{G}_p$ whenever $X, Y, Z \in \mathfrak{G}_p$ where \tilde{R} denotes the curvature tensor of M . Given $p \in M$ let $\tilde{K}(p)$ denote the extrinsic sectional curvature of \mathfrak{G}_p and let $K(p)$ denote the intrinsic sectional curvature of the corresponding leaf of \mathfrak{G} .

PROPOSITION 3.5. *Let \mathfrak{F} and \mathfrak{G} be orthogonal geodesic and minimal foliations of M with $\dim \mathfrak{G} = 2$ and with $\tilde{R}(X, Y)Z \in \mathfrak{G}_p$ whenever $X, Y, Z \in \mathfrak{G}_p$. Suppose one of the following is true:*

- (a) $K \geq 0$ and each leaf of \mathfrak{G} is compact;
- (b) $\dim M = 3$ and $\tilde{K} - K$ is constant on each leaf of \mathfrak{G} ;
- (c) each leaf of \mathfrak{G} is diffeomorphic to S^2 .

Then in a neighborhood of each point of M the metric may be written as in (i) or (ii) of Proposition 3.4. Furthermore, if condition (c) holds, the metric may be written as in (i).

The condition that $\tilde{R}(X, Y)Z \in \mathfrak{G}$ whenever $X, Y, Z \in \mathfrak{G}$ is automatically satisfied provided M has constant sectional curvature. In fact, for such manifolds we may strengthen Proposition 3.5.

THEOREM 3.6. *Let $\mathfrak{F} \neq 0$ and \mathfrak{G} be orthogonal geodesic and minimal distributions on a space M of constant sectional curvature. If $\dim \mathfrak{G} = 2$ then \mathfrak{G} is integrable iff M is flat and \mathfrak{F} and \mathfrak{G} are parallel.*

2. The relationship with Codazzi tensors. Let \mathfrak{F} be a codimension k distribution on a Riemannian manifold M with \mathfrak{S} the orthogonal distribution and $P: TM \rightarrow \mathfrak{S}$ the orthogonal projection. A local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ will be called adapted provided $e_1, e_2, \dots, e_k \in \mathfrak{S}$.

PROPOSITION 2.1. $S = P^k$ is a Codazzi tensor of type (k, k) iff \mathfrak{F} is integrable with totally geodesic leaves and \mathfrak{S} is minimal.

PROOF. Assume that S is a Codazzi tensor of type (k, k) and let $\{e_1, e_2, \dots, e_n\}$ be an adapted local orthonormal frame field. Equation (1) for $X_j = e_j, j = 1, 2, \dots, k$, and $X_{k+1} = e_\alpha, \alpha = k + 1, \dots, n$, becomes

$$0 = \left\langle \sum \nabla_{e_j} e_j, e_\alpha \right\rangle e_1 \wedge e_2 \wedge \dots \wedge e_k - \sum e_1 \wedge e_2 \wedge \dots \wedge \nabla_{e_\alpha} e_j \wedge \dots \wedge e_k$$

where, in each term, the summation is over $j = 1$ to $j = k$. Since α is arbitrary, the vanishing of the first term implies \mathfrak{S} is minimal. The second term may be written as

$$\sum (-1)^{k-j} \left\langle \nabla_{e_\alpha} e_\beta, e_j \right\rangle e_1 \wedge \dots \wedge e_{j-1} \wedge e_{j+1} \wedge \dots \wedge e_k \wedge e_\beta$$

where the summation is over $j = 1$ to $j = k$ and $\beta = k + 1$ to $\beta = n$. Again, since α is arbitrary, the vanishing of the second term implies \mathfrak{F} is geodesic.

