

COMPLETION OF AKAHORI'S CONSTRUCTION OF THE VERSAL FAMILY OF STRONGLY PSEUDO-CONVEX CR STRUCTURES

BY

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ABSTRACT. Let M be a compact smooth boundary of a strongly pseudo-convex domain of a complex manifold N with $\dim N \geq 4$. We established a sharp a priori estimate for the Laplacian operator associated with Akahori's subcomplex of the $T'N|_M$ -valued $\bar{\partial}_b$ -complex to construct the complex analytic versal family (in the sense of Kuranishi) of CR structures of class C^∞ on M .

Introduction. Since the epoch-making work of Kuranishi [6], it has been a fundamental method in deformation theories to apply the implicit function theorems to nonlinear partial differential equations. In spite of great use in the case of compact complex structures (cf. [6]), the Banach inverse mapping theorem seemed impossible to be applied to the deformation theory of strongly pseudo-convex CR structures on a compact boundary of a complex manifold because of the nonellipticity of the tangential Cauchy-Riemann complex (cf. [7]). Recently Akahori [2] made a new approach by introducing a certain subcomplex of T' -valued tangential Cauchy-Riemann complex. His approach relies on the power series method of Kodaira's and Spencer's early works, based on a certain coercive basic estimate for the subcomplex. So he constructed a versal family (in the sense of Kuranishi [7]) depending complex analytically on its parameters. However it was not shown if each CR structure in the family is of class C^∞ , whereas it remains unknown whether a CR structure of class C^k ($k < +\infty$) is a boundary structure of a complex manifold (cf. [4]).

The purpose of this paper is to complete Akahori's construction by showing that the CR structures are of class C^∞ . Since his construction relies essentially on the Banach inverse mapping theorem for the map $A: \phi \rightarrow \phi + D^*NR_2(\phi)$, it seems possible to obtain the C^∞ -ness by applying the Nash-Moser iteration method to the map A . In fact we can invert the differential of A at each point near 0 by the Neumann series of it. To show the convergence of the series with respect to every $\|\cdot\|_s$ -norm, we need a sharper estimate for the Neumann operator N than in [2], and it is established in §2 by careful commutator calculations from Akahori's and

Received by the editors June 22, 1981 and, in revised form, February 26, 1982.

1980 *Mathematics Subject Classification.* Primary 32G07; Secondary 32F30.

Key words and phrases. CR structure, deformation, strongly pseudo-convex boundary, versal family.

¹Partially supported by Grant-in-Aid for Scientific Research (No. 574076), Ministry of Education of Japan.

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0002-9947/82/0000-0579/\$03.50

Kuranishi's basic estimate (cf. [2]). The arrangement of this paper is as follows: In §1, we follow Akahori's construction, relying on the Banach inverse mapping theorem. This is needed for us to apply the Nash-Moser technique. In §2 we sharpen an a priori estimate in [2] for the solutions of the Neumann problem associated with the Laplacian for Akahori's subcomplex. The C^∞ -ness of CR structures in our family will be shown in §3 by applying the Nash-Moser iteration method.

I wish to thank Professor Takao Akahori for helpful discussions in preparing this manuscript. I also thank the referee for his valuable suggestions.

1. Akahori's construction of the versal family. Let M be a compact smooth real hypersurface of a complex manifold N of complex dimension n (≥ 4). The case that the CR structure ${}^\circ T''$ on M induced from the complex structure of N is strongly pseudoconvex interests us.

We fix a splitting

$$(1.1) \quad CTM = {}^\circ T'' + {}^\circ \bar{T}'' + F$$

and set $T' = {}^\circ \bar{T}'' + F$, following Kuranishi [7], where F denotes the complexification of a real line bundle.

Each CR structure at a finite distance from ${}^\circ T''$ is represented by an element of $\Gamma(M, T' \otimes ({}^\circ T'')^*)$ satisfying the integrability condition:

$$(1.2) \quad P(\phi) = \bar{\partial}_b \phi + R_2(\phi) + R_3(\phi) = 0$$

where

$$R_2(\phi)(X, Y) = [\phi(X), \phi(Y)]_{T'} - \phi([X, \phi(Y)]_{{}^\circ T''} + [\phi(X), Y]_{{}^\circ T''})$$

and

$$R_3(\phi)(X, Y) = -\phi([\phi(X), \phi(Y)]_{{}^\circ T''})$$

for $X, Y \in \Gamma(M, {}^\circ T'')$ (cf. [1]). We note that $R_2(\phi)$ and $R_3(\phi)$ are homogeneous of order 2 and 3 with respect to ϕ , respectively.

