TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 277, Number 1, May 1983

DISCRETE SERIES CHARACTERS AND FOURIER INVERSION ON SEMISIMPLE REAL LIE GROUPS

BY

REBECCA A. HERB[1]

ABSTRACT. Let G be a semisimple real Lie group. Explicit formulas for discrete series characters on noncompact Cartan subgroups are given. These formulas are used to give a simple formula for the Fourier transform of orbital integrals of regular semisimple orbits.

1. Introduction. Let G be a connected semisimple real Lie group with finite center. Let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G. Associated with each $\pi \in \hat{G}$ is its distributional character θ_π, an invariant eigendistribution on G. A central problem of harmonic analysis on G is the Fourier inversion problem of expanding an arbitrary invariant distribution in terms of these distributional characters.

An important class of invariant distributions on G arises as follows. Let γ be a regular (semisimple) element of G with centralizer G_γ. Let $d\dot{x}$ denote a G-invariant measure on the quotient G/G_γ, and set

$$\Lambda_\gamma(f) = \int_{G/G_\gamma} f(x\gamma x^{-1})\, d\dot{x}, \qquad f \in C_c^\infty(G).$$

Λ_γ is an orbital integral of the type appearing in the Selberg trace formula. $H = G_\gamma$ is a Cartan subgroup of G, and for fixed γ, $\Lambda_\gamma(f)$ is, up to a constant, the invariant integral $F_f^H(\gamma)$ defined by Harish-Chandra. The main theorem of this paper gives a Fourier inversion formula for the distribution $f \to F_f^H(\gamma)$.

Orbital integrals Λ_γ can be defined for other elements of G. If γ is semisimple, but not regular, it is known that there is a Cartan subgroup H of G containing γ and a differential operator D on H so that

$$\Lambda_\gamma(f) = \lim_{h \to \gamma} DF_f^H(h)$$

where the limit is taken through regular elements of H. If $u \in G$ is unipotent, $\Lambda_u(f)$ can also be obtained as the derivative of an invariant integral evaluated at a singular point, although the explicit differential operators needed are only known in some cases [**1**]. Thus Fourier inversion formulas for $F_f^H(h)$, $h \in H$ regular, will yield Fourier inversion formulas for many singular orbital integrals as well. In particular,

---

Received by the editors December 9, 1980 and, in revised form, April 5, 1982.
1980 *Mathematics Subject Classification*. Primary 22E30, 43A30.
[1]Supported in the part by NSF grant MCS 77-18723A03

0002-9947/82/0000-0364/$11.00

when $\gamma = 1$, $\Lambda_1(f) = f(1)$ is a singular orbital integral, and the Plancherel theorem can be obtained via Fourier inversion of regular orbital integrals.

When G has real rank one, Sally and Warner computed Fourier inversion formulas for Λ_γ, γ semisimple (including $\gamma = 1$) [**10**]. Barbasch used these results to compute Fourier inversion formulas for Λ_u, u unipotent [**1**].

For G of real rank greater than one, the methods used by Sally and Warner can be extended to obtain a Fourier inversion formula for Λ_γ when γ is in a dense subset of the set of regular elements [**5b**]. This formula is stated as Theorem 4.3 in this paper, and serves as a starting point. It contains certain complicated terms which can be interpreted as Fourier series in several variables. These series are not absolutely convergent, as the functions F_f^H have jump discontinuities, and have no obvious closed form. Thus this formula gives little insight into the functions F_f^H and is not suitable for applications.

In [**2**] and [**5d, e**], simple Fourier inversion formulas are obtained for certain averaged orbital integrals obtained as follows. Let H be a Cartan subgroup of G, W_I the Weyl group generated by reflections in imaginary roots of H. Then W_I acts on H, and for regular $\gamma \in H$, the averaged orbital integral is given by

$$\overline{\Lambda}_\gamma(f) = \sum_{w \in W_I} \Lambda_{w\gamma}(f).$$

In this paper, Fourier inversion formulas for Λ_γ are obtained using those for $\overline{\Lambda}_\gamma$. Heuristically, the idea is as follows. Consider weighted averages

$$\Lambda_\gamma^\kappa(f) = \sum_{w \in W_I} \kappa(w)\Lambda_{w\gamma}(f) \quad \text{where } \kappa(w) = \pm 1 \text{ for } w \in W_I.$$

For $\kappa \equiv 1$, $\Lambda_\gamma^\kappa = \overline{\Lambda}_\gamma$. For other suitable choices of κ, Shelstad proves in [**12a, b, c**] that Λ_γ^κ corresponds to an ordinary averaged orbital integral for a lower-dimensional group. Thus Fourier inversion formulas for Λ_γ^κ can be obtained as in [**5d**]. This is done for sufficiently many choices of κ that the original Λ_γ can be recovered as a sum of the Λ_γ^κ.

Theorem 1 of this paper gives a simple Fourier inversion formula for Λ_γ, γ regular. The proof uses ideas of Shelstad. But rather than identify explicitly the groups and correspondences involved in [**12**], it is simpler to use an identity for discrete series characters which accomplishes the same purpose of reducing the problem to the study of averaged orbital integrals. This identity is stated as Theorem 2. It is the explicit form of a character identity proved by Shelstad in [**12c**] as a corollary of her results on orbital integrals. We give an elementary direct proof of this theorem in §3.

A more general (fewer restrictions on G), but less explicit, Fourier inversion formula has been announced by Harish-Chandra [**3c**].

I would like to thank Diana Shelstad for explaining to me her work on orbital integrals and suggesting its applicability to the Fourier inversion problem.

2. Statement of theorems. Let G be as in §1. Let $\mathfrak{g}$ denote the Lie algebra of G, $\mathfrak{g}_{\mathbf{C}}$ its complexification. If $G_{\mathbf{C}}$ is the simply connected complex analytic group with Lie algebra $\mathfrak{g}_{\mathbf{C}}$, we assume that G is the subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{g}$.

Let K be a maximal compact subgroup of G and θ the corresponding Cartan involution. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ where $\mathfrak{k}$ is the Lie algebra of K. If H is a θ-stable Cartan subgroup of G with Lie algebra $\mathfrak{h}$, write $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$ where $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{h}_p = \mathfrak{h} \cap \mathfrak{p}$, and $H = H_K H_p$ where $H_K = H \cap K$ and $H_p = \exp(\mathfrak{h}_p)$. The set of roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$ will be denoted by $\Phi = \Phi(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Φ^+ denotes a choice of positive roots in Φ. The subsets of Φ taking real and pure imaginary values on $\mathfrak{h}$ will be denoted by $\Phi_R = \Phi_R(\mathfrak{g}, \mathfrak{h})$ and $\Phi_I = \Phi_I(\mathfrak{g}, \mathfrak{h})$, respectively. Let $r_I(H) = \frac{1}{2}[\Phi_I]$. For any root system Φ, $L(\Phi)$ denotes the weight lattice and $W(\Phi)$ the Weyl group of Φ.

As in **[5e]**, for any root system Φ spanned by strongly orthogonal roots, we say a root system $\varphi \subseteq \Phi$ is a two-structure for Φ if: (i) all simple factors of φ are of type A_1 or $B_2 \cong C_2$; (ii) if φ^+ is any set of positive roots for φ then $\{w \in W(\Phi) \mid w\varphi^+ = \varphi^+\}$ contains no elements of determinant -1.

Let $\mathfrak{T}(\Phi)$ denote the set of all two-structures for Φ. All elements of $\mathfrak{T}(\Phi)$ are conjugate under $W(\Phi)$. If Φ^+ is a choice of positive roots for Φ, let $\varphi^+ = \varphi \cap \Phi^+$ for any $\varphi \in \mathfrak{T}(\Phi)$. In **[5e]** a distinguished element $\varphi_0 \in \mathfrak{T}(\Phi)$ is chosen for each type of root system. Then each $\varphi \in \mathfrak{T}(\Phi)$ is assigned a sign $\varepsilon(\varphi : \Phi^+) = \det \sigma$ where σ is an element of $W(\Phi)$ satisfying $\sigma\varphi_0^+ = \varphi^+$. Because of (iii) above, $\det \sigma$ is well defined even though σ may not be unique. Alternatively, the signs $\varepsilon(\varphi : \Phi^+)$ are uniquely determined by:

(i) for all $\varphi \in \mathfrak{T}(\Phi)$ and $\sigma \in W$ such that $\sigma\varphi^+ \subseteq \Phi^+$,

$$\varepsilon(\sigma\varphi : \Phi^+) = \det \sigma \, \varepsilon(\varphi : \Phi^+);$$

(ii) $\sum_{\varphi \in \mathfrak{T}(\Phi)} \varepsilon(\varphi : \Phi^+) = 1$.

If Φ is simple, $\mathfrak{T}(\Phi)$ will consist of all root systems φ contained in Φ of type given by the following table which also gives the values of $[\mathfrak{T}(\Phi)]$ and $[L(\varphi) : L(\Phi)]$.