Conversely, if \mathfrak{F} is geodesic and \mathfrak{S} is minimal it is routine to show that S is a Codazzi tensor of type (k, k) . \square

Suppose that M is a compact oriented Riemannian manifold without boundary isometrically immersed into \mathbb{R}^N . Let Y denote the position vector field of M with II the second fundamental form of the immersion. Define a symmetric tensor A of type $(1, 1)$ on M by the equation $\langle A(X), Z \rangle = \langle \text{II}(X, Z), Y \rangle$, where on the right-hand side \langle, \rangle denotes the natural inner product on \mathbb{R}^N . If S is a Codazzi tensor of type (k, k) on M then [1]

$$(3) \quad \int_M (n - k) \text{trace } S + \text{trace}(S * A) dV = 0.$$

If \mathfrak{F} is an integrable distribution on M then each leaf of \mathfrak{F} is an immersed submanifold of \mathbb{R}^N . Given $p \in M$ let $\eta(p)$ denote the mean curvature normal vector at p for the corresponding immersed leaf of \mathfrak{F} . Applying the argument given in [1] for parallel distributions, we have

THEOREM 2.2. Suppose $\mathfrak{F} \neq 0$ is geodesic and \mathfrak{S} is minimal. Then

$$(4) \quad \int_M |\eta|^2 dV \geq \frac{\lambda_1 \text{vol } M}{n}.$$

Equality holds iff both M and the leaves of \mathfrak{F} are minimally immersed into a sphere of radius $R = (n/\lambda_1)^{1/2}$.

PROOF. Letting $S = P^k$ in equation (3) yields

$$(5) \quad \int_M \langle \eta, Y \rangle dV = -\text{vol } M$$

since $\text{trace } S = k!$ and $\text{trace}(S * A) = (n - k)k! \langle \eta, Y \rangle$. The Cauchy-Schwarz inequalities for \langle, \rangle and for integrals imply

$$\left(\int_M |\eta|^2 dV \right) \left(\int_M |Y|^2 dV \right) \geq (\text{vol } M)^2$$

with equality iff $\eta = cY$ for some constant c . Assume without loss of generality that $\int_M Y dV = \vec{0}$. For such an immersion, Reilly [9] showed that $\int_M |Y|^2 dV \leq (n \text{vol } M)/\lambda_1$ with equality iff M is minimally immersed into a sphere about the origin of radius $R = (n/\lambda_1)^{1/2}$. This gives inequality (4). If equality holds in (4) then M is minimally immersed into a sphere about the origin of radius $R = (n/\lambda_1)^{1/2}$ and $\eta = cY$, which implies each leaf of \mathfrak{F} is a minimal submanifold of the given sphere. On the other hand, if each leaf of \mathfrak{F} is minimally immersed into a sphere of radius $R = (n/\lambda_1)^{1/2}$ then $\eta = -(\lambda_1/n)^{1/2}N$ where N denotes the outward unit normal to the sphere. But then we clearly have equality in (4). \square

Under the hypotheses of Theorem 2.2 assume that \mathfrak{F} is integrable. Then each leaf of \mathfrak{F} is a submanifold of \mathbf{R}^n with mean curvature normal vector β . If H denotes the mean curvature normal vector of M as a submanifold of \mathbf{R}^n then we have $H = \frac{1}{n}[(n - k)\eta + k\beta]$. Since equation (5) is true with η replaced by H , it follows that $\int_M \langle \beta, Y \rangle dV = -\text{vol } M$ if $k \neq 0$. The argument of Theorem 2.2 then implies

$$(6) \quad \int_M |\beta|^2 dV \geq \frac{\lambda_1 \text{vol } M}{n}$$

with equality iff both M and the leaves of \mathfrak{F} are minimally immersed into a sphere of radius $R = (n/\lambda_1)^{1/2}$. Furthermore, equality in (6) is equivalent to equality in (4).

