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NONSTANDARD CONSTRUCTION OF THE STOCHASTIC INTEGRAL AND APPLICATIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS. I

BY

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ABSTRACT. R. M. Anderson has developed a nonstandard approach to Itô integration in which the Itô integral is interpreted as an internal Riemann-Stieltjes sum. In this paper we extend this approach to integration with respect to semimartingales. Lifting and pushing down theorems are proved for local martingales, semimartingales and other right-continuous processes on a Loeb space.

0. Introduction. In a recent paper [**12**], H. J. Keisler, using the nonstandard representation of Itô integration developed by Anderson [**2**], shows how to use nonstandard analysis to obtain simple existence proofs for solutions of Itô integral equations. The two parts of the present work generalize the results of Anderson and Keisler to semimartingales. In this Part I we give a nonstandard representation of semimartingale integration, and in Part II, we show how this representation may be applied to prove existence of solutions of semimartingale stochastic integral equations.

In Part I we develop lifting theorems for right-continuous processes with left limits and, in particular, for local martingales and semimartingales. The stochastic integral with respect to a semimartingale is then represented as an ordinary Riemann-Stieltjes integral (thus extending Anderson's construction of the Itô integral). In this nonstandard approach the construction of the stochastic integral with respect to a local martingale is almost identical to that of a stochastic process with sample paths of bounded variation, giving a relatively unified approach to the stochastic integral with respect to a semimartingale.

Some earlier work generalizing the nonstandard representation of the stochastic integral to a restrictive class of continuous martingales is contained in Panetta [**21**]. Also some of the results on liftings of general right-continuous processes were proved independently by K. D. Stroyan (see [**27**]).

While this paper was being put into final draft, we learned that a nonstandard approach to local martingale integration has been developed independently by T. L.

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Lindstrøm in the series of papers **[13–15]**, duplicating some of the main results of Part I and §8 of Part II. We think our treatment has these advantages.

(1) Our lifting theorem for local martingales (Theorem 5.6) is more general.

(2) Our results on quadratic variation (§6) imply the standard construction of the quadratic variation of a local martingale, whereas the latter is used by Lindstrøm to obtain his results.

(3) Some of the proofs, notably that of the continuity theorem for local martingales (Theorem 8.5(a)), are considerably shorter.

Furthermore, Lindstrøm's treatment of stochastic integration is restricted to locally L^2-martingales. This is avoided here by the L^1 form of Burkholder's inequality (Theorem 1.3). Lindstrøm's papers also contain material not developed here, in particular a nonstandard proof of Itô's formula.

The local martingale lifting theorem (Theorem 5.6) is used by Perkins **[23]** in the solution of a problem of Gilat concerning the distribution of local martingales of given absolute value.

In Part II we apply the nonstandard representation of stochastic integration to prove existence of solutions of stochastic integral equations of the form

$$y(t, \omega) = h(t, \omega) + \int f(s, \omega, y(\cdot, \omega))\, dz(s, \omega), \tag{0.1}$$

where h is a right-continuous process with left limits, z is a semimartingale and for each (s, ω), $f(s, \omega, \cdot)$ is a function on right-continuous paths with left limits, continuous in the uniform topology, such that $f(s, \omega, d)$ depends only on $d \restriction [0, s)$ (for exact details see Theorem 10.3 of Part II). The general method follows the pattern of Keisler; namely, lift h, f, z to appropriate internal H, F, Z, solve the internal difference equation

$$Y(\underline{t}, \omega) = H(\underline{t}, \omega) + \sum_{\underline{s}<\underline{t}} F(\underline{s}, \omega, Y(\cdot, \omega))\Delta Z(\underline{s}, \omega)$$

(the trivial step), then show that the standard part, y, of the Y thus obtained is a solution of the original integral equation. There are, however, additional difficulties that arise here, on account of the path dependent character of the coefficient f, and because the presence of jumps in the integrator z makes it necessary to choose a special lifting Z to ensure that internal integrals with respect to Z will have nice path properties.

Such path-dependent stochastic integral equations were considered by Metivier and Pellaumail **[19]** in the case where f is Lipschitz in the third variable.

An existence result for stochastic integral equations similar to those considered here has been proved independently by Jacod and Memin **[11]** using standard methods. Their result involves enlarging the probability space and showing that a solution exists on the enlarged space, with an argument that the new semimartingale z' of the equation on the enlarged space is in a reasonable sense "the same" as the original semimartingale. Our results show that when dealing with equations on a Loeb space the solution exists without changing z or the underlying space. If one has an equation on a general "adapted probability space", $(\Omega, \mathfrak{F}, P, \mathfrak{F}_t)$, one can form a nonstandard extension of the space and the coefficients of the equation to get an equation on a Loeb space, to which our theorem will apply (see Hoover-Keisler **[9]**, and also Remark 10.10 in Part II). This procedure may be regarded as a method of constructing a very general enlargement of the original space in a way that preserves almost all properties of interest. However, our point of view is that adapted Loeb spaces suffice for practical purposes. In particular we believe that typical physical processes can be modelled on them directly.

The contents of this paper are as follows:

Part I.

§1. Notation and conventions.

§2. We study different notions of nearstandardness for functions in the class D of right-continuous functions with left limits.

§3. We set up a framework for a nonstandard treatment of the general theory of processes by defining notions of adapted Loeb space, internal filtration, and standard part of an internal filtration.

§4. We use the results of §§2 and 3 to give lifting and pushing down theorems for processes with sample paths in D.

§5. We begin the nonstandard treatment of local martingales by defining notions of "S-martingale" and "S-local martingale" and proving a lifting and pushing down theorem that shows that these are the appropriate nonstandard analogues of "martingale" and "local martingale".

§6. We treat the internal quadratic variation of S-local martingales. In particular, we show that if X is an S-local martingale lifting of x, then the standard part of the internal quadratic variation of X is the usual quadratic variation of x.

§7. We show how a stochastic integral $\int h(s)dz(s)$, where $h(s)$ is predictable and z is a semimartingale can be represented as the standard part of an internal Riemann-Stieltjes sum

$$\sum_{\underline{s}<\underline{t}} H(\underline{s})\Delta Z(\underline{s}),$$

where Z is an appropriate lifting of z and H is a lifting of h with respect to a random measure associated with Z.

Part II.

§8. We deal with the special case of continuous local martingales. The key result of the section is a generalization of Keisler's "continuity theorem" in **[12]**. Essentially it characterizes S-continuous internal martingales as those internal martingales, whose quadratic variation is S-continuous. It follows from this characterization that if X is an S-local martingale with S-continuous sample paths almost surely, then for any bounded, adapted, internal H, the same is true of the internal stochastic integral $\Sigma_{\underline{s}<\underline{t}} H(\underline{s})\Delta X(\underline{s})$.

§9. We use the tools of §8 to show that any local martingale x has an S-local martingale lifting X such that, for any bounded, adapted, internal H, the integral $\Sigma_{\underline{s}<\underline{t}} H(\underline{s})\Delta X(\underline{s})$ has good sample path properties.

§10. We use the result of the previous section and a lifting lemma for path dependent coefficients to prove existence of solutions of stochastic integral equations of the form in (0.1).

This paper presupposes a basic knowledge of nonstandard analysis (see Loeb [**18**] and Stroyan-Luxemburg [**26**]) as well as familiarity with previous work in nonstandard probability, particularly Loeb [**16**] and the parts of Anderson [**1–3**] and Keisler [**12**] that deal with lifting theorems. Knowledge of the other parts of [**1**, **2** and **12**] would help but is not essential. We suppose, however, only the most elementary facts about martingales and the general theory of processes with the exception of the martingale inequalities of Burkholder-Gundy-Davis [**5**] (Theorem 1.3 below). Since many of the basic standard results on martingale stochastic integration follow from our results, this work may be considered as a self-contained treatment of the subject for persons familiar with nonstandard analysis.

1. Preliminaries. We work in an ω_1-saturated enlargement of a superstructure $V(S)$, where $S \supset \mathbf{R}$.

Notation 1.1. (1) As in Keisler [**12**], F, X, etc., stand for internal functions and processes, while f, x, etc., stand for standard ones. An exception is stopping times, which whether standard or not, are represented by capitals.

(2) Unless stated otherwise (M, ρ) is a complete separable metric space. Let $\mathrm{ns}({}^*M)$ denote the set of nearstandard points in ${}^*M$.

(3) The space of functions from $[0, \infty)$ to M which are right-continuous with left limits is denoted by $D(M)$, or just D, when there is no ambiguity.

(4) $\mathbf{N}$ denotes the set of natural numbers $\{1, 2, 3, \dots\}$ without 0 and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Elements of $\mathbf{N}_0$ are denoted by n, m, etc., while numbers of ${}^*\mathbf{N} - \mathbf{N}$ are usually denoted by η or γ. The Euclidean norm in $\mathbf{R}^d$ is always denoted by $\| \ \|$.

(5) T denotes a (fixed) internal S-dense subset of ${}^*[0, \infty)$ (S-dense means $\{{}^\circ\underline{t}\colon \underline{t} \in T,\ {}^\circ\underline{t} < \infty\} = [0, \infty)$) and $\mathrm{ns}(T) = \{\underline{t} \in T \mid {}^\circ\underline{t} < \infty\}$. The elements of T are represented by $\underline{s}$, $\underline{t}$, $\underline{u}$, etc., whereas real numbers in $[0, \infty)$ are denoted by s, t, u, etc. Normally $T = \{k\Delta t \mid k \in {}^*\mathbf{N}_0\}$ for some positive $\Delta t \approx 0$, although this assumption is not required in §§2 and 3. We will assume that $0 \in T$ unless indicated otherwise.

(6) If $(\Omega, \mathcal{Q}, \mu)$ is an internal measure space, the corresponding Loeb space is $(\Omega, L(\mathcal{Q}), L(\mu))$. That is, $L(\mu)$ is the unique measure extending ${}^\circ\mu$ to the σ-algebra, $\sigma(\mathcal{Q})$, generated by $\mathcal{Q}$, and $L(\mathcal{Q})$ is the $L(\mu)$-completion of $\sigma(\mathcal{Q})$. The same notation is used if μ is a signed measure and ${}^\circ\mu^+(\Omega) \wedge {}^\circ\mu^-(\Omega) < \infty$.

(7) If $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces and $f\colon E \to F$ is measurable with respect to the σ-fields $\mathcal{E}$ and $\mathcal{F}$, we say that f is $\mathcal{E}/\mathcal{F}$-measurable. $\square$

REMARK 1.2. (a) The permanence (or "overspill") principle is the following: Given any internal sequence of objects $\{R_\gamma \mid \gamma \in {}^*\mathbf{N}\}$ and internal set S such that $R_n \in S$ for every $n \in \mathbf{N}$, there is a γ_0 in ${}^*\mathbf{N} - \mathbf{N}$ such that $R_\gamma \in S$ for every $\gamma < \gamma_0$.

(b) The saturation (or countable comprehension) property is the following: Let $\{R_n \mid n \in \mathbf{N}\}$ be a sequence of internal objects and $\{S_m \mid m \in \mathbf{N}\}$ a sequence of internal sets. If for each $m \in \mathbf{N}$ there is $N_m \in \mathbf{N}$ such that for all $n \geq N_m$, $R_n \in S_m$, then $\{R_n \mid n \in \mathbf{N}\}$ can be extended to an internal sequence $\{R_\gamma \mid \gamma \in {}^*\mathbf{N}\}$, such that $R_\gamma \in \bigcap_m S_m$ for every $\gamma \in {}^*\mathbf{N} - \mathbf{N}$.

For convenience we will sometimes invoke this principle with some of the S_m's not internal but intersections of countably many internal sets. Clearly this extension is implied by the saturation principle itself. A simple application of this extended saturation principle follows.

Suppose $\{A_m \mid m \in \mathbf{N}\}$ is a sequence of internal subsets of $^*[0,\infty)$, $\underline{t} \in {}^*[0,\infty)$, and $\{\underline{t}_n \mid n \in \mathbf{N}\}$ is a subset of $^*[0,\infty)$ such that

(i) $\underline{t}_n \in A_m$ for $m \leqslant n$,

(ii) $\underline{t}_n \approx \underline{t}$ for each n.

Then, letting $S_1 = \{\underline{s} \in {}^*[0,\infty) \mid \underline{s} \approx \underline{t}\} = \bigcap_k \{\underline{s} \mid |\underline{s} - \underline{t}| < k^{-1}\}$, $S_{m+1} = A_m$, we see by the extended saturation principle that $\{\underline{t}_n \mid n \in \mathbf{N}\}$ can be extended to an internal sequence $\{\underline{t}_\gamma \mid \gamma \in {}^*\mathbf{N}\}$ such that $\underline{t}_\gamma \approx \underline{t}$ and $\underline{t}_\gamma \in \bigcap_m A_m$ for each $\gamma \in {}^*\mathbf{N} - \mathbf{N}$. $\square$

The only nonelementary result from standard probability that we use in this paper is the following martingale inequality of Burkholder-Gundy-Davis.

THEOREM 1.3 (BURKHOLDER-GUNDY-DAVIS [5, THEOREM 1.1]). *Let $x = (x_n \mid n \in \mathbf{N}_0)$ be a d-dimensional martingale, and let*

$$x^* = \sup \|x_n\|,$$

$$S(x) = \left(\sum_{n=0}^{\infty} \|x_n - x_{n-1}\|^2 \right)^{1/2} \qquad (x_{-1} \equiv 0).$$

Suppose Φ: $[0,\infty) \to [0,\infty)$ is a nondecreasing convex function such that $\Phi(0) = 0$ and $\Phi(2\lambda) \leqslant k\Phi(\lambda)$ for all $\lambda \in [0,\infty)$ and some constant k. Then there are constants $c_1, c_2 > 0$, depending only on k and d, such that

$$c_1 E[\Phi(S(x))] \leqslant E[\Phi(x^*)] \leqslant c_2 E[\Phi(S(x))]. \quad \square$$

Although only the one-dimensional version of the above result appears in [5], the general result follows immediately since the growth condition on Φ implies that there are constants k_1 and k_2 such that

$$k_1 \sum_{i=1}^{d} \Phi(x^{i^*}) \leqslant \Phi(x^*) \leqslant k_2 \sum_{i=1}^{d} \Phi(x^{i^*})$$

and

$$k_1 \sum_{i=1}^{d} \Phi(S(x^i)) \leqslant \Phi(S(x)) \leqslant k_2 \sum_{i=1}^{d} \Phi(S(x^i)).$$

Finally, recall that two stochastic processes x_i: $[0,\infty) \times \Omega \to M$ $(i = 1,2)$ are indistinguishable if $x_1(t) = x_2(t)$ for all $t \geqslant 0$ a.s.

2. The standard part in the J_1 topology. If (M, ρ) is a complete separable metric space, recall that the J_1 topology on $D(M)$ is the unique topology for which $\{U_n(f) \mid n \in \mathbf{N}\}$ forms a neighbourhood basis at $f \in D$, where $U_n(f) = \{g \in D \mid$ there is a strictly increasing continuous function λ: $[0,n] \to [0,\infty)$ such that $\lambda(0) = 0$, $\sup_{t \leqslant n} |\lambda(t) - t| < n^{-1}$ and $\sup_{t \leqslant n} \rho(g(\lambda(t)), f(t)) < n^{-1}\}$. It is well known that D equipped with the J_1 topology is Polish, that is, metrizable as a complete separable space. (If $[0,\infty)$ is replaced by $[0,1]$ and the above definition is modified slightly, the result may be found in Billingsley [4, Theorem 14.2], and the embedding procedure in Stone [25] then gives us the required result.) Our immediate goal is to give a simple description of the nearstandard points in $^*D$ and the standard part map, st, on ns($^*D$).

DEFINITIONS 2.1. Let $F \in {}^*D$ such that $F(\underline{t}) \in \mathrm{ns}({}^*M)$ for all $\underline{t}$ in ns($^*[0, \infty)$).

(a) F is of class SD if for each t in $[0, \infty)$ there are points $\underline{t}_1 \approx \underline{t}_2 \approx t$ such that if $\underline{s}_1 \approx \underline{s}_2 \approx t$, $\underline{s}_1 < \underline{t}_1$, and $\underline{s}_2 \geqslant \underline{t}_2$, then $F(\underline{s}_1) \approx F(\underline{t}_1^-)$ and $F(\underline{s}_2) \approx F(\underline{t}_2)$.

(b) F is of class SDJ if (a) holds with $\underline{t}_1 = \underline{t}_2$ and $F(\underline{t}) \approx F(0)$ for all $\underline{t} \approx 0$ in $^*[0, \infty)$.

(c) F is S-continuous (SC) if $F(\underline{t}_1) \approx F(\underline{t}_2)$ whenever $\underline{t}_1 \approx \underline{t}_2$ are points in ns(T). □

A function F: $T \to {}^*M$ is SD (SDJ, SC) on T if it is the restriction to T of an SD (SDJ, SC) function on $^*[0, \infty)$. Note that this is equivalent to requiring that F satisfy the appropriate clause of the above definition with $\underline{t}_i$, $\underline{s}_i$ ranging over ns(T) instead of ns($^*[0, \infty)$).

DEFINITION 2.2. The standard part of an SD function F on T is the function st(F) defined by

$$\mathrm{st}(F)(t) = \lim_{\substack{{}^\circ\underline{t} \downarrow t \\ \underline{t} \in T}} {}^\circ F(\underline{t}). \quad \square$$

PROPOSITION 2.3. *Suppose F: $T \to {}^*M$ is the restriction of a function in $^*D$ to T, and $F(\underline{t}) \in \mathrm{ns}({}^*M)$ for all $\underline{t}$ in* ns(T). *Then F is SD if and only if* st(F) *exists and belongs to D.*

PROOF. Suppose F is SD, and fix $\varepsilon > 0$ and $t \geqslant 0$. There is a $\underline{t} \approx t$ such that $F(\underline{s}) \approx F(\underline{t})$ for all $\underline{s} \approx t$ that satisfies $\underline{s} \geqslant \underline{t}$, and hence by the permanence principle for some δ in $(0, \infty)$, $|F(\underline{s}) - F(\underline{t})| < \varepsilon$ for all $\underline{s}$ in $[\underline{t}, \underline{t} + \delta) \cap T$. Therefore st($F$)($t$) exists. It is clear from its definition that st(F) is right-continuous. An argument similar to the above shows that for some δ' in $(0, \infty)$ and $\underline{t}' \approx t$, $|F(\underline{s}) - F(\underline{t}')| < \varepsilon$ whenever $\underline{s} \in (\underline{t}' - \delta', \underline{t}']$. It follows that $\lim_{s \uparrow t} \mathrm{st}(F)(s) = {}^\circ F(\underline{t}')$ and $\mathrm{st}(F) \in D$.

The proof of the converse is similar. □

REMARK 2.4. It follows easily from the above that if $M = \mathbf{R}$, $T = \{k\Delta t\colon k \in {}^*\mathbf{N}_0\}$ for some positive $\Delta t \approx 0$, and F: $T \to {}^*\mathbf{R}$ is a nondecreasing internal function such that $F(\underline{t}) \in \mathrm{ns}({}^*\mathbf{R})$ for all $\underline{t}$ in ns(T), then F is SD. □

The following result is an immediate consequence of the previous definitions and the above proof.