Φ	φ	$[\mathfrak{T}(\varphi)]$	$[L(\varphi) : L(\Phi)]$
A_1	A_1	1	1
B_{2n}	B_2^n	$(2n)!/n!2^n$	2^{n-1}
B_{2n+1}	$B_2^n \times B_1$	$(2n+1)!/n!2^n$	2^n
C_{2n}	C_2^n	$(2n)!/n!2^n$	1
C_{2n+1}	$C_2^n \times C_1$	$(2n+1)!/n!2^n$	1
D_{2n}	A_1^{2n}	$(2n)!/n!2^n$	2^{n-1}
E_7	A_1^7	5×3^3	2^3
E_8	A_1^8	$5^2 \times 3^4$	2^4
F_4	B_2^2	3^2	2
G_2	A_1^2	3	2

Let $W(G, H) = N_G(H)/H$ where $N_G(H)$ is the normalizer in G of H. For each $\alpha \in \Phi_R$, let $H_\alpha \in \mathfrak{h}_p$ be dual to $\check{\alpha} = 2\alpha/\langle \alpha, \alpha \rangle$ under the killing form. Let $\gamma_\alpha = \exp(\pi i H_\alpha) \in H_{\mathbf{C}}$. Then $\{\gamma_\alpha \mid \alpha \in \Phi_R\}$ generate the subgroup $Z(\mathfrak{h}_p) = K \cap \exp(i\mathfrak{h}_p)$ of H_K.

Let $L_H = C_G(H_p)$, the centralizer in G of H_p, and write $L_H = M_H H_p$ in its Langlands decomposition. Then $M = M_H$ is a reductive group with compact Cartan subgroup H_K. Let $\mathrm{Car}(M)$ denote a full set of θ-stable representatives of M-conjugacy classes of Cartan subgroups of M. These representatives can be chosen so that for $J \in \mathrm{Car}(M)$, $J_K^0 \subseteq H_K$. Each $J \in \mathrm{Car}(M)$ corresponds to a set R of strongly orthogonal singular imaginary roots in $\Phi_I^+(\mathfrak{g}, \mathfrak{h})$. Let ν be the corresponding Cayley transform. Then $\nu(\mathfrak{h}_k) = \mathfrak{j}_k + i\mathfrak{j}_p$ and νR spans $\Phi_R^+(\mathfrak{m}, \mathfrak{j})$. For each $J \in \mathrm{Car}(M)$, $\tilde{J} = JH_p$ is a Cartan subgroup of G. There may be distinct elements $J, J' \in \mathrm{Car}(M)$ with $\tilde{J}$ conjugate to $\tilde{J}'$ in G.

The unitary character group of H will be parameterized by pairs (b^*, μ) where $b^* \in \hat{H}_k$ and $\mu \in \mathfrak{h}_p^*$, the real dual of $\mathfrak{h}_p$. To each such pair (b^*, μ) there corresponds a tempered invariant eigendistribution $\theta(H, b^*, \mu)$ on G defined as in **[5b]**. If $b^* \in \hat{H}_k' = \{b^* \in \hat{H}_k \mid wb^* \neq b^* \text{ for } w \in W(\Phi_I), w \neq 1\}$, then $\theta(H, b^*, \mu)$ is, up to a sign, the character of a tempered unitary representation of G induced from a parabolic subgroup of G with split part H_p. If b^* is not regular, $\theta(H, b^*, \mu)$ also has a character-theoretic interpretation given in **[6]**.

Let G' be the set of regular semisimple elements of G and let $H' = H \cap G'$. For $h \in H'$, $f \in C_c^\infty(G)$, define

$$F_f^H(h) = \varepsilon_R(h)\Delta(h)\int_{G/H} f(xhx^{-1})\, d\dot{x}$$

where $\varepsilon_R(h)$ and $\Delta(h)$ are defined as in **[14]**. Let $\Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be the set of positive roots inherent in the definition of Δ. Assume all Haar measures are normalized as in **[5b]** and let $\mathrm{vol}(H_K)$ denote the total mass of H_K.

THEOREM 1. *Let $f \in C_c^\infty(G)$ and let $h_k h_p \in H'$ where $h_k \in H_K$ and $h_p \in H_p$. Then*

$$\begin{aligned} F_f^H(h_k h_p) &= \mathrm{vol}(H_K)^{-1}(2\pi)^{-\dim \mathfrak{h}_p} \\ &\times \sum_{J \in \mathrm{Car}(M)} \left(\frac{i}{2}\right)^{\dim \mathfrak{j}_p} \frac{(-1)^{r_I(J)}}{[W(M,J)][Z(\mathfrak{h}_p) \cap Z(\mathfrak{j}_p)]} \\ &\times \sum_{b^* \in \hat{J}_K} \int_{\mathfrak{h}_p^*} h_p^{-i\mu'} \int_{\mathfrak{j}_p^*} \theta(JH_p, b^*, \mu' \otimes \mu)(f) \\ &\times \sum_{w \in W(M, H_K)} \det wK(M, J, b^*, \mu, wh_k)\, d\mu\, d\mu'. \end{aligned}$$

Let $J \in \mathrm{Car}(M)$, and let R be the corresponding set of strongly orthogonal singular imaginary roots of $(\mathfrak{m}, \mathfrak{h}_k)$, ν the Cayley transform. Fix $b^* \in \hat{J}_K$, $\mu \in j_p^*$, and $h_k \in H_K \cap M'$. Then $K(M, J, b^*, \mu, h_k)$ is defined as follows. Let $\Phi = \Phi_R(\mathfrak{m}, \mathfrak{j})$ and $\Phi^+ = \Phi \cap \nu\Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Let φ_R be a two-structure for Φ with $\nu R \subseteq \varphi_R$.

Decompose $\varphi_R = \varphi_1 \cup \cdots \cup \varphi_k$ where the φ_j, $1 \leq j \leq k$, are simple root systems. Write $\mathfrak{j}_p = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$ where $1 \leq j \leq k$, $\mathfrak{a}_j = \sum_{\alpha \in \varphi_j} \mathbf{R} H_\alpha$. Then νh_k can be decomposed as $\nu h_k = j_0 a_1 \cdots a_k$ where $j_0 \in J_K$ and $a_j \in \exp(i\mathfrak{a}_j)$, $1 \leq j \leq k$. Let

$\mu_j \in \mathfrak{a}_j^*$ be the restriction of μ to $\mathfrak{a}_j$, b_j^* the restriction of b^* to $Z(\mathfrak{a}_j)$. Then

$$K(M, J, b^*, \mu, h_K) = \frac{\varepsilon(\varphi_R : \Phi^+)[\mathfrak{T}(\Phi)]}{[L(\varphi_R) : L(\Phi)]} \overline{b^*(j_0)} \prod_{j=1}^{k} K(\varphi_j^+, b_j^*, \mu_j, a_j)$$

is a product of factors corresponding to the simple root systems φ_j.

Let $\varphi = \varphi_j$ be one of these and write $\mathfrak{a} = \mathfrak{a}_j$, $\varphi^+ = \varphi \cap \Phi^+$. Suppose φ is of type A_1 and $\varphi^+ = \{\alpha\}$. Write $a \in \exp(i\mathfrak{a})'$ as $a = \exp(-i\theta H_\alpha)$ where $0 < |\theta| < \pi$. Each $\mu \in \mathfrak{a}^*$ is determined by $\mu(H_\alpha)$ which we also denote by μ. Define

$$S(\varphi^+, \mu, a) = \frac{2}{\|\alpha\|} \frac{\sinh \mu(\theta \mp \pi)}{\sinh \pi\mu}$$

where

$$\theta \mp \pi = \begin{cases} \theta - \pi & \text{if } 0 < \theta < \pi, \\ \theta + \pi & \text{if } -\pi < \theta < 0. \end{cases}$$

Then for $b^* \in Z(\mathfrak{a})\hat{}$,

$$K(\varphi^+, b^*, \mu, a) = S(\varphi^+, \mu, a) + b^*(\gamma_\alpha) S(\varphi^+, \mu, \gamma_\alpha a).$$

Suppose φ is of type B_2 and $\varphi^+ = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ where $\beta_1 = \frac{1}{2}(\alpha_1 + \alpha_2)$ and $\beta_2 = \frac{1}{2}(\alpha_1 - \alpha_2)$. Then exactly one of $\nu^{-1}\beta_i$, $i = 1, 2$, is compact. Write

$$\varepsilon(\varphi^+) = \begin{cases} 1 & \text{if } \nu^{-1}\beta_2 \text{ is compact}, \\ -1 & \text{if } \nu^{-1}\beta_1 \text{ is compact}. \end{cases}$$

