

THE DISPERSION OF THE COEFFICIENTS OF UNIVALENT FUNCTIONS

BY

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ABSTRACT. The Hayman T_a function for the asymptotic distribution of the coefficients of univalent functions has a continuous derivative which is closely related to the asymptotic behavior of coefficient differences.

1. Introduction. Suppose that S denotes the class of functions $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ which are univalent in the disk $\{|z| < 1\}$. We define

$$A_n = \sup\{|a_n| : f \in S\}.$$

FitzGerald's method (see Horowitz [9]) shows that $A_n \leq (1.05\dots)n$, and Hayman [7] showed that $A_n/n \rightarrow K_0$, where K_0 is some absolute constant. (However it is not even known if A_n is increasing.) A closely related result is Nehari's [11] proof that

$$(1) \quad |a_n| \leq 4nK_0 \operatorname{dist}(0, \mathbb{C} - f(|z| < 1)),$$

which was extended by Bombieri [1]. Recently FitzGerald [3] showed that if $f_n(z) = \sum_{m=1}^{\infty} a_{n,m} z^m$ in S has $|a_{n,n}| \sim A_n$ then $|a_{n,2}| \rightarrow 2$. Hamilton [5] gave a simpler proof of a more general result and noted that this implies Littlewood's conjecture is equivalent to Hayman's, i.e., $|a_n| \leq 4n \operatorname{dist}(0, \mathbb{C} - f(|z| < 1))$ for all $f \in S \Rightarrow K_0 = 1$. (See [4, 5, 6] for related results.)

This paper is concerned with extending asymptotic results of this type. Hayman [7] proves that if $f_n = \sum_{m=1}^{\infty} a_{n,m} z^m$ is a sequence in S such that

$$(2) \quad \lim_{n \rightarrow \infty} n^{-2} \left| f_n \left(1 - \frac{1}{n} \right) \right| = \lambda > 0,$$

then $a_{n,m}/n$ converges on a subsequence of $n \rightarrow \infty$ and as $m/n \rightarrow a > 0$ to a continuous function T_a . We prove that T_a has continuous derivative and as $n \rightarrow \infty$, $m/n \rightarrow a > 0$,

$$(3) \quad a_{n,m+1} - a_{n,m} \rightarrow T'_a.$$

This result has a number of interesting consequences.

THEOREM 1. Suppose that $f_n \in S$ and $a_{n,n} = A_n$; then as $n \rightarrow \infty$,

$$(4) \quad |a_{n,m+1}| - |a_{n,m}| \rightarrow K_0,$$

provided that $m/n \sim 1$.

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REMARK 1. We deduce from (3) and the Marty relation yet another proof of $a_{n,2} \rightarrow 2$. Also we prove

THEOREM 2. As $n \rightarrow \infty$,

$$(5) \quad A_{n+1} - A_n \rightarrow K_0.$$

REMARK 2. In particular A_n is an increasing sequence for large n .

REMARK 3. Relation (4) also shows that there is no essential difference between the odd and even coefficients, as was found by Bombieri [2] near the Koebe function.

We begin with a discussion of Hayman's asymptotic theory, concluding that section with a statement of some technical results. The following sections are devoted to proving these statements. Finally we obtain Theorems 1 and 2.

2. H -theory. We consider subsequences

$$(6) \quad f_n(z) = z + \sum_{m=2}^{\infty} a_{n,m} z^m$$

which satisfy (2). Then some subsequence of $n^{-2}f_n(1 - z/n)$ converges locally uniformly on the half plane $\{\operatorname{Re}(z) > 0\}$ to a function $\phi(z)$ such that:

$$(7) \quad \phi(z) \text{ is nonzero on } \{\operatorname{Re}(z) > 0\},$$

$$(8) \quad \phi(z) \text{ is univalent on } \{\operatorname{Re}(z) > 0\},$$

$$(9) \quad \lim_{x \rightarrow \infty} x^2 |\phi(x)| = \alpha \leq 1.$$

The class H consists of those functions ϕ which satisfy properties (7)–(9). Furthermore for each $\phi \in H$ there is a sequence f_n in S , satisfying (2), such that

$$n^{-2}f_n(1 - z/n) \rightarrow \phi(z) \quad \text{as } n \rightarrow \infty.$$

Hayman also shows that

$$(10) \quad \int_{-\infty}^{\infty} |\phi(x + iy)| dy \leq \frac{1}{2x}$$

for $\phi \in H$, $x > 0$. In particular the Fourier transform

$$(11) \quad T_a(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x + iy) e^{a(x+iy)} dy$$

exists as a continuous function of a . Hayman's main result is

THEOREM A. If $n^{-2}f_n(1 - z/n) \rightarrow \phi(z) \in H$ as $n \rightarrow \infty$, then $a_{n,m}/n \rightarrow T_a(\phi)$ as $n \rightarrow \infty$ and $m/n \rightarrow a > 0$.

