

TRACE CLASS SELF-COMMUTATORS

BY

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ABSTRACT. This paper extends earlier results of Berger and Shaw to all W^* algebras. The multiplicity of an operator in a W^* algebra is defined in terms of the trace on the W^* -algebra, and it is shown that if T is a hyponormal operator in such an algebra, the trace of its self-commutator is bounded by this multiplicity times the area of the spectrum of T , divided by π .

Let A be an operator on a Hilbert space H . The self-commutator of A , $[A^*, A]$, is defined by $[A^*, A] = A^*A - AA^*$. Earlier results provided bounds on the trace of a self-commutator valid for operators in the I_∞ von Neumann factor $B(H)$. Let G be a set of vectors in H and let $|G|$ denote the cardinality of G . Let T be a hyponormal operator in $B(H)$ such that the rational functions in T acting on G form a dense subspace of H . Berger and Shaw showed that $\text{tr}[T^*, T] \leq \pi^{-1} |G| \cdot \text{area}(\text{spectrum } T)$ [2]. The authors prove a version of this result meaningful in an arbitrary von Neumann algebra context.

Each section in this paper deals with a theorem that is an essential tool for the proof of the major result in §4.

Preliminaries. $\mathcal{Q}_H$ will denote a von Neumann algebra of operators on the Hilbert space H . All Hilbert spaces are complex and tr and τ are used to denote normal traces. When the symbols for the trace function are subscripted, the subscript refers to the algebra of operators upon which the function is acting. The same capital letter will represent both a space and the projection onto the space. The set of rational functions with poles off a compact set F will be denoted by $Q(F)$. If μ is a finite regular measure with compact support E , $Q^2(F, \mu)$ is the closure of rational functions with poles off F in $L^2(\mu)$ for any compact set F in E . Rational functions in an operator T will be denoted by $q(T)$ and polynomials in T by $p(T)$. P will represent the set of analytic polynomials and P' will be $P \setminus \{0\}$. Planar Lebesgue measure will be denoted by η . Other notations are standard; e.g., $B(H, K)$ is the set of bounded operators from H to K , and $B(H) = B(H, H)$.

The following definitions are fundamental. Let T be an operator and V a projection in $\mathcal{Q}_H$. Then $\langle v \rangle_T$ is the closure of the set $\{\sum_{i=1}^m p_i(T)v_i; p_i \in P, v_i \in V\}$ and $\langle v \rangle'_T$ is the closure of the set $\{\sum_{i=1}^m q_i(T)v_i; q_i \in Q(\text{sp}(T)), v_i \in V\}$.

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(i) If $\langle v \rangle_T = H$, then V is said to be polynomially cyclic for T . The polynomial multiplicity of T is the infimum of all values $\text{tr}(V)$ such that V is in $\mathcal{Q}_H$ and V is polynomially cyclic for T .

(ii) If $\langle v \rangle'_T = H$, V is said to be rationally cyclic for T . The infimum of all values $\text{tr}(V)$ such that V is in $\mathcal{Q}_H$ and V is rationally cyclic for T is the rational multiplicity of T .

(iii) If $\{x: \exists p \in P' \text{ such that } p(T)x \in \langle v \rangle_T\}$ is dense in H , then V is said to be effectually rationally cyclic for T . The effectual rational multiplicity of T is the infimum of all values $\text{tr}(V)$ such that V is effectually rationally cyclic for T .

The major result we present is the following theorem.

THEOREM 4.1. *Let A be a hyponormal operator in $\mathcal{Q}_H$ with effectual rational multiplicity $s < +\infty$. Then*

$$\text{tr}[A^*, A] \leq (s/\pi)\text{area}(\text{sp}(T)).$$

1. Let $\mathcal{Q}_H, \mathcal{Q}_K$ be von Neumann algebras in $B(H), B(K)$, respectively. Let $W \in B(H, K)$ such that $W^*\mathcal{Q}_K W \subseteq \mathcal{Q}_H$, $W\mathcal{Q}_H W^* \subseteq \mathcal{Q}_K$.

Let $\mathcal{Q}_{H,K}(W)$ be the weak closure of the set $\{BWA: B \in \mathcal{Q}_K, A \in \mathcal{Q}_H\}$. It follows that $\mathcal{Q}_{K,H}(W^*) = \mathcal{Q}_{H,K}(W)^*$. Writing $W = V\sqrt{W^*W}$, where V is the partial isometry from $R(W^*)$ to $R(W)$, $\mathcal{Q}_{H,K}(W) = \mathcal{Q}_K V \mathcal{Q}_H = V \mathcal{Q}_H$.

Let $\mathcal{U}$ be the von Neumann algebra in $B(H, K)$ consisting of operators

$$\begin{pmatrix} A & Y \\ Y' & B \end{pmatrix}$$

such that $A \in \mathcal{Q}_H, B \in \mathcal{Q}_K, Y' \in \mathcal{Q}_{H,K}(W)$ and $Y \in \mathcal{Q}_{K,H}(W^*)$.

We identify $\mathcal{Q}_H$ with the embedded algebra $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathcal{U}$ and $\mathcal{Q}_K$ with the embedded algebra $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ in $\mathcal{U}$.

LEMMA 1.1. *Let $\mathcal{Q}$ be a von Neumann algebra with a real valued trace τ . Let X be a positive hermitian operator and P a projection in $\mathcal{Q}$. Then X lies in trace class if and only if $(I - P)X(I - P)$ and PXP do.*

PROOF. The operator

$$\begin{aligned} X &= [P + (I - P)]X[P + (I - P)] \\ &= PXP + (I - P)X(I - P) + (I - P)XP + PX(I - P). \end{aligned}$$

Trace class operators form an ideal, therefore if X is in trace class so are PXP and $(I - P)X(I - P)$, $PX(I - P)$ and $(I - P)XP$.

Conversely, suppose

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

is a positive hermitian operator and X_{11} and X_{22} are in trace class. We can write $X = Y^*Y$, where Y is in W and

$$X_{11} = Y_{11}^*Y_{11} + Y_{12}^*Y_{21}, \quad X_{22} = Y_{21}^*Y_{21} + Y_{22}^*Y_{22}.$$

Thus, $Y_{11}^*Y_{11}$, $Y_{12}^*Y_{21}$, $Y_{21}^*Y_{21}$ and $Y_{22}^*Y_{22}$ are individually in trace class. Thus, each Y_{ij} is a Hilbert-Schmidt operator, so Y is. Thus, $X = Y^*Y$ is in trace class.

In the following three lemmas, W is an operator in $B(H, K)$, and $W = V\sqrt{W^*W}$ is its polar decomposition. Both $\mathcal{Q}_H$ and $\mathcal{Q}_K$ are von Neumann algebras, and $W^*\mathcal{Q}_K W \subset \mathcal{Q}_H$, $W\mathcal{Q}_H W^* \subset \mathcal{Q}_K$. Note that then $V^*\mathcal{Q}_K V \subset \mathcal{Q}_H$, $V\mathcal{Q}_H V^* \subset \mathcal{Q}_K$.

LEMMA 1.2. *If W has dense range and $\tau_{\mathcal{Q}_H}$ is a normal trace on $\mathcal{Q}_H$, then there is a normal trace $\tau_{\mathcal{Q}_K}$ on $\mathcal{Q}_K$ such that if P_H is a projection in $\mathcal{Q}_H$ contained in $\overline{R(W^*)}$, then $\tau_{\mathcal{Q}_K}(VP_H V^*) = \tau_{\mathcal{Q}_H}(P_H)$. In particular, if $E(\sigma)$ is the spectral projection for W^*W and $F(\sigma)$ is the spectral projection for WW^* , then*

$$\tau_{\mathcal{Q}_H}(E(\sigma)) = \tau_{\mathcal{Q}_K}(F(\sigma)).$$

PROOF. Since $R(V) = \overline{R(W)} = K$, $V^*\mathcal{Q}_K V$ in $\mathcal{Q}_H$ and $V\mathcal{Q}_H V^*$ in $\mathcal{Q}_K$ together imply $\mathcal{Q}_K = V\mathcal{Q}_H V^*$. It suffices to define $\tau_{\mathcal{Q}_K}$ on projections in $\mathcal{Q}_K$. The operator V is an isometry from $\overline{R(W^*)} = \overline{R\sqrt{W^*W}}$ onto K . Given a projection P_K in $\mathcal{Q}_K$, there exists a unique projection P_H in $\mathcal{Q}_H$, contained in $\overline{R\sqrt{W^*W}}$, such that $P_K = VP_H V^*$. If we define $\tau_{\mathcal{Q}_K}(P_K) = \tau_{\mathcal{Q}_K}(VP_H V^*) = \tau_{\mathcal{Q}_H}(P_H)$, $\tau_{\mathcal{Q}_K}$ is well defined. Moreover, $F(\sigma) = VE(\sigma)V^*$ and the result follows.