3. The integrability of \mathfrak{F} . Let \mathfrak{F} be a codimension k geodesic distribution on M with \mathfrak{F} the orthogonal distribution and $\{e_1, e_2, \dots, e_n\}$ an adapted local orthonormal frame field. Denote the dual coframing of M by $\{\theta^1, \theta^2, \dots, \theta^n\}$. If α is a q -form defined on the domain of $\{\theta^1, \theta^2, \dots, \theta^n\}$ then we write $\alpha \equiv 0 \pmod{\mathfrak{F}}$ (resp. $\pmod{\mathfrak{F}}$) provided α belongs to the exterior ideal generated by $\{\theta^1, \theta^2, \dots, \theta^k\}$ (resp. $\{\theta^{k+1}, \dots, \theta^n\}$). Define the connection 1-forms $\omega_j^i = -\omega_i^j$ by the equation $\omega_j^i(X) = \langle \nabla_X e_i, e_j \rangle$. The *first structural equation* of M then takes the form

$$(7) \quad d\theta^i = \sum \omega_j^i \wedge \theta^j.$$

The curvature 2-forms $\Omega_j^i = -\Omega_i^j$ are defined by the *second structural equation* of M ,

$$(8) \quad d\omega_j^i = \sum \omega_k^i \wedge \omega_k^j + \Omega_j^i.$$

If $R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$ denotes the curvature tensor of M then $\langle R(X, Y)e_i, e_j \rangle = \Omega_j^i(X, Y)$ and thus the sectional curvature of the 2-plane spanned by $\{e_i, e_j\}$ is $\langle R(e_i, e_j)e_i, e_j \rangle = -\Omega_j^i(e_i, e_j)$. Since \mathfrak{F} is geodesic

$$(9) \quad \omega_\alpha^i, d\omega_\alpha^i \equiv 0 \pmod{\mathfrak{F}}$$

where, for the remainder of this section, we adopt the convention that Roman indices run from 1 to k and Greek indices run from $k + 1$ to n .

Suppose now that \mathfrak{F} is integrable. With respect to any adapted coordinate system $(x^1, x^2, \dots, x^n): U \rightarrow \mathbf{R}^n$, the metric may be written as

$$ds^2 = \sum g_{ij}(x^1, \dots, x^n) dx^i \otimes dx^j + \sum g_{\alpha\beta}(x^{k+1}, \dots, x^n) dx^\alpha \otimes dx^\beta.$$

We strengthen the notion of an adapted local orthonormal frame field by requiring the domain of $\{e_1, e_2, \dots, e_n\}$ to belong to such a neighborhood U and by also requiring that $e_\alpha = \sum a_\alpha^\beta(x^{k+1}, \dots, x^n)(\partial/\partial x_\beta)$. It immediately follows that $[e_i, e_\alpha] \in \mathfrak{F}$. Now $\langle \nabla_{e_i} e_\alpha, e_\beta \rangle = \langle [e_i, e_\alpha] + \nabla_{e_\alpha} e_i, e_\beta \rangle = \langle \nabla_{e_\alpha} e_i, e_\beta \rangle = -\langle e_i, \nabla_{e_\alpha} e_\beta \rangle = 0$ so that

$$(10) \quad \omega_\beta^\alpha \equiv 0 \pmod{\mathfrak{F}}.$$

Assume that, in addition to being integrable, \mathfrak{F} is also minimal and let Π denote the second fundamental form of the leaves of \mathfrak{F} . Given an adapted local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ define symmetric tensors A_α of type $(1, 1)$ on the leaves of \mathfrak{F} by the equation $\Pi(X, Y) = \sum \langle A_\alpha(X), Y \rangle e_\alpha$. In terms of tensor products, A_α may be written as $A_\alpha = \sum \omega_\alpha^i \otimes e_i$ and the minimality of \mathfrak{F} implies $0 = \text{trace } A_\alpha = \sum \omega_\alpha^i(e_i)$. Given $p \in M$ we let $\|\Pi_p\|^2$ denote the square norm of Π at p for the corresponding leaf of \mathfrak{F} . The next result relates this square norm to the sectional curvature of M .