Consequently the problem of bounding the linear functional $T_1(\phi)$ on H is equivalent to finding $\overline{\lim}_{n \rightarrow \infty} A_n/n$, i.e., $K_0 = \sup\{|T_1(\phi)| : \phi \in H\}$. The main result of this paper is

THEOREM 3. For each $\phi \in H$ and positive a , the improper integral

$$(12) \quad V_a(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (x + iy)\phi(x + iy)e^{a(x+iy)} dy$$

exists as a continuous function of a . Furthermore if $f_n \in S$ and $n^{-2}f_n(1 - z/n) \rightarrow \phi(z)$ then

$$(13) \quad a_{n,m+1} - a_{n,m} \rightarrow V_a(\phi),$$

as $m/n \rightarrow a$ and $n \rightarrow \infty$.

REMARK 4. For fixed x , the function $(x + iy)\phi(x + iy)$ is not an L_1 function of y (as compared with (10)). Thus Theorem 3 is established in an indirect manner.

EXAMPLE 1. Suppose we choose

$$f_n(z) = \frac{z}{(1 - 2z \cos(t/n) + z^2)}.$$

Then it is not hard to show that

$$n^{-2}f_n(1 - z/2) \rightarrow 1/(z + t)^2.$$

Thus $T_a = \sin(at)/t$, $T'_a = \cos(at)$, while

$$a_{n,m+1} - a_{n,m} = \frac{\sin\{(m+1)t/n\} - \sin\{mt/n\}}{\sin(t/n)} \rightarrow \cos(at)$$

as $n \rightarrow \infty$ and $m/n \rightarrow a$.

COROLLARY 1. $T_a(\phi)$ has continuous derivative on $(0, \infty)$.

EXAMPLE 2. In [5] it was shown how H -theory easily implies Hayman's results on functions $f \in S$ such that

$$r^{-1}(1 - r)^2 |f(r)| \rightarrow \alpha > 0.$$

For then putting $f_n = f$ we find that

$$n^{-2}f_n(1 - z/n) \rightarrow \alpha e^{i\theta}/z^2 = \phi(z)$$

along some subsequence, where different θ may arise from different subsequences. Thus $T_a(\phi) = \alpha e^{i\theta}a$ and $T'_a(\phi) = \alpha e^{i\theta}$. Consequently by Theorem 3, $|a_{n+1}| - |a_n| \rightarrow \alpha$, which was the second result in [8].

3. Preliminary results. As we are assuming (2) we need to know the location of points of maximum modulus of $f(re^{i\theta})$ on $\{|z| = r\}$. Let $M(r, f) = \max_{\theta} |f(re^{i\theta})|$.

LEMMA 1. Suppose that $f_n \in S$ satisfies (2). Then for any positive x the maximum modulus of f_n on $\{|z| = 1 - x/n\}$ occurs at a point of argument $\eta(x)$ which satisfies for large n

$$(14) \quad \eta(x) \leq A(\lambda, x)/n,$$

where $A(\lambda, x)$ is a finite constant depending on λ and x only.²

Suppose that $n^{-2}|f_n(1 - x/n)| \rightarrow 0$ on some subsequence. Then the corresponding limit function $\phi(z) \in H$ satisfies $\phi(x) = 0$, which is impossible. Thus as $n \rightarrow \infty$,

$$(15) \quad n^{-2}|f_n(1 - x/n)| \geq A(x, \lambda) > 0.$$

We frequently use the following inequality of Hayman [7, p. 11].

²The symbols A , $A(\cdot, \cdot)$, etc. will be used to denote constants which depend on parameters shown.

LEMMA A. Suppose that $f \in S$ satisfies $M(r, f) = |f(r)| \geq \lambda/(1-r)^2$, $\lambda > 0$. Then

$$|f(re^{i\theta})| \leq \frac{A(\lambda, \varepsilon)}{(1-r)^{2-\varepsilon} |\theta|^\varepsilon}$$

for any $0 < \varepsilon < 2$.