PROPOSITION 2.5. *If F is SDJ on T, then for every t in $(0, \infty)$ there is a $\underline{t} \approx t$ such that if $\underline{u} \approx t$ and $\underline{u} < \underline{t}$, then $F(\underline{u}) \approx \mathrm{st}(F)(t^-)$, and if $\underline{u} \approx t$ and $\underline{u} \geqslant \underline{t}$ then $F(\underline{u}) \approx \mathrm{st}(F)(t)$. In particular if* st($F$) *is continuous at t then $F(\underline{u}) \approx F(\underline{v})$ for all $\underline{u} \approx \underline{v} \approx t$.* □

Although Proposition 2.3 shows SD to be the class of functions determined naturally by the standard part map st, SD does not appear to be the class of nearstandard functions for any topology on D. The class of nearstandard functions for the J_1 topology is SDJ, as we now show.

THEOREM 2.6. *The class of functions in* $^*D$ *which are nearstandard in the* J_1 *topology is SDJ and* $\mathrm{st}\,|_{SDJ}$ *is the standard part map for the* J_1 *topology.*

PROOF. Suppose F is nearstandard in the J_1 topology and $\mathrm{st}_J(F)=f$, where st_J is the standard part map for the J_1 topology. Clearly $^*f$ is SDJ and by the definition of the J_1 topology there is a γ in $^*\mathbf{N}-\mathbf{N}$ and a continuous, strictly increasing, internal mapping λ: $^*[0,\gamma]\to{}^*[0,\infty)$ such that $\lambda(\underline{t})\approx\underline{t}$, $\lambda(0)=0$ and $F(\lambda(\underline{t}))\approx{}^*f(\underline{t})$ for all $\underline{t}$ in $^*[0,\gamma]$. That F is SDJ now follows easily from the corresponding property of $^*f$. Moreover,

$$\begin{aligned}\mathrm{st}(F)(t)&=\lim_{^\circ\underline{t}\downarrow t}{}^\circ F(\underline{t})=\lim_{^\circ\underline{t}\downarrow t}{}^\circ F(\lambda(\underline{t}))\\&=\lim_{^\circ\underline{t}\downarrow t}{}^{\circ *}f(\underline{t})=f(t).\end{aligned}$$

Suppose F is SDJ and $\mathrm{st}(F)=f$. Fix N in $\mathbf{N}$ and let $0=t_0<\cdots<t_k=N$ contain all the points on $[0,N]$ where f has a jump greater than or equal to N^{-1} in magnitude. By Proposition 2.5 there are points $\underline{t}_i\approx t_i$ ($\underline{t}_0=0$) such that if $\underline{t}\approx t_i$, then $F(\underline{t})\approx f(t_i)$ whenever $\underline{t}\geqslant\underline{t}_i$, and $F(\underline{t})\approx f(t_i^-)$ whenever $\underline{t}<\underline{t}_i$. Define λ: $^*[0,N]\to{}^*[0,\infty)$ by setting $\lambda(t_i)=\underline{t}_i$ and interpolating linearly. Clearly $\lambda(\underline{t})\approx\underline{t}$ for all $\underline{t}$ in $^*[0,N]$. If $\underline{t}\approx t_i$ and $\underline{t}\geqslant t_i$, then $F(\lambda(\underline{t}))\approx f(t_i)\approx{}^*f(\underline{t})$, and if $\underline{t}\approx t_i$ and $\underline{t}<t_i$, then $F(\lambda(\underline{t}))\approx f(t_i^-)\approx{}^*f(\underline{t})$. If $\underline{t}\not\approx t_i$ for $i=0,\dots,k$ and $\underline{t}\leqslant N$, then $\rho(f(^\circ\underline{t}),f(^\circ\underline{t}^-))<N^{-1}$ and therefore by Proposition 2.5, $^*\rho(F(\lambda(\underline{t})),{}^*f(\underline{t}))<N^{-1}$. It follows that $F\in\bigcap_{N=1}^{\infty}{}^*U_N(f)$ and therefore $\mathrm{st}_J(F)=f$. □

The mapping "st" was used by Loeb [**16**] in his construction of the Poisson process. Indeed he observed in a preliminary draft of [**17**] that on the class of nondecreasing paths in $^*D(\mathbf{R})$, $\mathrm{st}(F)(t)=\sup_{\underline{t}\approx t}{}^\circ F(\underline{t})$ is the standard part map for the J_1 topology. The facts about SDJ contained in this section were also observed independently by K. D. Stroyan (see Stroyan and Bayod [**27**, Chapter 5]) and I. Schiopu.

The following is a simple, but useful, result on the nonstandard representation of Lebesgue-Stieltjes measures.

LEMMA 2.7. *Assume that* $T=\{k\Delta t\mid k\in{}^*\mathbf{N}_0\}$, *and* F: $T\to{}^*\mathbf{R}$ *satisfies the following conditions*:

(i) $F(\underline{t})\approx F(0)$ *for all* $\underline{t}\approx 0$.

(ii) $^\circ\Sigma_{\underline{s}<\underline{t}}\,|F(\underline{s}+\Delta t)-F(\underline{s})|<\infty$ *for all* $\underline{t}$ *in* $\mathrm{ns}(T)$.

Then F is of class SD and $f=\mathrm{st}(F)$ is of bounded variation on compacts. Moreover, if $N\in\mathbf{N}$ is fixed, λ_F is the internal signed measure on the internal subsets of $T\cap{}^*[0,N]$ defined by $\lambda_F(\{\underline{t}\})=F(\underline{t}+\Delta t)-F(\underline{t})$, and μ_f is the Lebesgue-Stieltjes measure induced by f, then $\mu_f(B)=L(\lambda_F)(\mathrm{st}^{-1}(B))$ for each Borel subset, B, of $[0,N)$.

PROOF. By Remark 2.4, $|F|(\underline{t})=\Sigma_{\underline{s}<\underline{t}}\,|F(\underline{s}+\Delta t)-F(\underline{s})|$ is of class SD. The first statement of the lemma is now immediate. Since $\mu_f(\{0\})=L(\lambda_F)(\mathrm{st}^{-1}(\{0\}))=0$ (by (i)), it suffices to prove the second statement for $B=(a,b]$ and this is obvious because $f=\mathrm{st}(F)$. □

2.8. *Further remarks.* Here, without proof, are characterizations of the nearstandard points in the other three topologies on D, which were also introduced in

Skorokhod [24]. (When defining the M_1 and M_2 topologies, we assume that M is a normed linear space.)

(a) M_1: $F \in {}^*D$ is nearstandard if and only if

(i) $F \in \mathrm{SD}$,

(ii) if $\underline{t} \approx 0$ then $F(\underline{t}) \approx F(0)$,

(iii) if $\underline{t}_1, \underline{t}_2 \approx t$, $\underline{t}_1 < \underline{s} < \underline{t}_2$, then $F(\underline{s}) \approx \alpha F(\underline{t}_1) + (1-\alpha)F(\underline{t}_2)$ for some $\alpha \in [0,1]$.

(b) J_2: F is nearstandard if and only if (i), (ii), and

(iv) for all t there are $\underline{t}_1, \underline{t}_2 \approx t$ such that for all $\underline{s} \approx t$,

$$F(\underline{s}) \approx F(\underline{t}_1) \quad \text{or} \quad F(\underline{s}) \approx F(\underline{t}_2).$$

(c) M_2: F is nearstandard if and only if (i), (ii), and

(v) for all t there are $\underline{t}_1, \underline{t}_2 \approx t$ such that for all $\underline{s} \approx \underline{t}$,

$$F(\underline{s}) \approx \alpha F(\underline{t}_1) + (1-\alpha)F(\underline{t}_2) \quad \text{for some } \alpha \in [0,1].$$

None of the topologies J_1, M_1, J_2, M_2 has the property that sums of nearstandard functions are nearstandard. This is equivalent to the fact that addition is not a continuous operation in these topologies. Note, however, that $F, G \in \mathrm{SD}$ implies $F + G \in \mathrm{SD}$. This suggests looking for a topology on D in which addition is continuous and the class of nearstandard points lies between SDJ and SD. $\square$

3. The nonstandard probability spaces. Our notation for nonstandard probability will be similar to that of Keisler [12], although the framework is more general.

Let $(\Omega, \mathcal{C}, \bar{P})$ denote an internal probability space and let $(\Omega, \mathcal{F}, P) = (\Omega, L(\mathcal{C}), L(\bar{P}))$. Internal expectation with respect to $\bar{P}$ is denoted by $\bar{E}$, and E denotes expectation with respect to P. The class of P-null sets of $\mathcal{F}$ is denoted by $\mathfrak{N}$.

DEFINITION 3.1. If T' is an internal S-dense subset of ${}^*[0,\infty)$, an internal filtration on T' is an internal nondecreasing collection of $*$-sub-σ-fields of $\mathcal{C}$, $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T'\}$. The standard part of $\{\mathfrak{B}_{\underline{t}}\}$ is the filtration $\{\mathcal{G}_t \mid t \geq 0\}$ defined by

$$\mathcal{G}_t = \Bigg(\bigcap_{\substack{{}^\circ\underline{t} > t \\ \underline{t} \in T'}} \sigma(\mathfrak{B}_{\underline{t}}) \Bigg) \vee \mathfrak{N}.$$

The 4-tuple $(\Omega, \mathcal{F}, P, \mathcal{G}_t)$ is called an adapted Loeb space. $\square$

It is easy to check that the standard part of an internal filtration is right-continuous and hence satisfies the "usual hypotheses" of Meyer [20, p. 248].

Let $\{\mathcal{C}_{\underline{t}} \mid \underline{t} \in T\}$ be a fixed internal filtration and let $\{\mathcal{F}_t\}$ be the standard part of $\{\mathcal{C}_{\underline{t}}\}$. Henceforth we will use the term "internal filtration" to refer only to internal filtrations whose standard part is $\{\mathcal{F}_t\}$.

THEOREM 3.2. *Assume M is a separable metric space and x is an M-valued $\mathcal{F}_t$-measurable random vector. If $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T'\}$ is an internal filtration, then there exists $\underline{t} \approx t$ and an internal $\mathfrak{B}_{\underline{t}}$-measurable $*$-random vector X: $\Omega \to {}^*M$ such that ${}^\circ X = x$ a.s. If M is a normed linear space and $\|x\|^p$ is integrable for some $p \geq 1$, we may take $\|X\|^p$ to be S-integrable.*

PROOF. Let $\{\underline{t}_n \mid n \in \mathbf{N}\}$ be a decreasing sequence in T' such that $0 < {}^\circ\underline{t}_n - t < n^{-1}$. Since x is $\sigma(\mathfrak{B}_{\underline{t}_n}) \vee \mathfrak{N}$-measurable, it follows easily from Keisler [12, Proposition 1.16] that there is a $\mathfrak{B}_{\underline{t}_n}$-measurable $*$-random vector X_n such that ${}^\circ X_n = x$ a.s.

By saturation, there is a $\underline{t}_\gamma \approx t$ ($\gamma \in {}^*\mathbf{N} - \mathbf{N}$) and a $\mathfrak{B}_{\underline{t}_\gamma}$-measurable X_γ such that ${}^\circ X_\gamma = x$ a.s. Take $X = X_\gamma$ and we are done.

If $\|x\|^p$ is integrable, we may take $\|X_n\|^p$ to be S-integrable (see the proof of Theorem 7 in Anderson **[2]**). Since $\bar{E}(\|X_n - X_m\|^p) \approx 0$ for n, m in $\mathbf{N}$, by saturation we may select γ in ${}^*\mathbf{N} - \mathbf{N}$, as above, so that $\bar{E}(\|X_\gamma - X_m\|^p) \approx 0$ for all $m \leqslant \gamma$. Hence $\|X_\gamma\|^p$ is S-integrable and $X = X_\gamma$ is the required lifting. $\square$

We will also use the following result on conditional expectations which was originally proved in a special case in Anderson **[2**, Theorem 12]. The proof, which appears in Panetta **[21]** and Perkins **[22]**, is trivial and hence omitted.

LEMMA 3.3. *If* $X\colon \Omega \to {}^*\mathbf{R}$ *is an S-integrable internal random variable and* $\mathfrak{D}$ *is a *-sub-σ-algebra of* $\mathcal{Q}$, *then* $\bar{E}(X \mid \mathfrak{D})$ *is S-integrable and* ${}^\circ\bar{E}(X \mid \mathfrak{D}) = E({}^\circ X \mid \sigma(\mathfrak{D}))$ *a.s.* $\square$

4. Lifting and pushing down theorems for processes in D. An internal stochastic process on T is an internal mapping $X\colon T \times \Omega \to {}^*M$ such that $X(\underline{t}, \cdot)$ is $\mathcal{Q}$-measurable for all $\underline{t}$ in T. For convenience we will assume that $T = \{k\Delta t \mid k \in {}^*\mathbf{N}_0\}$ for some positive infinitesimal Δt, throughout the remainder of this work.

DEFINITIONS 4.1. An internal stochastic process X is of class SD (SDJ, SC) if for almost all ω, the mapping

$$X(\cdot, \omega)\colon T \to {}^*M$$

is of class SD (SDJ, SC). If X is SD, a process, $\mathrm{st}(X)$, with sample paths in D, is defined by fixing x_0 in M and letting

$$\mathrm{st}(X)(t) = \begin{cases} \mathrm{st}(X(\cdot, \omega))(t) & \text{if } X(\cdot, \omega) \text{ is SD,} \\ x_0 & \text{otherwise.} \end{cases}$$

An SD (SDJ) lifting of a stochastic process $x\colon [0, \infty) \times \Omega \to M$ is an internal stochastic process X of class SD (SDJ) such that $\mathrm{st}(X)$ and x are indistinguishable. $\square$

We can of course replace T by any S-dense subset of ${}^*[0, \infty)$ in the above definitions.

The following lifting and pushing down theorem holds with or without the expressions in parentheses, with the understanding that $(M, \|\ \|)$ is a separable Banach space if the expressions in parentheses are included. A similar convention is used in Theorem 4.4.

THEOREM 4.2. *A stochastic process* $x\colon [0, \infty) \times \Omega \to M$ *has sample paths in D a.s.* (*and* $\{\|x(t)\|^p \mid t \leqslant m\}$ *is uniformly integrable for all m in* $\mathbf{N}$, *for some* $p \geqslant 1$) *if and only if it has an SDJ lifting, X* (*such that* $\|X(\underline{t})\|^p$ *is S-integrable for all* $\underline{t}$ *in* $\mathrm{ns}(T)$).

PROOF. ($\Leftarrow$) is trivial.

($\Rightarrow$) Clearly we may assume that $x(\cdot, \omega)\colon \Omega \to D$ by changing x on a null set. Since D with the J_1 topology is a separable metric space, the lifting theorem of R. M. Anderson (see Keisler **[12**, Proposition 1.16]) implies that there is an internal stochastic process $X'\colon {}^*[0, \infty) \times \Omega \to {}^*M$ such that $\mathrm{st}_J(X'(\cdot, \omega)) = x(\cdot, \omega)$ a.s., where st_J is the standard part map for the J_1 topology. By Theorem 2.6, X' is SDJ and $X = X' \restriction T \times \Omega$ is the desired lifting.

Suppose, in addition, that M is a separable Banach space and $\{\|x(t)\|^p \mid t \leq m\}$ is uniformly integrable for all m in $\mathbf{N}$ and some $p \geq 1$. Let Y be the SDJ lifting of x obtained above. Define

$$x^N = \begin{cases} x & \text{if } \|x\| \leq N, \\ N\|x\|^{-1}x & \text{if } \|x\| \geq N, \end{cases}$$

and define Y^N similarly. Then $x^N(\cdot, \omega) \in D$ and Y^N, since bounded, is an SDJ lifting of x^N such that $Y^N(\underline{t})$ is S-integrable for all $\underline{t}$. Fix a real $\varepsilon > 0$, $m \in \mathbf{N}$ and choose $M = M(m, \varepsilon)$ such that for all $s \leq m$,

$$\int I_{\{\|x(s)\| > M\}} \|x(s)\|^p \, dP \leq \varepsilon. \tag{4.1}$$

By Fatou's Lemma, if $s \leq m$, then

$$\int I_{\{\|x(s^-)\| > M\}} \|x(s^-)\|^p \, dP \leq \varepsilon. \tag{4.2}$$

Fix $t \leq m$, $\underline{t} \approx t$. Since Y^N is SDJ,

$$P\left(\{{}^\circ Y^N(\underline{t}) = x^N(t)\} \cup \{{}^\circ Y^N(\underline{t}) = x^N(t^-)\}\right) = 1.$$

It follows that

$$\begin{aligned} &\int I_{\{\|Y^N(\underline{t})\| \geq M+1\}} \|Y^N(\underline{t})\|^p \, d\overline{P} \\ &\quad \leq \int I_{\{Y^N(\underline{t}) \approx x^N(t), \|x^N(t)\| > M\}} \|x^N(t)\|^p \, dP \\ &\qquad + \int I_{\{Y^N(\underline{t}) \approx x^N(t^-), \|x^N(t^-)\| > M\}} \|x^N(t^-)\|^p \, dP + \varepsilon \\ &\quad \leq 3\varepsilon \quad (\text{by (4.1) and (4.2)}). \end{aligned}$$

By saturation we can obtain $\gamma \in {}^*\mathbf{N} - \mathbf{N}$ such that the above holds for Y^γ for every $m \in \mathbf{N}$, $\underline{t} \leq m$, $\varepsilon > 0$, and $M = M(m, \varepsilon)$. Then $X = Y^\gamma$ is an SDJ lifting of x such that $\|X(\underline{t})\|^p$ is S-integrable for every $\underline{t}$ in $\mathrm{ns}(T)$. □

DEFINITION 4.3. An internal stochastic process X: $T \times \Omega \to {}^*M$ is adapted with respect to an internal filtration $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ (or $\mathfrak{B}_{\underline{t}}$-adapted) if $X(\underline{t})$ is an internal $\mathfrak{B}_{\underline{t}}$-measurable random vector for each $\underline{t}$ in T. We say that a stochastic process x: $[0, \infty) \times \Omega \to M$ is $\mathfrak{F}_t$-adapted if $x(t, \cdot)$ is $\mathfrak{F}_t$-measurable for all $t \geq 0$. □

Theorem 4.2 has the following version for adapted processes. Analogous results for continuous processes and arbitrary stochastic processes may be found in Keisler **[12]**.

THEOREM 4.4. *Let $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ be an internal filtration. A process x: $[0, \infty) \times \Omega \to M$ is $\mathfrak{F}_t$-adapted and has almost all sample paths in D (and $\{\|x(t)\|^p \mid t \leq m\}$ is uniformly integrable for all m in* $\mathbf{N}$, *for some $p \geq 1$) if and only if x has an SDJ lifting, X, that is $\{\mathfrak{B}_{\underline{t} \vee \Delta' t} \mid \underline{t} \in T\}$-adapted for some positive infinitesimal $\Delta' t$ in T (and for which $\|X(\underline{t})\|^p$ is S-integrable for all $\underline{t}$ in* $\mathrm{ns}(T)$*).*

PROOF. ($\Leftarrow$) is trivial.

($\Rightarrow$) We shall only prove the case with the parentheses included as the proof of the case without them is then obvious.