Akahori [2] constructed a versal family of integrable elements of class C^k depending complex analytically on the parameters under the condition that the second cohomology group $H_{\bar{\partial}_b}^2(T')$ of the T' -valued $\bar{\partial}_b$ -complex vanishes. In this section, relying on his idea, we construct a family having the same properties without this condition. His idea is to introduce a subcomplex $(\Gamma(M, E_q), D)$ of the T' -valued $\bar{\partial}_b$ -complex and search solutions of the integrability condition in $\Gamma(M, E_1)$ based on the harmonic theory on $\Gamma(M, E_2)$ (cf. [2]).

(I) *Subcomplex $(\Gamma(M, E_q), D)$.* Let

$$E_{q,p} = \{u \in {}^\circ \bar{T}_p'' \otimes \Lambda^q({}^\circ T_p'')^* \mid (\bar{\partial}_b \tilde{u})_F(p) = 0 \text{ for any local extension } \tilde{u} \text{ of } u\},$$

where $(\)_F$ denotes the projection onto F -part according to the splitting (1.1). Then $E_q = \bigcup_{p \in M} E_{q,p}$ is a subbundle of $T' \otimes \Lambda^q({}^\circ T'')^*$ (cf. [2, Proposition 2.1]) and we have a subcomplex $(\Gamma(M, E_q), D)$ of $(\Gamma(M, T' \otimes \Lambda^q({}^\circ T'')^*), \bar{\partial}_b)$ with $D = \bar{\partial}_b|_{\Gamma(M, E_q)}$.

The natural injection induces the following, which assures no deformation is taken out of consideration or no obstruction is missed in our construction relying on this subcomplex instead of $\bar{\partial}_b$ -complex (cf. [2, Theorems 2.3 and 2.4]):

(1.3) a surjection $H_D^1(E.) \rightarrow H_{\bar{\partial}_b}^1(T')$,

(1.4) an isomorphism $H_{\bar{\partial}_b}^q(E.) \xrightarrow{\sim} H_{\bar{\partial}_b}^q(T')$ for $2 \leq q \leq n-1$, where $H_{\bar{\partial}_b}^q(E.)$ ($1 \leq q \leq n-1$) denotes the q th cohomology group of the complex $(\Gamma(M, E_q), D)$.

(II) *Preliminary from analysis.* We denote the Sobolev s -norm by $\|\cdot\|_s$ and by $\|\cdot\|'_s$, $\|\cdot\|''_s$ the norms introduced by Akahori [2] (cf. (2.1) and (2.2)).

(1.5) Akahori's and Kuranishi's basic estimate [2, Theorem 4.1]

$$C_1 \|\phi\|_{1/2}^2 \leq \|\phi\|'^2 \leq C_2 Q(\phi, \phi) \quad \text{for } \phi \in \Gamma(M, E_2),$$

where $Q(\phi, \phi) = \|D\phi\|^2 + \|D^*\phi\|^2 + \|\phi\|^2$.

This subelliptic estimate has the following consequences (cf. [5]): Let $\square$ be the Laplacian operator $D^*D + DD^*$.

(i) The complex vector space $H^2 = \{\phi \in \Gamma(M, E_2) \mid \square\phi = 0\}$ is finite dimensional.

(ii) There exists a linear map (the so-called Neumann operator) $N: \Gamma(M, E_2) \rightarrow \Gamma(M, E_2)$ such that

$$(1.6) \quad \square N\phi = \phi - H\phi, \quad \square N = N\square, \quad NH = HN = 0,$$

where H denotes the orthogonal projection onto H^2 .

(iii) For any integer $s \geq 0$,

$$(1.7) \quad \|N\phi\|''_s \leq C_s \|\phi\|_s \quad \text{for } \phi \in \Gamma(M, E_2),$$

(cf. [2, Corollary 5.2]).

In § 2 we will establish a sharper estimate than (1.7) (cf. Corollary 2.4).

As is well known, by introducing hermitian metrics along the fibres of T' and ${}^\circ T''$, we can also speak of the harmonic theory on $\Gamma(M, T' \otimes \Lambda^q({}^\circ T'')^*)$ ($1 \leq q \leq n-2$). In particular, we denote by $H_{\bar{\partial}_b}^q$ and $\dot{H}_{\bar{\partial}_b}^q$ the harmonic part and the projection on it, respectively.

(III) *Construction of the versal family.* The following proposition enables us to search solutions of integrability conditions relying on the subcomplex $(\Gamma(M, E_q), D)$.