LEMMA 1.3.1. *Let*

$$\mathcal{Q}_H \otimes M_2 = \begin{pmatrix} \mathcal{Q}_H & \mathcal{Q}_H \\ \mathcal{Q}_H & \mathcal{Q}_H \end{pmatrix}.$$

Let $\tilde{e}$ be the embedding map of $\mathcal{Q}_H$ into $\mathcal{Q}_H \otimes M_2$. That is,

$$\tilde{A} = \tilde{e}(A) = \begin{pmatrix} A & 0 \\ A & 0 \end{pmatrix} \quad \text{for } A \text{ in } \mathcal{Q}_H.$$

Let Q be a projection in $\mathcal{Q}_H \otimes M_2$. Then $Q = Q_1 + Q_2$, where Q_1 and Q_2 are orthogonal projections such that $Q_1 = V_1^ V_1$ and $Q_2 = V_2^* V_2$, where $V_1 V_1^*$ and $V_2 V_2^*$ are contained in $\tilde{e}(\mathcal{Q}_H)$.*

PROOF. Let Q be a projection in $\mathcal{Q}_H \otimes M_2$ and let $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. The operator $Q = QP + Q(I - P)$. Let $QP = W_1 M_1$ and $Q(I - P) = W_2 M_2$ be the polar decompositions for QP and $Q(I - P)$, respectively. The operator $M = \sqrt{PQP}$, $R(M_1)$ is contained in $R(P)$, $W_1^* W_1$ is contained in P , and $W_1 W_1^*$ is in Q . Since Q and P belong to $\mathcal{Q}_H \otimes M_2$, so do W_i and M_i for $i = 1, 2$. $W_1 W_1^* = R(QPQ)$ implies that $Q - W_1 W_1^* = Q - R(QPQ)$.

We claim $Q - W_1 W_1^*$ is contained in $(I - P)$. It suffices to show $(Q - W_1 W_1^*)P = 0$. We have $QP = W_1 M_1$ and $R(W_1 W_1^*) = P(W_1) = \overline{R(QP)} = \overline{R(QPQ)}$. If x is in $R(Q - W_1 W_1^*)$, then x is perpendicular to $R(W_1 W_1^*)$ and x is in $R(Q)$. Thus, $QPQ = 0$. Since $0 = QPQx = QPPQx = (PQ)^*(PQ)x$, we see $0 = PQx = Px$. We have $P(Q - W_1 W_1^*) = 0$ or, in other words, $(Q - W_1 W_1^*) + W_1 W_1^*$ is a projection orthogonal to P . Express Q as $Q = (Q - W_1 W_1^*) + W_1 W_1^*$. Since $(Q - W_1 W_1^*)$ is orthogonal to P it is of the form $\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}$. It is equivalent to a matrix of the form $\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}$ by the unitary transformation $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. The operator $W_1 W_1^*$ is equivalent to $W_1^* W_1$ which is contained in P . Let $Q_1 = (Q - W_1 W_1^*)$, $Q_2 = W_1 W_1^*$, $V_2 = W_1^*$, $V_1 V_1^* = Y$ and the proof is complete.

LEMMA 1.3.2. *Define $\tau_{\mathcal{Q}} = \tau_{\mathcal{Q}_H} + \tau_{\mathcal{Q}_K}$. If $W \in B(H, K)$ has dense range, then $\tau_{\mathcal{Q}}$ is a well-defined trace on $\mathcal{Q}$. If $\tau_{\mathcal{Q}_H}$ and $\tau_{\mathcal{Q}_K}$ are normal, so is $\tau_{\mathcal{Q}}$.*

PROOF. Once it is known that $\tau_{\mathcal{Q}}$ is well defined, the final remark is trivial. Let $W^* = V^*\sqrt{WW^*}$. We note that $R(V^*) = R(W^*)$ and $R(V) = R(W)$. The operator W is dense so $\overline{R(V)} = \overline{R(W)}$ and $VV^* = I_K$. Since $W\mathcal{Q}_H W^*$ is in $\mathcal{Q}_K$ and $W^*\mathcal{Q}_K W$ is in $\mathcal{Q}_H$, we find $V\mathcal{Q}_H V^*$ is contained in $\mathcal{Q}_K$ and $V^*\mathcal{Q}_K V$ is in $\mathcal{Q}_H$. Let B be in $\mathcal{Q}_K$. V^*BV is in $\mathcal{Q}_H$ implying $V(V^*BV)V^* = B$, and thus $V\mathcal{Q}_H V^* = \mathcal{Q}_K$. Using this fact and that $R(V^*) = R(W^*)$ and $R(V) = R(W)$, we find

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_H & \mathcal{Q}_H V^* \\ V\mathcal{Q}_H & V\mathcal{Q}_H V^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} \mathcal{Q}_H & \mathcal{Q}_H \\ \mathcal{Q}_H & \mathcal{Q}_H \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & V^* \end{pmatrix}.$$

Since

$$\begin{pmatrix} I & 0 \\ 0 & V^* \end{pmatrix}$$

is an isometry, it suffices to show $\tau_{\mathcal{Q}_H} \oplus \tau_{\mathcal{Q}_H}$ is well defined on the projections in $\mathcal{Q}_H \otimes M_2$. Clearly $\tau_{\mathcal{Q}_H} \oplus \tau_{\mathcal{Q}_H}$ is linear, and thus we need only prove that if $\tilde{Q}$ and R are two projections in $\mathcal{Q}_H \otimes M_2$ and $X^*X = \tilde{Q}$ and $XX^* = R$, then $(\tau_{\mathcal{Q}_H} \oplus \tau_{\mathcal{Q}_H})(\tilde{Q}) = (\tau_{\mathcal{Q}_H} \oplus \tau_{\mathcal{Q}_H})(R)$. If $X^*X = \tilde{Q}$ and $XX^* = R$, $R(X^*) = R(\tilde{Q})$ and thus $X\tilde{Q}X^* = R$. From Lemma 1.3.1 we have $\tilde{Q} = Q_1 + Q_2$, where the Q_i are orthogonal projections, each equivalent to an embedded projection. We have $X\tilde{Q}X^* = XQ_1X^* + XQ_2X^* = R_1 + R_2 = R$. Thus it suffices to show that $\text{tr}(Q_i) = \text{tr}(XQ_iX^*) = \text{tr}(R_i)$ for $i = 1, 2$. Since Q_1 and Q_2 are equivalent to an embedded projection, we may, without any loss of generality, assume our original projection $\tilde{Q}$ was embedded; i.e. $\tilde{Q} = e(Q)$ for Q in $\mathcal{Q}_H$. Let $X^*X = \tilde{Q}$ and $XX^* = R$. Since $X\tilde{Q}X^* = R$ and $R(X^*) = R(\tilde{Q})$, we have that

$$X^* = \begin{pmatrix} X_{11}^* & X_{21}^* \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix}.$$

Thus, $Q = X_{11}^*X_{11} + X_{21}^*X_{21}$ and

$$R = \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} \begin{pmatrix} X_{11}^* & X_{21}^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}X_{11}^* \\ X_{21}X_{21}^* \end{pmatrix}.$$

To show $\text{tr}(\tilde{Q}) = \text{tr}(R)$ we must show

$$(1) \quad \text{tr } \mathcal{Q}_H(X_{11}^*X_{11}) + \text{tr } \mathcal{Q}_H(X_{21}^*X_{21}) = \text{tr } \mathcal{Q}_H(X_{11}X_{11}^*) + \text{tr } \mathcal{Q}_H(X_{21}X_{21}^*).$$

Since $\text{tr } \mathcal{Q}_H$ is well defined on $\mathcal{Q}_H$ and X_{ij} and V_{ij}^* are in $\mathcal{Q}_H$ for $i, j = 1, 2$ we have that (1) is true.

THE TRANSFER THEOREM. Let T in $\mathcal{Q}_H$ and A in $\mathcal{Q}_K$ be hyponormal operators. Let W belong to $B(H, K)$ and satisfy the following:

- (i) $W^*\mathcal{Q}_K W \subset \mathcal{Q}_H$, $W\mathcal{Q}_K W^* \subset \mathcal{Q}_H$.
- (ii) W^*W and WW^* are trace class operators in $\mathcal{Q}_H$ and $\mathcal{Q}_K$, respectively.
- (iii) $\tau_{\mathcal{Q}_K}(VXV^*) = \tau_{\mathcal{Q}_H}(X)$ for all X in trace class in $\mathcal{Q}_H$.
- (iv) W has dense range.
- (v) $WT = AW$.