Then $a \in \exp(i\mathfrak{a})'$ can be written as $a = \exp(-i\Sigma_{j=1}^2 \theta_j H_{\alpha_j})$ where $0 < |\theta_j| < \pi$, $j = 1, 2$, and $|\theta_1| \neq |\theta_2|$. For $\mu \in \mathfrak{a}^*$ write $\mu_i = \mu(H_{\alpha_i})$. Define

$$S(\varphi^+, \mu, a) = \frac{4}{\|\alpha_1\|^2} \times \begin{cases} \dfrac{\sinh \mu_1\theta_1 \sinh \mu_2(\theta_2 \mp \pi)}{\sinh \pi\mu_1 \sinh \pi(\mu_1 + \mu_2)} & \text{if } |\theta_1| < |\theta_2|, \\[2ex] \dfrac{\sinh \mu_1(\theta_1 \mp \pi) \sinh \mu_2(\theta_2 \mp (1 + \mu_1\mu_2^{-1})\pi)}{\sinh \pi\mu_1 \sinh \pi(\mu_1 + \mu_2)} & \text{if } |\theta_1| > |\theta_2|. \end{cases}$$

Write $\varphi_l^+ = \{\alpha_1, \alpha_2\}$ and $\varphi_s^+ = \{\beta_1, \beta_2\}$. Define

$$S(\varphi_l^+, \mu, a) = \frac{4}{\|\alpha_1\|^2} \frac{\sinh \mu_1(\theta_1 \mp \pi) \sinh \mu_2(\theta_2 \mp \pi)}{\sinh \pi\mu_1 \sinh \pi\mu_2}$$

and

$$S(\varphi_s^+, \mu, a) = \frac{4\varepsilon(\varphi^+)}{\|\beta_1\|^2} \left[\frac{\sinh(\mu_1 + \mu_2)((\theta_1 + \theta_2)/2 \mp \pi) \sinh(\mu_1 - \mu_2)((\theta_1 - \theta_2)/2 \mp \pi)}{\sinh \pi(\mu_1 + \mu_2) \sinh \pi(\mu_1 - \mu_2)} + \frac{\sinh(\mu_1 + \mu_2)((\theta_1 + \theta_2)/2) \sinh(\mu_1 - \mu_2)((\theta_1 - \theta_2)/2)}{\sinh \pi(\mu_1 + \mu_2) \sinh \pi(\mu_1 - \mu_2)} \right].$$

Then for $b^* \in Z(\mathfrak{a})\hat{}$,

$$K(\varphi^+, \mu, b^*, a) = -2S(\varphi^+, \mu, a) + \tfrac{1}{2}S(\varphi_s^+, \mu, a)$$
$$+ b^*(\gamma_{\beta_1})\{-2S(\varphi^+, \mu, \gamma_{\beta_1}a) + \tfrac{1}{2}S(\varphi_s^+, \mu, \gamma_{\beta_1}a)\}$$
$$+ b^*(\gamma_{\alpha_1})2S(\varphi_l^+, \mu, \gamma_{\alpha_1}a).$$

Theorem 1 will be obtained by simplifying the formula for F_f^H given in Theorem 4.3. The main result needed is an identity involving constants which appear in discrete series character formulas. Let Φ be a root system spanned by strongly orthogonal roots, Φ^+ a choice of positive roots. We assume Φ comes equipped with a subset Φ_{CPT} called the set of compact roots. Let $L = L(\Phi)$, $W = W(\Phi)$, and $W_K = W(\Phi_{CPT})$. For each $\tau \in L$, $w \in W$, there is a constant $c(w:\tau:\Phi^+)$ which arises as follows.

Let G be a split real group satisfying our hypotheses with root system of type Φ. Then G has a compact Cartan subgroup T with Lie algebra $\mathfrak{t}$, and L can be identified with the lattice of elements $\tau \in i\mathfrak{t}^*$ for which $\xi_\tau(\exp H) = \exp(\tau(H))$, $H \in \mathfrak{t}_{\mathbf{C}}$, gives a well-defined character of $T_{\mathbf{C}}$. We assume that when Φ is identified with $\Phi(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$, Φ_{CPT} corresponds to the set of compact roots of $\mathfrak{t}$. Corresponding to each $\tau \in L$ there is an invariant eigendistribution θ_τ defined by Harish-Chandra in **[3a]** and **[3b]**. If $\tau \in L' = \{\tau \in L \mid \prod_{\alpha \in \Phi}\langle \alpha, \tau\rangle \neq 0\}$, θ_τ is, up to sign, the character of a discrete series representation of G.

Let A be a split Cartan subgroup of G with Lie algebra $\mathfrak{a} \subseteq \mathfrak{p}$. Let ν be a Cayley transform satisfying $\nu\mathfrak{t} = i\mathfrak{a}$. Let $\mathfrak{a}^+ = \{H \in a \mid \nu\alpha(H) > 0 \text{ for all } \alpha \in \Phi^+\}$. Then for $h \in A^+ = \exp(\mathfrak{a}^+)$,

$$\theta_\tau(h) = \Delta(h)^{-1} \sum_{w \in W} \det w c(w:\tau:\Phi^+)\xi_{w\tau}(\nu^{-1}h).$$

The constant $c(w:w^{-1}\tau:\Phi^+)$ depends only on the coset of W in W/W_K and we define

$$\bar{c}(\tau:\Phi^+) = \sum_{w \in W/W_K} c(w:w^{-1}\tau:\Phi^+).$$

Then it is proved in **[5c, e]** that

$$\bar{c}(\tau:\Phi^+) = \sum_{\varphi \in \mathfrak{T}(\Phi)} \varepsilon(\varphi:\Phi^+)\bar{c}(\tau:\varphi^+). \tag{2.1}$$

Since all simple factors of each two-structure φ for Φ are of type A_1 or B_2, $\bar{c}(\tau:\varphi^+)$ can be evaluated explicitly as in **[5c]**. Theorem 2 gives a formula for $c(w:\tau:\Phi^+)$ in terms of averaged constants $\bar{c}(w\tau:\Phi^+(\lambda))$ where the $\Phi(\lambda)$ are root systems contained in Φ.

Using reduction procedures described by Harish-Chandra in **[3a]**, knowledge of the constants $c(w:\tau:\Phi^+)$ is sufficient to determine discrete series character formulas on any Cartan subgroup of any group G satisfying our assumptions.

Let $\Lambda = \Lambda(\Phi)$ denote the root lattice of Φ. For $\lambda \in \Lambda$, let $\Phi(\lambda) = \{\beta \in \Phi \mid \langle \beta, \lambda \rangle / \langle \beta, \beta \rangle \in Z\}$. Then for $\lambda_1, \lambda_2 \in \Lambda$, $\Phi(\lambda_1) = \Phi(\lambda_2)$ if and only if $\lambda_1 - \lambda_2 \in \Lambda_0 = \{\lambda \in \Lambda \mid \langle \lambda, \beta \rangle / \langle \beta, \beta \rangle \in Z$ for all $\beta \in \Phi\}$. Define a homomorphism χ: $\Lambda \to Z/2Z$ by setting

$$\chi(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Phi_{CPT}, \\ 1 & \text{if } \alpha \notin \Phi_{CPT} \end{cases}$$

and extending linearly. This extension is well defined because if α, β, and $\alpha + \beta$ are all roots, then $\chi(\alpha + \beta) \equiv \chi(\alpha) + \chi(\beta) \pmod 2$. For $\lambda_1, \lambda_2 \in \Lambda$, define $\lambda_1 \equiv \lambda_2$ if $\chi(\lambda_1 - \lambda_2) = 0$. Then for $\lambda \in \Lambda$ and $w \in W(\Phi)$ we define

$$\kappa_\lambda(w) = \begin{cases} 1 & \text{if } w^{-1}\lambda \equiv \lambda, \\ -1 & \text{if } w^{-1}\lambda \not\equiv \lambda. \end{cases}$$

It is easy to check from the definitions that κ_λ depends only on the coset of λ in Λ/Λ_0 and of w in W/W_K.

Suppose Φ is simple. Then the root systems $\Phi(\lambda)$, $\lambda \in \Lambda$, will be all root systems $\Phi(\lambda) \subseteq \Phi$ of type given by the following table.

Φ	$\Phi(\lambda)$
A_1	A_1
B_n	$B_p \times B_{n-p}$, $0 \leqslant p \leqslant n$
C_n	$D_{2p} \times C_{n-2p}$, $0 \leqslant 2p \leqslant n$
D_{2n}	$D_{2p} \times D_{2(n-p)}$, $0 \leqslant p \leqslant n$
E_7	E_7, $D_6 \times A_1$
E_8	E_8, $E_7 \times A_1$, D_8
F_4	F_4, $B_3 \times B_1$, C_4
G_2	G_2, A_1^2

We note that when Φ is identified with $\Phi(\mathfrak{g}_C, \mathfrak{t}_C)$ as above, then $\{\kappa_\lambda \mid \lambda \in \Lambda\}$ is a subset of the set of "characters" κ of $\mathfrak{D}(T)$ considered by Shelstad in **[12a]** and that $\Phi(\lambda)$ is the root system of the endoscopic group H associated to κ_λ. The characters κ which are of the form κ_λ, $\lambda \in \Lambda$, are exactly those for which H "shares" both T and A with G.