PROPOSITION 1.1. *For $\phi \in \Gamma(M, E_1)$, $P(\phi) = D\phi + R_2(\phi)$ and $P(\phi)$ is in $\Gamma(M, E_2)$.*

PROOF. The first assertion follows from (1.2) because ${}^\circ \bar{T}''$ is closed under the bracket operation. By (1.2),

$$\begin{aligned} (P(\phi)(X, Y))_F &= (\bar{\partial}_b \phi(X, Y))_F + [\phi(X), \phi(Y)]_F \\ &\quad - \phi([X, \phi(Y)]_{{}^\circ T''} + [\phi(X), Y]_{{}^\circ T''})_F \\ &= 0 \quad \text{for } X, Y \in \Gamma(M, {}^\circ T''). \end{aligned}$$

(1.8)

$$\begin{aligned}
\bar{\partial}_b^* P(\phi)(X_1, X_2, X_3) &= \bar{\partial}_b P(\phi)(X_1, X_2, X_3) \\
&+ \sum_{j=1}^3 (-1)^{j+1} [\phi(X_j), P(\phi)(\cdots \hat{X}_j \cdots)]_{T'} \\
&+ \sum_{i,j=1}^3 (-1)^{i+j} P(\phi)([X_i, \phi(X_j)]_{\circ T''} \cdots \hat{X}_i \cdots \hat{X}_j \cdots) \\
&+ \sum_{i,j=1}^3 (-1)^{i+j} P(\phi)([\phi(X_i), X_j]_{\circ T''} \cdots \hat{X}_i \cdots \hat{X}_j \cdots) \\
&- \sum_{j=1}^3 (-1)^{j+1} \phi([X_j, P(\phi)(\cdots \hat{X}_j \cdots)]_{\circ T''}) \\
&- \sum_{j=1}^3 (-1)^{j+1} \phi([\phi(X_j), P(\phi)(\cdots \hat{X}_j \cdots)]_{\circ T''})
\end{aligned}$$

for $X_1, X_2, X_3 \in \Gamma(M, \circ T'')$. Since $\bar{\partial}_b^* P(\phi) = 0$ by [1, Theorem 4.10],

$$(\bar{\partial}_b P(\phi)(X_1, X_2, X_3))_F = 0. \quad \text{Q.E.D.}$$

Now we set about constructing the versal family.

First we recall the linear map $\mathcal{L}: \Gamma(M, T' \otimes (\circ T'')^*) \supset \text{Ker } \bar{\partial}_b \rightarrow \text{Ker } D \subset \Gamma(M, E_1)$ with $H_b^1 \mathcal{L} = H_b^1$, which implies (1.3). Let $\mathcal{H} = \mathcal{L}(H_T^1)$ (cf. [2, Theorem 2.3]).

We fix positive integers m, k with $m \geq n$ and $k \geq 2m$. If we set

$$(1.9) \quad A(\phi) = \phi + D^* N R_2(\phi) \quad \text{for } \phi \in \Gamma(M, E_1),$$

A can be extended to a complex analytic map of a neighbourhood of the origin of $\Gamma'_k(M, E_1)$ into itself by [2, Proposition 5.4] (also by Lemma 3.2(2)), where $\Gamma'_k(M, E_1)$ denotes the completion of $\Gamma(M, E_1)$ with respect to the norm $\|\cdot\|'_k$. Since the differential of A at the origin is clearly the identity map, there exist neighbourhoods W_1 and W_2 of the origin in $\Gamma'_k(M, E_1)$ such that A is an analytic isomorphism of W_1 onto W_2 by the inverse mapping theorem on Banach manifolds. Set $\phi = A^{-1}|_{\mathcal{H} \cap W_2}$; then ϕ satisfies

$$(1.10) \quad \phi(0) = 0,$$

$$(1.11) \quad A(\phi(t)) = t \text{ for } t \in \mathcal{H} \cap W_2.$$

PROPOSITION 1.2. *For sufficiently small t , $P(\phi(t)) = 0$ if and only if $HP(\phi(t)) = 0$.*

PROOF. Only if part is trivial because

$$(1.12) \quad P(\phi(t)) = D\phi(t) + R_2(\phi(t)) = HP(\phi(t)) + ND^*DP(\phi(t))$$

by (1.11) and (1.6).

We assume that $HP(\phi(t)) = 0$. By (1.8) and [7, Lemma 5.1], for any $s \geq 0$,

$$(1.13) \quad \|DP(\phi)\|'_s \leq c_s \|P(\phi)\|''_s \|\phi\|''_{m+s} \quad \text{for } \phi \in \Gamma(M, E_1).$$

Then, by (1.12) and (1.7),

$$\begin{aligned} \|P(\phi(t))\|''_{m-1} &\leq c_1 \|D^*DP(\phi(t))\|_{m-1} \leq c_2 \|DP(\phi(t))\|'_{m-1} \\ &\leq c_3 \|P(\phi(t))\|''_{m-1} \|\phi(t)\|''_{2m-1} \quad (\text{by (1.13)}) \\ &\leq c_4 \|P(\phi(t))\|''_{m-1} \|\phi(t)\|'_k. \end{aligned}$$

Hence, if t is small such that $\|\phi(t)\|'_k < 1/(2c_4)$, we have $P(\phi(t)) = 0$. Q.E.D.