We apply Lemma A to $e^{-i\eta}f_n(e^{i\eta}z)$, $r = 1 - x/n$ and $\varepsilon = 1$, noting

$$M(1 - x/n, f_n) \geq A(\lambda, x)(1 - (1 - x/n))^{-2},$$

to obtain

$$|f_n(1 - x/n)| \leq An^2/|\eta|,$$

which contradicts (15) unless (14) holds.

In general we want to consider the expression $a_{n,m} - e^{it/n}a_{n,m-1}$. We shall assume that $t = 0$ and $M((1 - 1/m), f_n) = |f_n(1 - 1/m)|$. Suppose that we have established Theorem 3 under this assumption. Now for general f_n satisfying (2), Lemma 1 shows that the maximum of $|f((1 - 1/m)e^{i\theta})|$ occurs at a point having argument η , $|\eta| \leq A(\lambda, x)/n$, $x = n/m \sim 1/a$. Thus the function $e^{-i\eta}f_n(e^{i\eta}z)$ satisfies condition (2) and has maximum modulus at $(1 - 1/m)$. Consequently

$$a_{n,m}e^{i(m-1)\eta} - a_{n,m-1}e^{i(m-2)\eta} \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} (x + iy)\phi(x + i(y - \nu))e^{a(x+iy)} dy$$

as if $n^{-2}f_n(1 - z/n) \rightarrow \phi(z)$; then by Lemma 1

$$n^{-2}e^{-i\eta}f_n(e^{i\eta}(1 - z/n)) \rightarrow \phi(x + i(y - \nu))$$

where $\eta n \rightarrow \nu$, $z = x + iy$. Changing the variable and noting that $e^{i(m-1)\eta} \rightarrow e^{i(m/n)\eta n} \rightarrow e^{iav}$ gives

$$(16) \quad e^{iav}(a_{n,m} - e^{i(-\nu)/n}a_{n,m-1}) \rightarrow \frac{e^{iav}}{2\pi} \int_{-\infty}^{\infty} (x + i(y + \nu))\phi(x + iy)e^{a(x+iy)} dy.$$

Now by Theorem A,

$$\frac{a_{n,m-1}}{n} \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x + iy)e^{a(x+iy)} dy = T_a,$$

and consequently (16) becomes

$$a_{n,m} - e^{-i\nu/n}a_{n,m-1} \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} (x + iy)\phi(x + iy)e^{a(x+iy)} dy + i\nu T_a.$$

Thus we obtain $a_{n,m} - a_{n,m-1} \rightarrow V_a$. For general t we have

$$(17) \quad a_{n,m} - e^{it/n}a_{n,m-1} \rightarrow V_a + itT_a.$$

Thus we have shown that to obtain (17) it suffices to assume the maximum of $|f_n((1 - 1/m)e^{i\theta})|$ occurs at $(1 - 1/m)$.

4. $a_{n,m} - a_{n,m-1}$ remains bounded. To show $a_{n,m} - a_{n,m-1}$ remains bounded we need modifications of Hayman's argument. These modified formulae and Hayman's lemmas will be needed to prove Theorem 3.

LEMMA 2. Suppose that $f \in S$ satisfies $M(1 - 1/m, f) = |f(1 - 1/m)|$. Then $|a_m - a_{m-1}| \leq A$, where A is an absolute constant.

Following Hayman we write

$$(18) \quad (m-1)a_{m-1} - \left(1 - \frac{1}{m}\right)ma_m = \frac{1}{2ni} \int_{|z|=\rho} \frac{(z - (1 - 1/m))}{z^m} f'(z) dz.$$

Next we integrate (18) with respect to ρ from $1 - 3/m$ to $1 - 2/m$. The left-hand side of (18) becomes $((m-1)/m)\{a_{m-1} - a_m\}$, while the right-hand side becomes

$$\frac{1}{2\pi} \int_{1-3/m}^{1-2/m} \int_{|z|=\rho} \frac{e^{-i(m-1)\theta}}{\rho^m} \left(\rho e^{i\theta} - \left(1 - \frac{1}{m}\right) \right) f'(\rho e^{i\theta}) \rho d\theta d\rho.$$