Given a fixed choice Φ^+ of positive roots for Φ, we will describe in §3 a way of assigning to each $\lambda \in \Lambda$ a sign $\varepsilon(\lambda : \Phi^+)$. For each $\lambda \in \Lambda$, let $\Phi^+(\lambda) = \Phi(\lambda) \cap \Phi^+$.

THEOREM 2. *Let $n = \operatorname{rank} \Phi$. Then for $w \in W$, $\tau \in L$,*

$$c(w : \tau : \Phi^+) = 2^{-n} \sum_{\lambda \in \Lambda/\Lambda_0} \varepsilon(\lambda : \Phi^+) \kappa_\lambda(w) \bar{c}(w\tau : \Phi^+(\lambda)).$$

Combining Theorem 2 with (2.1) and known values for $\bar{c}(\tau : \Phi^+)$ when Φ is of type A_1 or B_2 gives a complete description of the constants $c(w : \tau : \Phi^+)$. Much simpler formulas for these constants have been obtained in special cases, for example when the corresponding discrete series representation is holomorphic. Our formula can be simplified to agree with the formula of Hecht **[4]** in this case.

Although complicated, the formulas of Theorem 2 and (2.1) are well suited to simplifying the terms occurring in the Fourier inversion formula for F_f^H. For other discrete series character formulas the reader is referred to work of Hirai [**7**], Schmid [**11**], Martens [**8**], Midorikawa [**9**], and Vargas [**13**].

3. Proof of Theorem 2.

LEMMA 3.1. *Let $\lambda \in \Lambda$ and $s \in W(\Phi(\lambda))$. Then for all $w \in W$, $\kappa_\lambda(sw) = \kappa_\lambda(w)$.*

PROOF. It is enough to prove that for all $\alpha \in \Phi(\lambda)$ and $w \in W$, $\kappa_\lambda(s_\alpha w) = \kappa_\lambda(w)$ where s_α denotes the reflection corresponding to α. But $w^{-1}s_\alpha\lambda = w^{-1}\lambda - 2(\langle\alpha, \lambda\rangle/\langle\alpha, \alpha\rangle)w^{-1}\alpha \equiv w^{-1}\lambda$ since $\langle\alpha, \lambda\rangle/\langle\alpha, \alpha\rangle \in Z$ and $2n\beta \equiv 0$ for any $n \in Z$, $\beta \in \Phi$. The result follows from the definition of κ_λ.

LEMMA 3.2. *Let $\lambda \in \Lambda$, $v, w \in W$. Then $\kappa_{v\lambda}(w) = \kappa_\lambda(v^{-1})\kappa_\lambda(v^{-1}w)$.*

PROOF. This follows trivially from the definition of κ_λ.

We note that the above two lemmas appear in [**12a**] as Propositions 3.1 and 3.3.

We will now assign to each $\lambda \in \Lambda$ and system Φ^+ of positive roots for Φ a sign $\varepsilon(\lambda : \Phi^+)$. We will first do this for positive systems chosen as follows.

(3.3) If Φ is of type A_1, B_{2n+1}, D_{2n}, E_7, E_8, or G_2, pick Φ^+ so that all simple roots are noncompact. If Φ is of type B_{2n}, C_n, or F_4, pick Φ^+ so that all long simple roots are noncompact and all short simple roots are compact. If Φ is not simple, pick positive roots for each simple factor as above.

For Φ^+ chosen as above, if Φ is of type A_1, B_n, D_{2n}, E_7, E_8, or G_2, set $\varepsilon(\lambda : \Phi^+) = 1$ for all $\lambda \in \Lambda$. For Φ of type C_n, denote the simple roots for Φ^+ by $\{\alpha_1, \ldots, \alpha_n\}$ when $\langle\alpha_i, \alpha_{i+1}\rangle = -1$, $1 \leq i \leq n-1$, and $\langle\alpha_{n-1}, \alpha_n\rangle = -2$. Let $\Lambda_1 = \{\alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1} \mid 0 \leq p \leq k\}$ where $n = 2k$ or $2k+1$. If Φ is of type F_4, let $\Lambda_1 = \{0, \alpha_1, \alpha_2\}$ where α_1 is a short simple root and α_2 is a long simple root for Φ^+. In either case, for $\lambda \in \Lambda_1$, define $W(\lambda : \Phi^+) = \{\sigma \in W \mid \sigma\Phi^+(\lambda) \subseteq \Phi^+\}$. Then for any $\lambda \in \Lambda$ there are $\lambda_1 \in \Lambda_1$ and $\sigma \in W(\lambda_1 : \Phi^+)$ with $\Phi^+(\lambda) = \sigma\Phi^+(\lambda_1)$. Define $\varepsilon(\lambda : \Phi^+) = \det\sigma\,\kappa_{\lambda_1}(\sigma^{-1})$. For Φ of type F_4, λ_1 and σ are uniquely determined by λ. For Φ of type C_n, λ_1 is uniquely determined by λ. Each element of $W(\lambda_1 : \Phi^+)$ is of the form $\sigma = vs$ where $v \in W_K \cap W(\lambda_1 : \Phi^+)$ and $s \in \{\sigma \in W \mid \sigma\Phi^+(\lambda_1) = \Phi^+(\lambda_1)\}$. Again, v is uniquely determined by λ but s is not. However, $\det\sigma\,\kappa_{\lambda_1}(\sigma^{-1}) = \det v$ is well defined.

If Φ is not simple, write $\Phi^+ = \Phi_1^+ \cup \cdots \cup \Phi_s^+$ where the Φ_i are simple. Then for $\lambda_i \in \Lambda(\Phi_i)$, $1 \leq i \leq s$, define $\varepsilon(\lambda_1 + \cdots + \lambda_s : \Phi^+) = \prod_{i=1}^s \varepsilon(\lambda_i : \Phi_i^+)$.

If Φ^+ is chosen as in (3.3) and $v \in W$, define $\varepsilon(\lambda : v\Phi^+) = \kappa_\lambda(v)\varepsilon(v^{-1}\lambda : \Phi^+)$ for all $\lambda \in \Lambda$.

LEMMA 3.4. *Let $\lambda \in \Lambda$ and $\sigma \in W(\lambda : \Phi^+) = \{\sigma \in W \mid \sigma\Phi^+(\lambda) \subseteq \Phi^+\}$. Then for all $w \in W$, $\varepsilon(\sigma\lambda : \Phi^+)\kappa_{\sigma\lambda}(w) = \det\sigma\,\varepsilon(\lambda : \Phi^+)\kappa_\lambda(\sigma^{-1}w)$.*

PROOF. Using (3.2), it is enough to prove that $\varepsilon(\sigma\lambda : \Phi^+) = \kappa_\lambda(\sigma^{-1})\det\sigma\,\varepsilon(\lambda : \Phi^+)$ for all $\sigma \in W(\lambda : \Phi^+)$. Suppose first that Φ^+ has been chosen as in (3.3). Write $\varepsilon(\lambda : \Phi^+) = \varepsilon(\lambda)$. If Φ is of type C_n or F_4 and $\lambda \in \Lambda_1$, then (3.4) follows directly from the definition of $\varepsilon(\sigma\lambda)$. If λ is arbitrary, then $\Phi^+(\lambda) = \tau\Phi^+(\lambda_1)$ for some

$\lambda_1 \in \Lambda_1$ and $\tau \in W(\lambda_1, \Phi^+)$. If $\sigma \in W(\lambda : \Phi^+)$, then $\sigma\tau \in W(\lambda_1 : \Phi^+)$ and the result follows from using (3.2) and the definitions of $\varepsilon(\sigma\tau\lambda_1)$ and $\varepsilon(\tau\lambda_1)$.

If Φ is of type A_1, B_n, D_n, E_7, E_8, or G_2, $\varepsilon(\lambda) = 1$ for all λ so we must show that $\kappa_\lambda(\sigma^{-1}) = \det\sigma$ for all $\sigma \in W(\lambda : \Phi^+)$. Write $\sigma \in W(\lambda : \Phi^+)$ as $\sigma = s_{\alpha_1} \cdots s_{\alpha_k}$ where the α_i are simple roots and k is minimal. In this case each α_i is noncompact. The proof is by induction on $k = l(\sigma)$.