Then $\tau_{\mathcal{Q}_H}[A^*, A] \leq \tau_{\mathcal{Q}_H}[T^*, T]$.

PROOF. Let J_t be the subspace of $H \oplus K$ defined by $J_t = \{th \oplus Wh \mid h \in H\}$. J_t is invariant for $T \oplus A$ and therefore $(T \oplus A)|_{J_t}$ is a hyponormal operator. After proving

$$\operatorname{tr}[(T \oplus A)|_{J_t}]^*, (T \oplus A)|_{J_t}] = \operatorname{tr}[T^*, T] \quad \text{when } t \rightarrow 0,$$

we will estimate $\operatorname{tr}[(T \oplus A)|_{J_t}]^*, (T \oplus A)|_{J_t}]$.

Let $\mathfrak{U}$ be as in Lemma 1.3.2 and let tr and τ be $\tau_{\mathfrak{U}}$. Let Q be the map in $\mathfrak{U}$ defined by $Q(h \oplus k) = th \oplus Wh$. Then

$$QT = (T \oplus A)|_{J_t} Q \quad \text{and} \quad (Q^*Q)^{1/2} = t(I + W^*W/t^2)^{1/2} = t(I + M),$$

where M is a positive hermitian operator with finite trace. Thus, $Q = V_J t(I + M)$ and $(T \oplus A)|_{J_t} V_J(I + M) = V_J(I + M)T$ leads us to

$$\begin{aligned} \operatorname{tr}[(T \oplus A)|_{J_t}]^*, (T \oplus A)|_{J_t}] &= \operatorname{tr}[(V_J^*(T \oplus A)|_{J_t} V_J)^*, (V_J^*(T \oplus A)|_{J_t} V_J)] \\ &= \operatorname{tr}[T^*, T]. \end{aligned}$$

Let $\{E(\sigma)\}$ and $\{F(\sigma)\}$ be the spectral resolutions for W^*W and WW^* , respectively. If $Q = V_{J_t} M_{J_t}$ is the polar decomposition for Q , then $R(E(\sigma))$ is contained in $V_{J_t}^*$ and $E(\sigma)V_{J_t}^*$ is a partial isometry. Also, $(E(\sigma) \oplus 0)$ and $J_t(E(\sigma) \oplus F(\sigma))J_t$ are equivalent projections and

$$\tau_{\mathfrak{U}_n}(E(\sigma)) = \tau_{\mathfrak{U}}(E(\sigma) \oplus 0) = \tau(J_t(E(\sigma) \oplus F(\sigma))J_t) = \tau[(E(\sigma) \oplus F(\sigma))J_t].$$

We compute J_t explicitly.

$$\begin{aligned} J_t &= \overline{R(Q)} = \overline{R\begin{pmatrix} t & 0 \\ W & 0 \end{pmatrix}}, \quad Q = V\sqrt{Q^*Q}. \\ Q^*Q &= \begin{pmatrix} t^2I + W^*W & 0 \\ 0 & 0 \end{pmatrix}, \quad \sqrt{Q^*Q} = \begin{pmatrix} (t^2I + W^*W)^{1/2} & 0 \\ 0 & 0 \end{pmatrix}, \\ V &= Q(\sqrt{Q^*Q})^{-1} = \begin{pmatrix} t & 0 \\ W & 0 \end{pmatrix} \begin{pmatrix} (t^2I + W^*W)^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} t(t^2I + W^*W)^{-1/2} & 0 \\ W(t^2I + W^*W)^{-1/2} & 0 \end{pmatrix}, \end{aligned}$$

$$J_t = VV^*$$

$$\begin{aligned} &= \begin{pmatrix} t(t^2I + W^*W)^{-1/2} & 0 \\ W(t^2I + W^*W)^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} t(t^2I + W^*W)^{-1/2} & (t^2I + W^*W)^{-1/2}W^* \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} t^2(t^2I + W^*W)^{-1} & t(t^2I + W^*W)^{-1}W^* \\ tW(t^2I + W^*W)^{-1} & W(t^2I + W^*W)^{-1}W^* \end{pmatrix}. \end{aligned}$$

For $0 \in \sigma$, let

$$\begin{aligned} \gamma(\sigma) &= \operatorname{tr}\{J_t(E(\sigma) \oplus F(\sigma))J_t[T \oplus A|_{J_t}^*, T \oplus A|_{J_t}]J_t(E(\sigma) \oplus F(\sigma))J_t\}, \\ \gamma(\{0\}) &= \operatorname{tr}\{J_t(E(\{0\}) \oplus 0)J_t[T \oplus A|_{J_t}^*]J_t(E(\{0\}) \oplus 0)J_t\}. \end{aligned}$$

We will evaluate

$$(2) \quad \int_{[0, \|W\|]} d\gamma = \gamma(\{0\}) + \int_{(0, \|W\|]} d\gamma$$

since $\{J_t(E(\sigma) \oplus F(\sigma))J_t\}$ is a spectral resolution for J_t . From Lemma 1.1 we see it suffices to examine (2) in order to estimate $\text{tr}[(T \oplus A)|_{J_t}^*, (T \oplus A)|_{J_t}]$.

First consider

$$\begin{aligned} & (E(\sigma) \oplus F(\sigma))J_t(T^*T \oplus A^*A)J_t(E(\sigma) \oplus F(\sigma)) \\ &= (E(\sigma) \oplus F(\sigma))J_t(T^*T \oplus 0)J_t(E(\sigma) \oplus F(\sigma)) \\ &+ (E(\sigma) \oplus F(\sigma))J_t(0 \oplus A^*A)J_t(E(\sigma) \oplus F(\sigma)). \end{aligned}$$

We claim, as τ approaches 0, the trace of this expression goes to

$$\text{tr}[(0 \oplus A^*A)F(\sigma)] = \text{tr}_{\mathcal{Q}_K}(A^*AF(\sigma)).$$

If σ is a Borel set, bounded away from 0 in $(0, \|W\|]$, $E(\sigma)$ and $F(\sigma)$ are trace class projections. Utilizing the properties of the trace function and the fact that

$$(E(\sigma) \oplus F(\sigma))J_t = J_t(E(\sigma) \oplus F(\sigma)),$$

we find

$$\begin{aligned} & \text{tr}(E(\sigma) \oplus F(\sigma))J_t(T^*T \oplus 0)J_t(E(\sigma) \oplus F(\sigma)) - \text{tr}[J_t(T^*TE(\sigma) \oplus 0)J_t] \\ &= \text{tr}[J_t(T^*TE(\sigma) \oplus 0)] = \text{tr} \begin{pmatrix} t^2(t^2I + W^*W)^{-1}T^*TE(\sigma) & 0 \\ tW(t^2I + W^*W)^{-1}T^*TE(\sigma) & 0 \end{pmatrix} \\ &= \tau_{\mathcal{Q}_H}(t^2(t^2I + W^*W)^{-1}T^*TE(\sigma)) = \tau_{\mathcal{Q}_H}(t^2(t^2I + W^*W)^{-1}E(\sigma)T^*T). \end{aligned}$$

Since T^*T is a bounded operator, it suffices to show $t^2(t^2I + W^*W)^{-1}E(\sigma)$ goes to 0 in trace norm. Since this operator is positive, it suffices to show

$$\tau_{\mathcal{Q}_H}(t^2(t^2I + W^*W)^{-1}E(\sigma)) \rightarrow 0.$$

This will be done later. Note that

$$\tau_{\mathcal{Q}_H}(t^2(t^2I + W^*W)^{-1}E(\sigma)) = \text{tr}((E(\sigma) \oplus 0)J_t).$$

We further claim

$$\begin{aligned} & \text{tr}[(E(\sigma) \oplus F(\sigma))J_t(0 \oplus A^*A) - J_t(E(\sigma) \oplus F(\sigma))] \\ &= \text{tr}[J_t(0 \oplus A^*A)(E(\sigma) \oplus F(\sigma))J_t] = \text{tr}[J_t(0 \oplus A^*A)(E(\sigma) \oplus F(\sigma))] \end{aligned}$$

approaches $\text{tr}[0 \oplus A^*AF(\sigma)]$. In other words we want to show

$$\text{tr}[(I - J_t)(0 \oplus A^*AF(\sigma))]$$

goes to 0 as t goes to 0.

$$\begin{aligned} \text{tr}((I - J_t)(0 \oplus A^*AF(\sigma))) &= \text{tr} \begin{pmatrix} 0 & -t(t^2I + W^*W)^{-1}W^*A^*AF(\sigma) \\ 0 & W(t^2I + W^*W)^{-1}W^*A^*AF(\sigma) \end{pmatrix} \\ &= \tau_{\mathcal{Q}_K}(W(t^2I + W^*W)^{-1}W^*A^*AF(\sigma)). \end{aligned}$$