By Theorem 4.2 there is an SDJ lifting, X', of x such that $\|X'(\underline{t})\|^p$ is S-integrable for all $\underline{t}$ in ns(T). Let $\{t_i \mid i \in \mathbf{N}_0\}$ be a dense set of $[0, \infty)$ such that $x(t_i^-) = x(t_i)$ a.s., and $t_0 = 0$ ($x(0^-) \equiv x(0)$). Let $\underline{t}_i \in T$ satisfy $\underline{t}_i \approx t_i$ for $i \in \mathbf{N}$ and $\underline{t}_0 = 0$. Extend $\{\underline{t}_i \mid i \in \mathbf{N}_0\}$ to $^*\mathbf{N}_0$ by saturation so that $\{\underline{t}_i \mid i \in {}^*\mathbf{N}_0\} \subset T$, and let $0 = \underline{t}_0^n \leqslant \cdots \leqslant \underline{t}_n^n$ be the elements of $\{\underline{t}_i \mid i \leqslant n\}$ arranged in increasing order. Let $\underline{\delta}_n = \max\{\underline{t} \in T \mid \underline{t} \leqslant 2^{-n}\}$ for $n \in {}^*\mathbf{N}$. Since ${}^\circ X'(\underline{t}_i) = x(t_i)$ a.s. for all $i \in \mathbf{N}$ (by the a.s.-continuity of x at t_i), ${}^\circ X'(\underline{t}_i)$ is $\sigma(\mathfrak{B}_{\underline{t}_i+\underline{\delta}_n}) \vee \mathfrak{N}$-measurable and therefore (see the proof of Theorem 7 in Anderson [**2**]) there is an internal $\mathfrak{B}_{\underline{t}_i+\underline{\delta}_n}$-measurable random vector $Y^n(\underline{t}_i)$ such that $\bar{E}(\|X'(\underline{t}_i) - Y^n(\underline{t}_i)\|^p) \approx 0$. Define a sequence of internal processes $\{X^n \mid n \in \mathbf{N}\}$ by

$$X^n(\underline{t}) = \begin{cases} Y^n(\underline{t}_i^n) & \text{if } \underline{t} \in [\underline{t}_i^n, \underline{t}_{i+1}^n) \\ Y^n(\underline{t}_n^n) & \text{if } \underline{t} \geqslant \underline{t}_n^n \end{cases} \quad \text{for } 0 \leqslant i < n.$$

The following conditions are then satisfied for all n in $\mathbf{N}$:

(i) $\bar{E}(\max_{i \leqslant n} \|X'(\underline{t}_i^n) - X^n(\underline{t}_i^n)\|^p) < 2^{-n}$;

(ii) $X^n(\underline{t})$ is $\mathfrak{B}_{\underline{t}+\underline{\delta}_n}$-measurable for all $\underline{t}$ in T;

(iii) For each ω and all $0 \leqslant i < n$, $X^n(\cdot, \omega)$ is constant on $[\underline{t}_i^n, \underline{t}_{i+1}^n)$ and on $[\underline{t}_n^n, \infty)$.

By saturation we can obtain $\gamma \in {}^*\mathbf{N} - \mathbf{N}$ such that (i)–(iii) hold with γ in place of n. By (i) and (iii) (with $n = \gamma$), X^γ is an SDJ lifting of x such that $\|X^\gamma(\underline{t})\|^p$ is S-integrable for all $\underline{t}$ in ns(T), and, by (ii), $X^\gamma(\underline{t})$ is $\mathfrak{B}_{\underline{t}+\underline{\delta}_\gamma}$-measurable for all $\underline{t}$ in T. Therefore $X(\underline{t}) = X^\gamma((\underline{t} - \underline{\delta}_\gamma) \vee 0)$ is the required lifting of x with $\Delta' t = \underline{\delta}_\gamma \approx 0$. $\square$

REMARK 4.5. (a) The proofs of Theorems 4.2 and 4.4 may be modified to accommodate additional hypotheses on x. For example, if $M = \mathbf{R}$ and $x(\cdot, \omega)$ is a.s. nondecreasing then (in both Theorems 4.2 and 4.4) X may be chosen so that $X(\cdot, \omega)$ is nondecreasing for all ω. Note that if x is a.s. continuous, then by Proposition 2.5, $X(\cdot, \omega)$ is necessarily a.s. S-continuous.

(b) It is clear that $\Delta' t$ may be chosen to be zero in Theorem 4.4 if and only if $x(0)$ is $\sigma(\mathfrak{B}_0) \vee \mathfrak{N}$-measurable. $\square$

Recall that a stopping time U is a $[0, \infty]$-valued random variable such that $\{U \leqslant t\} \in \mathfrak{F}_t$ for all $t \geqslant 0$.

DEFINITION 4.6. A *-stopping time with respect to an internal filtration $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T'\}$ (or a $\mathfrak{B}_{\underline{t}}$-stopping time) is an internal mapping from Ω to $T' \cup \{\infty\}$ such that $\{V \leqslant \underline{t}\} \in \mathfrak{B}_{\underline{t}}$ for all $\underline{t}$ in $T' \cup \{\infty\}$ (here $\mathfrak{B}_\infty \equiv \mathcal{Q}$). If V is a $\mathfrak{B}_{\underline{t}}$-stopping time, let

$$\mathfrak{B}_V = \{A \in \mathcal{Q} \mid A \cap \{V = \underline{t}\} \in \mathfrak{B}_{\underline{t}} \quad \text{for all } \underline{t} \text{ in } T'\}. \quad \square$$

THEOREM 4.7. *Let $T' \subseteq T$ be an internal S-dense subset of* $^*[0, \infty)$ *that is closed under addition, and let $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T'\}$ be an internal filtration.*

(a) *A mapping U: $\Omega \to \bar{[0, \infty]}$ is a stopping time if and only if $U = {}^\circ V$ a.s. for some $\mathfrak{B}_{\underline{t}}$-stopping time, V.*

(b) *Suppose that* X: $T \times \Omega \to {}^*M$ *is an internal stochastic process of class SD*, $x = \mathrm{st}(X)$ *a.s., and* U: $\Omega \to [0, \infty)$ *is* $\mathcal{F}$*-measurable. Then there is an internal* $\mathcal{Q}$*-measurable mapping* V *from* Ω *to* T' *and a* P*-null set* N *such that if* $\omega \notin N$, *then* ${}^\circ V(\omega) = U(\omega)$ *and if, in addition,* $\underline{t} \approx U(\omega)$ *and* $\underline{t} \geqslant V(\omega)$, *then* ${}^\circ X(\underline{t}, \omega) = x(U(\omega), \omega)$. *If* U *is a stopping time,* V *may be chosen to be a* $\mathcal{B}_{\underline{t}}$*-stopping time and if* U *is a constant, then* V *may be chosen to be a constant.*

PROOF. (a) This argument is due to H. J. Keisler.

($\Rightarrow$) Define z: $(0, \infty) \times \Omega \to \{0, 1\}$ by

$$z(t) = \begin{cases} 1 & \text{if } t \geqslant U, \\ 0 & \text{if } t < U. \end{cases}$$

Then z is $\mathcal{F}_t$-adapted and has sample paths in D. By Theorem 4.4, z has an SDJ lifting

$$Z\colon T' \times \Omega \to \{0, 1\},$$

such that for all $\underline{t} \in T'$, $Z(\underline{t})$ is $\mathcal{B}_{\underline{t} \vee \Delta' t}$-measurable for some $\Delta' t \approx 0$ in T'. Let

$$V' = \min\{\underline{t}\colon Z(\underline{t}) = 1\} \quad (\min \varnothing = \infty).$$

Then V' is a $\mathcal{B}_{\underline{t} \vee \Delta' t}$-stopping time and ${}^\circ V' = U$ a.s. If we let $V = V' \vee \Delta' t$, then V is the desired $\mathcal{B}_{\underline{t}}$-stopping time.

($\Leftarrow$) Apply the preceding proof in reverse.

(b) Extend X to ${}^*[0, \infty) \times \Omega$ by setting $X(t, \omega) = X(\underline{t}, \omega)$ for $t \in [\underline{t}, \underline{t} + \Delta t)$, $\underline{t} \in T$. Let Y be a lifting of $x(U)$ and let U': $\Omega \to T'$ be a lifting of U. Since ${}^\circ Y = \mathrm{st}(X)({}^\circ U')$ a.s., we may choose a sequence $\{\underline{\varepsilon}_n \mid n \in \mathbf{N}\} \subset T$ such that $0 < {}^\circ \underline{\varepsilon}_n < n^{-1}$ and

$$P\Big(\sup_{0 < {}^\circ\varepsilon \leqslant {}^\circ\underline{\varepsilon}_n} {}^\circ\rho(Y, X(U' + \varepsilon)) \geqslant n^{-1} \Big) < n^{-1}.$$

By the permanence principle there exists an infinitesimal $\underline{\delta}_n$ in T' such that

$$\bar{P}\Big(\sup_{\underline{\delta}_n \leqslant \varepsilon \leqslant \underline{\varepsilon}_n} \rho(Y, X(U' + \varepsilon)) \geqslant n^{-1} \Big) < n^{-1}. \tag{4.3}$$

After extending $\{\underline{\delta}_n \mid n \in \mathbf{N}\}$ to ${}^*\mathbf{N}$ by ω_1-saturation, we may obtain γ in ${}^*\mathbf{N} - \mathbf{N}$ such that $\underline{\delta} = \max_{n \leqslant \gamma} \underline{\delta}_n \approx 0$ and $\underline{\delta} \in T'$. It follows from (4.3) that

$$N_1 = \Big\{ \omega \mid \sup_{\underline{\delta} \leqslant \varepsilon \approx 0} {}^\circ\rho(Y, X(U' + \varepsilon)) > 0 \Big\}$$

is a P-null set. Let $V = U' + \underline{\delta}$. Then V: $\Omega \to T'$ (since T' is closed under addition),

$$N = N_1 \cup \{\omega \mid {}^\circ V \neq U \quad \text{or} \quad {}^\circ Y \neq x(U)\}$$

is a null set, and if $\omega \notin N$, $\underline{t} \approx U(\omega)$ and $\underline{t} \geqslant V(\omega)$ then

$$\rho(x(U), {}^\circ X(\underline{t})) = {}^\circ\rho(Y, X(\underline{t})) \leqslant \sup_{\underline{\delta} \leqslant \varepsilon \approx 0} {}^\circ\rho(Y, X(U' + \varepsilon)) = 0.$$

Hence $V = U' + \underline{\delta}$ is the required mapping. If U is a stopping time, then by (a), U' may be chosen to be a $\mathcal{B}_{\underline{t}}$-stopping time. Therefore $V = U' + \underline{\delta}$ is also a $\mathcal{B}_{\underline{t}}$-stopping time (recall T' is closed under addition). Similarly if U is constant, V may be chosen to be a constant. $\square$

Although every stochastic process with sample paths a.s. in D has an SDJ lifting, if additional properties are required of the lifting X (as will be the case when we lift martingales in §5), then it may be harder to obtain liftings of class SDJ. The following result shows that an SDJ lifting may be obtained from an SD lifting by restricting the time parameter to a "coarser set". It arose from a suggestion of K. D. Stroyan.

PROPOSITION 4.8. *If X: $T \times \Omega \to {}^*M$ is of class SD, then there is a positive infinitesimal $\Delta' t$ in T such that if $T' = \{k\Delta' t \mid k \in {}^*\mathbf{N}\}$ then $X \restriction T' \times \Omega$ is of class SDJ.*

PROOF. Let Y be an SDJ lifting of $\mathrm{st}(X)$. By Theorem 4.7(b) for each n in $\mathbf{N}$ there exists $\underline{\delta}_n \approx n^{-1}$ ($\underline{\delta}_n \in T$) such that

$$°X(k\underline{\delta}_n) = °Y(k\underline{\delta}_n) = x(k/n) \quad \text{for } k = 1,\dots,n^2 \text{ a.s.}$$

Hence for all n in $\mathbf{N}$ we have $0 < \underline{\delta}_n < 2/n$, and

$$\bar{P}\Big(\max_{1 \leq k \leq n^2} \rho\big(X(k\underline{\delta}_n), Y(k\underline{\delta}_n)\big) > n^{-1}\Big) < n^{-1}. \tag{4.4}$$

By ω_1-saturation we may extend $\{\underline{\delta}_n \mid n \in \mathbf{N}\}$ internally to ${}^*\mathbf{N}$ and obtain γ in ${}^*\mathbf{N} - \mathbf{N}$ such that $\underline{\delta}_\gamma$ is a positive infinitesimal, and (4.4) holds with n replaced by γ. Since Y is SDJ it follows that $X \restriction T' \times \Omega$ is SDJ, where $T' = \{k\underline{\delta}_\gamma \mid k \in {}^*\mathbf{N}\}$. □

5. Local martingales. Recall that a stochastic process x: $[0,\infty) \times \Omega \to \mathbf{R}^d$ is a (d-dimensional) local martingale if x is an $\mathcal{F}_t$-adapted process with sample paths a.s. in $D(\mathbf{R}^d)$ and there is a sequence of stopping times $\{U_n\}$ increasing to ∞ a.s. such that $x(t \wedge U_n)$ is a uniformly integrable $\mathcal{F}_t$-martingale for all n. The sequence $\{U_n\}$ is said to reduce x. Let $\mathcal{L}^d$ denote the class of d-dimensional local martingales and let $\mathcal{L}_0^d = \{x \in \mathcal{L}^d \mid x(0) = 0\}$.

DEFINITIONS 5.1. Let $\{\mathcal{B}_{\underline{t}} \mid \underline{t} \in T\}$ be an internal filtration.

If M is a normed linear space, then an internal stochastic process X: $T \times \Omega \to {}^*M$ is locally S-integrable with respect to $\{\mathcal{B}_{\underline{t}}\}$ if there is a nondecreasing sequence of $\mathcal{B}_{\underline{t}}$-stopping times $\{V_n\}$, such that

$$\lim_{n\to\infty} °V_n = \infty \quad \text{a.s.,} \tag{5.1}$$

and

$$\|X(\underline{t} \wedge V_n)\| \text{ is } S\text{-integrable for each } \underline{t} \in T \cup \{\infty\}. \tag{5.2}$$

An internal stochastic process X: $T \times \Omega \to {}^*\mathbf{R}^d$ is a $*$-martingale with respect to $\{\mathcal{B}_{\underline{t}}\}$ (or a $\mathcal{B}_{\underline{t}}$-martingale) if $\{(X(\underline{t}), \mathcal{B}_{\underline{t}}) \mid \underline{t} \in T\}$ is an internal martingale. We say X is an S-local martingale with respect to $\{\mathcal{B}_{\underline{t}}\}$ if, in addition, there is a nondecreasing sequence of $*$-stopping times $\{V_n\}$ satisfying (5.1), (5.2), and

$$°X(V_n) = \mathrm{st}(X)(°V_n) \text{ a.s.} \quad \text{on } \{°V_n < \infty\} \text{ for all } n. \tag{5.3}$$

The sequence $\{V_n\}$ is said to reduce X. An S-martingale with respect to $\{\mathcal{B}_{\underline{t}}\}$ is a $\mathcal{B}_{\underline{t}}$-martingale X for which $X(\underline{t})$ is S-integrable for all $\underline{t}$ in $\mathrm{ns}(T)$. □

The first part of the following "pushing down" theorem shows that (5.3) makes sense.

THEOREM 5.2. (a) *If X is a $\mathfrak{B}_{\underline{t}}$-martingale, and* $°\bar{E}(\|X(V_n \wedge \underline{t})\|) < \infty$ *for all* $\underline{t} \in T \cup \{\infty\}$ *and some sequence of* $\mathfrak{B}_{\underline{t}}$-*stopping times*, $\{V_n\}$, *satisfying* (5.1), *then X is SD*.

(b) *If X is an S-martingale* (*respectively, an S-local martingale*) *with respect to* $\{\mathfrak{B}_{\underline{t}}\}$, *then* $\mathrm{st}(X)$ *is an* $\mathfrak{F}_t$-*martingale* (*respectively, local martingale*).

PROOF. (a) Since X is SD if each component of $X(\underline{t} \wedge V_n)$ is SD for all n, we may assume that $X(\underline{t})$ is $^*\mathbf{R}$-valued and $°\bar{E}(|X(\underline{t})|) < \infty$ for all $\underline{t}$ in T. If M and N are natural numbers and $\underline{t}_N \leqslant N < \underline{t}_N + \Delta t$, then by the martingale maximal inequality,

$$\bar{P}\Big(\max_{\underline{t} \leqslant N} |X(\underline{t})| \geqslant M\Big) \leqslant M^{-1}\bar{E}\big(|X(\underline{t}_N)|\big).$$

Hence $X(\cdot, \omega)$ is a finite function on $\mathrm{ns}(T)$ a.s. To show the main condition for membership in SD we use the upcrossing lemma (see Doob [**8**, p. 316]): If $U^N_{a,b}$ is the number of upcrossings of the interval $[a, b]$ completed by $X(\underline{t}, \omega)$ for $\underline{t} \leqslant N$ ($N \in \mathbf{N}$), then

$$\bar{E}\big(U^N_{a,b}\big) \leqslant \big(\bar{E}\big(|X(\underline{t}_N)|\big) + |a|\big)/b - a.$$

Therefore for a.a. ω, $X(\cdot, \omega) \restriction \mathrm{ns}(T)$ has only countably many upcrossings (and hence downcrossings) of nontrivial standard intervals with rational end points. Fix such on ω and let $t \in [0, \infty)$. By saturation we can find $\underline{t}_1, \underline{t}_2 \approx t$ so that every such crossing which occurs in the monad of t, occurs in the interval $(\underline{t}_1, \underline{t}_2)$. (We only need find $\underline{t}_1, \underline{t}_2 \approx t$ such that $(\underline{t}_1, \underline{t}_2)$ contains a given countable set of intervals contained in the monad of t.) Since X cannot change a noninfinitesimal amount without crossing a nontrivial standard rational interval this implies that for $\underline{s} \approx t$, if $\underline{s} \leqslant \underline{t}_1$, then $X(\underline{s}, \omega) \approx X(\underline{t}_1, \omega)$, and if $\underline{s} \geqslant \underline{t}_2$, then $X(\underline{s}, \omega) \approx X(\underline{t}_2, \omega)$. Ergo X is SD.

(b) Suppose X is an S-local martingale and $\{V_n\}$ reduces X. By truncating at an infinite integer we may assume that $\sup_{n,\omega} V_n(\omega) \in {}^*\mathbf{R}$. Since $°X(V_n) = \mathrm{st}(X)(°V_n)$ a.s. on $\{°V_n < \infty\}$, it is easy to see that $\mathrm{st}(X(\cdot \wedge V_n))(t) = \mathrm{st}(X)(t \wedge °V_n)$ for all $t \geqslant 0$ a.s. Hence $\mathrm{st}(X)$ will be a local martingale if we show that $\mathrm{st}(X_n)$ is a uniformly integrable martingale for all n in $\mathbf{N}$, where $X_n(\cdot) = X(\cdot \wedge V_n)$. Since V_n is internally bounded, the optional stopping theorem (see Doob [**8**, p. 300]) implies that $X_n(\underline{t}) = \bar{E}(X(V_n) \mid \mathfrak{B}_{\underline{t}})$ a.s. and therefore

$$\begin{aligned} \mathrm{st}(X_n)(t) &= \lim_{°\underline{t} \downarrow t} °X_n(\underline{t}) \quad \text{a.s.} \\ &= \lim_{°\underline{t} \downarrow t} °\bar{E}\big(X(V_n) \mid \mathfrak{B}_{\underline{t}}\big) \quad \text{a.s.} \\ &= \lim_{°\underline{t} \downarrow t} E\big(°X(V_n) \mid \sigma(\mathfrak{B}_{\underline{t}})\big) \quad \text{a.s.} \quad \text{(Lemma 3.3)} \\ &= E\big(°X(V_n) \mid \mathfrak{F}_t\big) \quad \text{a.s.,} \end{aligned} \tag{5.4}$$

where we have used the reverse martingale convergence theorem (see Doob [**8**, p. 328, Theorem 4.2]) in the last. Hence $\mathrm{st}(X_n)$ is a uniformly integrable martingale.