Consequently

$$(19) \quad a_{m-1} - a_m = \frac{m}{m-1} \{I_1 + I_2\},$$

where for $j = 1, 2$ we have

$$(20) \quad I_j = \frac{1}{2\pi} \iint_{\mathfrak{D}_j} \frac{e^{-i(m-1)\theta}}{\rho^m} \left(\rho e^{i\theta} - \left(1 - \frac{1}{m}\right) \right) f'(\rho e^{i\theta}) \rho d\theta d\rho,$$

and $\mathfrak{D}_1 = \{z: 1 - 3/m \leq |z| \leq 1 - 2/m, \delta_2/n \leq \text{Arg}(z) \leq \delta_1/n, \text{ for some } \delta_2 < 0 < \delta_1\}$, and $\mathfrak{D}_2 = \{1 - 3/m \leq |z| \leq 1 - 2/m\} - \mathfrak{D}_1$. Now let us define $M_1 = M(1 - 1/m, f)$ and regions $\epsilon_k \subset \mathfrak{D}_1 \cup \mathfrak{D}_2$ by

$$(21) \quad \epsilon_k: 2^{-k}M_1 \leq |f(z)| \leq 2^{1-k}M_1.$$

The following lemma of Hayman [8, p. 238] is useful.

LEMMA B. If $M_k = 2^{1-k}M_1$ and $G_k(R) = M_k^2 R^2 / (M_k^2 + R^2)$, then for $1 - 3/m \leq \rho \leq 1 - 2/m$,

$$(22) \quad \int_0^{2\pi} \left| \rho e^{i\theta} - \left(1 - \frac{1}{m}\right) \right|^2 G_k(|f(\rho e^{i\theta})|) d\theta \leq A m 2^{-k/4},$$

where A is an absolute constant.

Now returning to (20) we have

$$|I_j| \leq A \iint_{\mathfrak{D}_j} \left| \rho e^{i\theta} - \left(1 - \frac{1}{m}\right) \right| |f'(\rho e^{i\theta})| \rho d\theta d\rho$$

as $\rho^{-m} \leq e^{-3}$. Consequently

$$(23) \quad |I_j| \leq A \sum_{k=1}^{\infty} \left(\iint_{\mathfrak{D}_j \cap \epsilon_k} \left| \frac{f'(z)}{f(z)} \right|^2 \rho d\rho d\theta \right)^{1/2} \times \left(\iint_{\mathfrak{D}_j \cap \epsilon_k} \left| z - \left(1 - \frac{1}{m}\right) \right|^2 |f(z)|^2 d\theta \rho d\rho \right)^{1/2}.$$

Now

$$\begin{aligned} \iint_{\epsilon_k} \left| \frac{f'(z)}{f(z)} \right|^2 \rho d\rho d\theta &\leq M_{k+1}^{-2} \iint_{\epsilon_k} |f'(z)|^2 \rho d\rho d\theta \\ &\leq M_{k+1}^{-2} \text{Area}(f(\epsilon_k)) \leq M_{k+1}^{-2} \pi M_k^2 = 4\pi. \end{aligned}$$

On the other hand, for $z \in \varepsilon_k$,

$$|f(z)|^2 \leq \frac{2M_k^2 |f(z)|^2}{M_k^2 + |f(z)|^2}.$$

Thus

$$(24) \quad |I_j| \leq A \sum_{k=1}^{\infty} \left(\iint_{\mathfrak{D}_j \cap \varepsilon_k} \left| z - \left(1 - \frac{1}{m} \right) \right|^2 G_k(|f(z)|) \rho \, d\rho \, d\theta \right)^{1/2}.$$

Thus by Lemma B,

$$|I_j| \leq A \sum_{k=1}^{\infty} \left(\int_{1-3/m}^{1-2/m} A m 2^{-k/4} \rho \, d\rho \right)^{1/2} \leq A,$$

which by (19) completes the proof of Lemma 2.

5. Estimating I_2 . The formulae of §4 enable us to estimate I_2 . Let $\delta = \min(\delta_1, -\delta_2)$ (see (21)).

LEMMA 3. With f_n satisfying (2) and I_2 defined by (20), $m/n \sim a > 0$,

$$|I_2| \leq A(\lambda, a) \delta^{-1/8}.$$

Now the definition of $\mathfrak{D}_2$ and Lemma A, with $\varepsilon = 1$, implies that for $|\theta| \geq \delta/n$, $1 - 3/m \leq \rho \leq 1 - 2/m$,

$$(25) \quad |f(\rho e^{i\theta})| \leq A(\lambda, a) m^2 / \delta,$$

where we used Lemma 1 to ensure that $|f(\rho)| \geq A(\lambda, a)/(1 - \rho)^2$ in the range $1 - 3/m \leq \rho \leq 1 - 2/m$, $m/n \sim a$. Thus for