If $k = 0$ the result is obvious. Suppose $l(\sigma) = k \geq 1$. Because $s_{\alpha_1} \cdots s_{\alpha_k}$ is a reduced expression for σ, $\sigma_0 = s_{\alpha_2} \cdots s_{\alpha_k} \in W(\lambda : \Phi^+)$ also and $s_{\alpha_1} \in W(\sigma_0\lambda : \Phi^+)$. Using (3.2) and the induction hypothesis, $\kappa_\lambda(\sigma^{-1}) = \kappa_{\sigma_0\lambda}(s_{\alpha_1})\kappa_\lambda(\sigma_0^{-1}) = \det\sigma_0\, \kappa_{\sigma_0\lambda}(s_{\alpha_1})$. Further $s_{\alpha_1}\Phi^+(\sigma_0\lambda) \subseteq \Phi^+$ implies that $\alpha_1 \notin \Phi(\sigma_0\lambda)$ so that $2(\langle\sigma_0\lambda, \alpha_1\rangle/\langle\alpha_1, \alpha_1\rangle)\alpha_1 \not\equiv 0$ and $s_{\alpha_1}\sigma_0\lambda \not\equiv \sigma_0\lambda$. Thus $\kappa_{\sigma_0\lambda}(s_{\alpha_1})\det\sigma_0 = -\det\sigma_0 = \det\sigma$.

If $v\Phi^+$ is another choice of positive roots, $v^{-1}W(\lambda : v\Phi^+)v = W(v^{-1}\lambda : \Phi^+)$ and the result follows by using the first case, (3.2), and the definition of $\varepsilon(\lambda : v\Phi^+)$.

THEOREM 2. *Let $w \in W$, $\tau \in L$. Let $n = \operatorname{rank}\Phi$. Then*

$$c(w : \tau : \Phi^+) = 2^{-n} \sum_{\lambda \in \Lambda/\Lambda_0} \varepsilon(\lambda : \Phi^+)\kappa_\lambda(w)\bar{c}(w\tau : \Phi^+(\lambda)).$$

Before proceeding with the proof of Theorem 2, we will illustrate it in the cases $n = 1, 2$, where all values of $c(w : \tau : \Phi^+)$ are known **[5a]**. Note that in these cases all the signs $\varepsilon(\lambda : \Phi^+)$ are one.

Case I. Suppose $\Phi^+ = \{\alpha\}$ is of type A_1. Then for $w \in W$, $\tau \in L$,

$$c(w : w^{-1}\tau : \Phi^+) = \begin{cases} 1 & \text{if } \langle\alpha, \tau\rangle < 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{c}(\tau : \Phi^+) = c(1 : \tau : \Phi^+) + c(s_\alpha : s_\alpha\tau : \Phi^+)$$
$$= \begin{cases} 2 & \text{if } \langle\alpha, \tau\rangle < 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this case $\Lambda/\Lambda_0 = \{0\}$ so that Theorem 2 just says $c(w : w^{-1}\tau : \Phi^+) = \frac{1}{2}\bar{c}(\tau : \Phi^+)$.

Case II. Suppose $\Phi^+ = \{\alpha, \beta, \gamma, \delta\}$ is of type B_2 where δ is the compact root and $\alpha = \gamma + \delta$, $\beta = \gamma - \delta$. In this case $\Lambda/\Lambda_0 = \{0, \gamma + \Lambda_0\}$ where $\Phi(\gamma) = \{\gamma, \delta\}$ is of type $A_1 \times A_1$ and

$$\kappa_\gamma(w) = \begin{cases} 1 & \text{if } w \in W_K \cup s_\gamma W_K = W_0, \\ -1 & \text{if } w \in s_\alpha W_K \cup s_\beta W_K = s_\alpha W_0. \end{cases}$$

If $\tau \in L$ is written as $\tau = n\alpha + m\beta$, then the nonzero values of $c(w : w^{-1}\tau : \Phi^+)$, $\bar{c}(\tau : \Phi^+)$, and $\bar{c}(\tau : \Phi^+(\gamma))$ are given by the following table. Since $c(w : w^{-1}\tau : \Phi^+)$ is constant on cosets of W/W_0, only values for $w = 1$, s_α are given.

	$0 > m > n$	$0 > n > m$	$0 > -n > m$
$c(1 : \tau : \Phi^+)$	1	1	2
$c(s_\alpha : s_\alpha\tau : \Phi^+)$	1	-1	0
$\bar{c}(\tau : \Phi^+)$	4	0	4
$\bar{c}(\tau : \Phi^+(\gamma))$	0	4	4

Clearly in each case $c(w : w^{-1}\tau : \Phi^+) = \frac{1}{4}[\bar{c}(\tau : \Phi^+) + \kappa_\gamma(w)\bar{c}(\tau : \Phi^+(\gamma))]$.

Case III. Suppose $\Phi^+ = \{\alpha, \beta, \gamma, \alpha', \beta', \gamma'\}$ is of type G_2 where α is simple and short, β is simple and long, $\alpha + \beta = \gamma$, and α', β', γ' are orthogonal to α, β, γ, respectively. We assume that γ, γ' are the compact roots. Then $\Lambda/\Lambda_0 = \{0, \alpha, \beta, \gamma\}$ and $W = W_K \cup s_\alpha W_K \cup s_\beta W_K$. For each $\lambda \in \Lambda/\Lambda_0$, $\Phi^+(\lambda)$ and values of $\kappa_\lambda(w)$ are given in the following table.

	0	α	β	γ
$\Phi^+(\lambda)$	Φ^+	$\{\alpha, \alpha'\}$	$\{\beta, \beta'\}$	$\{\gamma, \gamma'\}$
$\kappa_\lambda(1)$	1	1	1	1
$\kappa_\lambda(s_\alpha)$	1	1	-1	-1
$\kappa_\lambda(s_\beta)$	1	-1	1	-1

For fixed $\tau \in L$, write $\tau = n\alpha + m\alpha'$. Then the nonzero values of the $c(w : w^{-1}\tau : \Phi^+)$ and $\bar{c}(\tau : \Phi^+(\lambda))$ are as follows, verifying the formula of Theorem 2.

	$0 > 3m > n$	$m > n > 3m$	$m < n < 0$	$0 < n < -m$	$-m < n < -3m$
$c(1 : \tau : \Phi^+)$	2	2	4	2	2
$c(s_\alpha : s_\alpha\tau : \Phi^+)$	2	0	0	-2	0
$c(s_\beta : s_\beta\tau : \Phi^+)$	0	-2	0	0	2
$\bar{c}(\tau : \Phi^+)$	4	0	4	0	4
$\bar{c}(\tau : \Phi^+(\alpha))$	4	4	4	0	0
$\bar{c}(\tau : \Phi^+(\beta))$	0	0	4	4	4
$\bar{c}(\tau : \Phi^+(\gamma))$	0	4	4	4	0

PROOF. We first assume that Φ^+ has been chosen as in (3.3). Write $\varepsilon(\lambda : \Phi^+) = \varepsilon(\lambda)$. For $\tau \in L'$, Theorem 2 will be proved by induction on n, the rank of Φ. When $n = 1, 2$, it has been verified above that Theorem 2 holds. We assume that $n \geqslant 3$. The induction step will make use of an identity of Harish-Chandra **[3a]** which relates the constants $c(w : \tau : \Phi^+)$ to constants for root systems of rank $n - 1$ as follows.

Let $\alpha \in S$, the set of simple roots for Φ^+. Let $\Phi_\alpha = \{\beta \in \Phi \mid \langle \beta, \alpha \rangle = 0\}$ and $\Phi_\alpha^+ = \Phi^+ \cap \Phi_\alpha$. It is easy to check in each case that Φ_α^+ is a set of positive roots for Φ_α satisfying (3.3). For $\tau \in L'$, let τ_α be the unique element in the weight lattice L_α of Φ_α satisfying $\langle \tau_\alpha, \beta \rangle = \langle \tau, \beta \rangle$ for all $\beta \in \Phi_\alpha$. The Weyl group W_α of Φ_α can be considered as a subgroup of W.