LEMMA 3.1. *Suppose \mathfrak{F} and \mathfrak{G} are orthogonal geodesic and minimal foliations of M . Given $p \in M$ let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis for $T_p M$ with $e_1, e_2, \dots, e_k \in \mathfrak{F}_p$ and $e_{k+1}, \dots, e_n \in \mathfrak{G}_p$. Then*

$$(11) \quad \|\Pi_p\|^2 = -\sum \langle R(e_i, e_\alpha)e_i, e_\alpha \rangle.$$

PROOF. Assume that $\{e_1, e_2, \dots, e_n\}$ has been extended to an adapted local orthonormal frame field. Then

$$\begin{aligned} d\omega_\alpha^i(e_i, e_\alpha) &= e_i[\omega_\alpha^i(e_\alpha)] - e_\alpha[\omega_\alpha^i(e_i)] - \omega_\alpha^i([e_i, e_\alpha]) \\ &= -e_\alpha[\omega_\alpha^i(e_i)] - \omega_\alpha^i(\nabla_{e_i} e_\alpha) + \omega_\alpha^i(\nabla_{e_\alpha} e_i) \\ &= -e_\alpha[\omega_\alpha^i(e_i)] + \|A_\alpha(e_i)\|^2 + \langle A_\alpha(\nabla_{e_\alpha} e_i), e_i \rangle. \end{aligned}$$

By the second structural equation

$$\begin{aligned} -\langle R(e_i, e_\alpha)e_i, e_\alpha \rangle &= \Omega_\alpha^i(e_i, e_\alpha) \\ &= -e_\alpha[\omega_\alpha^i(e_i)] + \|A_\alpha(e_i)\|^2 + 2\langle A_\alpha(\nabla_{e_\alpha} e_i), e_i \rangle \\ &\quad - \sum \omega_\beta^i(e_i)\omega_\alpha^\beta(e_\alpha). \end{aligned}$$

Since the right-hand side of (11) is independent of the choice of $e_1, e_2, \dots, e_k \in \mathfrak{F}$, we may assume that at p , e_i is an eigenvector of A_α . Thus at p

$$(12) \quad -\sum \langle R(e_i, e_\alpha)e_i, e_\alpha \rangle = \|A_\alpha\|^2.$$

Summation over α then gives equation (11). \square

Lemma 3.1 may also be derived from Corollary 1 of [8] or by using the second variation formula for minimal immersions. Since II vanishes iff \mathfrak{F} is geodesic we immediately have

THEOREM 3.2. *Let \mathfrak{F} and \mathfrak{S} be orthogonal geodesic and minimal distributions on a space M of nonnegative sectional curvature. Then \mathfrak{S} is integrable iff \mathfrak{F} and \mathfrak{S} are parallel.*

For the next result it will be convenient to change our notation slightly. We now let $\tilde{\nabla}$ denote the Riemannian connection of M with ∇ the Riemannian connection for the leaves of \mathfrak{S} . Let D denote the connection induced in the normal bundle to the leaves of \mathfrak{S} by $\tilde{\nabla}$. If $\{e_1, e_2, \dots, e_n\}$ is an adapted local orthonormal frame field then equation (10) implies $De_\alpha = 0$, that is $\{e_{k+1}, \dots, e_n\}$ is a local parallel framing of the normal bundle to the leaves of \mathfrak{S} . Given vector fields $X, Y, Z \in \mathfrak{S}$ the covariant derivative, $D_X \text{II}$, of II in the direction of X is defined by

$$(D_X \text{II})(Y, Z) = D_X[\text{II}(Y, Z)] - \text{II}(\nabla_X Y, Z) - \text{II}(Y, \nabla_X Z).$$

It is easy to see that this definition depends only on the values of X, Y , and Z at each particular point of M . We say each leaf of \mathfrak{S} has parallel second fundamental form provided $D\text{II} = 0$.