$$k < [(\log^+(\delta) + \log^+ A(\lambda, a))/\log(2)] = k_0,$$

$\varepsilon_k \cap \mathfrak{D}_2 = \emptyset$. Consequently by (23),

$$(26) \quad I_2 \leq A \sum_{k=k_0}^{\infty} \left(\iint_{\mathfrak{D}_2} \left| z - \left(1 - \frac{1}{m} \right) \right|^2 G_k(|f_n(z)|) \rho \, d\rho \, d\theta \right)^{1/2} \leq A 2^{-k_0/8}$$

by Lemma B. Thus by (25) and (26), $I_2 \leq A(a, \lambda) \delta^{-1/8}$ which proves Lemma 2.

6. Estimating I_1 . To obtain asymptotic expressions for I_1 we take a sequence $f_n \in S$ such that $n^{-2} f_n(1 - z/n) \rightarrow \phi(z)$. Then we write $f_n = f$, and use the results of §§4 and 5. Integrating by parts,

$$(27) \quad \int_{\delta_2/n}^{\delta_1/n} \frac{e^{-i(m-1)\theta}}{\rho^m} \left(\rho e^{i\theta} - \left(1 - \frac{1}{m} \right) \right) f'(\rho e^{i\theta}) \, d\theta \\ = \int_{\delta_2/n}^{\delta_1/n} \frac{(-i)}{\rho^m} e^{-im\theta} \left(\rho e^{i\theta} - \left(1 - \frac{1}{m} \right) \right) f(\rho e^{i\theta}) \\ + \int_{\delta_2/n}^{\delta_1/n} \frac{(m-1)}{\rho^m} \{ \rho e^{-i(m-1)\theta} - e^{-im\theta} f(\rho e^{i\theta}) \} \, d\theta,$$

where the two terms of the right-hand side are denoted by J_1 and J_2 , respectively, and

$$(28) \quad I_1 = \frac{1}{2\pi} \int_{1-3/m}^{1-2/m} (J_1 + J_2) d\rho.$$

LEMMA 4. *With the above assumptions,*

$$\left| \int_{1-3/m}^{1-2/m} J_1 d\rho \right| \leq \frac{A(a, \lambda, \epsilon)}{\delta^{1-\epsilon}},$$

where $\delta = \min(\delta_1, -\delta_2) > 2$, $m/n \sim a > 0$.

Now

(29)

$$\begin{aligned} & \left| \int_{1-3/m}^{1-2/m} \frac{e^{-i(m/n)\delta_j}}{\rho^m} \left(\rho e^{i\delta_j/n} - \left(1 - \frac{1}{m}\right) \right) f(\rho e^{i\delta_j/n}) d\rho \right| \\ & \leq A n^{-1} \max_{1-3/m \leq \rho \leq 1-2/m} |f(\rho e^{i\delta_j/n})| \max_{1-3/m \leq \rho \leq 1-2/m} \left| \rho e^{i\delta_j/n} - \left(1 - \frac{1}{m}\right) \right|, \end{aligned}$$

using $\rho^{-m} \leq e^3$, $m/n \sim a$. By Lemmas 1 and B,

$$(30) \quad |f(\rho e^{i\delta_j/n})| \leq \frac{A(a, \lambda, \epsilon)}{\delta_j^{2-\epsilon}} n^2$$

and

(31)

$$\begin{aligned} \max_{1-3/m \leq \rho \leq 1-2/m} \left| \rho e^{i\delta_j/n} - \left(1 - \frac{1}{m}\right) \right| & \leq A \left| \left(1 - \frac{3}{m}\right) \left(1 + \frac{i\delta_j}{n}\right) - \left(1 - \frac{1}{m}\right) \right| \\ & = A \left| \frac{2}{m} - \frac{i\delta_j}{n} \right| \leq \frac{A}{n} \delta_j, \end{aligned}$$

provided $\delta_j \geq 2$. Substituting (30) and (31) in (29) gives

$$\left| \int_{1-3/m}^{1-2/m} J_1 d\rho \right| \leq \sum_{j=1}^2 \frac{A}{\delta_j^{1-\epsilon}} < \frac{A}{\delta^{1-\epsilon}},$$

as required.

Next we estimate the second term in (28).