For $w \in W$ and $\tau \in L$, define

$$c\left(w : w^{-1}\tau : \Phi_\alpha^+\right) = \begin{cases} c\left(w_1 : w_1^{-1}\tau_\alpha : \Phi_\alpha^+\right) & \text{if } w = w_1 w_2,\ w_1 \in W_\alpha,\ w_2 \in W_K, \\ 0 & \text{if } w \notin W_\alpha W_K. \end{cases}$$

Let $s_\alpha \in W$ be the reflection corresponding to α. Then for $\tau \in L'$:

$$c(w : \tau : \Phi^+) + c(s_\alpha w : \tau : \Phi^+) = c(w : \tau : \Phi_\alpha^+) + c(s_\alpha w : \tau : \Phi_\alpha^+); \tag{3.5}$$

$$\bar{c}(\tau : \Phi^+) + \bar{c}(s_\alpha \tau : \Phi^+) = 2\bar{c}(\tau_\alpha : \Phi_\alpha^+). \tag{3.6}$$

Identity (3.5) together with the fact that $c(w : w^{-1}\tau : \Phi^+) = 0$ if $\tau \in L^+ = \{\tau \in L \mid \langle \tau, \alpha \rangle > 0 \text{ for all } \alpha \in \Phi^+\}$ determines the constants $c(w : \tau : \Phi^+)$ inductively **[5a]**. Thus if we define

$$s(w : \tau : \Phi^+) = 2^{-n} \sum_{\lambda \in \Lambda/\Lambda_0} \varepsilon(\lambda)\kappa_\lambda(w)\bar{c}(w\tau : \Phi^+(\lambda)),$$

to prove Theorem 2 for $\tau \in L'$ it is only necessary to verify that $s(w : w^{-1}\tau : \Phi^+) = 0$ for $\tau \in L^+$ and that for all $\alpha \in S$,

(3.7)

$$s(w : w^{-1}\tau : \Phi^+) + s(s_\alpha w : w^{-1}\tau : \Phi^+) = c(w : w^{-1}\tau : \Phi^+_\alpha) + c(s_\alpha w : w^{-1}\tau : \Phi^+_\alpha).$$

The first identity follows from the fact that for $\tau \in L^+$ and any $\lambda \in \Lambda$, $\langle \tau, \alpha \rangle > 0$ for all $\alpha \in \Phi^+(\lambda) \subseteq \Phi^+$ so that $\bar{c}(\tau : \Phi^+(\lambda)) = 0$. The second identity requires a detailed analysis of the terms occurring in (3.7).

Fix $\tau \in L'$, $w \in W$, and a simple root α and denote the left-hand-side of (3.7) by LHS. Then

$$\mathrm{LHS} = 2^{-n} \sum_{\lambda \in \Lambda/\Lambda_0} \varepsilon(\lambda)\left[\kappa_\lambda(w)\bar{c}(\tau : \Phi^+(\lambda)) + \kappa_\lambda(s_\alpha w)\bar{c}(s_\alpha \tau : \Phi^+(\lambda))\right].$$

If $\alpha \notin \Phi(\lambda)$, then since α is simple, $s_\alpha \in W(\lambda : \Phi^+)$. Using (3.4),

$$\begin{aligned}\varepsilon(s_\alpha\lambda)&\left[\kappa_{s_\alpha\lambda}(w)\bar{c}(\tau : \Phi^+(s_\alpha\lambda)) + \kappa_{s_\alpha\lambda}(s_\alpha w)\bar{c}(s_\alpha\tau : \Phi^+(s_\alpha\lambda))\right]\\ &= -\varepsilon(\lambda)\left[\kappa_\lambda(s_\alpha w)\bar{c}(s_\alpha\tau : \Phi^+(\lambda)) + \kappa_\lambda(w)\bar{c}(\tau : \Phi^+(\lambda))\right],\end{aligned}$$

so that if $s_\alpha\lambda + \Lambda_0 \neq \lambda + \Lambda_0$, the corresponding terms in LHS cancel, and if $s_\alpha\lambda + \Lambda_0 = \lambda + \Lambda_0$, the corresponding term is zero.

Define $\Lambda(\alpha) = \{\lambda \in \Lambda \mid \alpha \in \Phi(\lambda)\}$. If $\lambda \in \Lambda(\alpha)$, then by (3.1), $\kappa_\lambda(s_\alpha w) = \kappa_\lambda(w)$ and α is a simple root of $\Phi^+(\lambda)$ so that using (3.6),

$$\mathrm{LHS} = 2^{-n} \sum_{\lambda \in \Lambda(\alpha)/\Lambda_0} \varepsilon(\lambda)\kappa_\lambda(w)2\bar{c}(\tau : \Phi^+(\lambda)_\alpha).$$

For any $\lambda \in \Lambda(\alpha)$, $\lambda_\alpha = \lambda - \langle \lambda, \alpha \rangle \alpha / \langle \alpha, \alpha \rangle \in \{\lambda \in \Lambda \mid \langle \lambda, \alpha \rangle = 0\}$ which can be identified with $\Lambda(\Phi_\alpha)$, the root lattice of Φ_α. The mapping $\lambda + \Lambda_0 \to \lambda_\alpha + \Lambda_0(\Phi_\alpha)$ gives a homomorphism of $\Lambda(\alpha)/\Lambda_0$ onto $\Lambda(\Phi_\alpha)/\Lambda_0(\Phi_\alpha)$ which is bijective if $\alpha \in \Lambda_0$ and has kernel $\{\Lambda_0, \alpha + \Lambda_0\}$ if $\alpha \notin \Lambda_0$. It can be checked case by case (only $\Phi = C_n$ or F_4 are nontrivial) that for $\lambda \in \Lambda(\alpha)$, $\varepsilon(\lambda) = \varepsilon(\lambda + \alpha)$, and that these signs agree with the sign $\varepsilon(\lambda_\alpha) = \varepsilon(\lambda_\alpha : \Phi^+_\alpha)$. Finally, note $\Phi(\lambda)_\alpha = \Phi(\lambda + \alpha)_\alpha = \Phi_\alpha(\lambda_\alpha)$ and that $\kappa_{\lambda+\alpha}(w) = \kappa_\lambda(w)$ if and only if $w^{-1}\alpha \equiv \alpha$.

Case I. Suppose $\alpha \in \Lambda_0$. (Since rank $\Phi \geqslant 3$, this occurs only when $\Phi = C_n$ and α is the long simple root.) Then

$$\mathrm{LHS} = 2^{-n+1} \sum_{\lambda \in \Lambda(\Phi_\alpha)/\Lambda_0(\Phi_\alpha)} \varepsilon(\lambda)\kappa_\lambda(w)\bar{c}(\tau : \Phi^+_\alpha(\lambda)).$$

If $w \in W_\alpha W_K$, then LHS is equal to $c(w : w^{-1}\tau : \Phi^+_\alpha)$ by the induction hypothesis and $s_\alpha w \notin W_\alpha W_K$ so that $c(s_\alpha w : w^{-1}\tau : \Phi^+_\alpha) = 0$. Conversely, if $w \notin W_\alpha W_K$, then $c(w : w^{-1}\tau : \Phi^+_\alpha) = 0$, but $s_\alpha w \in W_\alpha W_K$ and LHS $= c(s_\alpha w : w^{-1}\tau : \Phi^+_\alpha)$.

Case II. Suppose $\alpha \notin \Lambda_0$. Then

$$\begin{aligned}\mathrm{LHS} &= 2^{-n+1} \sum_{\lambda \in \Lambda(\Phi_\alpha)/\Lambda_0(\Phi_\alpha)} \varepsilon(\lambda)\bar{c}(\tau : \Phi^+_\alpha(\lambda))(\kappa_\lambda(w) + \kappa_{\lambda+\alpha}(w))\\ &= \begin{cases} 0 & \text{if } w^{-1}\alpha \not\equiv \alpha,\\ 2c(w : w^{-1}\tau : \Phi^+_\alpha) & \text{if } w^{-1}\alpha \equiv \alpha.\end{cases}\end{aligned}$$

In this case, $w \in W_\alpha W_K$ if and only if $s_\alpha w \in W_\alpha W_K$ and $c(w : w^{-1}\tau : \Phi_\alpha^+) = c(s_\alpha w : w^{-1}\tau : \Phi_\alpha^+)$. Further, if $w^{-1}\alpha \not\equiv \alpha$, then $w \notin W_\alpha W_K$ so that $c(w : w^{-1}\tau : \Phi_\alpha^+) = 0$.

This concludes the proof of Theorem 2 for $\tau \in L'$.

Let $\mathfrak{F}$ denote the real vector space spanned by Φ. Let $\mathfrak{F}^+$ be any component of $\mathfrak{F}' = \{\tau \in \mathfrak{F} \mid \prod_{\alpha\in\Phi}\langle \tau, \alpha\rangle \neq 0\}$. Define $c(w : \mathfrak{F}^+ : \Phi^+) = c(w : \tau : \Phi^+)$ where $\tau \in \mathfrak{F}^+ \cap L'$. This is independent of the choice of τ **[3a]**. Let $\tau \in L^s = L \setminus L'$. Let $W(\tau) = \{w \in W \mid w\tau = \tau\}$. Let $\mathfrak{F}^+$ be a component of $\mathfrak{F}'$ with $\tau \in \mathrm{cl}(\mathfrak{F}^+)$. Then it follows from the definitions of $c(w : \tau : \Phi^+)$ and $\bar{c}(\tau : \Phi^+)$ that

$$c(w : \tau : \Phi^+) = \frac{1}{[W(\tau)]} \sum_{v \in W(\tau)} c(w : v\mathfrak{F}^+ : \Phi^+)$$

and

$$\bar{c}(\tau : \Phi^+) = [W(\tau)]^{-1} \sum_{v \in W(\tau)} \bar{c}(v\mathfrak{F}^+ : \Phi^+).$$