If we set $T = \{t \in \mathcal{H} \cap W_2 \mid \|\phi(t)\|'_k < 1/(2c_4), HP(\phi(t)) = 0\}$, T is an analytic set and $\{\phi(t) \mid t \in T\}$ is a family of CR structures of class C^{k-n} , by Sobolev lemma, depending complex analytically on $t \in T$.

We can observe that this family has the same property of versality as in [3] by the same method and considering the following lemma which implies (1.4) for $q = 2$.

LEMMA 1.3. *There exists a linear map $\mathcal{L}_2: H_{T'}^2 \rightarrow \text{Ker } D \subset \Gamma(M, E_2)$ satisfying: (1) $H\phi = H\mathcal{L}_2 H_b^2 \phi$ for $\phi \in \Gamma(M, E_2)$ with $D\phi = 0$; (2) $H_b^2 \mathcal{L}_2 = H_b^2$.*

PROOF. It is shown for any ϕ in $\text{Ker } \bar{\partial}_b$ that there exists an element θ in $\Gamma(M, {}^\circ\bar{T}'' \otimes ({}^\circ T'')^*)$ satisfying $\phi - \bar{\partial}_b \theta \in \Gamma(M, E_2)$ (cf. [2]). Since $H_{T'}^2$ is finite dimensional, we have a linear map θ of $H_{T'}^2$ into $\Gamma(M, {}^\circ\bar{T}'' \otimes ({}^\circ T'')^*)$ such that $\psi - \bar{\partial}_b \theta(\psi) \in \Gamma(M, E_2)$ for $\psi \in H_{T'}^2$.

If we set $\mathcal{L}_2 \psi = \psi - \bar{\partial}_b \theta(\psi)$, then

$$\mathcal{L}_2 H_b^2 \phi = \phi - \bar{\partial}_b \bar{\partial}_b^* N_b \phi - \bar{\partial}_b \theta(H_b^2 \phi) = \phi - \bar{\partial}_b \mathcal{L}(\bar{\partial}_b^* N_b \phi - \theta(H_b^2 \phi))$$

for $\phi \in \Gamma(M, E_2)$ with $D\phi = 0$, where $\mathcal{L}$ is an operator of $\Gamma(M, T' \otimes ({}^\circ T'')^*)$ into $\Gamma(M, {}^\circ\bar{T}'' \otimes ({}^\circ T'')^*)$ introduced in [2]. Thus (1) follows, since ϕ and $\mathcal{L}_2 H_b^2 \phi$ are both in $\Gamma(M, E_2)$ and $\mathcal{L}(\bar{\partial}_b^* N_b \phi - \theta(H_b^2 \phi))$ in $\Gamma(M, E_1)$. (2) is clear. Q.E.D.

2. Sharper a priori estimate for $\square$. In this section we establish a priori estimate for the solutions of the Neumann problem associated with $\square$ to obtain a sharper estimate for N than (1.7).

Let $\{(U_k, h_k)\}_{k \in \Lambda}$ be an atlas of M and $\{\rho_k\}_{k \in \Lambda}$ be a partition of unity subordinate to $\{U_k\}_{k \in \Lambda}$. If $U \in \{U_k\}_{k \in \Lambda}$, we let $(e_1, \dots, e_{n-1})$ be a moving frame of ${}^\circ T''|_U$ such that $[e_i, \bar{e}_j]_F = \sqrt{-1} \delta_{ij} e_n$, where e_n denotes a real moving frame of $F|_U$. On U , $\phi \in \Gamma(M, {}^\circ\bar{T}'' \otimes \Lambda^q({}^\circ T'')^*)$ can be written as $\phi|_U = \sum_{\alpha, I} \phi_{\alpha, I} \bar{e}_\alpha \wedge (e^*)^I$ where $((e^*)^1, \dots, (e^*)^{n-1})$ is the dual frame of $({}^\circ T'')^*|_U$, $I = \{i_1 < \dots < i_q\}$ and $(e^*)^I = (e^*)^{i_1} \wedge \dots \wedge (e^*)^{i_q}$. With this expression we introduce a Sobolev norm into $\Gamma(M, E_q)$ by

$$\|\phi\|_s^2 = \sum_k \sum_{\alpha, I} \|P_s \rho_k \phi_{\alpha, I}\|^2$$

for each real number s , where $P_s = \chi'_k T_s \chi_k$, χ'_k and χ_k are in $C_0^\infty(U_k)$ with $\chi_k \equiv 1$ on $\text{Supp } \rho_k$, $\chi'_k \equiv 1$ on $\text{Supp } \chi_k$, and T_s denotes the pseudodifferential operator of the symbol $(1 + |\xi|^2)^{s/2}$. With this Sobolev norm we introduce the norms $\|\cdot\|'_s$ and $\|\cdot\|''_s$ as follows:

$$(2.1) \quad \|\phi\|_s'^2 = \sum_k \sum_{i, \alpha, I} \{ \|P_s e_i \rho_k \phi_{\alpha, I}\|^2 + \|P_s \bar{e}_i \rho_k \phi_{\alpha, I}\|^2 \} + \|\phi\|_s^2,$$

$$(2.2) \quad \|\phi\|_s'^2 = \sum_k \sum_{i,j,\alpha,I} \left\{ \|P_s e_i e_j \rho_k \phi_{\alpha,I}\|^2 + \|P_s e_i \bar{e}_j \rho_k \phi_{\alpha,I}\|^2 \right. \\ \left. + \|P_s \bar{e}_i e_j \rho_k \phi_{\alpha,I}\|^2 + \|P_s \bar{e}_i \bar{e}_j \rho_k \phi_{\alpha,I}\|^2 \right\} + \|\phi\|_s'.$$

The following properties of these norms, obtained by standard arguments, are essential in this section.

For a real number s ,

$$(2.3) \quad \|\phi\|_{s+1/2}^2 \leq C \|\phi\|_s'^2 + C_s \|\phi\|_s^2,$$

$$(2.4) \quad \|\phi\|_{s+1/2}'^2 \leq C \|\phi\|_s''^2 + C_s \|\phi\|_s'^2,$$

$$(2.5) \quad \|\phi\|_s^2 \leq \epsilon \|\phi\|_{s+1/2}^2 + C_{s,\epsilon} \|\phi\|_{s-1/2}^2 \quad \text{for any } \epsilon > 0,$$

$$(2.6) \quad \|\phi\|_s'^2 \leq \epsilon \|\phi\|_{s+1/2}'^2 + C_{s,\epsilon} \|\phi\|_{s-1/2}'^2 \quad \text{for any } \epsilon > 0,$$

where C and C_s are constants independent on s and dependent on s , respectively.

PROPOSITION 2.1. *There exists a constant $C > 0$ satisfying the following condition: For each real $s \geq 0$ we can find C_s such that*

$$\|\phi\|_s'^2 \leq C \|\square \phi\|_{s-1/2}^2 + C_s \|\phi\|_s^2 \quad \text{for } \phi \in \Gamma(M, E_2).$$

PROOF. At first let ϕ satisfy $\text{Supp } \phi \subset U_k$. Then we have

$$\|P_s e \phi\| \leq \|e P_s \phi\| + \|[P_s, e] \phi\| \leq \|e P_s \phi\| + c_s \|\phi\|_s,$$

where e represents one of $e_1^k, \dots, e_{n-1}^k$ or $\bar{e}_1^k, \dots, \bar{e}_{n-1}^k$, and, by (1.5), $\|e P_s \phi\|^2 \leq \|P_s \phi\|'^2 \leq C_2 Q(P_s \phi, P_s \phi)$.

Moreover we have

$$Q(P_s \phi, P_s \phi) \leq |(P_s(\square + 1)\phi, P_s \phi)| + c_s \|\phi\|_s Q(P_s \phi, P_s \phi)^{1/2},$$

derived from the following formula:

$$(AP\phi, AP\phi) = ([A, P]\phi, AP\phi) + (AP^*\phi, [P^*, A]\phi) + \|[P^*, A]\phi\|^2 \\ + (A\phi, [[P^*, A], P]\phi) + (PA^*A\phi, P\phi)$$

where $A = D$ or D^* , $P = P_s$. Because $[A, P]$, $P^* - P$ and $[[P^*, A], P]$ are of order s , $s - 1$ and $2s - 1$, respectively.

Then

$$Q(P_s \phi, P_s \phi) \leq |(P_s(\square + 1)\phi, P_s \phi)| + \epsilon c_s Q(P_s \phi, P_s \phi) + (c_s/4\epsilon) \|\phi\|_s^2 \\ \text{for any } \epsilon > 0.$$

Let $\epsilon = 1/(2c_s)$; then

$$Q(P_s \phi, P_s \phi) \leq 2 |(P_s(\square + 1)\phi, P_s \phi)| + c_s' \|\phi\|_s^2.$$

Accordingly we get

$$\|\phi\|_s'^2 \leq c \sum_{k \in \Lambda} |(P_s(\square + 1)\rho_k \phi, P_s \rho_k \phi)| + c_s \|\phi\|_s^2 \quad \text{for } \phi \in \Gamma(M, E_2).$$