LEMMA 5. *Suppose that $n^{-2}f_n(1 - (x + iy)/n) \rightarrow \phi(x + iy) \in H$ as $n \rightarrow \infty$. Then*

$$\begin{aligned} & \frac{1}{2\pi} \int_{1-3/m}^{1-2/m} \int_{\delta_2/n}^{\delta_1/n} \frac{(m-1)e^{-im\theta}}{m-1} \{ \rho e^{i\theta} - 1 \} f_n(e^{i\theta}) \rho d\theta d\rho \\ & = -\frac{1}{2\pi} \int_{\delta_2}^{\delta_1} (x + iy) \phi(x + iy) e^{a(x+iy)} dy + o(1), \end{aligned}$$

as $m/n \rightarrow a > 0$ and $n \rightarrow \infty$. The error term is bounded by

$$A(\delta_1, \delta_2, a) \left\{ \left| \frac{m}{n} - a \right| + \frac{1}{n} + \max_{\delta_2 \leq y \leq \delta_1} \left| \frac{f_n}{n^2} \left(\left(1 - \frac{x}{n} \right) e^{-iy/n} \right) - \phi(x + iy) \right| \right\} \\ + A(a, x, \lambda, \epsilon) \delta^{\epsilon-1}.$$

New variables $\rho = 1 - x/n$ and $\theta = -y/n$ are substituted into the double integral. This becomes

$$\frac{1}{2\pi} \int_{2/n}^{3n/m} \int_{\delta_2}^{\delta_1} (m-1) \left(1 - \frac{x}{n} \right)^{-n(m/n)x} e^{i(m/n)y} \left\{ \left(1 - \frac{x}{n} \right) e^{-iy/n} - 1 \right\} \\ \times f_n \left(\left(1 - \frac{x}{n} \right) e^{-iy/n} \right) \frac{dx dy}{n^2},$$

which is

$$-\frac{1}{2\pi} \int_{2/a}^{3/a} \int_{\delta_2}^{\delta_1} a e^{ax} e^{ia y} \{x + iy\} \frac{f_n((1 - x/n)e^{-iy/n})}{n^2} dx dy + \epsilon_1,$$

where the error term satisfies

$$\epsilon_1 < A \int_{2/a}^{3/a} \int_{\delta_2}^{\delta_1} a \left| \left(\frac{m}{n} - a \right) + \frac{1}{n} \right| e^{ax} |x + iy| \frac{|f_n|}{n^2} dx dy \\ < A \max \left(\delta_1, -\delta_2, \frac{3}{a} \right) \left| \frac{m}{n} - a \right| + \frac{1}{n} \int_{\delta_2}^{\delta_1} \frac{|f_n|}{n^2} dx \\ < A \max \left(\delta_1, -\delta_2, \frac{3}{a} \right) \left\{ \left| \frac{m}{n} - a \right| + \frac{1}{n} \right\}.$$

Thus the double integral is equal to

$$(32) \quad -\frac{a}{2\pi} \int_{2/a}^{3/a} \int_{\delta_2}^{\delta_1} (x + iy) e^{a(x+iy)} \phi(x + iy) dx dy + \epsilon_1 + \epsilon_2,$$

with the second error term satisfying

$$\epsilon_2 < A \left(\max \left(\delta_1, -\delta_2, \frac{3}{a} \right) \right)^2 \max_{\delta_2 \leq y \leq \delta_1} \left\{ \left| \frac{f_n((1 - x/n)e^{-iy/n})}{n^2} - \phi(x + iy) \right| \right\}.$$

Finally we show that integrating with respect to x in (32) is redundant. From Cauchy's theorem for any $x_1 \leq x_2$,

$$\int_{\delta_2}^{\delta_1} (x_1 + iy) \phi(x_1 + iy) e^{a(x_1+iy)} dy = \int_{\delta_2}^{\delta_1} (x_2 + iy) \phi(x_2 + iy) e^{a(x_2+iy)} dy + \epsilon_3,$$

where

$$\epsilon_3 < \sum_{j=1}^2 \left| \int_{x_1}^{x_2} (x + i\delta_j) \phi(x + i\delta_j) e^{a(x+i\delta_j)} dx \right| \\ < \sum_{j=1}^2 (x_2 - x_1) (x_2 + |\delta_j|) e^{ax_2} \max_{x_1 \leq x \leq x_2} |\phi(x + i\delta_j)|.$$

However by Lemmas 1 and B, letting $n^{-2}f_n \rightarrow \phi$,

$$\max_{x_1 \leq x \leq x_2} |\phi(x + i\delta_j)| \leq \frac{A(\lambda, x_1, x_2, \epsilon)}{\delta_j^{2-\epsilon}}.$$

Thus

$$\epsilon_3 < A(x_1, x_2, \lambda, \epsilon, a)/\delta^{1-\epsilon}$$

provided $\delta_1, \delta_2 \geq 1$. Combining these results proves the lemma.