We have seen that with respect to any adapted coordinate system (x^1, x^2, \dots, x^n) : $U \rightarrow \mathbf{R}^n$ the metric takes the form

$$ds^2 = \sum g_{ij}(x^1, x^2, \dots, x^n) dx^i \otimes dx^j + \sum g_{\alpha\beta}(x^{k+1}, \dots, x^n) dx^\alpha \otimes dx^\beta.$$

Furthermore, it is routine to verify that \mathfrak{S} is minimal iff $\det(g_{ij})$ is independent of x^{k+1}, \dots, x^n . Certainly this happens if each g_{ij} is independent of x^{k+1}, \dots, x^n , indeed then both \mathfrak{F} and \mathfrak{S} are parallel. Consider the other extreme in which each g_{ij} is a function of x^{k+1}, \dots, x^n only. Such foliations are characterized by the following result:

THEOREM 3.3. *Suppose \mathfrak{F} is geodesic and \mathfrak{S} is integrable. Then about any point of M there exists an adapted coordinate system such that*

$$ds^2 = \sum g_{ij}(x^{k+1}, \dots, x^n) dx^i \otimes dx^j + \sum g_{\alpha\beta}(x^{k+1}, \dots, x^n) dx^\alpha \otimes dx^\beta$$

iff each leaf of \mathfrak{S} is a flat submanifold of M with parallel second fundamental form.

PROOF. Assume each leaf of \mathfrak{S} is a flat submanifold of M with parallel second fundamental form. Let (y^1, y^2, \dots, y^n) : $U \rightarrow \mathbf{R}^n$ be an adapted coordinate system and choose an adapted local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ in U such that $\nabla_{e_i} e_j = 0$. This can be done since each leaf of \mathfrak{S} is locally isometric to \mathbf{R}^k . Equivalently, in terms of the connection 1-forms we choose the frame field such that $\omega_j^i \equiv 0 \pmod{\mathfrak{F}}$. If $\{\theta^1, \theta^2, \dots, \theta^n\}$ denotes the dual coframing of M then this implies $\nabla \theta^j = 0$ and $d\theta^j \equiv 0 \pmod{\mathfrak{F}}$. Since each leaf of \mathfrak{S} has parallel second fundamental form

$$0 = (D_X \text{II})(Y, Z) = \sum \langle (\nabla_X A_\alpha) Y, Z \rangle e_\alpha$$

so that $\nabla A_\alpha = 0$. Writing $\omega_\alpha^i = \sum b_{\alpha j}^i \theta^j$ we then have

$$0 = \nabla A_\alpha = \nabla \left(\sum \omega_\alpha^i \otimes e_i \right) = \nabla \left(\sum b_{\alpha j}^i \theta^j \otimes e_i \right) = \sum db_{\alpha j}^i \otimes \theta^j \otimes e_i$$

where $db_{\alpha j}^i$ is restricted to the leaves of \mathfrak{F} . Thus each $b_{\alpha j}^i$ is a function of y^{k+1}, \dots, y^n only and $d\omega_{\alpha}^i \equiv 0 \pmod{\mathfrak{F}}$. The second structural equation now implies $\Omega_{\alpha}^i \equiv 0 \pmod{\mathfrak{F}}$.

The first structural equation implies $d\theta^{\beta}(e_i, e_{\alpha}) = 0$ so that writing $\omega_k^j = \sum b_{k\beta}^j \theta^{\beta}$ we have $d\omega_k^j(e_i, e_{\alpha}) = db_{k\alpha}^j(e_i)$. On the other hand, it follows from the second structural equation that $d\omega_k^j(e_i, e_{\alpha}) = \Omega_k^j(e_i, e_{\alpha})$. Thus $db_{k\alpha}^j(e_i) = \Omega_k^j(e_i, e_{\alpha}) = \Omega_{\alpha}^j(e_j, e_k) = 0$ and each $b_{k\alpha}^j$ is a function of y^{k+1}, \dots, y^n only.