Then we have

$$\|\phi\|_s'^2 \leq c \sum_{k \in \Lambda} |(P_s \rho_k(\square + 1)\phi, P_s \rho_k \phi)| + c'_s \|\phi\|_s' \|\phi\|_s,$$

from the formula:

$$(PAA^*\rho\phi, P\rho\phi) - (P\rho AA^*\phi, P\rho\phi) = (PA[A^*, \rho]\phi, P\rho\phi) + (P[A, \rho]A^*\phi, P\rho\phi)$$

where A and P are as above, $\rho = \rho_k$. Because $A[A^*, \rho]$ and $[A, \rho]A^*$ are differential operators of order 1 whose principal terms are generated by e 's, we have

$$\|\phi\|_s'^2 \leq c\epsilon \|\phi\|_{s+1/2}^2 + (c/4\epsilon) \|(\square + 1)\phi\|_{s-1/2}^2 + c'_s \|\phi\|_s' \|\phi\|_s,$$

by generalized Schwarz inequality.

Therefore our proposition is proven by the same trick as above and (2.3).

PROPOSITION 2.2. *There exists a constant $C > 0$ satisfying the following condition: For each real $s \geq 0$ we can find C_s such that*

$$\|\phi\|_s'^2 \leq C \|\square\phi\|_s^2 + C_s \|\phi\|_s'^2 \quad \text{for } \phi \in \Gamma(M, E_2).$$

PROOF. Let e and e' be as in the proof of the previous proposition. First we show the following estimate:

$$(2.7) \quad \|e'e\phi\|_s^2 \leq c |(P_s(\square + 1)\phi, e^*P_s e\phi)| + c' \|\phi\|_{s+1/2}'^2 + c_s \|\phi\|_s' \|\phi\|_s'$$

for ϕ with $\text{Supp } \phi \subset U_k$, where e^* denotes the adjoint of e .

Since $\|e'e\phi\|_s^2 \leq 2\|P_s e\phi\|_s'^2 + c_s \|\phi\|_s'^2$, by (1.5) we have

$$\|e'e\phi\|_s^2 \leq 2C_2 Q(P_s e\phi, P_s e\phi) + c_s \|\phi\|_s'^2.$$

Let us estimate the difference $Q(P_s e\phi, P_s e\phi) - (P_s(\square + 1)\phi, e^*P_s e\phi)$ by using the formula:

$$\begin{aligned} (APe\phi, APe\phi) - (PA^*A\phi, e^*Pe\phi) &= ([A, P]e\phi, APe\phi) + (P[A, e]\phi, APe\phi) \\ &\quad + (eA\phi, [[P^*, A], P]e\phi) + (P^*eA\phi, [P^*, A]e\phi) \\ &\quad + (P[A^*, e]A\phi, Pe\phi) + ([P, e]A^*A\phi, Pe\phi) \end{aligned}$$

where $A = D$ or D^* , $P = P_s$. Since

$$\begin{aligned} (P[A, e]\phi, APe\phi) &= ([A, e]P\phi, APe\phi) + ([P, [A, e]]\phi, APe\phi), \\ (P[A^*, e]A\phi, Pe\phi) &= ([A^*, e]PA\phi, Pe\phi) + ([P, [A^*, e]]A\phi, Pe\phi), \end{aligned}$$

and $[A, e]$ and $[A^*, e]$ are differential operators of order 1, we have

$$|(P[A, e]\phi, APe\phi)| \leq (c\|\phi\|_{s+1} + c_s \|\phi\|_s) Q(Pe\phi, Pe\phi)^{1/2}$$

and

$$|(P[A^*, e]A\phi, Pe\phi)| \leq (c\|\phi\|_{s+1/2}' + c_s \|\phi\|_s') \|\phi\|_{s+1/2}'$$

by generalized Schwarz inequality and [7, Lemma 5.1]. The other terms are estimated by $c_s \|\phi\|_s' Q(Pe\phi, Pe\phi)^{1/2} + c'_s \|\phi\|_s' \|\phi\|_s'$ as in the proof of Proposition 2.1.

Hence, by (2.3) and (2.4), we have

$$\begin{aligned}
 Q(P_s e\phi, P_s e\phi) &\leq |(P_s(\square + 1)\phi, e^* P_s e\phi)| + c\|\phi\|_{s+1/2}' Q(P_s e\phi, P_s e\phi)^{1/2} \\
 &\quad + c'\|\phi\|_{s+1/2}'^2 + c_s\|\phi\|_s' Q(P_s e\phi, P_s e\phi)^{1/2} + c_s'\|\phi\|_s''\|\phi\|_s' \\
 &\leq |(P_s(\square + 1)\phi, e^* P_s e\phi)| + (\varepsilon c + \varepsilon' c_s)Q(P_s e\phi, P_s e\phi) \\
 &\quad + (c' + c/4\varepsilon)\|\phi\|_{s+1/2}'^2 + (c_s/4\varepsilon')\|\phi\|_s'^2 + c_s'\|\phi\|_s''\|\phi\|_s'
 \end{aligned}$$

for any $\varepsilon > 0$ and $\varepsilon' > 0$.