7. The proof of Theorem 3. From Lemmas 5 and 6, substituting into (28) we have

$$I_1 = -\frac{1}{2\pi} \int_{\delta_2}^{\delta_1} (x + iy)\phi(x + iy)e^{a(x+iy)} dy + \epsilon_4 + \epsilon_5,$$

where $\epsilon_4 < A(a, x, \lambda, \epsilon)/\delta^{1-\epsilon}$, and $\epsilon_5 \rightarrow 0$ as $n \rightarrow \infty$. Thus by Lemma 3 and (19), we have shown

LEMMA 6. *With the above notation*

$$a_{n,m} - a_{n,m-1} = \frac{1}{2\pi} \int_{\delta_2}^{\delta_1} (x + iy)\phi(x + iy)e^{a(x+iy)} dy + \epsilon_6 + \epsilon_7,$$

where $\epsilon_6 < A(a, x, \lambda, \epsilon)/\delta^{1/8}$ and $\epsilon_7 \rightarrow 0$ as $n \rightarrow \infty$.

Now in Lemma 2 it was shown that $|a_m - a_{m-1}|$ is bounded. Let $n = n_q$ be a subsequence such that

$$(33) \quad \lim_{q \rightarrow \infty} a_{n_q, m+1} - a_{n_q, m} \rightarrow V,$$

where the subsequence will also depend on how $m/n \rightarrow a$. (The limit V may possibly depend on the subsequence.) Then by Lemma 6,

$$(34) \quad V = \frac{1}{2\pi} \int_{-\delta_2}^{\delta_1} (x + iy)\phi(x + iy)e^{a(x+iy)} dy + \epsilon_6 + \epsilon_7,$$

where $\epsilon_6 \rightarrow 0$ as $\delta \rightarrow \infty$, and $\epsilon_7 \rightarrow 0$ as $n \rightarrow \infty$. This proves that the improper integral exists. Thus by Lemma 6 every convergent subsequence of $a_{n,m+1} - a_{n,m}$ converges to the same limit once we have checked that V_a is a continuous function of a .

LEMMA 7. V_a is a continuous function of a .

For any b near a by (34)

$$V - V_b = \frac{1}{2\pi} \int_{\delta_2}^{\delta_1} (x + iy)\phi(x + iy)\{e^{a(x+iy)} - e^{b(x+iy)}\} dy + \epsilon_8,$$

where $\epsilon_8 \rightarrow 0$ as $\delta \rightarrow \infty$. Thus

$$|V - V_b| \leq A(\delta_1, \delta_2, x) |a - b| + \epsilon_8,$$

which shows that $V_b \rightarrow V$ as $b \rightarrow a$. This completes the proof of Lemma 8 and consequently of Theorem 3. The last comments in §3 mean that we have also proved (17).

8. Proof of Corollary 1. This result is proved by showing that $T'_a(\phi) = V_a$, and then by appealing to Theorem 3. Now let $n^{-2}f_n(1 - z/n) \rightarrow \phi(z)$ as $n \rightarrow \infty$ and $1/c < a < b < c$. If m, p are integers such that $m/n \rightarrow a, p/n \rightarrow b$ then

$$(35) \quad \frac{T_a - T_b}{(a - b)} = \frac{a_{n,m}/n - a_{n,p}/n + \varepsilon_1}{m/n - p/n + \varepsilon_1},$$

where $\varepsilon_1 \rightarrow 0$, uniformly in c , as $n \rightarrow \infty$. Now by Theorem 3,

$$(36) \quad a_{n,m} = a_{n,p} + (m - p)V_a + (m - p)\varepsilon_2,$$

where $|\varepsilon_2| < \varepsilon_3 + \varepsilon_4$, $\varepsilon_3 \rightarrow 0$ as $a \rightarrow b$ and $\varepsilon_4 \rightarrow 0$ as $n \rightarrow \infty$. Thus by (35) and (36)

$$\frac{T_a - T_b}{a - b} = \frac{((m - p)/n)(V_a + \varepsilon_2) + \varepsilon_1}{(m - p)/n + \varepsilon_1} \rightarrow V_a + \varepsilon_5,$$

as $n \rightarrow \infty$ and $|\varepsilon_5| < \varepsilon_3$. Thus as $a \rightarrow b$, $(T_a - T_b)/(a - b) \rightarrow V_a$, which proves Corollary 1.