Thus, since A^*A is bounded, it suffices to show $F(\sigma)W(t^2I + W^*W)^{-1}W^* \rightarrow 0$ in trace norm. But

$$W(t^2I + W^*W)^{-1}W^* = WW(t^2I + WW^*)^{-1},$$

which commutes with $F(\sigma)$. Thus, $F(\sigma)W(t^2I + W^*W)^{-1}W^*$ is positive, and it suffices to show $\tau_{\mathcal{Q}_K}(F(\sigma)WW^*(t^2I + WW^*)^{-1}) \rightarrow 0$. By the above, we have

$$\tau_{\mathcal{Q}_K}(F(\sigma)WW^*(t^2I + WW^*)^{-1}) = \text{tr}[(I - J_t)(0 \oplus F(\sigma))].$$

Both $\text{tr}[(E(\sigma) \oplus 0)J_t]$ and $\text{tr}[(0 \oplus F(\sigma))J_t - (0 \oplus F(\sigma))]$ approaching 0 is equivalent to $\text{tr}[(E(\sigma) \oplus F(\sigma))J_t - (0 \oplus F(\sigma))]$ approaching 0; however by Lemmas 1.2 and 1.3.2 the last expression is equal to $\text{tr}_{\mathcal{Q}_K}(F(\sigma)) - \text{tr}_{\mathcal{Q}_K}(0 \oplus F(\sigma))$, which is 0. Thus, it suffices to show $\text{tr}(J_t(E(\sigma) \oplus 0))$ goes to 0 as t does to support both claims. Recall

$$J_t = \begin{pmatrix} t^2(t^2I + W^*W)^{-1} & t(t^2I + W^*W)^{-1}W^* \\ tW(t^2I + W^*W)^{-1} & W(t^2I + W^*W)^{-1}W^* \end{pmatrix}.$$

Thus,

$$J_t(E(\sigma) \oplus 0) = \begin{pmatrix} t^2(t^2I + W^*W)^{-1}E(\sigma) & 0 \\ tW(t^2I + W^*W)^{-1}E(\sigma) & 0 \end{pmatrix}$$

and

$$\begin{aligned} \text{tr}_{\mathcal{Q}_K} J_t(E(\sigma) \oplus 0) &= \text{tr}_{\mathcal{Q}_H} [t^2(t^2I + W^*W)^{-1}E(\sigma)] \\ &= \int_{0^+}^{\|W^*W\|} \frac{t^2}{t^2 + \lambda^2} d\tau(E(\lambda)E(\sigma)). \end{aligned}$$

Since $t^2/(t^2 + \lambda^2) \leq 1$, by the Lebesgue dominated convergence theorem

$$\lim_{t \rightarrow 0} \int_{0^+}^{\|W^*W\|} \frac{t^2}{t^2 + \lambda^2} d\tau(E(\lambda)E(\sigma)) = 0.$$

Now, let us consider

$$(3) \quad \text{tr}[(E(\sigma) \oplus F(\sigma))(T \oplus A)J_t(T^* \oplus A^*)(E(\sigma) \oplus F(\sigma))].$$

Writing each operator in (3) in matrix form, we find (3) equal to

$$\text{tr}_{\mathcal{Q}_H}(E(\sigma)t^2T(t^2I + W^*W)^{-1}T^*E(\sigma)) + \text{tr}_{\mathcal{Q}_K}(F(\sigma)AW^*W(t^2I + W^*W)^{-1}AF(\sigma)).$$

This expression is equal to

$$(4) \quad \int_{0^+}^{\|W^*W\|} \frac{t^2}{t^2 + \lambda^2} d\text{tr}_{\mathcal{Q}_H}(TE(\lambda)T(E(\sigma))) + \int_{0^+}^{\|W^*W\|} \frac{t^2}{t^2 + \lambda^2} d\tau_{\mathcal{Q}_K}(AF(\lambda)A^*F(\sigma)).$$

Both $t^2/(t^2 + \lambda^2)$ and $\lambda^2/(t^2 + \lambda^2)$ are bounded by 1. By applying the Lebesgue dominated convergence theorem we see that as $t \rightarrow 0^+$, (4) approaches

$$\begin{aligned} \int_{0^+}^{\|W^*W\|} d\tau_{\mathcal{Q}_K}(AF(\lambda)A^*F(\sigma)) &= \tau_{\mathcal{Q}_K}(AF(0, \|W\|]A^*F(\sigma)) \\ &= \tau_{\mathcal{Q}_K}(AA^*F(\sigma)) = \tau_{\mathcal{Q}_K}(AA^*F(\sigma)) = \tau_{\mathcal{Q}_K}[F(\sigma)AA^*F(\sigma)]. \end{aligned}$$

Next, consider

$$(E(\{0\}) \oplus 0)[J_t(T^*T \oplus A^*A)J_t - (T \oplus A)J_t(T^* \oplus A^*)](E(\{0\}) \oplus 0).$$

Using the fact that $[tE(\{0\}) \oplus 0]$ is contained in J_t we find the operator

$$(E(\{0\}) \oplus 0)[J_t(T^*T \oplus A^*A)J_t - (T \oplus A)J_t(T^* \oplus A^*)](E(\{0\}) \oplus 0)$$

in matrix form is

$$\begin{pmatrix} E(\{0\})t^2(t^2I + W^*W)^{-1}T^*TE(\{0\}) & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} E(\{0\})t^2T(t^2I + W^*W)^{-1}T^*E(\{0\}) & 0 \\ 0 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \tau_{\mathcal{Q}}[(E(\{0\}) \oplus 0)J_t(T^*T \oplus A^*A)J_t - (T \oplus A)J_t(T^* \oplus A^*)(E(\{0\}) \oplus 0)] \\ = \tau_{\mathcal{Q}_H}[E(\{0\})t^2(t^2I + W^*W)^{-1}T^*TE(\{0\}) \\ - E(\{0\})Tt^2(t^2I + W^*W)^{-1}T^*E(\{0\})]. \end{aligned}$$

Set

$$\begin{aligned} Y &= [E(\{0\})t^2(t^2I + W^*W)^{-1}T^*TE(\{0\}) - E(\{0\})Tt^2(t^2I + W^*W)^{-1}T^*E(\{0\})] \\ &= Y - E(\{0\})TT^*E(\{0\}) + E(\{0\})TT^*E(\{0\}) \\ &= E(\{0\})[T^*, T]E(\{0\}) + E(\{0\})T\left(\int_{0^+}^{\|W^*W\|} 1 - \frac{t^2}{t^2 + \lambda^2} dE(\lambda)\right)T^*E(\{0\}) \\ &= E(\{0\})[T^*, T]E(\{0\}) + E(\{0\})T\left(\int_{0^+}^{\|W^*W\|} \frac{\lambda^2}{t^2 + \lambda^2} dE(\lambda)\right)T^*E(\{0\}). \end{aligned}$$

Thus

$$\begin{aligned} \text{tr}_{\mathcal{Q}_H}(Y) &= \text{tr}_{\mathcal{Q}_H}[E(\{0\})[T^*, T]E(\{0\})] \\ &\quad + \int_{0^+}^{\|W^*W\|} \frac{\lambda^2}{t^2 + \lambda^2} d\tau_{\mathcal{Q}_H}(E(\{0\})TE(\lambda)T^*E(\{0\})). \end{aligned}$$

The latter quantity is finite since

$$\begin{aligned} (5) \quad \text{tr}[T^*, T] &= \text{tr}_{\mathcal{Q}}[(I - E(\{0\}))[(T \oplus A)|_{J^*}^*, (T \oplus A)|_L](I - E(\{0\}))] \\ &\quad + \tau_{\mathcal{Q}_H}[E(\{0\})[T^*, T]E(\{0\})] \\ &\quad + \int_{0^+}^{\|W^*W\|} \frac{\lambda^2}{t^2 + \lambda^2} d\tau_{\mathcal{Q}_H}(E(\{0\})TE(\lambda)T^*E(\{0\})). \end{aligned}$$

The left-hand side of (5) is positive and finite. The first two terms on the right-hand side of the equation are positive, thus the last quantity, which is also positive, must

be finite as well. Since the integrand increases monotonically to 1 as $t \rightarrow 0^+$, we see

$$\int_{0^+}^{\|W^*W\|} 1 d\tau_{\mathcal{Q}_H}(E(\{0\})TE(\lambda)T^*E(\{0\}))$$

is finite. Thus $\tau_{\mathcal{Q}_H}(E(\{0\})TE(\lambda)T^*E(\{0\}))$ is a finite measure.