The above computations imply $d\theta^i = \sum r_{j\alpha}^i dy^{\alpha} \wedge \theta^j$ where each $r_{j\alpha}^i$ is a function of y^{k+1}, \dots, y^n only. In matrix form this may be written as $d\theta = R \wedge \theta$ where θ is the column vector with i th entry θ^i and R is the $k \times k$ matrix with (i, j) entry $\sum r_{j\alpha}^i dy^{\alpha}$. We wish to find a nonsingular $k \times k$ matrix $A = (a_j^i)$ of functions depending only on y^{k+1}, \dots, y^n such that $d(A\theta) = 0$. Since

$$d(A\theta) = dA \wedge \theta + Ad\theta = dA \wedge \theta + (AR) \wedge \theta = (dA + AR) \wedge \theta,$$

A must satisfy the system of partial differential equations

$$(13) \quad dA = -AR.$$

The integrability conditions for this system are given by taking the exterior derivative of both sides of (13).

$$0 = d(dA) = -d(AR) = -(dA \wedge R + AdR) = A(R \wedge R - dR)$$

or equivalently, $dR = R \wedge R$. Taking the exterior derivative of $d\theta = R \wedge \theta$ yields

$$0 = d(d\theta) = dR \wedge \theta - R \wedge d\theta = (dR - R \wedge R) \wedge \theta$$

which implies $dR = R \wedge R$. Thus the system is integrable and if p is any fixed point in the domain of $\{e_1, e_2, \dots, e_n\}$ then we may find a solution A to (13) with $A(p)$ the identity matrix. In particular, A is nonsingular in a neighborhood of p . By the Poincaré Lemma we may find functions x^1, x^2, \dots, x^k in a neighborhood of p such that $dx^i = \sum a_j^i \theta^j$. Equivalently, by inverting A there exist functions c_j^i of y^{k+1}, \dots, y^n only such that $\theta^i = \sum c_j^i dx^j$. Since $ds^2 = \sum \theta^i \otimes \theta^i + \sum \theta^{\alpha} \otimes \theta^{\alpha}$, the desired coordinate system is given by (x^1, x^2, \dots, x^n) where $x^{\alpha} = y^{\alpha}$.

The converse is routine. \square

The special coordinate systems described in Theorem 3.3 together with the corresponding decomposition of the Riemannian metric have some interesting implications for the topology of M . First note that if $x = (x^1, x^2, \dots, x^n): U \rightarrow \mathbf{R}^n$ and $\bar{x} = (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n): \bar{U} \rightarrow \mathbf{R}^n$ are two such coordinate systems, then on $U \cap \bar{U}$ the change of coordinates $\bar{x}^i = \bar{x}^i(x^1, x^2, \dots, x^k)$, $i = 1, 2, \dots, k$, is given by an *affine function* of x^1, x^2, \dots, x^k . This, together with a theorem of R. Blumenthal [2], implies there exists a submersion of \tilde{M} , the simply connected cover of M , into \mathbf{R}^k which has as fibres the leaves of the natural lifting of \mathfrak{F} to \tilde{M} . In particular, M cannot be compact with finite fundamental group. (A similar result has been proved by R. Blumenthal [3] in a different context.)

Since the change of coordinates described above is given by an affine function it follows that \mathfrak{F} (considered as a k -dimensional vector bundle over M) has a discrete structural group. In particular, $\text{Pont}^i(\mathfrak{F}) = 0$ for $i > 0$ where Pont^i denotes the forms in the Pontrjagin algebra of degree i . Since \mathfrak{F} is a Riemannian foliation the Pasternack vanishing theorem implies $\text{Pont}^i(\mathfrak{F}) = 0$ for $i > n - k$. Consequently,

$\text{Pont}^i(M) \equiv \text{Pont}^i(TM) = \text{Pont}^i(\mathfrak{F} + \mathfrak{G}) = 0$ for $i > n - k$. (The author would like to thank Robert Gardner for suggesting the consideration of the Pontrjagin classes.)

For the remainder of this section we assume $\dim \mathfrak{G} = 2$. Given $p \in M$ let $K(p)$ denote the intrinsic sectional curvature at p of the corresponding leaf of \mathfrak{G} .