9. Proof of Theorem 1.

LEMMA 9. Suppose that $\phi(z) \in H$ and $f_n \in S$ such that $n^{-2}f_n(1 - z/n) \rightarrow \phi(z)$ as $n \rightarrow \infty$. Then the limit of $(|a_{n,m+1}| - |a_{n,m}|)_{n \rightarrow \infty}$, $m/n \rightarrow a$ exists as a continuous function of a , provided $T_a \neq 0$,

$$a_{n,m+1} - e^{it/n}a_{n,m} \rightarrow V_a - itT_a.$$

Thus

$$||a_{n,m+1}| - |a_{n,m}|| \rightarrow \min_{-\infty < t < \infty} |V_a - itT_a|,$$

and the sign of the limit of $|a_{n,m+1}| - |a_{n,m}|$ is determined by which side of the line $V_a - itT_a$ contains 0. Notice that at points a such that $T_a = 0$ we can have a discontinuity (see Example 1). Also $|a_{n,m+1}| - |a_{n,m}| \rightarrow 0$ if and only if $V_a \perp T_a$. Similarly to Corollary 1 we obtain

COROLLARY 2. $|T_a|$ is a C^1 function of a on $\{a: T_a \neq 0\}$. For these points a , if $m/n \rightarrow a$ as $n \rightarrow \infty$ we have $|a_{n,m+1}| - |a_{n,m}| \rightarrow d|T_a|/da$.

We can now prove Theorem 1. Let f_n be a function in S such that $|a_{n,n}| = A_n$. It is not clear that the maximum of $|f_n((1 - 1/n)e^{i\theta})|$ occurs near $(1 - 1/n)$ so we consider

$$f_n^*(z) = e^{-i\sigma}f_n(e^{i\sigma}z),$$

where $|f_n((1 - 1/n)e^{i\sigma(n)})| = M(1 - 1/n, f_n)$. The standard method (see Hayman [7, p. 3]) shows that

$$\lim_{n \rightarrow \infty} n^{-2}M\left(1 - \frac{1}{n}, f_n\right) = \lambda > 0,$$

and thus we may apply our analysis to f_n^* . Thus for any subsequence of n such that $n^{-2}f_n^*(1 - z/n) \rightarrow \phi(z)$, for some $\phi(z) \in H$, by Corollary 2, $|a_{n,m+1}| - |a_{n,m}| \rightarrow d|T_a(\phi)|/da$ as $n \rightarrow \infty$ along the same subsequence. Also $|a_{n,n}/n| = A_n/n \rightarrow K_0$.

Thus $|T_1(\phi)| = K_0$. Then we prove

LEMMA 10. *Suppose that $|T_1(\phi)| = K_0$. Then $d|T_1(\phi)|/da = K_0$.*

As $K_0 = \overline{\lim}(A_n/n)$, $\overline{\lim}|a_{n,m}/n| \leq K_0(m/n) \rightarrow K_0 a$ and thus $|T_a(\phi)| \leq K_0 a$ for all a . However $|T_1(\phi)| = K_0$ and $|T_a(\phi)|$ is smooth near 1 (by Corollary 2), which implies that

$$\frac{d}{da} |T_a|_{a=1} = K_0.$$

Thus we have shown that any convergent subsequence of $|a_{n,m+1}| - |a_{n,m}|$ converges to K_0 as $n \rightarrow \infty$, $m/n \rightarrow 1$. This proves the theorem.

REMARK 5. Lucas [10] shows that

$$||a_{n+2}| - 2|a_{n+1}| + |a_n|| \leq An^{1-\sqrt{2}},$$

but this is not strong enough to prove Theorem 1.

10. Proof of Theorem 2. Suppose that f_n is a subsequence in S with $|a_{n,n}| = A_n$. Then by Theorem 1 we have

$$(37) \quad |a_{n,n+1}| - A_n \rightarrow K_0$$

and

$$(38) \quad A_{n+1} - |a_{n+1,n}| \rightarrow K_0$$

as $n \rightarrow \infty$.

Since $|a_{n,n+1}| \leq A_{n+1}$ and $|a_{n+1,n}| \leq A_n$ we deduce

$$\liminf_{n \rightarrow \infty} (A_{n+1} - A_n) \geq K_0$$

from the first limit relation and

$$\limsup_{n \rightarrow \infty} (A_{n+1} - A_n) \leq K_0$$

from the second limit relation. This proves Theorem 2.

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