Now

$$\begin{aligned} Y &= Y - E(\{0\})TE\{0\}T^*E(\{0\}) + E(\{0\})TE(\{0\})T^*E(\{0\}) \\ &= [(TE(\{0\}))^*, TE\{0\}] + E(\{0\})TE(\{0\})T^*E(\{0\}) \\ &\quad - E(\{0\})Tt^2(t^2I + W^*W)^{-1}T^*E(\{0\}). \end{aligned}$$

We know $WTx = AWx = 0$ for x in $E(\{0\})$; therefore

$$\begin{aligned} \tau_{\mathcal{Q}_H}(Y) &= \tau_{\mathcal{Q}_H}[(TE(\{0\}))^*, TE(\{0\})] + \tau_{\mathcal{Q}_H}(E(\{0\})TE(\{0\})T^*E(\{0\})) \\ &\quad - E(\{0\})T\left(\int_0^{\|W^*W\|} \frac{t^2}{t^2 + \lambda^2} dE(\lambda)\right)T^*E(\{0\}), \end{aligned}$$

and

$$\begin{aligned} \tau_{\mathcal{Q}_H}\left\{\int_{[0]} 1 dE(\{0\})TE(\lambda)T^*E(\{0\}) - \int_{0^+}^{\|W^*W\|} \frac{t^2}{t^2 + \lambda^2} dE(\{0\})TE(\lambda)T^*E(\{0\})\right\} \\ = -\int_{0^+}^{\|W^*W\|} \frac{t^2}{t^2 + \lambda^2} d\tau_{\mathcal{Q}_H}[E(\{0\})TE(\lambda)T^*E(\{0\})]. \end{aligned}$$

As we have shown above, $\tau_{\mathcal{Q}_H}[E(\{0\})TE(\lambda)T^*E(\{0\})]$ is a finite measure. Since $t^2/(t^2 + \lambda^2) \leq 1$, by the Lebesgue dominated convergence theorem this integral approaches 0 as t goes to 0^+ . Thus as $t \rightarrow 0^+$, $\text{tr}(Y) \rightarrow \text{tr}[E(\{0\})T^*, TE(\{0\})]$.

Our previous result was

$$\begin{aligned} (6) \quad \text{tr}_{\mathcal{Q}_K}(F(\sigma)[A^*, A]F(\sigma)) \\ = \lim_{t \rightarrow 0^+} \left\{ (E(\sigma) \oplus F(\sigma)) \left[(T \oplus A)|_{J_t}^*, (T \oplus A)|_{J_t} \right] (E(\sigma) \oplus F(\sigma)) \right\} \end{aligned}$$

for $\sigma \subset (0, \|W\|]$. In particular, if we divide the interval $(0, \|W\|]$ into half-open intervals of the form $\sigma_n = (\|W\|/(n+1), \|W\|/n]$, equation (6) is true for each σ_n .

Thus, we have

$$\begin{aligned} \text{tr}[A^*, A] + \text{tr}[E(\{0\})T^*, TE(\{0\})] \\ = \sum \text{tr}(F(\sigma_n)[A^*, A]F(\sigma_n)) + \text{tr}[E(\{0\})T^*, TE(\{0\})]. \end{aligned}$$

By Fatou's theorem this is bounded by

$$\begin{aligned} \lim_{t \rightarrow 0^+} \sum_n \text{tr} \left\{ (E(\sigma_n) \oplus F(\sigma_n)) \left[(T \oplus A)|_{J_t}^*, (T \oplus A)|_{J_t} \right] (E(\sigma_n) \oplus F(\sigma_n)) \right\} \\ + \text{tr}[E(\{0\})T^*, TE(\{0\})]. \end{aligned}$$

Using the property of normality of our trace, this expression becomes

$$\begin{aligned}
& \lim_{t \rightarrow 0^+} \operatorname{tr} \sum_n \left\{ (E(\sigma_n) \oplus F(\sigma_n)) \left[(T \oplus A) \big|_{J_t} \right] (E(\sigma_n) \oplus F(\sigma_n)) \right\} \\
& \quad + \lim_{t \rightarrow 0^+} \operatorname{tr} \left[E(\{0\}) t^2 (t^2 I + W^* W)^{-1} T^* T E(\{0\}) \right. \\
& \quad \quad \left. - E(\{0\}) T t^2 (t^2 I + W^* W)^{-1} T^* E(\{0\}) \right] \\
& = \lim_{t \rightarrow 0^+} \operatorname{tr} (E(0, \|W\|) \oplus F(0, \|W\|)) \left[(T \oplus A) \big|_{J_t}^*, (T \oplus A) \big|_{J_t} \right] \\
& \quad \times (E(0, \|W\|) \oplus F(0, \|W\|)) \\
& \quad + \lim_{t \rightarrow 0^+} \operatorname{tr} E(\{0\}) \left[E(\{0\}) t^2 (t^2 I + W^* W)^{-1} T^* T E(\{0\}) \right. \\
& \quad \quad \left. - E(\{0\}) T t^2 (t^2 I + W^* W)^{-1} T^* E(\{0\}) \right] E(\{0\}) \\
& \leq \lim_{t \rightarrow 0^+} \left\{ \operatorname{tr} (E(0, \|W\|) \oplus F(0, \|W\|)) \left[(T \oplus A) \big|_{J_t}^*, (T \oplus A) \big|_{J_t} \right] \right. \\
& \quad \quad \left. \times (E(0, \|W\|) \oplus F(0, \|W\|)) \right\} \\
& \quad + \operatorname{tr} \{ (E\{0\} \oplus 0) [J_t(T^* T \oplus A^* A) J_t - (T \oplus A) J_t(T^* \oplus A^*)] (E\{0\} \oplus 0) \} \\
& = \lim_{t \rightarrow 0^+} \left\{ \operatorname{tr} (E(0, \|W\|) \oplus F(0, \|W\|)) \left[(T \oplus A) \big|_{J_t}^*, (T \oplus A) \big|_{J_t} \right] \right. \\
& \quad \times (E(0, \|W\|) \oplus F(0, \|W\|)) \\
& \quad \quad \left. + \operatorname{tr} (E(\{0\}) \oplus 0) \left[(T \oplus A) \big|_{J_t}^*, (T \oplus A) \big|_{J_t} \right] (E(\{0\}) \oplus 0) \right\} \\
& = \lim_{t \rightarrow 0^+} \left[(T \oplus A) \big|_{J_t}^*, (T \oplus A) \big|_{J_t} \right] = \operatorname{tr} [T^*, T].
\end{aligned}$$

2. In this section we prove the existence of an intertwining operator, W , which will enable us to transfer properties of an operator in one von Neumann algebra to an operator in another von Neumann algebra. To facilitate the proof of the intertwining theorem we use the following lemma.

LEMMA 2.1. *Let G be the finite union of multiconnected domains, each bounded by finitely many smooth Jordan curves. Let A be an operator in $\mathcal{Q}_K$ whose spectrum lies in G , and let F be a projection in $\mathcal{Q}_K$. For z in $G \setminus \operatorname{sp}(A)$, let W_z be the following operator in $B\{(Q^2(\chi_G \eta)) \otimes F, K\}$: $W_z(h \otimes x) = \langle h, \kappa_z \rangle (A - zI)^{-1} x$, where κ_z is the Bergman kernel for G [1]. Then $W_z^* \mathcal{Q}_K W_z$ is contained in $B(Q^2(\chi_G \eta)) \otimes F \mathcal{Q}_K F$ and $W_z(B(Q^2(\chi_G \eta)) \otimes F \mathcal{Q}_K F) W_z^*$ is in $\mathcal{Q}_K$.*

PROOF. It suffices to prove the lemma for the operator Y_z , where $Y_z(h \otimes x) = \langle h, \kappa_z \rangle x$, since $(A - zI)^{-1}$ belongs to $\mathcal{Q}_K$. Let us call $\langle h, \kappa_z \rangle = L_{\kappa_z}(h)$. Thus $Y_z(h \otimes x) = L_{\kappa_z}(h)x$, and $Y_z^* x = \kappa_z \otimes x$.