If X is an S-martingale, then by Theorem 4.7(b), X is an S-local martingale that is reduced by a sequence of constant times $\{\underline{t}_n\}$, where $\underline{t}_n \approx n$. The above argument shows that $\mathrm{st}(X)(t \wedge n)$ is a uniformly integrable martingale for all n in $\mathbf{N}$ and hence $\mathrm{st}(X)$ is an $\mathfrak{F}_t$-martingale. □

Part (b) of the above result is false if one had not included (5.3) in the definition of an S-local martingale. Indeed, one can construct a locally S-integrable, $^*\mathbf{R}$-valued, $\mathfrak{B}_t$-martingale X such that $x = \mathrm{st}(X)$ satisfies the following conditions:

(i) $x(t) = \Delta x(1) I_{\{t \geqslant 1\}}$,

(ii) $E(|\Delta x(1)|) = \infty$,

(iii) $\Delta x(1)$ is independent of $\mathfrak{F}_{1^-}$.

Such a process x cannot be a local martingale. To construct such an X, let X have a jump of $\pm 1/n(\omega)$ at $\underline{t} = 1$ and then a jump of $\pm n(\omega)$ at $\underline{t} = 1 + \Delta t$. The distribution of $n(\omega) \in {}^*\mathbf{N}$ is chosen so that it is a.s. finite but $^\circ\bar{E}(n(\omega)) = \infty$, and each of the $\pm$ signs are chosen independently of $n(\omega)$ with equal probability. Note that X is locally S-integrable since it knows in advance the size of the second jump.

Definition 5.3. If x is an $\mathfrak{F}_t$-martingale (respectively, local martingale) and $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ is an internal filtration, then a $\mathfrak{B}_{\underline{t}}$-martingale lifting (respectively, $\mathfrak{B}_{\underline{t}}$-local martingale lifting) is an SDJ lifting of x, X, such that X is an S-martingale (respectively, an S-local martingale) with respect to $\{\mathfrak{B}_{\underline{t}}\}$. □

Notation 5.4. If $t \in {}^*[0, \infty)$, $[t]$ is the greatest element of T satisfying $[t] \leqslant t$. More generally if $T' \subset T$, let

$$[t]^{T'} = \begin{cases} \max\{\underline{t} \in T' \mid \underline{t} \leqslant t\} & \text{if this set is nonempty,} \\ \min T' & \text{otherwise.} \end{cases} \quad \square$$

It is not difficult to obtain an SD lifting of a local martingale x, that is an S-local martingale. Given such a lifting one would want to use Proposition 4.8 to obtain an SDJ lifting. However, to retain the internal martingale property one must also change the internal filtration when applying Proposition 4.8. This is handled by the following lemma:

Lemma 5.5. *Let X be a d-dimensional S-local martingale with respect to an internal filtration $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ and let $T' = \{k\Delta' t \mid k \in {}^*\mathbf{N}\}$, where $\Delta' t$ is a positive infinitesimal in T. Then there is a *-stopping time W with respect to $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ such that $^\circ W = \infty$ a.s. and $X'(\underline{t}) = X([\underline{t}]^{T'} \wedge W)$ is an S-local martingale with respect to the internal filtration $\{\mathfrak{B}'_{\underline{t}} \mid \underline{t} \in T\}$, where $\mathfrak{B}'_{\underline{t}} = \mathfrak{B}_{[\underline{t}]^{T'}}$.*

Proof. Let $\{V_n\}$ be a sequence of *-stopping times that reduces X. Clearly we may assume $^\circ V_n < \infty$ (Theorem 4.7(b) implies there are $\underline{t}_n \approx n$ such that $V_n \wedge \underline{t}_n$ also reduces X). By Theorem 4.7(b) there is a nondecreasing sequence of T'-valued *-stopping times with respect to $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T'\}$, $\{V_m^n \mid m \leqslant n\}$, such that for each $m \leqslant n$ in $\mathbf{N}$, $^\circ V_m^n = {}^\circ V_m$ a.s. and $^\circ X(V_m^n) = {}^\circ X(V_m)$ a.s. It follows easily that $^\circ X(V_m^n \wedge V_n) = {}^\circ X(V_m)$ a.s. for all $m \leqslant n$, since for a.a. ω in $\{V_n < V_m^n\}$ we have $^\circ V_n = {}^\circ V_m$ and therefore

$$^\circ X(V_n) = \mathrm{st}(X)(^\circ V_n) = \mathrm{st}(X)(^\circ V_m) = {}^\circ X(V_m).$$

The S-integrability of $\|X(V_n)\|$ implies

$$\bar{E}\Big(\max_{m \leqslant n} \|X(V_m^n \wedge V_n) - X(V_m)\|\Big) < 2^{-n}. \tag{5.5}$$

We also have

$$\bar{P}\Big(\max_{m \leqslant n} |V_m^n - V_m| \geqslant 2^{-n}\Big) < 2^{-n}. \tag{5.6}$$

By saturation we may internally extend $\{((\{V_m^n \mid m \leqslant n\},\ V_n) \mid n \in \mathbf{N}\}$ to $^*\mathbf{N}$ and obtain γ in $^*\mathbf{N} - \mathbf{N}$ such that $\{V_m^\gamma \mid m \leqslant \gamma\}$ is a nondecreasing sequence of *-stopping times with respect to $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T'\}$, $\{V_m \mid m \leqslant \gamma\}$ is a nondecreasing sequence of *-stopping times with respect to $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$, and (5.5) and (5.6) hold with n replaced by γ. If $W = V_\gamma$, then $^\circ W = \infty$ a.s. since $W \geqslant V_n$ for all n in $\mathbf{N}$. It is easy to check that $X'(\underline{t}) = X([\underline{t}]^{T'} \wedge W)$ is a *-martingale with respect to $\{\mathfrak{B}'_{\underline{t}} \mid \underline{t} \in T\}$. Since $V_m^\gamma \in T'$, (5.5) with γ in place of n implies that for all m in $\mathbf{N}$, $\| X'(V_m^\gamma)\| = \| X(V_m^\gamma \wedge V_\gamma)\|$ is S-integrable and

$$\begin{aligned} {}^\circ X'(V_m^\gamma) &= {}^\circ X(V_m^\gamma \wedge V_\gamma) \\ &= {}^\circ X(V_m) \quad \text{a.s.} \\ &= \operatorname{st}(X)({}^\circ V_m) \quad \text{a.s.} \\ &= \operatorname{st}(X')({}^\circ V_m^\gamma) \quad \text{a.s. (by (5.6)).} \end{aligned}$$

Since $\lim_{m\to\infty} {}^\circ V_m^\gamma = \infty$ a.s. (by (5.6)) and V_m^γ is a *-stopping time with respect to $\{\mathfrak{B}'_{\underline{t}} \mid \underline{t} \in T\}$ we see that $\{V_m^\gamma \mid m \in \mathbf{N}\}$ reduces X' and the result is proved. □

THEOREM 5.6. *If x is a d-dimensional $\mathfrak{F}_t$-martingale (respectively, local martingale), there is an internal filtration $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ and a $\mathfrak{B}_{\underline{t}}$-martingale (respectively, $\mathfrak{B}_{\underline{t}}$-local martingale) lifting of x.*

PROOF. We deal only with the local martingale case, as the martingale case is the same except that the reducing stopping times should be chosen to be constant.

Suppose that x is a local martingale reduced by $\{U_n\}$, where $U_n < \infty$. For each n in $\mathbf{N}$ let Y_n be an S-integrable lifting of $x(U_n)$ such that $\| Y_n(\omega)\| \leqslant \eta$ for some $\eta \in {}^*\mathbf{N} - \mathbf{N}$. Let $X_n(\underline{t})$ be an internal stochastic process such that $X_n(\underline{t}) = \bar{E}(Y_n \mid \mathcal{C}_{\underline{t}})$ (recall that $\{\mathcal{C}_{\underline{t}}\}$ is the internal filtration used to define $\{\mathfrak{F}_t\}$) for all $\underline{t}$ in T, $\bar{P}$-a.s. and $\sup_{(\underline{t},\omega)} \| X_n(\underline{t}, \omega)\| \leqslant \eta$. Then X_n is an S-martingale, and therefore is SD by Theorem 5.2(a). It follows as in formula (5.4) that X_n is an SD lifting of $x(\cdot \wedge U_n)$. By Theorem 4.7(b) there is a sequence of $\mathcal{C}_{\underline{t}}$-stopping times, $\{V_m\}$, such that ${}^\circ V_m = U_m$ a.s., and

(5.7) $\qquad$ for a.a. ω, if ${}^\circ\underline{t} = U_m$ and $\underline{t} \geqslant V_m$, then ${}^\circ X_m(\underline{t}) = x(U_m)$.

By considering $\max_{i \leqslant m} V_i$ we may assume $\{V_m\}$ is nondecreasing. We now show that

(5.8) $\qquad$ if $m \leqslant n$, $\quad \bar{E}(\| X_n(V_m) - X_m(V_m)\|) \approx 0.$

Indeed, if m and n are fixed, by Theorem 4.7(b) there is a *-stopping time V such that $V \geqslant V_m$, ${}^\circ V = U_m$ a.s. and ${}^\circ X_n(V) = x(U_m)$ a.s. (recall that $\operatorname{st}(X_n) = x(\cdot \wedge U_n)$ a.s.). The S-integrability of $X_n(V)$ and $X_m(V)$, together with (5.7) implies that

(5.9) $\qquad \bar{E}(\| X_n(V) - X_m(V)\|) \approx 0.$

By optional sampling (see Doob [**8**, p. 302]) we have $X_n(V_m) = \bar{E}(X_n(V) \mid \mathcal{C}_{V_m})$ a.s. and $X_m(V_m) = \bar{E}(X_m(V) \mid \mathcal{C}_{V_m})$ a.s. (recall that X_n is internally bounded) and hence (5.8) follows from (5.9). Since X_n is an $\mathcal{C}_{\underline{t}}$-martingale satisfying (5.8), by saturation there is an $\mathcal{C}_{\underline{t}}$-martingale, X_γ, ($\gamma \in {}^*\mathbf{N} - \mathbf{N}$) such that (5.8) holds if n is replaced by γ. The maximal inequality for martingales implies that for all m in $\mathbf{N}$,

$$\max_{\underline{t} \in T} \| X_\gamma(\underline{t} \wedge V_m) - X_m(\underline{t} \wedge V_m)\| \approx 0 \quad \text{a.s.}$$

Since X_m is an SD lifting of $x(\cdot \wedge U_m)$, it follows that X_γ is an SD lifting of x. Moreover, (5.8) implies that $\|X_\gamma(V_m)\|$ is S-integrable for all m in $\mathbf{N}$ and ${}^\circ X_\gamma(V_m) = {}^\circ X_m(V_m) = x({}^\circ V_m)$ a.s. Choose $T' = \{k\Delta' t \mid k \in {}^*\mathbf{N}\}$, as in Proposition 4.8, such that $X_\gamma \restriction T' \times \Omega$ is SDJ. If $\mathfrak{B}_{\underline{t}} = \mathcal{Q}_{[\underline{t}]^{T'}}$ and $X(\underline{t}) = X_\gamma([\underline{t}]^{T'} \wedge W)$, where W is as in Lemma 5.5, then by Lemma 5.5 and the choice of T', X is a $\mathfrak{B}_{\underline{t}}$-local martingale lifting of x. $\square$

Remarks 5.7. (a) The previous result would be false if we did not allow ourselves the freedom of selecting an internal filtration $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$. It is easy to see that for any given internal filtration $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ and local martingale x we can always find an S-local martingale with respect to $\{\mathfrak{B}_{\underline{t}}\}$ that is an SD lifting of x, but it need not be of class SDJ. Indeed, a trivial application of the maximal inequality for martingales shows that if x has one $\mathfrak{B}_{\underline{t}}$-local martingale lifting, then every SD lifting of x that is an S-local martingale with respect to $\{\mathfrak{B}_{\underline{t}}\}$ is of class SDJ. It is easy to construct an example of an S-martingale X with respect to $\{\mathfrak{B}_{\underline{t}}\}$ that is not SDJ (let X have two jumps of size ± 1 in the same monad) and hence st(X) is an $\mathfrak{F}_t$-martingale with no $\mathfrak{B}_{\underline{t}}$-martingale lifting.

(b) Suppose that $x \in \mathcal{L}^d$ and h: $[0, \infty) \times \Omega \to M$ is an adapted process with sample paths in $D(M)$ a.s. By Theorem 3.2, $h(0)$ is $\sigma(\mathcal{Q}_{\Delta' t}) \vee \mathfrak{N}$-measurable for some infinitesimal $\Delta' t$ in T. It is clear from the previous proof that in Theorem 5.6 one can assume $\mathfrak{B}_0 \supset \mathcal{Q}_{\Delta' t}$. Therefore by Theorem 4.4 and Remark 4.5(b) one can find an internal filtration $\{\mathfrak{B}_{\underline{t}}\}$, a $\mathfrak{B}_{\underline{t}}$-local martingale lifting of x and a $\mathfrak{B}_{\underline{t}}$-adapted SDJ lifting of h. This observation will prove useful in §7.

(c) Suppose X is a $\mathfrak{B}_{\underline{t}}$-local martingale lifting of $x \in \mathcal{L}^d$ and x is reduced by $\{U_n\}$ where $U_n < \infty$. We claim there is a sequence of $\mathfrak{B}_{\underline{t}}$-stopping times $\{V_n\}$ reducing X such that ${}^\circ V_n = U_n$ a.s. Let $\{V_n'\}$ reduce X and be bounded by some infinite α, let X_n be an S-integrable lifting of $x(U_n)$, and choose nondecreasing $\mathfrak{B}_{\underline{t}}$-stopping times W_n such that ${}^\circ W_n = U_n$ a.s. and ${}^\circ X(W_n) = {}^\circ X_n$ a.s. (see Theorem 4.7(b)). For each $n \in \mathbf{N}$, we have

$$\lim_{m \to \infty} {}^\circ \bar{E}\left(\| X(W_n \wedge V_m') - X_n \|\right) = \lim_{m \to \infty} E\left(\| x(U_n \wedge {}^\circ V_m') - x(U_n) \|\right) = 0.$$

(Note that $X(W_n \wedge V_m')$ is S-integrable by optional sampling.) Therefore, there is a sequence $\{m_k\}$ increasing to ∞ such that

$$\sup_{n \leqslant k} \bar{E}\left(\| X(W_n \wedge V_{m_k}') - X_n \|\right) < 2^{-k}. \tag{5.10}$$

By saturation we may extend $\{V_m'\}$ to a nondecreasing internal sequence of $\mathfrak{B}_{\underline{t}}$-stopping times. Let $V_n = W_n \wedge V_{m_\gamma}'$ for some $\gamma \in {}^*\mathbf{N} - \mathbf{N}$ for which (5.10) holds with γ in place of k. Then since ${}^\circ V_{m_\gamma}' = \infty$ a.s., it is easy to see that $\{V_n\}$ is the required sequence. $\square$

6. Quadratic variation.

Notation 6.1. Let Y_i: $T \times \Omega \to {}^*\mathbf{R}^d$ $(i = 1, 2)$ be internal and let y: $[0, \infty) \times \Omega \to \mathbf{R}^d$.

(i) If $z \in [0, \infty)$, $\tau_y(z) = \inf\{t \mid \| y(t) \| > z\}$ $(\inf \varnothing = \infty)$.

(ii) If $z \in {}^*[0, \infty)$, $T_{Y_i}(z) = \min\{\underline{t} \in T \mid \| Y_i(\underline{t}) \| > z\}$ $(\min \varnothing = \infty)$.

(iii) Let $Y_i^*(\underline{t}, \omega) = \max_{\underline{s} \leqslant \underline{t}} \| Y_i(\underline{s}, \omega) \|$.

(iv) Let $|Y_i|(\underline{t}, \omega) = \Sigma_{\underline{s}<\underline{t}} \|\Delta Y_i(\underline{s}, \omega)\|$ where $\Delta Y_i(\underline{s}) = Y_i(\underline{s} + \Delta t) - Y_i(\underline{s})$. Let $\lambda^{(j)}(Y_i, \omega)$ be the internal measure on $(T, \mathcal{C})$ ($\mathcal{C}$ is the set of all internal subsets of T) defined by $\lambda^{(j)}(Y_i, \omega)(\{\underline{t}\}) = \|\Delta Y_i(\underline{t}, \omega)\|^j$ for $j = 1$ or 2.

(v) Let $[Y_1, Y_2]_{\underline{t}} = Y_1(0) \cdot Y_2(0) + \Sigma_{\underline{s}<\underline{t}} \Delta Y_1(\underline{s}) \cdot \Delta Y_2(\underline{s})$, where $\cdot$ denotes the scalar product. □

Let $x \in \mathfrak{L}^d$. If $t > 0$ is fixed and $Q = \{t_0, \ldots, t_L\}$ is a finite subset of $[0, t]$ with $0 = t_0 < \cdots < t_L = t$, let $\|Q\| = \sup_{i \leq L} |t_i - t_{i-1}|$ and $S_t(x, Q) = \|x(0)\|^2 + \Sigma_{i=1}^{L} \|x(t_i) - x(t_{i-1})\|^2$. It follows from Doléans-Dade [7] that $S_t(x, Q)$ converges in probability to a limit, $[x, x]_t$, as $\|Q\|$ approaches zero. One may choose a version of the process $[x, x]_t$ with sample paths in D. If y is another d-dimensional local martingale then $[x, y]$ is defined by $[x, y] = 1/2([x + y, x + y] - [x, x] - [y, y])$.

If X is the local martingale lifting of x obtained in Theorem 5.6, we shall show that $[X, X]$ is a lifting of $[x, x]$. Indeed, we will prove directly that $S_t(x, Q)$ converges in probability to $\mathrm{st}([X, X])(t)$. The following two technical lemmas are used to show that $\mathrm{st}([X, X])$ makes sense.

LEMMA 6.2. *Let Y: $T \times \Omega \to {}^*\mathbf{R}^d$ be an SDJ lifting of a standard process y: $[0, \infty) \times \Omega \to \mathbf{R}^d$. For every $z \in [0, \infty)$, there exists $z' \approx z$ such that for a.a. ω in $\{\tau_y(z) < \infty\}$, ${}^\circ T_Y(z') = \tau_y(z)$ and ${}^\circ Y(T_Y(z')) = y(\tau_y(z))$.*

PROOF. If $z \in [0, \infty)$, we claim that $\tau_y(z) = \lim_{m \to \infty} {}^\circ T_Y(z + m^{-1})$ a.s., and that for almost all ω in $\{\tau_y(z) < \infty\}$,

$$y\big(\tau_y(z)\big) = \lim_{m \to \infty} {}^\circ Y\big(T_Y(z + m^{-1})\big). \tag{6.1}$$

If $L = \lim_{m \to \infty} {}^\circ T_Y(z + m^{-1})$, then clearly $\tau_y(z) \leq L$ a.s. If ω is fixed such that $\mathrm{st}(Y(\cdot, \omega)) = y(\cdot, \omega)$ and $\tau_y(z) < \infty$ (if $\tau_y(z) = \infty$, then $\tau_y(z) = L$ is immediate), and t exceeds $\tau_y(z)$, then $\|y(u)\| > z$ for some u in $[\tau(z), t)$. Therefore ${}^\circ\|Y(\underline{u})\| > z$ for some $\underline{u} \approx u$. It follows that ${}^\circ T_Y(z + m^{-1}) \leq {}^\circ \underline{u} < t$ for large enough m and hence $L < t$, proving the first claim. If ${}^\circ T_Y(z + m^{-1}) > \tau_y(z)$ for all m in $\mathbf{N}$ or ${}^\circ T_Y(z + m^{-1}) = \tau_y(z) = 0$ for large enough m, then (6.1) is immediate from the definition of $\mathrm{st}(Y)$. Assume that $0 < \tau_y(z) = {}^\circ T_Y(z + m^{-1}) < \infty$ for large enough m. Since $\|Y(T_Y(z + m^{-1}))\| > z + m^{-1}$ and $\|y({}^\circ T_Y(z + m^{-1})^-)\| = \|y(\tau_y(z)^-)\| \leq z$, it follows that ${}^\circ Y(T_Y(z + m^{-1})) = y(\tau_y(z))$ for large m whenever $Y(\cdot, \omega)$ is SDJ and $\mathrm{st}(Y(\cdot, \omega)) = y(\cdot, \omega)$. Hence the second claim is proved.