PROPOSITION 3.4. *Let \mathfrak{F} and \mathfrak{G} be orthogonal geodesic and minimal foliations of M with $\dim \mathfrak{G} = 2$. If each leaf of \mathfrak{G} has parallel second fundamental form, then in a neighborhood of any point of M there exists an adapted coordinate system such that either*

$$(i) \, ds^2 = \sum g_{ij}(x^1, x^2) dx^i \otimes dx^j + \sum g_{\alpha\beta}(x^3, \dots, x^n) dx^\alpha \otimes dx^\beta \text{ or}$$

$$(ii) \, ds^2 = \sum g_{ij}(x^3, \dots, x^n) dx^i \otimes dx^j + \sum g_{\alpha\beta}(x^3, \dots, x^n) dx^\alpha \otimes dx^\beta \text{ with } \det(g_{ij}) \text{ constant.}$$

Furthermore, if M is compact and connected, the choice may be made unambiguously between (i) and (ii) for all of M .

PROOF. We first show that if $K(p) \neq 0$ then the leaf L of \mathfrak{G} through p is totally geodesic. Since L has parallel second fundamental form it suffices to show $\Pi_p = 0$. Let $\{e_1, e_2, \dots, e_n\}$ be an adapted orthonormal frame field in a neighborhood of p and define symmetric tensors A_α on L as before. We have seen that $\nabla A_\alpha = 0$ which implies the eigenspaces of A_α are parallel distributions of L . Since \mathfrak{G} is minimal, $\text{trace } A_\alpha = 0$ so that if some A_α is nonzero at p then $K(p) = 0$. Thus, if $K(p) \neq 0$ then $\Pi_p = 0$ and the result follows.

We now show that if $K(p) \neq 0$ then K is constant on the leaf of \mathfrak{G} through p . Assume $K(p) \neq 0$ and choose an adapted coordinate system $(y^1, y^2, \dots, y^n): U \rightarrow \mathbf{R}^n$ about p such that $K \neq 0$ on U and $y^i(p) = y^\alpha(p) = 0$. It follows from the above paragraph that \mathfrak{F} and \mathfrak{G} are parallel in U so that K is a function of y^1, y^2 only. Since the leaf of \mathfrak{F} through p in U is given as the locus $F = \{q \mid y^1(q) = y^2(q) = 0\}$, it follows that K is constant on F . A connectedness argument now implies K is constant on the entire leaf of \mathfrak{G} through p .

Now given any p in M let V be a neighborhood containing p such that in V each leaf of \mathfrak{F} intersects each leaf of \mathfrak{G} . If $K = 0$ on V then by Theorem 3.3 there exists an adapted coordinate system about p such that the metric may be written as in (ii). If $K \neq 0$ at some point in V then the above results imply \mathfrak{F} and \mathfrak{G} are parallel in V and the metric may be written as in (i).

Finally, if M is compact and connected then each leaf of \mathfrak{F} intersects each leaf of \mathfrak{G} [7] and thus either $K = 0$ on M or \mathfrak{F} and \mathfrak{G} are both parallel. \square

Given $X, Y, Z \in \mathfrak{G}_p$ let $[\tilde{R}(X, Y)Z]^\mathfrak{G}$ denote the component of $\tilde{R}(X, Y)Z$ tangent to \mathfrak{G}_p where \tilde{R} denotes the curvature tensor of M . The Codazzi equations for the leaves of \mathfrak{G} state that

$$(14) \quad [\tilde{R}(X, Y)Z]^\mathfrak{G} = (D_Y \Pi)(X, Z) - (D_X \Pi)(Y, Z).$$

In particular, a necessary condition for the leaves of \mathfrak{G} to have parallel second fundamental form is that $\tilde{R}(X, Y)Z \in \mathfrak{G}_p$ whenever $X, Y, Z \in \mathfrak{G}_p$. Given $p \in M$ let $\tilde{K}(p)$ denote the extrinsic sectional curvature of the 2-plane \mathfrak{G}_p .