Let C be an operator in $B(Q^2(\chi_G \eta))$ and let M belong to $\mathcal{Q}_K$. We compute

$$\begin{aligned} Y_z(C \otimes FMF)Y_z^* &= \langle C\kappa_z, \kappa_z \rangle FMF \in \mathcal{Q}_K, \\ Y_z^*MY_z &= (L_z, L_z^*) \otimes FMF \in B(Q^2(\chi_G \eta)) \otimes F\mathcal{Q}_KF. \end{aligned}$$

INTERTWINING THEOREM. *Let G be a finite union of multiconnected domains each bounded by a finite union of smooth disjoint Jordan curves. Let F be a finite projection in $\mathcal{Q}_K$ and let $T = T_z \otimes F$ be contained in $B(Q^2(\chi_G \eta)) \otimes F\mathcal{Q}_KF$. Let A in $\mathcal{Q}_K$ be such that $\text{sp}(A)$ is contained in G and*

$$(6') \quad \{q_\alpha(A)y : y \in F \text{ and } q_\alpha \in Q(\text{sp}(T_z \otimes F))\}^- = K.$$

We define $\text{tr}_{Q \otimes F}(x \otimes M) = \text{tr}(x)\text{tr}_{\mathcal{Q}_K}(M)$ for $M \in F\mathcal{Q}_KF$, and the first trace is the counting trace on $B(Q^2)$. Then there exists a W in $B(Q^2(\chi_G \eta) \otimes F, K)$ such that:

- (i) W has dense range,
- (ii) $WT = AW$,
- (iii) WW^* and W^*W are in trace class.
- (iv) $W\{B(Q^2(\chi_G \eta)) \otimes F\mathcal{Q}_KF\}W^*$ is in $\mathcal{Q}_K$ and $W^*\mathcal{Q}_KW$ is in $B(Q^2(\chi_G \eta)) \otimes F\mathcal{Q}_KF$.

PROOF. For u in $Q^2(\chi_G \eta) \otimes F$, define $\hat{u}: G \times F \rightarrow \mathbb{C}$ by $\hat{u}(z, x) = \langle u, \kappa_z \otimes x \rangle$, where κ_z is the Bergman kernel of G . The function $\hat{u}$ is conjugate linear in x , linear in u , and, since κ_z is conjugate analytic in z , u is analytic in z .

Let Γ be a finite union of smooth Jordan curves bounding $\text{sp}(A)$ in G . The map $z \rightarrow \kappa_z$ is strongly continuous for z on Γ . Thus

$$(7) \quad |\hat{u}(z, x)| \leq \|u\| \|\kappa_z \otimes x\| \leq \|u\| M_\Gamma \|x\| \quad \text{where } M_\Gamma = \sup_{z \in \Gamma} \|\kappa_z\|.$$

For fixed z , $\hat{u}(z, \cdot)$ is a bounded conjugate linear form on F and is therefore given as an inner product. There exists a unique h_z^u in F such that $\hat{u}(z, \cdot) = \langle h_z^u, \cdot \rangle$.

In this case

$$(8) \quad \langle h_z^u, x \rangle = \langle u, \kappa_z \otimes x \rangle.$$

The vector h_z^u , which is linear in u by the uniqueness of the Riesz representation theorem, is analytic in z since $u(z, \cdot)$ is. From (7) and (8) we see that $\|h_z^u\| \leq \|u\| \cdot M_\Gamma$. For $u \in Q^2(\chi_G \eta) \otimes F$ of the form $f \otimes x$, $h_z^u = f(z)x$.

Define

$$Wu = -\frac{1}{2\pi i} \int_\Gamma (A - zI)^{-1} h_z^u dz,$$

where Γ is a finite union of smooth Jordan curves bounding $\text{sp}(A)$ in G . We have

$$W(f \otimes x) = -\frac{1}{2\pi i} \int_\Gamma (A - zI)^{-1} h_z^{f \otimes x} dz = -\frac{1}{2\pi i} \int_\Gamma f(z)(A - zI)^{-1} x dz = f(A)x.$$

Thus, by (6'), W has dense range. We will now show W is bounded:

$$\begin{aligned} \|Wu\|^2 &= \left\| -\frac{1}{2\pi i} \int_\Gamma (A - zI)^{-1} h_z^u dz \right\|^2 \leq K_\Gamma \|(A - zI)^{-1}\| \cdot \|h_z^u\| \\ &\leq K_\Gamma \sup_{z \in \Gamma} \|(A - zI)^{-1}\| \|u\| \|\Gamma\| < +\infty, \end{aligned}$$

where $K_\Gamma = (\text{length of } \Gamma)/2\pi$.

Note that for $f \otimes x$ in $Q^2(\chi_G \eta) \otimes F$,

$$W(T_z \otimes F)(f \otimes x) = W(T_z f \otimes x) = W(zf \otimes x) = AF(A)x = AW(f \otimes x).$$

Thus $W(T_z \otimes F) = AW$. W is the uniform limit of sums of the form W_z where $W_z(h \otimes x) = \langle h, \kappa_z \rangle (A - zI)^{-1}x$. From Lemma 2.1 we see

$$W[B(Q^2(\chi_G \eta)) \otimes F\mathcal{Q}_K F]W^*$$

is contained in $\mathcal{Q}_K$. Similar reasoning shows $W^*\mathcal{Q}_K W$ is in $B(Q^2(\chi_G \eta)) \otimes F\mathcal{Q}_K F$. To show (iii), we compute the following:

$$\begin{aligned} \|W^*W\|_{\text{tr}_{Q \otimes F}} &= \left\| \frac{1}{(2\pi)^2} \iint W_z^* W_{z'} dz dz' \right\|_{\text{tr}_{Q \otimes F}} \\ &\leq \frac{1}{(2\pi)^2} \iint \|W_z^* W_{z'}\|_{\text{tr}} d|z| d|z'| \\ &= \frac{1}{(2\pi)^2} \iint \|L_z^* L_{z'} \otimes F(A - zI)^{-1*} (A - z'I)^{-1} F\|_{\text{tr}_{Q \otimes F}} d|z| d|z'| \\ &= \frac{1}{(2\pi)^2} \iint |\langle \kappa_z, \kappa_{z'} \rangle| \|F(A - zI)^{-1*} (A - z'I)^{-1} F\|_{\text{tr}_{\mathcal{Q}_K}} d|z| d|z'| \\ &\leq M \cdot \text{tr}_{\mathcal{Q}_K}(F). \end{aligned}$$

Also,

$$\begin{aligned} \|WW^*\|_{\text{tr}_{\mathcal{Q}_K}} &= \left\| \frac{1}{(2\pi)^2} \iint W_z W_z^* dz dz' \right\|_{\text{tr}_{\mathcal{Q}_K}} \\ &\leq \frac{1}{(2\pi)^2} \iint \|W_z W_z^*\|_{\text{tr}_{\mathcal{Q}_K}} d|z| d|z'| \\ &= \frac{1}{(2\pi)^2} \iint |\langle \kappa_{z'}, \kappa_z \rangle| (A - z'I)^{-1} F(A - zI)^{-1*} \| \text{tr}_{\mathcal{Q}_K} d|z| d|z'| \\ &\leq M \cdot \text{tr}_{\mathcal{Q}_K}(F). \end{aligned}$$

3. Let T belong to $\mathcal{Q}_H$ and let E be a finite projection in $\mathcal{Q}_H$. We will denote the space spanned by $\{p(T)E : p \in P\}$ by V . Clearly V is invariant with respect to T . Note that $V\mathcal{Q}_H|_V$ is again a von Neumann algebra. Let τ' be the restriction of $\tau_{\mathcal{Q}_H}$ to $V\mathcal{Q}_H|_V$.

LEMMA 3.1. *Let T be an invertible operator in $\mathcal{Q}_H$, and let E be a finite projection in $\mathcal{Q}_H$. Then the orthogonal projection whose range is $R(TET^{-1})$ is finite with trace equal to the trace of E .*

PROOF. Let Q be the orthogonal projection whose range is $\overline{R(TET^{-1})}$. We claim there is an isometry in $\mathcal{Q}_H$ from E onto $R(TET^{-1})$. Once this is established, the finiteness of E implies Q is finite, and $\text{tr}(Q) = \text{tr}(E)$.

Consider the polar decomposition of the operator TE in $\mathcal{Q}_H$. $TE = V\sqrt{ET^*TE}$, where U is in $\mathcal{Q}_H$. U is an isometry from $\overline{R((TE)^*)}$ onto $\overline{R(TE)}$. $\overline{R(TE)} = \overline{R(TET^{-1})}$ and $\overline{R((TE)^*)} = E$. Therefore, $Q = UEU^*$ implies Q is finite. Indeed, $E = U^*QU$, and so $\text{tr}(E) = \text{tr}(Q)$.

COROLLARY 3.2. *Let T be invertible. The projection P_k , whose range is $\overline{R(T^KE)}$, is finite for every K .*

Recall that V is the closure of the space $\{p(T)E: p \in P\}$.