We may now choose a sequence $(m_n \in \mathbf{N})$ increasing to ∞ such that if Y_1 and Y_2 are liftings of $\tau_y(z)$ and $y(\tau_y(z))$, respectively, then the following conditions are satisfied for all n in $\mathbf{N}$:

$$\bar{P}\big(Y_1 \leq m_n, |T_Y\big(z + m_n^{-1}\big) - Y_1| > 2^{-n}\big) < 2^{-n}, \tag{6.2}$$

$$\bar{P}\big(Y_1 \leq m_n, \|Y\big(T_Y\big(z + m_n^{-1}\big)\big) - Y_2\| > 2^{-n}\big) < 2^{-n}. \tag{6.3}$$

By the permanence principle there is a γ in ${}^*\mathbf{N} - \mathbf{N}$ such that (6.2) and (6.3) hold when n is replaced by γ. The result now follows with $z' = z + m_\gamma^{-1}$. □

LEMMA 6.3. (a) *If Z is a nonnegative internal random variable, then Z is S-integrable if and only if there is an internal function* Φ: $^*[0,\infty) \to {}^*[0,\infty)$ *such that* $^\circ\bar{E}(\Phi(Z)) < \infty$ *and the following conditions hold*:

(i) Φ *is* (*internally*) *increasing and convex.*

(ii) $\Phi(0) = 0$ *and* $\sup_{x \geqslant \gamma} x/\Phi(x) \approx 0$ *for all* γ *in* $^*\mathbf{N} - \mathbf{N}$.

(iii) $\Phi(2u) \leqslant 4\Phi(u)$ *for all* $u \geqslant 0$.

(b) *If X is a d-dimensional $\mathfrak{B}_{\underline{t}}$-martingale, V is a $\mathfrak{B}_{\underline{t}}$-stopping time for some internal filtration $\{B_{\underline{t}}\}$, and $p \geqslant 1$, then $X^*(V)^p$ is S-integrable if and only if* $([X, X]_V)^{p/2}$ *is.*

PROOF. (a) ($\Leftarrow$) If Φ is as above and $\gamma \in {}^*\mathbf{N} - \mathbf{N}$, then

$$^\circ\int ZI_{\{Z>\gamma\}}\,d\bar{P} \leqslant {}^\circ\Big(\sup_{x\geqslant\gamma} x/\Phi(x)\Big)\,{}^\circ\bar{E}(\Phi(Z)) = 0,$$

whence the S-integrability of Z.

($\Rightarrow$) This is the nonstandard statement of Lemma 5.1 in Burkholder, Davis and Gundy [5], and the proof given there goes through with only minor changes.

(b) Suppose $X^*(V)^p$ is S-integrable. Choose Φ as in (a) such that $^\circ\bar{E}(\Phi(X^*(V)^p)) < \infty$. Clearly $\Psi(x) = \Phi(x^p)$ satisfies the hypotheses of Theorem 1.3, so that $^\circ\bar{E}(\Phi([X, X]_V^{p/2})) < \infty$. Therefore $[X, X]_V^{p/2}$ is S-integrable by (a). The proof of the converse is similar. $\square$

THEOREM 6.4. *Let Y be a d-dimensional $\mathfrak{B}_{\underline{t}}$-martingale and let $\{V_n\}$ be a nondecreasing sequence of $\mathfrak{B}_{\underline{t}}$-stopping times satisfying* (5.1) *and* (5.2). *Then*:

(a) $[Y, Y]$ *is* $S\tilde{D}$.

(b) *There is a nondecreasing sequence of $\mathfrak{B}_{\underline{t}}$-stopping times $\{W_n\}$ such that*

(i) $\lim_{n\to\infty} {}^\circ W_n = \infty$ *and* $^\circ W_n < \infty$ *a.s.*,

(ii) $Y^*(W_n)$ *and* $[Y, Y]_{W_n}^{1/2}$ *are S-integrable, and*

(iii) $Y^*(W_n - \Delta t) \leqslant n$ *for all* ω *in* $\{W_n > 0\}$.

If in addition Y is SDJ and $^\circ Y(V_n) = \mathrm{st}(Y)(^\circ V_n)$ *for a.a.* ω *in* $\{^\circ V_n < \infty\}$, *then we may also choose W_n so that*

(iv) $^\circ Y(W_n) = \mathrm{st}(Y)(^\circ W_n)$ *and* $^\circ[Y, Y]_{W_n} = \mathrm{st}([Y, Y])(^\circ W_n)$ *a.s.*

(*We shall see in Theorem* 7.18 *that under the additional hypotheses on Y assumed in* (iv), $[Y, Y]$ *is in fact SDJ.*)

PROOF. Clearly (a) will follow from (b) and Remark 2.4, because (i) and (ii) imply that $^\circ[Y, Y]_{\underline{t}} < \infty$ for all $\underline{t}$ in ns(T) a.s.

Choose $\underline{n} \in T$ such that $\underline{n} \approx n$. Let $W_n = V_n \wedge T_Y(n) \wedge \underline{n}$. Clearly both (i) and (iii) hold (recall that Y is SD by Theorem 5.2). To show (ii), note that if $Y^*(W_n) > n$, then $Y^*(W_n) = \|Y(W_n)\|$. Therefore if $\gamma \in {}^*\mathbf{N} - \mathbf{N}$, then

$$^\circ\int Y^*(W_n) I_{\{Y^*(W_n)>\gamma\}}\,d\bar{P} = {}^\circ\int \|Y(W_n)\|\, I_{\{\|Y\|(W_n)>\gamma\}}\,d\bar{P}. \tag{6.4}$$

The S-integrability of $Y(V_n \wedge \underline{n})$ implies that $Y(W_n)$ is also S-integrable by the optional sampling theorem. It follows that (6.4) equals zero and hence $Y^*(W_n)$ is S-integrable. By Lemma 3.6(b), $[Y, Y]_{W_n}^{1/2}$ is also S-integrable.

If Y is SDJ and $°Y(V_n) = \mathrm{st}(Y)(°V_n)$ a.s., change the definition of W_n as follows. By Lemma 6.2 and Theorem 4.7(b) there is an $n' \approx n$ and an $\underline{n} \approx n$ ($\underline{n} \in T$) such that $°T_Y(n') = \tau_y(n)$ and $°Y(T_Y(n')) = y(\tau_y(n))$ a.s. on $\{\tau_y(n) < \infty\}$, and $°Y(\underline{n}) = y(n)$ a.s. Let $W_n = T_Y(n') \wedge V_n \wedge \underline{n}$. Then $°Y(W_n) = y(°W_n)$ a.s. and the above argument goes through without change. The following lemma shows that $°[Y, Y]_{W_n} = \mathrm{st}([Y, Y])(°W_n)$ a.s. and hence completes the proof.

LEMMA 6.5. *If (Y_1, Y_2) is an SDJ S-local martingale, where Y_i is d-dimensional for $i = 1, 2$, and V is a *-stopping time (all with respect to an internal filtration $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$), then $°Y_i(V) = \mathrm{st}(Y_i)(°V)$ a.s. on $\{°V < \infty\}$ for either $i = 1$ or $i = 2$ implies that $°[Y_1, Y_2]_V = \mathrm{st}([Y_1, Y_2])(°V)$ a.s. on $\{°V < \infty\}$. Conversely if $°[Y_1, Y_1]_V = \mathrm{st}([Y_1, Y_1])(°V)$ a.s. on $\{°V < \infty\}$ then $°Y_1(V) = \mathrm{st}(Y_1)(°V)$ a.s. on $\{°V < \infty\}$. Moreover this converse holds if Y_1 is only SD.*

PROOF. ($\Rightarrow$) Bear in mind that we are free to use Theorem 6.4 except for the last part of Theorem 6.4(b)(iv). Assume that $°Y_1(V) = \mathrm{st}(Y_1)(°V)$ a.s. on $\{°V < \infty\}$, and let $\underline{\delta}$ be a positive infinitesimal. Then

$$\begin{aligned}([Y_1, Y_2]_{V+\underline{\delta}} - [Y_1, Y_2]_V) &= \sum_{V \leq \underline{s} < V+\underline{\delta}} \Delta Y_1(\underline{s}) \cdot \Delta Y_2(\underline{s}) \\ &\leq \Big(\sum_{V \leq \underline{s} < V+\underline{\delta}} \|\Delta Y_1(\underline{s})\|^2 \Big)^{1/2} \Big(\sum_{V \leq \underline{s} < V+\underline{\delta}} \|\Delta Y_2(\underline{s})\|^2 \Big)^{1/2} \\ &= ([Y_1, Y_1]_{V+\underline{\delta}} - [Y_1, Y_1]_V)^{1/2} ([Y_2, Y_2]_{V+\underline{\delta}} - [Y_2, Y_2]_V)^{1/2}.\end{aligned}$$

Note that $[Y_1, Y_2]$ is SD by the previous theorem and the fact that

$$[Y_1, Y_2] = 1/2([Y_1 + Y_2, Y_1 + Y_2] - [Y_1, Y_1] - [Y_2, Y_2]).$$

Since $\underline{\delta}$ was an arbitrary positive infinitesimal, the result will follow if we show that $[Y_1, Y_1]_{V+\delta} \approx [Y_1, Y_1]_V$ a.s. on $\{°V < \infty\}$. If $\{W_n\}$ is the sequence obtained in Theorem 6.4 with X replaced by Y_1, then by Theorem 1.3 there is a real constant c for which

$$\begin{aligned}&°\bar{E}\Big(([Y_1, Y_1]_{(V+\underline{\delta})\wedge W_n} - [Y_1, Y_1]_{V \wedge W_n})^{1/2}\Big) \\ (6.5) \qquad &\leq c\, °\bar{E}\Big(\max_{V \wedge W_n \leq \underline{u} \leq (V+\underline{\delta}) \wedge W_n} \|Y_1(\underline{u}) - Y_1(V \wedge W_n)\| \Big) \\ &= cE\Big(°\max_{V \wedge W_n \leq \underline{u} \leq (V+\underline{\delta}) \wedge W_n} \|Y_1(\underline{u}) - Y_1(V \wedge W_n)\| \Big),\end{aligned}$$

since the integrand is bounded by $2Y_1^*(W_n)$, which is S-integrable. Recalling that Y_1 is SDJ and $°Y_1(V \wedge W_n) = \mathrm{st}(Y_1)(°(V \wedge W_n))$ a.s., we see that (6.5) is zero, and therefore by letting n approach ∞ we have $[Y_1, Y_1]_{V+\delta} \approx [Y_1, Y_1]_V$ a.s. on $\{°V < \infty\}$, as required.

The proof of the converse is similar to the above using the other inequality from Theorem 1.3. □

Notation 6.6. If $T' = \{\underline{t}_0, \ldots, \underline{t}_L\}$ is a *-finite subset of T ($0 = \underline{t}_0 < \underline{t}_1 < \cdots < \underline{t}_L$) and $Y\colon T \times \Omega \to {}^*\mathbf{R}^d$ is internal, let $[Y, Y]^{T'}_{\underline{t}_j} = \|Y(0)\|^2 + \sum_{i=1}^{j} \|Y(\underline{t}_i) - Y(\underline{t}_{i-1})\|^2$. □

Recall the notation $S_t(x, Q)$ introduced at the beginning of this section.

THEOREM 6.7. *Let X be a d-dimensional SDJ S-local martingale with respect to an internal filtration $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$. If $x = \operatorname{st}(X)$, then for each $t \geq 0$, $S_t(x, Q)$ converges in probability to* $\operatorname{st}([X, \bar{X}])(t)$ *as* $\|Q\|$ *approaches zero.*

We need the following lemma.

LEMMA 6.8. *Let X be as in Theorem 6.7. If T' is a *-finite S-dense subset of T such that $0 \in T'$, then*

$$\sup_{\underline{t} \in \operatorname{ns}(T')} {}^\circ\left|[X, X]_{\underline{t}} - [X, X]^{T'}_{\underline{t}}\right| = 0 \quad a.s.$$

PROOF. If $X = (X_1, \ldots, X_d)$, then $[X, X]_t = \sum_{i=1}^d [X_i, X_i]_t$ and therefore we may assume $d = 1$. Let $\{W_n\}$ be a sequence of *-stopping times that satisfy conditions (i)–(iv) in Theorem 6.4. An elementary computation shows that for $\underline{t}$ in T',

$$[X, X]^{T'}_{\underline{t}} - [X, X]_{\underline{t}} = 2 \sum_{\substack{\underline{s} < \underline{t} \\ \underline{s} \in T}} \left(X(\underline{s}) - X\left([\underline{s}]^{T'}\right)\right)\Delta X(\underline{s}). \tag{6.6}$$

Let $Z(\underline{t})$ be defined to be the right side of (6.6) for all $\underline{t}$ in T. Then

$$[Z, Z]_{\underline{t}} = 4\int_T I_{\{\underline{s}<\underline{t}\}}\left(X(\underline{s}) - X\left([\underline{s}]^{T'}\right)\right)^2 d\lambda^{(2)}(X).$$

In particular, since $X^*(W_n - \Delta t) \leq n$ on $\{W_n > 0\}$ (by Theorem 6.4(b)),

$$[Z, Z]^{1/2}_{W_n} \leq 4\, n\left([X, X]_{W_n}\right)^{1/2}$$

and therefore $[Z, Z]^{1/2}_{W_n}$ is S-integrable. It follows that $\|\Delta Z(\underline{t}) I_{\{\underline{t} < W_n\}}\|$ is internally integrable and hence $Z(\underline{t} \wedge W_n)$ is a $\mathfrak{B}_{\underline{t}}$-martingale. Using Theorem 1.3, we obtain c in $\mathbf{R}$ such that

$$\begin{aligned}
{}^\circ\bar{E}\left(Z^*(W_n)\right) &\leq c\,{}^\circ\bar{E}\left(\left([Z, Z]_{W_n}\right)^{1/2}\right) \\
&= 2cE\left(\left({}^\circ\int_T I_{\{\underline{s} < W_n\}}\left(X(\underline{s}) - X\left([\underline{s}]^{T'}\right)\right)^2 d\lambda^{(2)}(X)\right)^{1/2}\right) \\
&\leq 4c\left(N^{-1}E\left({}^\circ[X, X]^{1/2}_{W_n}\right) + nE\left(\left({}^\circ\lambda^{(2)}(X)(A_N)\right)^{1/2}\right)\right)
\end{aligned} \tag{6.7}$$

for all N in $\mathbf{N}$, where

$$A_N(\omega) = \left\{\underline{s} \in T \mid \underline{s} < W_n, \left|X(\underline{s}) - X\left([\underline{s}]^{T'}\right)\right| > 2N^{-1}\right\}.$$

If $T_i(\omega)$ is the ith time for which $|X(\underline{t}) - X(\underline{t} - \Delta t)| > N^{-1}$ ($T_i(\omega) = \infty$ if no such time exists) and $T_i' = \min\{\underline{t} \in T' \cup \{\infty\} \mid \underline{t} \geq T_i\}$, then T_i and T_i' are both

*-stopping times, and, since X is SDJ, $A_N \subset \bigcup_{i\in\mathbf{N}}[T_i \wedge W_n, T_i' \wedge W_n)$ a.s. This implies that for some real constant c,

$$E\left({}^\circ\lambda^{(2)}(X)(A_N)^{1/2}\right) \leqslant \sum_{i\in\mathbf{N}} {}^\circ\bar{E}\left(([X,X]_{T_i'\wedge W_n} - [X,X]_{T_i\wedge W_n})^{1/2}\right)$$

$$\leqslant c\sum_{i\in\mathbf{N}} {}^\circ\bar{E}\left(\max_{T_i\wedge W_n\leqslant \underline{u}\leqslant T_i'\wedge W_n} |X(\underline{u}) - X(T_i\wedge W_n)|\right)$$

(by Theorem 1.3)

$$= c\sum_{i\in\mathbf{N}} {}^\circ E\left(\max_{T_i\wedge W_n\leqslant \underline{u}\leqslant T_i'\wedge W_n} |X(\underline{u}) - X(T_i\wedge W_n)|\right)$$

(since $X^*(W_n)$ is S-integrable)

$$= 0$$

(the last because X is SDJ and $|X(T_i) - X(T_i - \Delta t)| > N^{-1}$ a.s. on $\{{}^\circ T_i < \infty\}$). Substituting this into (6.7) and letting N approach ∞, we get ${}^\circ Z^*(W_n) = 0$ a.s. The result follows by letting n approach infinity. □

Proof of Theorem 6.7. Fix t in $[0,\infty)$ and $\underline{t}$ in T such that $\underline{t} \approx t$, ${}^\circ X(\underline{t}) = x(t)$ a.s., and ${}^\circ[X,X]_{\underline{t}} = \mathrm{st}([X,X])(t)$ a.s. (see Theorem 4.7). If T' is a *-finite subset of $T \cap [0,\underline{t}]$ containing 0 and $\underline{t}$, then the previous result implies that whenever $\|T'\| \approx 0$,

$$\bar{P}\left(|[X,X]_{\underline{t}}^{T'} - [X,X]_{\underline{t}}| > 2^{-N}\right) < 2^{-N} \tag{6.8}$$

for all N in $\mathbf{N}$. Hence, by the permanence principle there is a sequence of positive reals $\{\varepsilon_N\}$ such that (6.8) holds whenever $\|T'\| < \varepsilon_N$. Let $Q = \{t_0, t_1, \ldots, t_L\}$ $(0 = t_0 < t_1 < \cdots < t_L = t)$ satisfy $\|Q\| < \varepsilon_N$ and choose $\underline{t}_i \approx t_i$ $(\underline{t}_i \in T)$ such that ${}^\circ X(\underline{t}_i) = x(t_i)$ a.s., $\underline{t}_0 = 0$, and $\underline{t}_L = \underline{t}$. If $T' = \{\underline{t}_0, \ldots, \underline{t}_L\}$, then $\|T'\| < \varepsilon_N$ and

$$\begin{aligned} {}^\circ[X,X]_{\underline{t}}^{T'} &= {}^\circ\|X(0)\|^2 + \sum_{i=0}^{L-1} {}^\circ\|X(\underline{t}_{i+1}) - X(\underline{t}_i)\|^2 \\ &= S_t(x,Q) \quad \text{a.s.} \end{aligned}$$

Since $\mathrm{st}([X,X])(t) = {}^\circ[X,X]_{\underline{t}}$ a.s., (6.8) implies that

$$P\left(|S_t(x,Q) - \mathrm{st}([X,X])(t)| > 2^{-N}\right) < 2^{-N},$$

and hence the result. □

Note that the above result is not true if X is not SDJ.

In view of the previous theorem, we make the following definition.