PROPOSITION 3.5. *Let \mathfrak{F} and \mathfrak{S} be orthogonal geodesic and minimal foliations of M with $\dim \mathfrak{S} = 2$ and with $\tilde{R}(X, Y)Z \in \mathfrak{S}_p$ whenever $X, Y, Z \in \mathfrak{S}_p$. Suppose one of the following is true:*

- (a) $K \geq 0$ and each leaf of \mathfrak{S} is compact;
- (b) $\dim M = 3$ and $\tilde{K} - K$ is constant on each leaf \mathfrak{S} ;
- (c) each leaf of \mathfrak{S} is diffeomorphic to S^2 .

Then in a neighborhood of each point of M the metric may be written as in (i) or (ii) of Proposition 3.4. Furthermore, if condition (c) holds, the metric may be written as in (i).

PROOF. Let Π denote the second fundamental form of a leaf L of \mathfrak{S} . It suffices to show that conditions (a) and (b) imply Π is parallel and that condition (c) implies $\Pi = 0$. By considering the double cover if necessary we assume L is oriented. Let $\{e_1, e_2, \dots, e_n\}$ be an adapted orthonormal frame field defined in a neighborhood of a point of L . Equation (14) then implies each A_α is a Codazzi tensor of type $(1, 1)$ on L . Since $\text{trace } A_\alpha = 0$, Theorem 1 of [10] becomes

$$\Delta k_\alpha^2 = 4k_\alpha^2 K + \|\nabla A_\alpha\|^2$$

where Δ denotes the Laplacian of L and k_α denotes the maximum eigenvalue of A_α . If (a) holds then summing over α gives the global inequality $\Delta \|\Pi\|^2 \geq 2\|D\Pi\|^2$. Since the integral of the left-hand side over L is zero, it follows that Π is parallel.

It is routine to show that a symmetric Codazzi tensor of type $(1, 1)$ on L with constant eigenvalues must be parallel. If $\dim M = 3$ the Gauss curvature equation becomes $\tilde{K} = K + k_3^2$. Thus condition (b) implies A_3 and hence Π is parallel.

Suppose (c) holds. Since L is simply connected we may find globally defined parallel orthonormal sections $\{e_3, e_4, \dots, e_n\}$ of the normal bundle to L . However, a trace zero symmetric Codazzi tensor of type $(1, 1)$ on L must be identically zero [5]. Thus each A_α is zero which implies $\Pi = 0$ on L .

If M has constant sectional curvature c then $\langle \tilde{R}(X_1, X_2)X_3, X_4 \rangle = c \det(\langle X_i, X_j \rangle)$ and it immediately follows that $\tilde{R}(X, Y)Z \in \mathfrak{S}_p$ whenever $X, Y, Z \in \mathfrak{S}_p$. For such manifolds we can strengthen Proposition 3.5 as follows:

THEOREM 3.6. *Let $\mathfrak{F} \neq 0$ and \mathfrak{S} be orthogonal geodesic and minimal distributions on a space M of constant sectional curvature. If $\dim \mathfrak{S} = 2$ then \mathfrak{S} is integrable iff M is flat and \mathfrak{F} and \mathfrak{S} are parallel.*

PROOF. In light of Theorem 3.2 it suffices to show that if M has constant sectional curvature $c < 0$ then \mathfrak{S} cannot be integrable. Assume then that \mathfrak{S} is integrable and let $\{e_1, e_2, \dots, e_n\}$ be an adapted local orthonormal frame field. Equation (12) becomes $k_\alpha^2 = -c$ where k_α is the maximum eigenvalue of A_α . This implies A_α has constant eigenvalues and is thus parallel on each leaf of \mathfrak{S} . But then each leaf of \mathfrak{S} has parallel second fundamental form and the metric may be written as in either (i) or (ii) of Proposition 3.4. However, (i) implies M splits locally as a Riemannian product, a clear contradiction since the sectional curvature of M is negative for all 2-planes. If the metric can be written as in (ii) then each leaf of \mathfrak{S} is flat. However, this is also a contradiction since the Gauss curvature equation implies $K \leq \tilde{K} = c < 0$. Therefore, \mathfrak{S} cannot be integrable. \square

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