LEMMA 3.3. *The operator V belongs to $\mathcal{Q}_H$ and is the strong limit of an increasing sequence of finite projections in $\mathcal{Q}_H$.*

PROOF. Let E_0 be the space E , and let E_n be the space spanned by E , $T[E], \dots, T^n[E]$ for each n . Since E and T belong to $\mathcal{Q}_H$, E_n is a finite projection in $\mathcal{Q}_H$ for every n . The E_n increase strongly to V and the conclusion follows.

LEMMA 3.4. *Let T be hyponormal and assume $\tau_{\mathcal{Q}_H}[T^*, T] < +\infty$. Then $T = T_0 \oplus T_n$ where $H = H_0 \oplus H_n$. $T_0 = T|_{H_0}$ is a completely nonnormal operator and $T_n = T|_{H_n}$ is a normal operator. Furthermore, H_0 and H_n lie in $\mathcal{Q}_H$. If σ is a Borel subset of $\mathbb{R}$ which is bounded away from 0, and $E(\sigma)$ is the corresponding spectral projection for $[T_0^*, T_0]$, then $E(\sigma)$ is a finite projection in $\mathcal{Q}_H$.*

LEMMA 3.5. *Let V be an invariant space for T , a hyponormal operator, where T and V belong to $\mathcal{Q}_H$. Let $H = V_0 \oplus V_n$ where V_0 is the subspace of H upon which TV is completely nonnormal and V_n is the subspace where TV is normal. Then T is normal on V_n also.*

PROOF. Clearly V_n is contained in V . For u in V_n we have $\|VT^*u\| = \|TVu\| = \|Tu\| \geq \|T^*u\|$ since T is hyponormal. Since V is a projection, $\|T^*u\| \geq \|VT^*u\|$. For u in V_n , $VT^*u = T^*u$. Since V_n is contained in V , T and its adjoint agree with TV and its adjoint, respectively. Thus T is normal on V_n .

THE SUBSPACE DOMINANCE THEOREM. *Let the hyponormal operator T be effectually rationally cyclic with respect to V . Then $\text{tr}[T^*, T] \leq \text{tr}[(T|_V)^*, (T|_V)]$.*

PROOF. We may assume that $\text{tr}[(T|_V)^*, (T|_V)]$ is finite. Since T is effectually rationally cyclic with respect to the invariant space V , $\{x: p \in P', p(T)x \in V\}^\perp = H$. Polynomials with rational coefficients are dense in the set of all polynomials so there is a countable subset P'' of P' such that $\{x: p \in P'', p(T)x \in V\}^\perp = H$. For $p \in P''$, define $V_p = \{x \in H: p(T)x \in V\}$. Let $V_1 = V$. Clearly V_p is invariant for T and there are a countable number of V_p . Set $s_0 = 1$, $s_1 = p_1$, $s_2 = p_1 p_2$, $s_3 = p_1 p_2 p_3$, etc. We have $V = V_{s_0}$ and $V_{s_i} = \{x \in H: s_i(T)x \in V\}$. Trivially V_{s_i} contains V_{p_i} for every i . The V_{s_i} are nested and increase strongly to the identity. Each polynomial s_i contains s_{i-1} as a factor. By choosing P'' suitably, we may assume that s_n/s_{n-1} is a monic linear polynomial. Thus $V = V$ and $V_{n+1} = \{x: (T - a_{n+1})x \in V_n\}$, where $a_{n+1} \in \mathbb{C}$. We will prove the following chain of inequalities.

$$(9) \quad \text{tr}[T^*, T] \leq \lim_n \left(\text{tr}[(TV_n)^*, (TV_n)] \leq \text{tr}[(T|_V)^*, (T|_V)] \right).$$

Since $V_n = \{x: (I - V)p(T)x = 0 \text{ for some } p \in P'\}$, V_n is the kernel of an operator in $\mathcal{Q}_H$ and so belongs to $\mathcal{Q}_H$.

In proving the first inequality in (9), we assume $\text{tr}[(TV_n)^*, (TV_n)]$ is frequently finite, otherwise the result follows trivially. By Lemma 3.4 there exists a sequence of finite orthogonal projections whose partial sum increases to the identity on the space on which TV_n is completely nonnormal. Let $(V_n)_0$ be the space upon which $T|_{V_n}$ is completely nonnormal. It suffices to calculate the trace of the self-commutators of the operators restricted to $(V_n)_0$. Since the V_n are increasing up to I_H , without loss of generality we may assume the sum of the sequence of finite orthogonal projections is increasing up to I_H .

Let $\{E_i\}$ be a sequence of finite orthogonal projections such that $\sum_{i=1}^N E_i$ is increasing up to I , the identity. Since τ is a normal trace, we find

$$\text{tr}[T^*, T] = \sum_{i=1}^{\infty} \text{tr}(E_i[T^*, T]E_i)$$

and

$$\text{tr}[(TV_n)^*, (TV_n)] = \sum_{i=1}^{\infty} \text{tr}(E_i[(TV_n)^*, (TV_n)]E_i).$$

Next it will be shown that

$$\lim_n \text{tr}(E_i(TV_n)(TV_n)^*E_i) = \text{tr}(E_iTT^*E_i)$$

and

$$\lim_n \text{tr}(E_i(TV_n)^*(TV_n)E_i) = \text{tr}(E_iT^*TE_i).$$

These equalities will imply

$$\lim_n \text{tr}(E_i[(TV_n)^*, (TV_n)]E_i) = \text{tr} E_i[T^*, T]E_i.$$

We have

$$E_i(TV_n)(TV_n)^*E_i = E_iTV_nT^*E_i \quad \text{and} \quad \langle E_iTV_nT^*E_ix, x \rangle = \|V_nT^*E_ix\|.$$

Setting $Y = T^*E_ix$, we find $\|V_nT^*E_ix\| = \|V_ny\|$. The $\{V_n\}$ are increasing up to H and therefore $\{E_iTV_nT^*E_i\}$ are increasing up to $E_iTT^*E_i$. Since τ is normal,

$$\lim_n \tau(E_iTV_nT^*E_i) = \tau(E_iTT^*E_i).$$

We will now show $\lim_n \tau(E_i(TV_n)^*(TV_n)E_i) = \tau(E_iT^*TE_i)$. Since

$$\begin{aligned} \{\text{tr}(E_i(TV_n)^*(TV_n)E_i)\} &= \{\text{tr}(E_iV_nT^*TV_nE_i)\} \\ &= \text{tr}\{(V_nE_iV_nT^*T)\} = \text{tr}(TV_nE_iV_nT^*) \end{aligned}$$

and $\text{tr}(E_iT^*TE_i) = \text{tr}(TE_iT^*)$, it suffices to prove $\lim_n \text{tr}(TV_nE_iV_nT^*) = \text{tr}(TE_iT^*)$. Consider the following set of equivalent relationships:

$$(10) \quad \lim_n \text{tr}(TV_nE_iV_nT^*) = \text{tr}(TE_iT^*),$$

$$(11) \quad \lim_n \text{tr}(TV_nE_iV_nT^*) - \text{tr}(TE_iT^*) = 0,$$

(12) $\text{tr}[T(E_i - V_n E_i V_n)T^*]$ is approaching 0,

(13) $\text{tr}(T[(I - V_n)E_n T^*]) + \text{tr}(TV_n)[E_i(I - V_n)T^*]$ is approaching 0.

It therefore suffices to prove (13) true.

Clearly

$$\text{tr}(TV_n(E_i(I - V_n)T^*)) = \text{tr}[(T^*TV_n E_i)(E_i(I - V_n))].$$

If A and B belong to a von Neumann algebra with a trace defined on it, $|\text{tr}(AB)|^2 \leq |\text{tr} AA^*| \cdot |\text{tr} B^*B|$; therefore,

$$|\text{tr}(T^*TV_n E_i)(E_i(I - V_n))|^2 \leq \text{tr}(T^*TV_n E_i V_n T^* T) \cdot \text{tr}[(I - V_n)E_i(I - V_n)].$$

Since E_i belongs to trace class and T^* , T and V are all bounded operators, $T^*TV_n E_i V_n T^* T$ belongs to trace class. We claim $\lim_n \text{tr}(E_i(I - V_n)E_i)$ goes to 0. Since the V_n are increasing up to I and τ is normal,

$$\begin{aligned} \left\{ \lim_n \text{tr}[E_i(I - V_n)E_i] \right\} &= \left\{ \text{tr}(E_i I E_i) - \lim_n \text{tr}(E_i V_n E_i) \right\} \\ &= \left\{ \text{tr}(I E_i) - \lim_n \text{tr}(V_n E_i) \right\} = 0. \end{aligned}$$

The value

$$\tau(T(I - V_n)E_i T^*) = \text{tr}(T^*T(I - V_n)E_i) = \text{tr}(E_i T^*T(I - V_n)E_i)$$

and

$$|\text{tr}(E_i T^*T(I - V_n)E_i)|^2 \leq \text{tr}(T^*T E_i T^* T) \cdot \text{tr}(E_i(I - V_n)E_i).$$

Using an argument similar to the previous one, $\text{tr}(T^*T E_i T^* T) < +\infty$ and $\text{tr}[E_i(I - V_n)E_i]$ approaches 0. Therefore,

$$\lim_n \text{tr}(E_i[(TV_n)^*, (TV_n)]E_i) = \text{tr}(E_i[T^*, T]E_i).$$

Using this we see

$$\begin{aligned} \text{tr}[T^*, T] &= \text{tr} \sum_{i=1}^{\infty} E_i[T^*, T]E_i = \sum_{i=1}^{\infty} \text{tr}(E_i[T^*, T]E_i) \\ &= \sum_{i=1}^{\infty} \lim_n \text{tr}(E_i[(TV_n)^*, (TV_n)]E_i) \leq \lim_n \sum_{i=1}^{\infty} \text{tr}(E_i[(TV_n)^*, (TV_n)]E_i) \\ &= \lim_n \text{tr}[(TV_n)^*, TV_n]. \end{aligned}$$

The inequality is a consequence of Fatou's theorem.