Definition 6.9. If x and y are d-dimensional local martingales, let (X,Y) be a $\mathfrak{B}_{\underline{t}}$-local martingale lifting of (x,y) for some internal filtration $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$. Then let

$$[x,y]_t = \mathrm{st}([X,Y])(t). \quad \square$$

Since $[X,Y] = 1/2([X+Y, X+Y] - [X,X] - [Y,Y])$, it follows from Theorem 6.4 that $\mathrm{st}([X,Y])$ exists. Moreover, Theorem 6.7 shows that the above definition is independent of the choice of the lifting (X,Y) (at least up to indistinguishability), and agrees with the classical definition described at the beginning of this section.

7. Stochastic integration.

DEFINITIONS 7.1. A process of bounded variation a: $[0,\infty) \times \Omega \to \mathbf{R}^d$ is an $\mathcal{F}_t$-adapted process whose sample paths belong to D, are of bounded variation on bounded intervals, and satisfy $a(0) = 0$. Let $\mathcal{V}_0^d$ denote the set of such processes and let $|a|(t)$ denote the variation of a on $[0, t]$.

A d-dimensional semimartingale, z, is an $\mathcal{F}_t$-adapted, $\mathbf{R}^d$-valued process with sample paths in D such that $z(t) - z(0) = x(t) + a(t)$ for some $x \in \mathcal{L}_0^d$ and $a \in \mathcal{V}_0^d$. The set of d-dimensional semimartingales is denoted by $\mathcal{S}^d$ and $\mathcal{S}_0^d = \{z \in \mathcal{S}^d \mid z(0) = 0\}$. $\square$

Note that if $z \in \mathcal{S}^d$, the decomposition $z - z(0) = x + a$ with $x \in \mathcal{L}_0^d$ and $a \in \mathcal{V}_0^d$ need not be unique.

NOTATION 7.2. (a) The σ-field of predictable sets in $[0,\infty) \times \Omega$ is denoted by $\mathcal{P}$. That is to say, $\mathcal{P}$ is the σ-field on $[0,\infty) \times \Omega$ generated by the set of all $\mathcal{F}_t$-adapted, left-continuous processes.

(b) Suppose M is a normed linear space with norm $\|\ \|$. If $x \in \mathcal{L}_0^d$, let

$$\mathcal{L}_{\mathrm{loc}}(x, M) = \Big\{h\colon [0,\infty) \times \Omega \to M \mid h \text{ is predictable and } E\Big(\Big(\int_0^{R_n} \|h(s)\|^2 d[x,x]_s\Big)^{1/2}\Big) < \infty \text{ for some sequence of stopping times } \{R_n\} \text{ increasing to } \infty \text{ a.s.}\Big\},$$

and if $a \in \mathcal{V}_0^d$, let

$$\mathcal{L}_s(a, M) = \Big\{h\colon [0,\infty) \times \Omega \to M \mid h \text{ is predictable and } \int_0^t \|h(s)\| d|a|_s < \infty \text{ for all } t \geq 0 \text{ a.s.}\Big\}$$

(the "s" stands for "Stieltjes integrable"). Finally, for $z \in \mathcal{S}_0^d$ define

$$\mathcal{L}(z, M) = \big\{h\colon [0,\infty) \times \Omega \to M \mid h \text{ is predictable and for some } (a,x) \in \mathcal{V}_0^d \times \mathcal{L}_0^d,\ z = a + x \text{ and } h \in \mathcal{L}_{\mathrm{loc}}(x, M) \cap \mathcal{L}_s(a, M)\big\}. \quad \square$$

If $x \in \mathcal{L}_0^1$, the classical stochastic integral $h \cdot x(t) = \int_0^t h(s)\,dx(s)$ may be defined for $h \in \mathcal{L}_{\mathrm{loc}}(x, \mathbf{R})$ as the unique element of $\mathcal{L}_0^1$ such that $[h \cdot x, y]_t = \int_0^t h(s) d[x, y]_s$ for all bounded martingales y (see Meyer [**20**, p. 341]). More generally, if $z \in \mathcal{S}_0^1$, the classical stochastic integral $h \cdot z(t) = \int_0^t h(s) dz(s)$ may be defined for $h \in \mathcal{L}(z, \mathbf{R})$ by $h \cdot z(t) = h \cdot x(t) + \int_0^t h(s) da(s)$, where $x \in \mathcal{L}_0^1$, $a \in \mathcal{V}_0^1$, $z = a + x$ and $h \in \mathcal{L}_{\mathrm{loc}}(x, \mathbf{R}) \cap \mathcal{L}_s(a, \mathbf{R})$ (see Jacod [**10**]). If $z \in \mathcal{S}_0^d$ and h is a predictable process taking values in $\mathbf{R}^{k \times d}$ (the normed linear space of $k \times d$ matrices over $\mathbf{R}$ with the Euclidean norm) such that $h_{ij} \in \mathcal{L}(z_j, \mathbf{R})$ for $j = 1, \ldots, d$, then $h \cdot z \in \mathcal{S}_0^k$ is defined by $(h \cdot z)_i = \sum_{j=1}^d h_{ij} \cdot z_j$. It is possible to define $h \cdot z$ for a larger class of integrands h (see Jacod [**10**]).

We shall obtain a nonstandard representation of $h \cdot z$ for $z \in \mathcal{S}_0^d$ and $h \in \mathcal{L}(z, \mathbf{R}^{k\times d})$, independently of the classical construction, by first obtaining appropriate liftings, Z and H, of z and h, respectively, and then defining $h \cdot z(t) = \mathrm{st}(\Sigma_{\underline{s}<\cdot} H(\underline{s})\Delta Z(\underline{s}))(t)$.

We will need the following result on predictable processes, that follows easily from Dellacherie and Meyer [**6**, IV, Theorem 67] and a monotone class argument. If M is a normed linear space, we let

$$V'(M) = \Big\{h\colon [0,\infty)\times\Omega \to M \mid h(t,\omega) = h_0(\omega)I_{\{0\}}(t) + \sum_{i=1}^{n-1} h_i(\omega)I_{(t_i,t_{i+1}]}(t), \text{ where } h_i \text{ is bounded and } \mathcal{F}_{t_i}\text{-measurable and } 0 = t_1 < t_2 < \cdots < t_n \leqslant \infty\Big\}.$$

LEMMA 7.3. *If M is a separable normed linear space and V is a vector space of bounded predictable M-valued processes that contains $V'(M)$ and is closed under bounded pointwise convergence, then V is the space of all bounded, M-valued, predictable processes.* □

DEFINITION 7.4. Let $\{\mathfrak{B}_{\underline{t}}\}$ be an internal filtration. If $a \in \mathcal{V}_0^d$, a $\mathfrak{B}_{\underline{t}}$-$BV$ lifting of a is a $\mathfrak{B}_{\underline{t}}$-adapted process, A, such that A and $|A|$ (see Notation 6.1) are SDJ liftings of a and $|a|$, respectively. If $a \in \mathcal{V}_0^d$ and $x \in \mathcal{L}^d$, then a $\mathfrak{B}_{\underline{t}}$-semimartingale lifting of $(a; x)$ is a pair $(A; X)$ such that X is a $\mathfrak{B}_{\underline{t}}$-local martingale lifting of x, A is a $\mathfrak{B}_{\underline{t}}$-$BV$ lifting of a, and (A, X) is SDJ. □

LEMMA 7.5. *Suppose $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ is an internal filtration and $Y\colon T\times\Omega \to {}^*\mathbf{R}^d$ is a $\mathfrak{B}_{\underline{t}}$-adapted, SD lifting of $y \in \mathcal{V}_0^d$ such that ${}^\circ Y(0) = y(0)$ a.s. There is a positive infinitesimal $\Delta' t$ in T such that if T'' is an internal S-dense subset of $T' = \{k\Delta' t \mid k \in {}^*\mathbf{N}_0\}$, then $Y([\underline{t}]^{T''})$ is a $\mathfrak{B}_{[\underline{t}]^{T''}}$-$BV$ lifting of y.*

PROOF. By Proposition 4.8, there is a positive infinitesimal $\widehat{\Delta t}$ in T such that if $\hat{T} = \{\widehat{k\Delta t} \mid k \in {}^*\mathbf{N}_0\}$, then $Y\restriction \hat{T}\times\Omega$ is SDJ (we may include 0 in $\hat{T}$ since ${}^\circ Y(0) = y(0)$ a.s.). Let $\tilde{Y}$ be an SDJ lifting of $|y|$. By Theorem 4.7(b) there is a $\Delta_n t$ in $\hat{T}$ such that ${}^\circ\Delta_n t = 2^{-n}$ and for $k = 0,\dots,2^{2n}$, ${}^\circ\tilde{Y}(k\Delta_n t) = |y|(k2^{-n})$ and ${}^\circ Y(k\Delta_n t) = y(k2^{-n})$ a.s. Therefore, for each $0 \leqslant k_1 \leqslant k_2 < 2^{2n}$,

$$\begin{aligned}{}^\circ\sum_{j=k_1}^{k_2} \|Y((j+1)\Delta_n t) - Y(j\Delta_n t)\| &= \sum_{j=k_1}^{k_2} \|y((j+1)2^{-n}) - y(j2^{-n})\| \\ &\leqslant {}^\circ\big(\tilde{Y}((k_2+1)\Delta_n t) - \tilde{Y}(k_1\Delta_n t)\big).\end{aligned}$$

By permanence there is a positive infinitesimal $\Delta' t \in \hat{T}$ such that if $T' = \{k\Delta' t \mid k \in {}^*\mathbf{N}_0\}$, then for a.a. ω and all $k_1\Delta' t \leqslant k_2\Delta' t$ in $\mathrm{ns}(T')$.

$$(7.1)\qquad {}^\circ\sum_{j=k_1}^{k_2} \|Y((j+1)\Delta' t) - Y(j\Delta' t)\| \leqslant {}^\circ\tilde{Y}((k_2+1)\Delta' t) - {}^\circ\tilde{Y}(k_1\Delta' t).$$

If $Y'(\underline{t}) = Y([\underline{t}]^{T'})$, then Y' is SDJ since $T' \subset \hat{T}$. Since $\tilde{Y}$ is SDJ, it follows from (7.1) that $|Y'|$ is SDJ. Moreover, by (7.1) we also have

$$\mathrm{st}(|Y'|)(t) \leqslant |y|(t)$$

for all $t \geqslant 0$ a.s. Therefore Y' is a $\mathcal{B}_{[\underline{t}]^{T'}}$-$BV$ lifting of y because the converse inequality is obvious. If T'' is an internal $\bar{S}$-dense subset of T' and $Y''(\underline{t}) = Y([\underline{t}]^{T''})$, then Y'' is clearly a $\mathcal{B}_{[\underline{t}]^{T''}}$-$BV$ lifting of y because $|Y''|(\underline{t}) \leqslant |Y'|(\underline{t})$. □

THEOREM 7.6. *If $(a, x) \in \mathcal{V}_0^d \times \mathcal{L}^n$ and h: $[0, \infty) \times \Omega \to \mathbf{R}^m$ is an adapted process with sample paths a.s. in D, there is an internal filtration, $\{\mathcal{B}_{\underline{t}}\}$, a $\mathcal{B}_{\underline{t}}$-semimartingale lifting of $(a; x)$, $(A; X)$, and a $\mathcal{B}_{\underline{t}}$-adapted SDJ lifting of h, H, such that (H, A, X) is SDJ.*

PROOF. By Theorem 5.6 and Remark 5.7(b), there is an internal filtration $\{\mathcal{B}'_{\underline{t}}\}$, a $\mathcal{B}'_{\underline{t}}$-local martingale lifting of x, X', and a $\mathcal{B}'_{\underline{t}}$-adapted SDJ lifting of (h, a), (H', A'). Proposition 4.8 implies that for some positive infinitesimal $\Delta' t$ in T, if $T' = \{k\Delta' t \mid k \in {}^*\mathbf{N}_0\}$, then $(H', A', X') \restriction T' \times \Omega$ is SDJ (we may include zero in T' since ${}^\circ(H', A', X')(0) = (h, a, x)(0)$ a.s.). By Lemma 7.5 we may also choose T' so that $A(\underline{t}) = A'([\underline{t}]^{T'})$ is a $\mathcal{B}'_{[\underline{t}]^{T'}}$-$BV$ lifting of a. Let $\mathcal{B}_{\underline{t}} = \mathcal{B}'_{[\underline{t}]^{T'}}$ and $H(\underline{t}) = H'([\underline{t}]^{T'})$. Choose W as in Lemma 5.5 so that $X(\underline{t}) = X'([\underline{t}]^{T'} \wedge W)$ is a $\mathcal{B}_{\underline{t}}$-local martingale lifting of x. Then (H, A, X) is SDJ by the choice of T' and hence $(A; X)$ and H are the required liftings. □

REMARK 7.7. For our immediate goal of defining the stochastic integral, the existence of a semimartingale lifting of $(a; x)$ is the only part of Theorem 7.6 that will be needed. The part of the above result that deals with the auxiliary process h will be useful in solving stochastic differential equations in §10. □

Having obtained a lifting of a semimartingale, the next step is to lift the integrand h in $h \cdot z$. To this end we define a measure on $T \times \Omega$ with respect to which we will lift h.

Notation 7.8. If $B \subset T \times \Omega$, let $B(\omega) = \{\underline{t} \mid (\underline{t}, \omega) \in B\}$ and $B(\underline{t}) = \{\omega \mid (\underline{t}, \omega) \in B\}$. If A and X are internal ${}^*\mathbf{R}^d$-valued processes, define an internal random measure $\lambda(A; X)$ on $(T, \mathcal{C})$ by $\lambda(A; X)(\{\underline{t}\}) = \|\Delta A(\underline{t})\| + \|\Delta X(\underline{t})\|^2$. Let

$$S_n = \min\{\underline{t} \in T \mid \lambda(A; X)([0, \underline{t}]) \geqslant n\} \wedge n,$$

and let $\mu(A; X)$ be the internal subprobability measure on $\mathcal{C} \times \mathcal{Q}$ defined by

$$\mu(A; X)(B) = \bar{E}\left(\sum_{n \in {}^*\mathbf{N}} \lambda(A; X)\big(B(\omega) \cap [0, S_n)\big) n^{-1} 2^{-n} \right).$$

If $\{\mathcal{B}_{\underline{t}} \mid \underline{t} \in T\}$ is an internal filtration, let $\bar{\mathcal{P}}(\mathcal{B}.)$ be the internal σ-algebra of $\mathcal{B}_{\underline{t}}$-adapted subsets of $T \times \Omega$ (here A is $\mathcal{B}_{\underline{t}}$-adapted if I_A is). Let $L_{(A;X)}(\bar{\mathcal{P}}(\mathcal{B}.))$ denote the trace of the $L(\mu(A; X))$-completion of $\sigma(\bar{\mathcal{P}}(\mathcal{B}.))$ on $\mathrm{ns}(T) \times \Omega$. If there is no ambiguity the dependence on A, X and $\{\mathcal{B}_{\underline{t}}\}$ is suppressed and we simply write $L(\bar{\mathcal{P}})$. If M is a normed linear space with norm $\| \ \|$, let $L(A; X, M, \mathcal{B}.)$ denote the set of $\mathcal{B}_{\underline{t}}$-adapted processes H: $T \times \Omega \to {}^*M$ such that

for a.a. ω and all $\underline{t}$ in $\mathrm{ns}(T)$,

$$\text{(7.2)} \qquad {}^\circ\!\int I_{\{\underline{s} < \underline{t}\}} \|H(\underline{s})\| \, d\lambda^{(1)}(A) = \int I_{\{\underline{s} < \underline{t}\}} {}^\circ\|H(\underline{s})\| \, dL(\lambda^{(1)}(A)),$$

and there is a nondecreasing sequence of *-stopping times $\{V_n\}$ such that

(7.3) $$\text{for a.a. } \omega,\ {}^\circ V_n < \infty,\ \lim_{n\to\infty} {}^\circ V_n = \infty \text{ and } \operatorname{st}(X)({}^\circ V_n) = {}^\circ X(V_n),$$

and

(7.4) $$\begin{aligned} &{}^\circ \bar{E}\left(\left(\int I_{\{\underline{s}<V_n\}} \|H(\underline{s})\|^2 \, d\lambda^{(2)}(X)\right)^{1/2}\right) \\ &\qquad = E\left(\left(\int I_{\{\underline{s}<V_n\}} {}^\circ\|H(\underline{s})\|^2 \, dL(\lambda^{(2)}(X))\right)^{1/2}\right) < \infty. \end{aligned}$$

(The definitions of $\lambda^{(j)}$ are given in Notation 6.1.) □

Note that an internal process is $\bar{\mathcal{P}}(\mathfrak{B}.)$-measurable if and only if it is $\mathfrak{B}_{\underline{t}}$-adapted.

DEFINITION 7.9. If $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ is an internal filtration, A and X are $\mathfrak{B}_{\underline{t}}$-adapted, $^*\mathbf{R}^d$-valued processes, M is a Hausdorff space and $h\colon [0,\infty) \times \Omega \to M$, then a weak $(A;\ X, \mathfrak{B}.)$-lifting (or a weak $(A;\ X)$-lifting, if there is no ambiguity) of h is a $\mathfrak{B}_{\underline{t}}$-adapted process $H\colon T \times \Omega \to {}^*M$ such that

(7.5) $${}^\circ H(\underline{t}, \omega) = h({}^\circ\underline{t}, \omega) \quad L(\mu(A;\ X))\text{-a.s.}$$

We call H an $(A;\ X, \mathfrak{B}.)$-lifting (or an $(A;\ X)$-lifting) of h if, in addition, M is a normed linear space and $H \in L(A;\ X, M, \mathfrak{B}.)$. □

At this point we establish a pair of results which will help us deal with the measures $L(\mu(A;\ X))$. Assume that $\lim_{n\to\infty} {}^\circ S_n = \infty$ a.s.

(1) We claim that if $\{G_m\}$ is a sequence of sets in $\mathcal{C} \times \mathcal{Q}$, then

$$\lim_{m\to\infty} {}^\circ\mu(A;\ X)(G_m) = 0$$

if and only if ${}^\circ\lambda(A;\ X)(G_m(\omega) \cap [0, S_n))$ converges in probability to zero as $m \to \infty$, for all $n \in \mathbf{N}$.

Let

$$Y_m = \sum_{n \in {}^*\mathbf{N}} \lambda(A;\ X)(G_m(\omega) \cap [0, S_n))n^{-1}2^{-n}.$$

Then ${}^\circ\mu(A;\ X)(G_m) = {}^\circ\bar{E}(Y_m)$, and since $0 \leqslant Y_m \leqslant 1$, we see that

$$\lim_{m\to\infty} {}^\circ\mu(A;\ X)(G_m) = 0$$

if and only if ${}^\circ Y_m$ converges in probability to zero. Note that if $N \in \mathbf{N}$, then

$$\begin{aligned} P({}^\circ Y_m > 2^{-N+1}) &\leqslant P\left(\sum_{n=1}^{N} {}^\circ\lambda(A;\ X)(G_m(\omega) \cap [0, S_n))n^{-1}2^{-n} > 2^{-N}\right) \\ &\leqslant P({}^\circ Y_m > 2^{-N}). \end{aligned}$$

Therefore ${}^\circ Y_m$ converges in probability to zero if and only if the same is true of ${}^\circ\lambda(A;\ X)(G_m(\omega) \cap [0, S_n))$, for each $n \in \mathbf{N}$, and the claim is proved.

(2) We claim that if G is in the $L(\mu(A;\ X))$-completion of $\sigma(\mathcal{C} \times \mathcal{Q})$, then

$$L(\mu(A;\ X))(G) = 0$$

if and only if for almost all ω, $L(\lambda(A;\ X))(G(\omega) \cap \operatorname{ns}(T)) = 0$.