Let $\varepsilon = 1/(4c)$ and $\varepsilon' = 1/(4c_s)$; then

$$Q(P_s e\phi, P_s e\phi) \leq 2|(P_s(\square + 1)\phi, e^* P_s e\phi)| + c\|\phi\|_{s+1/2}'^2 + c_s\|\phi\|_s''\|\phi\|_s'.$$

Thus we get (2.7).

Next, by (2.7), we have

$$\begin{aligned}
 \|\phi\|_s''^2 &= \sum_{k \in \Lambda} \sum_{e, e'} \|e' e \rho_k \phi\|_s^2 \\
 &\leq \sum_{k \in \Lambda} \sum_{e, e'} c |(P_s(\square + 1)\rho_k \phi, e^* P_s e \rho_k \phi)| \\
 &\quad + c'\|\phi\|_{s+1/2}'^2 + c_s\|\phi\|_s''\|\phi\|_s' \quad \text{for } \phi \in \Gamma(M, E_2).
 \end{aligned}$$

Since

$$|(P_s(\square + 1)\rho_k \phi, e^* P_s e \rho_k \phi)| \leq |(P_s \rho_k(\square + 1)\phi, e^* P_s e \rho_k \phi)| + c_s\|\phi\|_s'\|\phi\|_s''$$

and $\|e^* P_s e \rho_k \phi\| \leq c\|\rho_k \phi\|_s' + c_s\|\phi\|_s'$, we have

$$\begin{aligned}
 \|\phi\|_s''^2 &\leq c\|(\square + 1)\phi\|_s\|\phi\|_s'' + c_s\|(\square + 1)\phi\|_s\|\phi\|_s' \\
 &\quad + c'\|\phi\|_{s+1/2}'^2 + c_s'\|\phi\|_s'\|\phi\|_s''.
 \end{aligned}$$

By the same trick as above we have

$$\|\phi\|_s''^2 \leq c\|\square \phi\|_s^2 + c'\|\phi\|_{s+1/2}'^2 + c_s\|\phi\|_s'^2.$$

Consequently we infer our proposition from (2.3) and Proposition 2.1.

THEOREM 2.3. *There exists a constant $C > 0$ satisfying the following condition: For each integer $s \geq 0$ we can find C_s such that*

$$\|\phi\|_s''^2 \leq C\|\square \phi\|_s^2 + C_s\|\phi\|_s''^2 \quad \text{for } \phi \in \Gamma(M, E_2).$$

PROOF. By (2.6) and Proposition 2.2,

$$\|\phi\|_s''^2 \leq c\|\square \phi\|_s^2 + \frac{1}{4}(C + C_s)\|\phi\|_{s+1/2}'^2 + c_s\|\phi\|_{s-1/2}'^2,$$

where C and C_s are constants in (2.4). Then

$$\|\phi\|_s''^2 \leq c\|\square \phi\|_s^2 + \frac{1}{4}\|\phi\|_s''^2 + c_s\|\phi\|_{s-1/2}'^2 \quad \text{by (2.4).}$$

Repeating this process, we have

$$\|\phi\|_s''^2 \leq c\|\square \phi\|_s^2 + \frac{1}{4}\|\phi\|_s''^2 + \frac{1}{8}\|\phi\|_{s-1/2}'^2 + \cdots + (1/2)^{2s}\|\phi\|_1''^2 + c_s\|\phi\|_{1/2}'^2.$$

Thus we have our theorem.

By Theorem 2.3 and (1.7) we have

COROLLARY 2.4. $\|N\phi\|_s'' \leq C\|\phi\|_s + C_s\|\phi\|$ for $\phi \in \Gamma(M, E_2)$ and $s \in \mathbb{Z}^+$.

3. C^∞ -ness of the versal family. The purpose of this section is to show that each $\phi(t)$ in the versal family constructed in §1 is of class C^∞ . We deduce this result by applying the Nash-Moser inverse mapping theorem (cf. [7, Theorem 8.1]) to the map A (cf. (1.9)).

For $\omega \in \Gamma(M, E_1)$, let R_ω be a differential operator on $\Gamma(M, E_1)$ given by

$$(3.1) \quad \begin{aligned} 2R_\omega(\phi)(X, Y) = & [\omega(X), \phi(Y)]_{T'} + [\phi(X), \omega(Y)]_{T'} \\ & - \omega([X, \phi(Y)]_{\circ T''} + [\phi(X), Y]_{\circ T''}) \\ & - \phi([X, \omega(Y)]_{\circ T''} + [\omega(X), Y]_{\circ T''}) \end{aligned}$$

for $X, Y \in \Gamma(M, \circ T'')$.