In order to complete the proof of this theorem we need to establish

$$\lim_n \text{tr}[(TV_n)^*, (TV_n)] \leq \text{tr}[(T|_{V_n})^*, (T|_{V_n})].$$

Let $B_{n+1} = T|_{V_{n+1}}$, where $V_{n+1} = \{x: (T - a_{n+1}I)x \in V_n\}$. Clearly B_{n+1} is effectually rationally cyclic with respect to V_n . It suffices to show

$$(14) \quad \text{tr}[B_{n+1}^*, B_{n+1}] \leq \text{tr}[B_n^*, B_n] \quad \text{for arbitrary } n.$$

Equivalently, it is enough to show that if K is an invariant space for T , a hyponormal operator in $\mathcal{Q}_H$, and $\{x: Tx \in K\}^\perp = H$, then $\text{tr}[T^*, T] \leq \text{tr}[(TK)^*, TK]$.

Without loss of generality we may assume $N(T) = 0$. To see this, let $K' = \{x + n: x \in K, n \in N(T)\}$. The space N is a reducing space for T since T is hyponormal. Since $TK = TK'$ and $(TK) = (TK')^*$, $[(TK)^*, (TK)] = [(TK')^*, (TK')]$. Furthermore, TH contained in K implies TH is in K' . The space K' is invariant for T since both N and K are invariant. It suffices, therefore, to prove $\text{tr}[T^*, T] \leq \text{tr}[(TK')^*, (TK')]$. Further, we may assume $N(T)$ is in K . Thus, N is a reducing space for $T|_{K^\perp}$. Since both T and $T|_{K^\perp}$ are normal on N , the foregoing does not affect the trace of either commutator.

By Lemma 3.4 there exist $\{E_n\}$, finite projections increasing up to K . Let $F_n = \{x: Tx \in E_n\}$. Our claim is that F_n is a finite projection for every n . Consider $B_n = T|_{F_n}$. Since $N(T) = 0$, $N(B_n)$ must equal 0. The operator $TF_n|_{F_n}$ has no nullity and $R(TF_n)$ is contained in E_n . Thus $TF_n = E_n TF_n = S_n R_n$ where R_n is a positive hermitian operator and S_n is a partial isometry. Since T has no nullity, $N(R_n) = I - F_n$. Also, $R(S_n)$ is contained in $R(E_n)$ and $\overline{R(S_n^*)} = N(S_n) = \overline{R(R_n)}$. We conclude that $F_n = \overline{R(R_n)} = \overline{R(S_n)}$. Since $\overline{R(S_n^*)} = S_n^* S_n$, $S_n S_n^* = \overline{R(S_n)}$ and $S_n S_n^*$ is contained in E_n , F_n is equivalent to a subspace of E_n . For each n , E_n is finite implies F_n is finite. The F_n are increasing up to the identity on H .

Let $U_n = \{x + y: x \in K, y \in F_n\}$. TH is contained in K implying U_n is invariant for T . Clearly K is in U_n and $U_n - K$ is in F_n . Thus $U_n - K$ is finite and $\text{tr}[(T|_{U_n})^*, (T|_{U_n})] = \text{tr}[(T|_K)^*]$. The U_n are increasing up to the identity. By an argument similar to the one used to prove the first inequality in (9), we have

$$\text{tr}[(T|_K)^*, (T|_K)] = \lim_n \text{tr}[(T|_{U_n})^*, (T|_{U_n})] \leq \text{tr}(T^*, T).$$

Combining the two inequalities we have $\text{tr}[T^*, T] \leq \text{tr}[(T|_\nu)^*, (T|_\nu)]$.

4. In this final section we prove our major result. Utilizing the results of the preceding sections we prove that the trace of the self-commutator of a hyponormal operator is bounded above by a multiplicity factor times the area of the spectrum of the operator. Formally, we have

THEOREM 4.1. *Let A be a hyponormal operator in $\mathcal{Q}_H$ with effectually rational multiplicity s . Then $[A^*, A]$ is in trace class and $\text{tr}[A^*, A] \leq (s/\pi)\text{area}(\text{sp}(A))$. If f is a function analytic on $\text{sp}(A)$ and if $f(A)$ is hyponormal, then*

$$\text{tr}[f(A)^*, f(A)] \leq \frac{s}{\pi} \int_{\text{sp}(A)} |f'|^2 d\eta.$$

PROOF. Let U be a bounded open set in $\mathbb{C}$ such that $\text{sp}(A) \subset U$, $\eta(U) - \eta(\text{sp}(A))$ is small and U is bounded by a finite number of smooth Jordan curves. Let H' be the space spanned by $\{\sum_{i=1}^n q_i(A)e_i: e_i \in E \text{ and } q_i \in Q(U)\}$. Clearly H' is in $\mathcal{Q}_H$. Let $B = A|_{H'}$. From [16] we see that B has its spectrum contained in U . B is hyponormal and rationally cyclic with respect to E on H' . By the subspace dominance theorem, $[\text{tr } A^*, A] \leq \text{tr}[B^*, B]$.

Let $T = T_z \otimes E$ be the operator acting on the Hilbert space $K = Q^2(\chi_U \eta) \otimes E$. From the results in the I_∞ case [7], we know

$$\begin{aligned} \operatorname{tr}[T^*, T] \operatorname{tr}[T_z^* \otimes E, T_z \otimes E] &= \operatorname{tr}[T_z^* T_z \otimes E - T_z T_z^* \otimes E] \\ &= \operatorname{tr}[T_z^*, T_z] \cdot \operatorname{tr}(E) = (\kappa_E / \pi) \eta(U), \end{aligned}$$

where $\operatorname{tr}(E) = \kappa_E$.

To complete the proof we need to exhibit W in $B(K, H')$, an intertwiner between T and B that satisfies the following hypotheses of the intertwining theorem Z:

- (i) W^*W and WW^* are trace class operators.
- (ii) W has dense range.
- (iii) $WT = BW$.
- (iv) $V\mathcal{Q}_H V^*$ is in $\mathcal{Q}_K$ and $V^*\mathcal{Q}_K V$ is contained in $\mathcal{Q}_{H'}$, where $W = V\sqrt{W^*W}$.
- (v) $\tau_{\mathcal{Q}_{H'}}(VXV^*) = \tau_{\mathcal{Q}_K}(X)$ for all X in trace class in $\mathcal{Q}_K$.

The map $W: Q^2(\chi_U, \eta) \times E \rightarrow H'$ defined by $W(f \otimes e_\alpha) = f(A)e_\alpha$ satisfies properties (i)–(v). The range of W is dense in H' since B is rationally cyclic with respect to E on H . Thus

$$(15) \quad \operatorname{tr}[A^*, A] \leq (\kappa_E / \pi) \eta(U).$$

If we let E vary over all the finite projections in $\mathcal{Q}_H$ with respect to which A is effectually rationally cyclic, we find, upon taking the infimum of both sides of (15), that $\operatorname{tr}[A^*, A] \leq (s/\pi) \eta(U)$. Let $\eta(U)$ approach $\eta(\operatorname{sp}(A))$ and the first result is proved.

As for the second statement in the theorem, since any f analytic on J can be uniformly approximated by rationals, $W[f(T)] = f(B)W$; therefore, we have an intertwining map, and coupled with the I_∞ result that $\operatorname{tr}[f(T)^*, f(T)] = \frac{1}{\pi} \int |f'|^2 d\eta$ [1], the proof is essentially the same as the above.

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