Choose a nonincreasing sequence, $\{G_m\}$, and a nondecreasing sequence, $\{F_m\}$, of sets in $\mathcal{C} \times \mathcal{Q}$ such that $F_m \subset G \subset G_m$ and $\lim_{m\to\infty} {}^\circ\mu(A;\, X)(G_m - F_m) = 0$. By the above result, we have (note that $G_m - F_m$ is nonincreasing in m)

$$\lim_{m\to\infty} {}^\circ\lambda(A;\, X)\big((G_m(\omega) - F_m(\omega)) \cap [0, S_n)\big) = 0 \quad \text{a.s. for all } n \in \mathbf{N}.$$

It follows that for a.a. ω and all $n \in \mathbf{N}$,

$$\begin{aligned} L(\lambda(A;\, X))\big(G(\omega) \cap [0, S_n)\big) &= \lim_{m\to\infty} {}^\circ\lambda(A;\, X)\big(G_m(\omega) \cap [0, S_n)\big) \\ &= \lim_{m\to\infty} {}^\circ\lambda(A;\, X)\big(F_m(\omega) \cap [0, S_n)\big). \end{aligned}$$

Therefore

$$\begin{aligned} L(\mu(A;\, X))(G) = 0 &\Leftrightarrow \lim_{m\to\infty} {}^\circ\mu(A;\, X)(G_m) = \lim_{m\to\infty} {}^\circ\mu(A;\, X)(F_m) = 0 \\ &\Leftrightarrow \lim_{m\to\infty} {}^\circ\lambda(A;\, x)\big(G_m(\omega) \cap [0, S_n)\big) \\ &= \lim_{m\to\infty} {}^\circ\lambda(A;\, X)\big(F_m(\omega) \cap [0, S_n)\big) = 0 \quad \text{a.s.} \end{aligned}$$

for all $n \in \mathbf{N}$ (by the previous result and the monotonicity of $\{F_m\}$ and $\{G_m\}$)

$$\Leftrightarrow L(\lambda(A;\, X))\big(G(\omega) \cap \mathrm{ns}(T)\big) = 0 \quad \text{a.s.}$$

(by the above equality), and the claim is proved.

In particular if h is predictable (say), then (7.5) is equivalent to

$$(7.5)' \qquad {}^\circ H(\underline{t}, \omega) = h({}^\circ\underline{t}, \omega) \quad L(\lambda(A;\, X))\text{-a.s. on } \mathrm{ns}(T) \text{ for a.a. } \omega.$$

LEMMA 7.10. *Let A and X be ${}^*\mathbf{R}^d$- and ${}^*\mathbf{R}^k$-valued internal processes, respectively, such that $|A| + [X, X]$ is SD and a.s. S-continuous at zero. If $\{\mathcal{B}_{\underline{t}} \mid \underline{t} \in T\}$ is an internal filtration and $\mathrm{st}(\underline{t}, \omega) = ({}^\circ\underline{t}, \omega)$ for $(\underline{t}, \omega) \in \mathrm{ns}(T) \times \Omega$, then $\mathrm{st}^{-1}(B) \in L_{(A;X)}(\overline{\mathcal{P}}(\mathcal{B}.))$ for all $B \in \mathcal{P}$.*

PROOF. If $B = (t_1, t_2] \times C$ for $C \in \mathcal{F}_{t_1}$, then by Theorems 3.2 and 4.7(b), there exist $\underline{t}_i \approx t_i$ and $D \in \mathcal{B}_{\underline{t}_1}$ such that $P(C \Delta D) = 0$ and ${}^\circ|A|(\underline{t}_i) + {}^\circ[X, X]_{\underline{t}_i} = \mathrm{st}(|A| + [X, X])(t_i)$ a.s. $(i = 1, 2)$. Let $F = ((\underline{t}_1, \underline{t}_2] \cap T) \times D$. We claim that $L(\mu(A;\, X))(\mathrm{st}^{-1}(B) \Delta F) = 0$. Let $\{G_n\}$ be a decreasing sequence of sets in $\mathcal{Q}$ such that $C \Delta D \subset \bigcap_n G_n$ and $\lim_{n\to\infty} P(G_n) = 0$, and let $\underline{\delta}_n \approx n^{-1}$. If $H_n = T \times G_n$ and $K_n = (\bigcup_{i=1}^2 (\underline{t}_i, \underline{t}_i + \underline{\delta}_n)) \times \Omega$, then

$$\mathrm{st}^{-1}(B) \Delta F \subset \Big(\bigcap_n H_n\Big) \cup \Big(\bigcap_n K_n\Big).$$

Clearly ${}^\circ\lambda(A;\, X)(H_n(\omega))$ converges in probability to zero and the same is true of ${}^\circ\lambda(A;\, X)(K_n(\omega))$ by the choice of $\underline{t}_i$. It follows that $\lim_{n\to\infty} {}^\circ\mu(A;\, X)(H_n \cup K_n) = 0$ (see the above remark) and the claim is proved. In particular we see that $\mathrm{st}^{-1}(B) \in L(\overline{\mathcal{P}})$ because $F \in \overline{\mathcal{P}}$.

If $B = \{0\} \times C$ for $C \in \mathcal{F}_0$, then an argument similar to the above shows that $\mathrm{st}^{-1}(B)$ is an $L(\mu(A;\, X))$-null set because of the S-continuity of $|A| + [X, X]$ at zero. Therefore $\mathrm{st}^{-1}(B) \in L(\overline{\mathcal{P}})$.

The lemma follows immediately since $\mathcal{P}$ is the σ-field generated by

$$\{\{0\} \times C \mid C \in \mathcal{F}_0\} \cup \{(t_1, t_2] \times C \mid 0 \leq t_1 < t_2, C \in \mathcal{F}_{t_1}\}$$

(see Dellacherie and Meyer [**6**, p. 200]). □

THEOREM 7.11. *Suppose* $(a, x) \in \mathcal{V}_0^d \times \mathcal{L}^k$ *and* $(A;\, X)$ *is a* $\mathcal{B}_{\underline{t}}$*-semimartingale lifting of* $(a;\, x)$ *for some internal filtration* $\{\mathcal{B}_t \mid \underline{t} \in T\}$.

(a) *If M is a separable metric space and* h: $[0, \infty) \times \Omega \to M$ *is predictable, then h has a weak* $(A;\, X)$*-lifting.*

(b) *If M is a separable normed linear space and* $h \in L_{\mathrm{loc}}(x, M) \cap L_s(a, M)$, *then h has an* $(A;\, X)$*-lifting, H, such that* $\sup_{(\underline{t},\omega)} \|H(\underline{t}, \omega)\| \in {}^*\mathbf{R}$.

PROOF. (a) If h: $[0, \infty) \times \Omega \to M$ is predictable, by Lemma 7.10 $h({}^\circ\underline{t}, \omega)$ is $L(\overline{\mathcal{P}})$-measurable. (Note that $[X, X]$ is S-continuous at zero by Lemma 6.5 with $V \equiv 0$.) By Anderson's lifting theorem (see Keisler [**12**, Proposition 1.16]) there is a $\overline{\mathcal{P}}$-measurable internal function H: $T \times \Omega \to {}^*M$ such that

$${}^\circ H(\underline{t}, \omega) = h({}^\circ\underline{t}, \omega) \quad L(\mu(A;\, X))\text{-a.s.}$$

Clearly H is the required weak $(A;\, X)$-lifting.

(b) If h is bounded and predictable, the existence of a bounded $(A;\, X)$-lifting of h follows as in (a) (use the *-stopping times $\{W_n\}$ obtained in Theorem 6.4 to satisfy (7.3) and (7.4)).

If $h \in L_{\mathrm{loc}}(x, M) \cap L_s(a, M)$, let $h_n = hI_{\{\|h\| \leq n\}}$. Then each h_n has a bounded $(A;\, X)$-lifting H_n by (a). Let $\{R_m\}$ be a sequence of stopping times increasing to ∞ such that $R_m \leq m$ and $E((\int_0^{R_m} \|h(s)\|^2\, d[x, x]_s)^{1/2}) < \infty$, and let $\{W_m\}$ be the sequence of *-stopping times obtained in Theorem 6.4. By Theorem 4.7(b) there is a sequence of *-stopping times $\{V_m'\}$ such that ${}^\circ V_m' = R_m$ a.s. and ${}^\circ X(V_m') = x(R_m)$ a.s. By considering $\max\{V_i' \mid i \leq m\}$ we may assume the $\{V_m'\}$ are nondecreasing. Let $V_m = W_m \wedge V_m'$. Clearly (7.3) holds. By taking a subsequence if necessary we may assume that for $m \leq n$,

$$E\left(\left(\int_0^{{}^\circ V_m} \|h_n - h_m\|^2\, d[x, x]\right)^{1/2}\right) < 2^{-m} \tag{7.6}$$

and

$$P\left(\int_0^m \|h_n - h_m\|\, d|a| \geq 2^{-m}\right) < 2^{-m}. \tag{7.7}$$

Note that

$$
\begin{aligned}
{}^{\circ}\bar{E}&\left(\left(\int I_{\{\underline{s}<V_m\}}\|H_n - H_m\|^2\, d\lambda^{(2)}(X)\right)^{1/2}\right)\\
&= E\left(\left({}^{\circ}\int I_{\{\underline{s}<V_m\}}\|H_n - H_m\|^2\, d\lambda^{(2)}(X)\right)^{1/2}\right)\\
&\qquad\qquad\qquad\qquad\qquad\left(\text{since } [X, X]_{V_m}^{1/2} \text{ is } S\text{-integrable}\right)\\
&= E\left(\left(\int I_{\{\underline{s}<V_m\}}\|h_n({}^{\circ}\underline{s}) - h_m({}^{\circ}\underline{s})\|^2\, dL(\lambda^{(2)}(X))\right)^{1/2}\right)\\
&\qquad\qquad\qquad\text{(see the second remark following Definition 7.9)}\\
&\leqslant E\left(\left(\int_0^{{}^{\circ}V_m}\|h_n(s) - h_m(s)\|^2\, d[x, x]_s\right)^{1/2}\right) \quad \text{(Lemma 2.7)}\\
&< 2^{-m}.
\end{aligned}
$$

Therefore we have

$$
(7.8) \qquad \bar{E}\left(\left(\int I_{\{\underline{s}<V_m\}}\|H_n - H_m\|^2\, d\lambda^{(2)}(X)\right)^{1/2}\right) < 2^{-m}, \quad \text{for all } m \leqslant n.
$$

A similar argument shows that

$$
(7.9) \qquad \bar{P}\left(\int I_{\{\underline{s}<m\}}\|H_n - H_m\|\, d\lambda^{(1)}(A) > 2^{-m}\right) < 2^{-m}, \quad \text{for all } m \leqslant n.
$$

By saturation there is a $\mathfrak{B}_t$-adapted process H_γ $(\gamma \in {}^*\mathbf{N} - \mathbf{N})$ such that $\sup_{(\underline{t},\omega)}\|H_\gamma(\underline{t}, \omega)\| \in {}^*\mathbf{R}$ and (7.8) and (7.9) hold with γ in place of n. An easy computation using (7.8) (with $n = \gamma$) shows that H_γ and $\{V_n\}$ satisfy (7.4). Similarly (7.9) shows that (7.2) holds. Therefore $H_\gamma \in L(A; X, M, \mathfrak{B}.)$. Finally use both (7.8) and (7.9) (with $n = \gamma$) to see that for a.a. ω and all m in $\mathbf{N}$,

$$
\begin{aligned}
&\int I_{\{\underline{s}<V_m\wedge m\}}\|{}^{\circ}H_\gamma(\underline{s}) - h({}^{\circ}\underline{s})\|\, dL(\lambda(A; X))\\
&\qquad\leqslant \lim_{n\to\infty} \int I_{\{\underline{s}<V_m\}}\|{}^{\circ}H_n(\underline{s}) - h({}^{\circ}\underline{s})\|\, dL(\lambda^{(2)}(X))\\
&\qquad\qquad + \int I_{\{\underline{s}<m\}}\|{}^{\circ}H_n(\underline{s}) - h({}^{\circ}\underline{s})\|\, dL(\lambda^{(1)}(A))\\
&\qquad\leqslant \lim_{n\to\infty} \int_0^{{}^{\circ}V_m}\|h_n - h\|\, d[x, x] + \int_0^m \|h_n - h\|\, d|a|\\
&\qquad\qquad\qquad\qquad\qquad\text{(by Lemma 2.7 and the choice of } H_n)\\
&\qquad= 0 \quad (\text{by (7.6) and (7.7)}).
\end{aligned}
$$

Therefore H_γ satisfies (7.5)′ and hence is an internally bounded $(A; X)$-lifting of h. □

Notation 7.12. Vectors in $\mathbf{R}^d$ are interpreted as column vectors for the purpose of matrix multiplication. If H: $T \times \Omega \to {}^*\mathbf{R}^{k\times d}$ and Z: $T \times \Omega \to {}^*\mathbf{R}^d$ are internal processes, define $H \cdot Z \colon T \times \Omega \to {}^*\mathbf{R}^k$ by

$$H \cdot Z(\underline{t}) = \sum_{\underline{s}<\underline{t}} H(\underline{s})\Delta Z(\underline{s}). \quad \square$$

Having found appropriate liftings, Z and H, of z and h, we would like to define the stochastic integral $\int_0^t h(s)\,dz(s)$ as $\mathrm{st}(H \cdot Z)(t)$. The following lemma establishes several properties of $H \cdot Z$ including the fact that it is SDJ, so that the above definition is possible.

LEMMA 7.13. *Let* $(A;\, X)$ *be a* $\mathfrak{B}_{\underline{t}}$*-semimartingale lifting of* $(a, x) \in \mathcal{V}_0^d \times \mathcal{L}_0^d$ *for some internal filtration* $\{\mathfrak{B}_{\underline{t}} \mid \underline{t} \in T\}$ *and suppose* $\{H, H', H_n \mid n \in \mathbf{N}\} \subset L(A;\, X, \mathbf{R}^{k\times d}, \mathfrak{B}.)$. *Let* $Z = A + X$.

(a) *If* $\|H\|$ *is internally bounded,* $H \cdot X(\underline{t})$ *is an S-local martingale with respect to* $\{\mathfrak{B}_{\underline{t}}\}$.

(b) *If* $L(\mu(A;\, X))(\{{}^\circ H(\underline{t}, \omega) \neq {}^\circ H'(\underline{t}, \omega)\}) = 0$, *then* ${}^\circ H \cdot Z(\underline{t}) = {}^\circ H' \cdot Z(\underline{t})$ *for all* $\underline{t}$ *in* $\mathrm{ns}(T)$ *a.s.*

(c) *If* ${}^\circ H_n(\underline{s}, \omega)$ *converges in measure* (*with respect to* $L(\mu(A;\, X))$) *to* ${}^\circ H(\underline{s}, \omega)$ *and* $\|H_n\| \leqslant \|G\|$ *for some* $G \in L(A;\, X, \mathbf{R}^{k\times d}, \mathfrak{B}.)$, *then* ${}^\circ\sup_{\underline{t}\leqslant m}\|H_n \cdot Z(\underline{t}) - H \cdot Z(\underline{t})\|$ *converges to zero in probability as n approaches* ∞ *for all* m *in* $\mathbf{N}$.

(d) *For a.a.* ω, *if* $\underline{t} \in \mathrm{ns}(T)$ *and* $\Delta Z(\underline{t}) \approx 0$ *then* $\Delta(H \cdot Z)(\underline{t}) \approx 0$.

(e) *If H is an* $(A;\, X)$*-lifting of h, and* $h \in L_{\mathrm{loc}}(x, \mathbf{R}^{k\times d}) \cap L_s(a, \mathbf{R}^{k\times d})$, *then* $H \cdot Z$ *is SDJ.*

(f) *In addition to the hypotheses of* (e), *assume that* $(A';\, X')$ *is a* $\mathfrak{B}'_{\underline{t}}$*-semimartingale lifting of* $(a;\, x)$ *and* H' *is an* $(A';\, X')$*-lifting of h. If* $Z' = A' + X'$, *then* $\mathrm{st}(H \cdot Z)$ *and* $\mathrm{st}(H' \cdot Z')$ *are indistinguishable.*

PROOF. Let $\{V_n\}$ and $\{V_n'\}$ be the sequence of *-stopping times that satisfy both (7.3) and (7.4) for H and H', respectively.

(a) Since $\|H\|$ is internally bounded, $Y(\underline{t}) \equiv H \cdot X(\underline{t})$ is a $\mathfrak{B}_{\underline{t}}$-martingale. Note that (7.4) implies that $[Y, Y]_{V_n}^{1/2}$ is S-integrable. Therefore Lemma 6.3(b) implies that $Y^*(V_n)$ is S-integrable. It follows from (7.4) that for a.a. ω

$$(7.10) \qquad {}^\circ\!\int I_{\{\underline{s}<\underline{t}\}}\|H(\underline{s})\|^2\,d\lambda^{(2)}(X) = \int I_{\{\underline{s}<\underline{t}\}}{}^\circ\|H(\underline{s})\|^2\,dL(\lambda^{(2)}(X))$$

for all $\underline{t}$ in $\mathrm{ns}(T)$. If $\underline{\delta}_m \approx m^{-1}$, then

$$\begin{aligned}
&\mathrm{st}([Y, Y])({}^\circ V_n) - {}^\circ[Y, Y]_{V_n} \\
&\quad = \lim_{m\to\infty} {}^\circ[Y, Y]_{V_n+\underline{\delta}_m} - {}^\circ[Y, Y]_{V_n} \quad \text{a.s.} \\
&\quad \leqslant \lim_{m\to\infty} {}^\circ\!\int I(V_n \leqslant \underline{s} < V_n + \underline{\delta}_m)\|H(\underline{s})\|^2\,d\lambda^{(2)}(X) \\
&\quad = \lim_{m\to\infty} \int I(V_n \leqslant \underline{s} < V_n + \underline{\delta}_m){}^\circ\|H(\underline{s})\|^2\,dL(\lambda^{(2)}(X)) \quad \text{a.s.} \quad (\text{by } (7.10)) \\
&\quad = 0 \quad \text{a.s.,}
\end{aligned}$$

since Lemma 6.5 and the fact that ${}^\circ X(V_n) = \mathrm{st}(X)({}^\circ V_n)$ imply

$${}^\circ[X, X]_{V_n} = \mathrm{st}([X, X])({}^\circ V_n) \quad \text{a.s.}$$

The converse of Lemma 6.5 shows that $^\circ Y(V_n) = \mathrm{st}(Y)(^\circ V_n)$ (recall that Y is SD by Theorem 5.2) and hence Y is an S-local martingale.

(b) If $U_n = V_n \wedge V_n'$, then $H \cdot X(\underline{t} \wedge U_n)$ and $H' \cdot X(\underline{t} \wedge U_n)$ are $\mathfrak{B}_{\underline{t}}$-martingales and therefore for some $c \in \mathbf{R}$,

$$^\circ \bar{E}\Big(\max_{\underline{t} \leqslant U_n} \| H \cdot X(\underline{t}) - H' \cdot X(\underline{t}) \| \Big)$$

$$\leqslant c^\circ \bar{E}\left(\left(\int I_{\{\underline{t} < U_n\}} \| H(\underline{t}) - H'(\underline{t}) \|^2 \, d\lambda^{(2)}(X) \right)^{1/2} \right) \qquad \text{(Theorem 1.3)}$$

$$= 0$$

by (7.4) and the fact that $^\circ H = {}^\circ H'$ $L(\mu(A;\, X))$-a.s. (see also the second remark following Definition 7.9). Moreover, if $\underline{n} \approx n$, $\underline{n} \in T$, then

$$^\circ \sup_{\underline{t} \leqslant \underline{n}} \| H' \cdot A(\underline{t}) - H \cdot A(\underline{t}) \| \leqslant {}^\circ(\| H' - H \| \cdot |A|)(\underline{n}) = 0 \quad \text{a.s.}$$

by (7.2). Letting n approach ∞ in the above inequalities, we obtain (b).