For $\lambda \in \Lambda$, let $\mathfrak{F}'(\lambda) = \{\tau \in \mathfrak{F} \mid \prod_{\alpha\in\Phi(\lambda)}\langle \tau, \alpha\rangle \neq 0\}$. Let $\mathfrak{F}^+(\lambda)$ be the component of $\mathfrak{F}'(\lambda)$ containing $\mathfrak{F}^+$. Define $U = U(\lambda, \tau) = \{w \in W(\tau) \mid w\mathfrak{F}^+ \subseteq \mathfrak{F}^+(\lambda)\}$, and $V = V(\lambda, \tau) = \{w \in W(\Phi(\lambda)) \mid w\tau = \tau\}$. Then $W(\tau) = VU$ and $U \cap V = \{1\}$. For $w \in U$ and $\tau' \in \mathfrak{F}^+ \cap L$, $\bar{c}(w\tau' : \Phi^+(\lambda)) = \bar{c}(\mathfrak{F}^+(\lambda) : \Phi^+(\lambda))$. Thus for $\tau \in L^s$, $\tau' \in \mathfrak{F}^+ \cap L'$, using Theorem 2 for $\tau' \in L'$,

$$\begin{aligned} c(w : w^{-1}\tau : \Phi^+) &= 2^{-n} \sum_{\lambda\in\Lambda/\Lambda_0} \varepsilon(\lambda)\kappa_\lambda(w)[W(\tau)]^{-1} \sum_{v\in W(\tau)} \bar{c}(v\tau' : \Phi^+(\lambda)) \\ &= 2^{-n} \sum_{\lambda\in\Lambda/\Lambda_0} \varepsilon(\lambda)\kappa_\lambda(w)[U][W(\tau)]^{-1} \sum_{v\in V} \bar{c}(v\mathfrak{F}^+(\lambda) : \Phi^+(\lambda)) \\ &= 2^{-n} \sum_{\lambda\in\Lambda/\Lambda_0} \varepsilon(\lambda)\kappa_\lambda(w)\bar{c}(\tau : \Phi^+(\lambda)). \end{aligned}$$

This finishes the proof of Theorem 2 for Φ^+ chosen as in (3.3). Let $v \in W$. Then using (3.2) and the definition of $\varepsilon(\lambda : v\Phi^+)$,

$$\begin{aligned} c(w : \tau : v\Phi^+) &= c(v^{-1}w : \tau : \Phi^+) \\ &= 2^{-n} \sum_{\lambda\in\Lambda/\Lambda_0} \varepsilon(\lambda : \Phi^+)\kappa_\lambda(v^{-1}w)\bar{c}(v^{-1}w\tau : \Phi(\lambda) \cap \Phi^+) \\ &= 2^{-n} \sum_{\lambda\in\Lambda/\Lambda_0} \varepsilon(v^{-1}\lambda : \Phi^+)\kappa_\lambda(v)\kappa_\lambda(w)\bar{c}(v^{-1}w\tau : \Phi(v^{-1}\lambda) \cap \Phi^+) \\ &= 2^{-n} \sum_{\lambda\in\Lambda/\Lambda_0} \varepsilon(\lambda : v\Phi^+)\kappa_\lambda(w)\bar{c}(w\tau : \Phi(\lambda) \cap v\Phi^+). \end{aligned}$$

4. Proof of Theorem 1. Let $H \in \mathrm{Car}(G)$. We will use the notation established in §2. Let $\mathrm{Car}'(M) = \mathrm{Car}(M) \setminus \{H_K\}$. Fix $J \in \mathrm{Car}(M)$ with Lie algebra $\mathfrak{j} = \mathfrak{j}_k + \mathfrak{j}_p$. Let W be the Weyl group of $\Phi = \Phi(\mathfrak{m}_C, \mathfrak{j}_C)$, W_R and W_I the subgroups corresponding to Φ_R and Φ_I. Let $W_\sigma = \{w \in W \mid w\mathfrak{j} = \mathfrak{j}\}$. Then W_R, W_I, and $W(M, J)$ are subgroups of W_σ, and as in **[5d]**, we define

(4.1) $\quad w(\mathfrak{m}, \mathfrak{j}) = [W_R][W(M_J, J_K)]/[W(M, J)] = [W_R][W_I]/[W_\sigma].$

Let $\Gamma(\mathfrak{j}_p)$ be a full set of representatives for W_R orbits in $Z(\mathfrak{j}_p)$. The representatives can be chosen so that $\Gamma(\mathfrak{j}_p) \subseteq H_K^0$. For $\gamma \in \Gamma(\mathfrak{j}_p)$, let

$$\Phi_R(\gamma) = \{\alpha \in \Phi_R \mid \xi_\alpha(\gamma) = 1\}, \quad \Phi_R^+(\gamma) = \Phi_R(\gamma) \cap \nu\Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}).$$

Let $W_R(\gamma) = W(\Phi_R(\gamma))$ and let $W_K(\gamma)$ be the subgroup of $W_R(\gamma)$ generated by reflections in the roots α of $\Phi_R(\gamma)$ for which $\nu^{-1}\alpha$ is a compact root of $(\mathfrak{m}, \mathfrak{h}_k)$. Let $\mathfrak{j}_p^+(\gamma) = \{H \in \mathfrak{j}_p \mid \alpha(H) > 0 \text{ for all } \alpha \in \Phi_R^+(\gamma)\}$. Then for $\gamma \in \Gamma(\mathfrak{j}_p)$, $h \in \exp(i\mathfrak{j}_p)$, and $\mu \in \mathfrak{j}_p^*$, we define

$$(4.2) \qquad I_J^H(\gamma : h : \mu) = [W_K(\gamma)]^{-1} \sum_{v \in W_R(\gamma)} \det v \int_{\mathfrak{j}_p^+(\gamma)} \sum_{\tau \in L(\Phi_R)} \overline{\xi_\tau(\gamma v h)}$$
$$\times c(v : v^{-1}\tau : \Phi_R^+(\gamma)) \exp(\tau - i\mu)(H)\, dH.$$

Let $\mathcal{S}(H, J)$ be the set of all sequences $J_0, J_1, \ldots, J_l$ where $J_0 = H$, $\tilde{J}_l = \tilde{J} = JH_p$, and for $1 \leqslant i \leqslant l$, $J_i \in \mathrm{Car}'(M_{\tilde{J}_{i-1}})$. For $0 \leqslant i \leqslant l$, let $\mathfrak{a}_i = (\mathfrak{j}_i)_p$. Then $\mathfrak{j}_p = \oplus\sum_{i=1}^l \mathfrak{a}_i$. For $h_k \in H_K$, write $\nu(h_k) = h_0 h_1 \cdots h_l$ where $h_0 \in J_K$ and for $1 \leqslant i \leqslant l$, $h_i \in \exp(i\mathfrak{a}_i)$. For $\mu \in j_p^*$, write μ_i for the restriction of μ to $\mathfrak{a}_i$.

THEOREM 4.3. *There is a dense open set $H^* \subseteq H'$ so that for $h = h_k h_p \in H^*$, $f \in C_c^\infty(G)$,*

$$F_f^H(h) = \frac{(-1)^{r_l(H)}[W(M, H_K) : W(M^0, H_K^0)]}{\mathrm{vol}(H_K)(2\pi)^{\dim \mathfrak{h}_p}} \sum_{w \in W(M^0, H_K^0)} \det w$$
$$\times \sum_{J \in \mathrm{Car}(M)} (4\pi)^{-\dim \mathfrak{j}_p}[W(M_J, J_K)]^{-1} \sum_{b^* \in \hat{J}_K} \int_{\mathfrak{h}_p^*} h_p^{-i\mu'} \int_{\mathfrak{j}_p^*} \theta(\tilde{J}, b^*, \mu' \otimes \mu)(f)$$
$$\times \sum_{S \in \mathcal{S}(H, J)} (-1)^l I(S : b^* : \mu : w h_k)\, d\mu\, d\mu'.$$

For $S \in \mathcal{S}(H, J)$, $b^* \in \hat{J}_K$, $\mu \in \mathfrak{j}_p^*$, and $\nu h_k = h_0 h_1 \cdots h_l \in H_K$,

$$I(S : b^* : \mu : h_k) = \sum_{\gamma_1 \in \Gamma(\mathfrak{a}_1)} \cdots \sum_{\gamma_l \in \Gamma(\mathfrak{a}_l)} \overline{b^*(\gamma_1 \cdots \gamma_l h_0)} \prod_{i=1}^{l} w_i I_{J_i}^{\tilde{J}_{i-1}}(\gamma_i : h_i : \mu_i).$$

Here $w_i = w(\mathfrak{m}_{i-1}, \mathfrak{j}_i)/[Z(\mathfrak{a}_i) \cap Z(\tilde{\mathfrak{a}}_{i-1})]$ where $\tilde{\mathfrak{a}}_i = \oplus\sum_{j=0}^i \mathfrak{a}_j$, $0 \leqslant i \leqslant l$.