LEMMA 3.1. $R_\omega(\phi)$ is in $\Gamma(M, E_2)$.

PROOF. We observe by a direct calculation that $R_2(\omega + \phi) = R_2(\omega) + R_\omega(\phi) + R_2(\phi)$. So the lemma follows from Proposition 1.1.

LEMMA 3.2. For any real $s \geq 0$,

$$(1) \quad \|R_\omega \phi\|_s \leq C\|\omega\|_m' \|\phi\|_s' + C_s\|\omega\|_{m+s}' \|\phi\|',$$

$$(2) \quad \|R_2(\omega + \phi) - R_2(\omega) - R_\omega \phi\|_s \leq C_s\|\phi\|_{m+s}' \|\phi\|_m'.$$

PROOF. (1) follows from (3.1) and [7, Lemma 5.1]. Since $R_2(\omega + \phi) = R_2(\omega) + R_\omega \phi + R_2(\phi)$, (2) follows from

$$\|R_2(\phi)\|_s \leq c\|\phi\|_m' \|\phi\|_s' + c_s\|\phi\|_{m+s}' \|\phi\|'$$

by [7, Lemma 5.1]. Q.E.D.

LEMMA 3.3. There is a constant $C > 0$ satisfying the following condition: For each $s \in \mathbb{Z}^+$ we can find C_s such that

$$\|D^*NR_\omega \phi\|_s' \leq C\|\omega\|_m' \|\phi\|_s' + C_s\|\omega\|_{m+s}' \|\phi\|'.$$

PROOF.

$$\|D^*NR_\omega \phi\|_s' \leq c\|NR_\omega \phi\|_s'' + c_s\|NR_\omega \phi\|_s' \leq c\|R_\omega \phi\|_s + c_s\|R_\omega \phi\|_{s-1/2},$$

by (2.4) and Corollary 2.4. Then

$$\|D^*NR_\omega \phi\|_s' \leq c\|R_\omega \phi\|_s + \frac{1}{4}\|R_\omega \phi\|_s + c_s\|R_\omega \phi\|_{s-1}$$

by (2.5). Repeating this process we have

$$\|D^*NR_\omega \phi\|_s' \leq c\|R_\omega \phi\|_s + c_s\|R_\omega \phi\|.$$

Hence we have our lemma by Lemma 3.2(1).

Let E be the Fréchet space $\Gamma(M, E_1)$ with the fundamental system of norms $\{\|\cdot\|_s' \mid s = 0, 1, 2, \dots\}$. The usual smoothing operator $R(u)$ ($u \in \mathbb{R}$, $u > 0$) on E has the following properties with respect to $\|\cdot\|_s'$ -norms: for $s \leq r$,

$$(i) \quad \|R(u)\phi\|_r' \leq c_{r,s}u^{r-s+1/2}\|\phi\|_s',$$

$$(ii) \quad \|\phi - R(u)\phi\|_s' \leq c_{r,s}u^{s-r+1/2}\|\phi\|_r'.$$

These properties of $R(u)$ are enough for the Nash-Moser iteration method to be available.

Set $E(r, a) = \{\phi \in E \mid \|\phi\|'_r < a\}$. Let a be a real number such that $E(k, a) \subset W_1$ (cf. §1) and $a < 1/(C + C_m)$ where C and C_m are constants in Lemma 3.3.

Let A'_ω be the differential of A at $\omega \in E(k, a)$; then $A'_\omega \phi = \phi + D^*NR_\omega \phi$. By Lemma 3.3 we have

$$\|A'_\omega \phi - \phi\|'_s \leq C \|\omega\|'_m \|\phi\|'_s + C_s \|\omega\|'_{m+s} \|\phi\|'_s.$$

Since $(C + C_m)\|\omega\|'_{2m} < 1$, we infer from [7, Proposition 8.1] that A'_ω is invertible and

$$\|(A'_\omega)^{-1}\psi\|'_s \leq C \|\psi\|'_s + C_s \|\omega\|'_{m+s} \|\psi\|'_m.$$

It is clear that A satisfies conditions (II.1) and (II.2) of [7, Theorem 8.1] by Lemma 3.2 and (1.7).

Hence the Nash-Moser iteration method implies that we can find $k_1 \in \mathbb{Z}^+$, $a_1 > 0$, and a map $S: E(k_1, a_1) \rightarrow E(k, a)$ such that $A(S(\psi)) = \psi$ for all $\psi \in E(k_1, a_1)$ (cf. [7, Theorem 8.1]). Moreover if we set $W = \mathcal{H} \cap W_2 \cap E(k_1, a_1)$, W is a neighbourhood of the origin in $\mathcal{H}$ and $S(t)$ coincides with $\phi(t)$ for $t \in W$. Consequently $\phi(t)$ is in $\Gamma(M, E_1)$ for $t \in W$.

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