(c) Choose a real sequence $\{\varepsilon_n\}$ decreasing to zero such that

$$\mu(A;\, X)(\| H_n - H \| > \varepsilon_n) < \varepsilon_n \tag{7.11}$$

for each $n \in \mathbf{N}$. Extend $\{H_n\}$ by saturation so that for all n in $^*\mathbf{N}$ $\| H_n \| \leqslant \| G \|$, (7.11) holds, and H_n is a $^*\mathbf{R}^{k \times d}$-valued, $\mathfrak{B}_{\underline{t}}$-adapted process. If $\gamma \in {}^*\mathbf{N} - \mathbf{N}$, then by (7.11), $^\circ H_\gamma(\underline{t}, \omega) = {}^\circ H(\underline{t}, \omega)$ $L(\mu(A;\, X))$-a.s. and $H_\gamma \in L(A;\, X,\, \mathbf{R}^{k \times d}, \mathfrak{B}.)$ since $\| H_\gamma \| \leqslant \| G \|$. By (b) we have $^\circ(H_\gamma \cdot Z(\underline{t})) = {}^\circ(H \cdot Z(\underline{t}))$ for all $\underline{t}$ in $\mathrm{ns}(T)$ a.s. It follows easily that $^\circ \sup_{\underline{t} \leqslant m} \| H_n \cdot Z(\underline{t}) - H \cdot Z(\underline{t}) \|$ converges to zero in probability as n approaches ∞ for each m in $\mathbf{N}$.

(d) If $\| H \|$ is uniformly bounded, this is obvious. For the general case let $H_n = H I_{\{\| H \| \leqslant n\}}$. It follows from (c) that for a.a. ω if $\underline{t} \in \mathrm{ns}(T)$ and $^\circ \Delta(H \cdot Z)(\underline{t}) \neq 0$ then $^\circ \Delta(H_n \cdot Z)(\underline{t}) \neq 0$ for some large value of n and hence $^\circ \Delta Z(\underline{t}) \neq 0$.

(e) Let

$$V = \big\{ h\colon [0, \infty) \times \Omega \to \mathbf{R}^{k \times d} \mid h \text{ bounded, predictable and } H \cdot Z \text{ is SDJ for each } (A;\, X)\text{-lifting } H \text{ of } h \big\}.$$

If

$$h(t, \omega) = h_0(\omega) I_{\{0\}}(t) + \sum_{i=1}^{n-1} h_i(\omega) I_{(t_i, t_{i+1}]}(t) \in V'(\mathbf{R}^{k \times d}),$$

an easy application of Theorems 3.2 and 4.7(b) shows that h has a bounded $(A;\, X)$-lifting, H, of the form

$$H(\underline{t}, \omega) = \sum_{i=1}^{n-1} H_i(\omega) I_{(\underline{t}_i, \underline{t}_{i+1}]}(\underline{t}),$$

where $\underline{t}_i \approx t_i$, $1 \leqslant i \leqslant n$. Clearly $H \cdot Z$ is SDJ and hence the same is true for every $(A;\, X)$ lifting of h (by (b)). Therefore $V'(\mathbf{R}^{k \times d}) \subset V$. It follows easily from (c) that V is closed under bounded pointwise convergence. Moreover, V is clearly a vector space by (d). Therefore Lemma 7.3 implies the result if h is bounded. More generally, if $h \in L_{\mathrm{loc}}(x, \mathbf{R}^{k \times d}) \cap L_s(a, \mathbf{R}^{k \times d})$ let H_n' be a bounded $(A;\, X)$-lifting of

$hI_{\{\|h\| \leq n\}}$ such that $\|H'_n\| \leq \|H\|$. By (c), ${}^\circ\sup_{t \leq m} \|H'_n \cdot Z(\underline{t}) - H \cdot Z(\underline{t})\|$ converges to zero in probability as n approaches ∞ for each m in $\mathbf{N}$. Since each $H'_n \cdot Z$ is SDJ by the above, we see that $H \cdot Z$ is SDJ.

(f) Let $V = \{h: [0, \infty) \times \Omega \to \mathbf{R}^{k \times d} \mid h$ bounded, predictable; if H and H' are $(A;$ $X)$- and $(A';\ X')$-liftings of h, respectively, then $\mathrm{st}(H \cdot Z)$ and $\mathrm{st}(H' \cdot Z')$ are indistinguishable$\}$. If $h \in V'(\mathbf{R}^{k \times d})$ and H and H' are the $(A;\ X)$- and $(A';$ $X')$-liftings, respectively, of h obtained in the proof of (e), then clearly $\mathrm{st}(H \cdot Z) = \mathrm{st}(H' \cdot Z')$. It follows from (b) that $V'(\mathbf{R}^{k \times d}) \subset V$. An application of (c) shows that V is closed under bounded pointwise convergence and, since V is clearly a vector space, Lemma 7.3 implies the result for bounded h. The usual truncation argument now completes the proof. $\square$

DEFINITION 7.14. Let $z \in \mathbb{S}_0^d$ and $h \in L(z, \mathbf{R}^{k \times d})$ (see Notation 7.2). If $h \in L_{\mathrm{loc}}(x, \mathbf{R}^{k \times d}) \cap L_s(a, \mathbf{R}^{k \times d})$ for some $(a, x) \in \mathcal{V}_0^d \times \mathcal{L}_0^d$ such that $z = a + x$, first choose an internal filtration $\{\mathcal{B}_t\}$ and a $\mathcal{B}_t$-semimartingale lifting, $(A;\ X)$, of $(a;\ x)$ (by Theorem 7.6) and then choose an $(A;\ X)$-lifting, H, of h (by Theorem 7.11). Define an $\mathcal{F}_t$-adapted process, $h \cdot z$, with sample paths in $D(\mathbf{R}^k)$ by

$$h \cdot z(t) = \mathrm{st}(H \cdot Z)(t). \quad \square$$

The above definition is possible by Lemma 7.13(e) and is independent of the choice of H, $(A;\ X)$ and $\{\mathcal{B}_t\}$ (up to indistinguishability) by Lemma 7.13(f). It remains to show that $h \cdot z$ is independent of the choice of a and x, and coincides with the classical stochastic integral, which we denote by $\int_0^t h(s)\,dz(s)$.

In what follows we fix z, a, x, $\{\mathcal{B}_t\}$, A and X as in the above definition.

THEOREM 7.15. (a) *If* h_n, h, $g \in L_{\mathrm{loc}}(x, \mathbf{R}^{k \times d}) \cap L_s(a, \mathbf{R}^{k \times d})$ *and satisfy* $\lim_{n \to \infty} h_n(t, \omega) = h(t, \omega)$ *and* $\|h_n(t, \omega)\| \leq \|g(t, \omega)\|$ *for all* (t, ω), *then for each* m *in* $\mathbf{N}$, $\sup_{t \leq m} \|h_n \cdot z(t) - h \cdot z(t)\|$ *converges to zero in probability as n approaches* ∞.

(b) *If* $h \in L_{\mathrm{loc}}(x, \mathbf{R}^{k \times d}) \cap L_s(a, \mathbf{R}^{k \times d})$, *then* $h \cdot z(t)$ *and* $\int_0^t h(s)\,dz(s)$ *are indistinguishable.*

PROOF. (a) is immediate from Lemma 7.13(c).

(b) Let

$$V = \left\{h: [0, \infty) \times \Omega \to \mathbf{R}^{k \times d} \mid h \text{ bounded and predictable,} \right.$$
$$\left. \text{and } h \cdot z(t) \text{ is indistinguishable from } \int_0^t h(s)\,dz(s)\right\}.$$

It is easy to check that $V'(\mathbf{R}^{k \times d}) \subset V$ (see the proof of Lemma 7.13(e)) and that V is a vector space. By (a) and the corresponding result for $\int_0^t h(s)\,dz(s)$, we see that V is closed under bounded pointwise convergence. Lemma 7.3 now implies the result if h is bounded. The proof is completed by the usual truncation argument (which involves a further application of (a) and the corresponding result for the classical definition of the stochastic integral). $\square$

We will now show that the stochastic integral $h \cdot z$ is well defined for $h \in \mathcal{L}(z, \mathbf{R}^{k \times d})$, i.e., that $h \cdot z$ is independent of the choice of the decomposition $z = a + x$ for which $h \in \mathcal{L}_{\mathrm{loc}}(x, \mathbf{R}^{k \times d}) \cap \mathcal{L}_s(a, \mathbf{R}^{k \times d})$.

THEOREM 7.16. *Suppose $z = a' + x'$, where $(a', x') \in \mathcal{V}_0^d \times \mathcal{L}_0^d$, and $(h \cdot z)'(t)$ is defined as before, only with (a', x') in place of (a, x). Then $(h \cdot z)$ and $(h \cdot z)'$ are indistinguishable whenever they both are defined.*

PROOF. Let $\{\mathfrak{B}'_t\}$ be an internal filtration and let $(A_1;\ X_1, X'_1)$ be a $\mathfrak{B}'_t$-semimartingale lifting of $(a;\ x, x')$. Then $A'_1 = X_1 - X'_1 + A_1$ is a $\mathfrak{B}'_t$-adapted SDJ lifting of a'. Lemma 7.5 implies that for some positive infinitesimal $\Delta' t$ in T, if $T' = \{k\Delta' t \mid k \in {}^*\mathbf{N}_0\}$ then $A'(\underline{t}) = A'_1([\underline{t}]^{T'})$ is a $\mathfrak{B}'_{[\underline{t}]^{T'}}$-$BV$ lifting of a. Let $\mathfrak{B}_{\underline{t}} = \mathfrak{B}'_{[\underline{t}]^{T'}}$. By Lemma 5.5 there is a $\mathfrak{B}'_{\underline{t}}$-stopping time W such that ${}^\circ W = \infty$ a.s. and

$$(X(\underline{t}), X'(\underline{t})) = \left(X_1\left([\underline{t}]^{T'} \wedge W\right), X'_1\left([\underline{t}]^{T'} \wedge W\right)\right)$$

is a $\mathfrak{B}_{\underline{t}}$-local martingale lifting of (x, x'). Clearly $A(\underline{t}) = A_1([\underline{t}]^{T'})$ is a $\mathfrak{B}_{\underline{t}}$-$BV$ lifting of a and (A, A', X, X') is SDJ. Therefore $(A, A';\ X, X')$ is a $\mathfrak{B}_{\underline{t}}$-semimartingale lifting of $(a, a';\ x, x')$ and $A' + X' = A + X$ on ns(T) a.s. Let

$$h \in L_{\mathrm{loc}}(x, \mathbf{R}^{k\times d}) \cap L_{\mathrm{loc}}(x', \mathbf{R}^{k\times d}) \cap L_s(a, \mathbf{R}^{k\times d}) \cap L_s(a', \mathbf{R}^{k\times d}).$$

Then $h \in L_{\mathrm{loc}}((x, x'),\ \mathbf{R}^{k\times d}) \cap L_s((a, a'),\ \mathbf{R}^{k\times d})$ and therefore h has an $(A, A';\ X, X')$-lifting H, which is clearly both an $(A;\ X)$- and $(A';\ X')$-lifting of h. It follows that for a.a. ω

$$(h \cdot z) = \mathrm{st}(H \cdot (A + X)) = \mathrm{st}(H \cdot (A' + X')) = (h \cdot z)'. \quad \square$$

Hence $h \cdot z$ is consistently defined for $h \in \mathcal{L}(z, \mathbf{R}^{k\times d})$. We note that it is not true in general that if $h \in \mathcal{L}_s(a) \cap \mathcal{L}_{\mathrm{loc}}(x)$ for some decomposition $z = a + x$ then $h \in \mathcal{L}_s(a') \cap \mathcal{L}_{\mathrm{loc}}(x')$ for all decompositions $z = a' + x'$. A simple counterexample may be constructed by considering the decomposition $z \equiv 0 = y - y$ where y is both a local martingale and a process of bounded variation (for example let

$$P\left(\Delta y(1 - n^{-1}) = \pm n^{-2}\right) = \tfrac{1}{2},$$

where the jumps are independent).

One advantage of representing the stochastic integral as an internal Riemann-Stieltjes sum is that several properties of $h \cdot z$ become obvious when viewed internally. For example, consider the proof of the following well-known result.

THEOREM 7.17. *If $x, y \in \mathcal{L}_0^1$ and $h \in \mathcal{L}_{\mathrm{loc}}(x, \mathbf{R})$, then $h \cdot x \in \mathcal{L}_0^1$ and $[h \cdot x, y]_t = h \cdot [x, y](t)$ for all $t \geq 0$ a.s.*

PROOF. Choose an internal filtration $\{\mathfrak{B}_{\underline{t}}\}$ and a $\{\mathfrak{B}_{\underline{t}}\}$-local martingale lifting, (X, Y), of (x, y). Let H be an internally bounded $(0;\ X)$-lifting of h. Note that if $A \in \sigma(\mathcal{C})$ and $A \subset [0, \underline{t})$, then

$$L\left(\lambda^{(1)}([X, Y])\right)(A) \leq \left(L\left(\lambda^{(1)}([X, X])\right)(A)\,{}^\circ[Y, Y]_{\underline{t}}\right)^{1/2} \tag{7.12}$$

since this is obvious for sets in $\mathcal{C}$. It follows from (7.12) that $L(\mu([X, Y];\ 0))$ is absolutely continuous with respect to $L(\mu(0;\ X))$, and H is also an $([X, Y];\ 0)$-lifting of h (to verify (7.2), use (7.12) to check that for a.a. ω and all $\underline{t}$ in ns(T) if $\gamma \in {}^*\mathbf{N} - \mathbf{N}$, then ${}^\circ\!\int I_{\{s<\underline{t}, |H(\underline{s})|>\gamma\}} |H(\underline{s})|\, d\lambda^{(1)}([X, Y]) = 0$). Lemma 7.13(a) and Theorem 5.2 imply that $h \cdot x$ is a local martingale and hence $h \cdot x \in \mathcal{L}_0^1$. By Lemma 7.13(d), $(H \cdot X, Y)$ is SDJ because (X, Y) and $H \cdot X$ are SDJ by Lemma 7.13(e).

Therefore $(H \cdot X, Y)$ is a local martingale lifting of $(h \cdot x, y)$ (see Lemma 7.13(a)). Therefore, for a.a. ω and all $t \geqslant 0$,

$$\begin{aligned}[h \cdot x, y]_t &= \mathrm{st}([H \cdot X, Y])(t) \quad \text{(Definition 6.9)} \\ &= \mathrm{st}(H \cdot [X, Y])(t) \\ &= \lim_{{}^\circ \underline{t} \downarrow t} \int I_{\{\underline{s} < \underline{t}\}} h({}^\circ \underline{s})\, dL(\lambda^{(1)}([X, Y])) \\ &\qquad\qquad (H \text{ is an } ([X, Y]; 0)\text{-lifting of } h) \\ &= (h \cdot [x, y])(t),\end{aligned}$$

where in the last line we have used Lemma 2.7 (note that $[X, Y]$ is S-continuous at zero because $[X, X]$ is, and

$${}^\circ |[X, Y]|(\underline{t}) \leqslant {}^\circ([X, X]_{\underline{t}}[Y, Y]_{\underline{t}})^{1/2} < \infty \text{ for all } \underline{t} \text{ in } \mathrm{ns}(T) \text{ a.s.}). \quad \square$$

Note that this gives another proof that our definition of $h \cdot x$ coincides with the classical one because according to the standard definition, $h \cdot x$ is the unique element of $\mathfrak{L}_0^1$ such that $[h \cdot x, y] = h \cdot [x, y]$ a.s. for all bounded martingales y.

We close this section with the proof of the following result that was promised earlier.

THEOREM 7.18. *If $Y = (Y_1, Y_2)$ is an SDJ S-local martingale with respect to an internal filtration $\{\mathfrak{B}_{\underline{t}}\}$ (each Y_i is d-dimensional), then $[Y_1, Y_2]$ is SDJ.*

PROOF. Let $y_i = \mathrm{st}(Y_i)$. We claim that Y_1 is a $(0;\ Y_2)$-lifting of $y_1(s^-)$. Let $\{W_n\}$ be the sequence of *-stopping times obtained for Y in Theorem 6.4. Since $Y_1^*(W_n - \Delta t) \leqslant n$ on $\{W_n > 0\}$ and $[Y_2, Y_2]_{W_n}^{1/2}$ is S-integrable, clearly

$${}^\circ \bar{E}\left(\left(\int I_{\{\underline{s} < W_n\}} \|Y_1\|^2(\underline{s})\, d\lambda^{(2)}(Y_2)\right)^{1/2}\right)$$
$$= E\left(\left(\int I_{\{\underline{s} < W_n\}} {}^\circ\|Y_1\|^2(\underline{s})\, dL(\lambda^{(2)}(Y_2))\right)^{1/2}\right).$$

Let $\{T_i \mid i \in \mathbf{N}\}$ be a sequence of *-stopping times such that

$$\{\underline{t} \in \mathrm{ns}(T) \mid {}^\circ\|\Delta Y(\underline{t} - \Delta t)\| > 0\} \subset \{T_i\} \quad \text{a.s.}$$

To prove the claim it remains to show that ${}^\circ Y_1(\underline{s}) = y_1({}^\circ \underline{s}^-)$ $L(\mu(0;\ Y_2))$-a.s. To this end note that

$$L(\mu(0;\ Y_2))({}^\circ Y_1(\underline{s}) \neq y_1({}^\circ \underline{s}^-))$$
$$\leqslant L(\mu(0;\ Y_2))(\{(\underline{s}, \omega) \mid \underline{s} \approx T_i(\omega) \text{ and } \underline{s} \geqslant T_i(\omega) \text{ for some } i\}).$$

Since ${}^\circ Y_2(T_i) = y_2({}^\circ T_i)$ a.s. on $\{{}^\circ T_i < \infty\}$, Lemma 6.5 implies that ${}^\circ[Y_2, Y_2]_{T_i} = \mathrm{st}([Y_2, Y_2])({}^\circ T_i)$ a.s. on $\{{}^\circ T_i < \infty\}$ and hence, the above expression equals zero and the claim is proved.

An elementary induction argument shows that

$$Y_1(\underline{t}) Y_2(\underline{t}) = Y_1^{\prime} \cdot Y_2(\underline{t}) + Y_2^{\prime} \cdot Y_1(\underline{t}) + [Y_1, Y_2]_{\underline{t}}, \tag{7.13}$$

where the product on the left is the scalar product and Y_i^t is the transpose of Y_i. Since Y_1^t is a $(0;\ Y_2)$-lifting of $y_1(s^-)^t$, $Y_1^t \cdot Y_2$ is SDJ by Lemma 7.13(e). By symmetry $Y_2^t \cdot Y_1$ is also SDJ. Now use the fact that Y is SDJ together with Lemma 7.13(d) to see that $[Y_1, Y_2] = Y_1Y_2 - Y_1^t \cdot Y_2 - Y_2^t \cdot Y_1$ is SDJ. □

Note that by taking standard parts on both sides of (7.13) we obtain the well-known integration by parts formula

$$y_1 y_2(t) = y_1^- \cdot y_2(t) + y_2^- \cdot y_1(t) + [y_1, y_2]_t,$$

where $y_i^-(s) = y_i(s^-)^t$.

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