PROOF. This result is proved in the same way as Theorem 3.7 of **[5d]**, using Theorem 3.4 of **[5d]**. Note that for $J \in \mathrm{Car}'(M)$, $W(M_J^0, J_K^0) \subseteq \nu(W(M^0, H_K^0))$, so that all Weyl group sums can be combined. The subset H^* of H' for which the formula is valid will be smaller than that for Theorem 3.4 **[5d]**, as when each remainder term is analyzed, it is necessary to eliminate certain elements.

The remainder of this section will be devoted to evaluating the integrals I_J^H, $H \in \mathrm{Car}(G)$, $J \in \mathrm{Car}'(M_H)$, which occur in the inversion formula. Sally and Warner derived a simple formula for $I_J^H(\gamma : h : \mu)$ in **[10]** when Φ is of type A_1. For rank $\Phi \geqslant 2$. Theorem 2 will be used to express I_J^H in terms of factors of the type which occur in the Fourier inversion of stable orbital integrals, which are then simplified as in **[5d, e]** into products of rank one and two type factors.

Fix $\gamma \in \Gamma(\mathfrak{j}_p)$. Write $\Phi = \Phi_R(\gamma)$, $\Phi_{CPT} = \{\alpha \in \Phi \mid \nu^{-1}\alpha$ is a compact root of $(\mathfrak{m}, \mathfrak{h}_k)\}$, $\mathfrak{a}^+ = \mathfrak{j}_p^+(\gamma)$, $W = W_R(\gamma) = W(\Phi)$, and $W_K = W(\Phi_{CPT})$. We will use the notation of §§2 and 3. Let Λ_1 be a subset of Λ so that for each $\lambda \in \Lambda$ there is a unique $\lambda_1 \in \Lambda_1$ with $\lambda \in w\lambda_1 + \Lambda_0$ for some $w \in W$. Then every coset $\lambda + \Lambda_0$ can be represented by one of the form $\sigma\lambda + \Lambda_0$ where $\lambda \in \Lambda_1$ and $\sigma \in W(\lambda : \Phi^+) = \{\sigma \in W \mid \sigma\Phi^+(\lambda) \subseteq \Phi^+\}$. This representation is unique only up to an element of $W_1(\Phi : \Phi^+(\lambda)) = \{\sigma \in W \mid \sigma\Phi^+(\lambda) = \Phi^+(\lambda)\}$. Fix $\lambda \in \Lambda_1$ and $\varphi = \varphi(\lambda) \in \mathfrak{T}(\Phi(\lambda))$. Write $\varphi = \varphi_1 \cup \cdots \cup \varphi_s$ where the φ_i are simple. Let $\mathfrak{a}_i = \sum_{\alpha \in \varphi_i} \mathbf{R} H_\alpha$, $1 \leqslant i \leqslant s$. Then $\mathfrak{j}_p = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_s$. Write $T_0 = \exp(i\mathfrak{j}_p)$ and $T_j = \exp(i\mathfrak{a}_j)$, $1 \leqslant j \leqslant s$. Then $T_0 = T_1 \cdots T_s$, but the product need not be direct. Thus for $\tau \in L(\varphi)$, ξ_τ need not be well defined on T_0, although it is on $\tilde{T}_0 = T_1 \times \cdots \times T_s$, the abstract direct product, by $\xi_\tau(\eta_1, \ldots, \eta_s) = \prod_{j=1}^s \xi_{\tau_j}(\eta_j)$ where τ_j is the restriction of τ to $\mathfrak{a}_j$. Let $E = \{(\eta_1, \ldots, \eta_s) \in \tilde{T}_0 \mid \prod_{j=1}^s \eta_i = 1\}$. Write $L_R = L(\Phi_R(\mathfrak{m}, \mathfrak{j}))$. Then $[E] = [L(\varphi) : L_R]$, and for $\tau \in L(\varphi)$,

$$\sum_{(\eta_1, \ldots, \eta_s) \in E} \xi_\tau(\eta_1, \ldots, \eta_s) = \begin{cases} [E] & \text{if } \tau \in L_R, \\ 0 & \text{if } \tau \notin L_R. \end{cases} \tag{4.4}$$

For $\mu \in \mathfrak{j}_p^*$, let μ_j denote the restriction of μ to $\mathfrak{a}_j$, $1 \leqslant j \leqslant s$. For $h \in T_0$, write $h = \prod_{j=1}^s h_j$ where $h_j \in T_j$. This decomposition is unique only up to componentwise multiplication by an element of E. Define

$$P(\varphi^+, \mu, h) = \sum_{(\eta_1, \ldots, \eta_s) \in E} \prod_{i=1}^{s} S(\varphi_i^+, \mu_i, \eta_i h_i) \tag{4.5}$$

where the $S(\varphi_i^+, \mu_i, h_i)$ are defined as in Theorem 1. Define

$$c(\lambda) = \varepsilon(\varphi : \Phi^+(\lambda))\varepsilon(\lambda : \Phi^+)2^n[E][W_1(\Phi : \Phi^+(\lambda))][W_1(\Phi(\lambda) : \varphi^+)]$$

where $n = \operatorname{rank}\Phi$, and for any root system Φ and subset ψ, $W_1(\Phi : \psi) = \{w \in W(\Phi) \mid w\psi = \psi\}$.

To shorten formulas, we use the following notation. For $b^* \in \hat{J}_K$, $\mu \in \mathfrak{j}_p^*$, $h_k \in H'_K$, we say $g(b^*, \mu, h_k) \equiv g'(b^*, \mu, h_k)$ if for $h_p \in H'_p$,

$$\sum_{b^* \in \hat{J}_K} \int_{\mathfrak{h}_p^*} h_p^{-i\mu'} \int_{\mathfrak{j}_p^*} \theta(\tilde{J}, b^*, \mu \otimes \mu')(f) g(b^*, \mu, h_k)\, d\mu\, d\mu'$$

$$= \sum_{b^* \in \hat{J}_K} \int_{\mathfrak{h}_p^*} h_p^{-i\mu'} \int_{\mathfrak{j}_p^*} \theta(\tilde{J}, b^*, \mu \otimes \mu')(f) g'(b^*, \mu, h_k)\, d\mu\, d\mu',$$

where g and g' are functions of (b^*, μ, h_k) for which the sums and integrals above converge. Because of the transformation rule for $\theta(\tilde{J}, b^*, \mu \otimes \mu')(f)$, for $w \in W_R(\mathfrak{m}, \mathfrak{j})$,

$$g(b^*, \mu, h_k) \equiv g(\eta(w) \otimes wb^*, w\mu, h_k) \tag{4.6}$$

where $\eta(w) \in \hat{J}_K$ is defined as in **[5d]**. Recall that $(\eta(w) \otimes wb^*)(j_k) = b^*(j_k)$ if $wj_k = j_k$, $j_k \in J_K$.

LEMMA 4.7 *For $h_K \in H_K$, decompose $vh_K = h_0 h_1$ where $h_0 \in J_K$ and $h_1 \in \exp(i\mathfrak{j}_p)$. Then*

$$\overline{b^*(\gamma h_0)}\, I_J^H(\gamma : h_1 : \mu) \equiv (\pi i)^n [W_K]^{-1} \overline{b^*(\gamma h_0)} \times \sum_{\lambda \in \Lambda_1} c(\lambda)^{-1} \sum_{v \in W} \det v\, \kappa_\lambda(v) P(\varphi^+(\lambda), \mu, \gamma v h_1).$$

PROOF. Using (4.2) and Theorem 2,

$$\overline{b^*(\gamma h_0)}\, I_J^H(\gamma : h_1 : \mu) \equiv [W_K]^{-1} 2^{-n} \overline{b^*(\gamma h_0)} \sum_{\lambda \in \Lambda_1} I(\lambda, \mu, \gamma h_1)$$

where for $\lambda \in \Lambda_1$ and $j \in \exp(i\mathfrak{j}_p)$,

$$I(\lambda, \mu, j) = [W_1(\Phi : \Phi^+(\lambda))]^{-1} \sum_{\sigma \in W(\lambda : \Phi^+)} \sum_{v \in W} \det v$$
$$\times \int_{\mathfrak{a}^+} \sum_{\tau \in L_R} \overline{\xi_\tau(vj)}\, \varepsilon(\sigma\lambda : \Phi^+) \kappa_{\sigma\lambda}(v) \bar{c}(\tau : \sigma\Phi^+(\lambda)) \exp(\tau - i\mu)(H)\, dH.$$

Fix $\lambda \in \Lambda_1$. Write $W_0 = W(\lambda : \Phi^+)$, $W_1 = W_1(\Phi : \Phi^+(\lambda))$, and $\varepsilon(\sigma\lambda : \Phi^+) = \varepsilon(\sigma\lambda)$. Then using (3.4),

$$I(\lambda, \mu, j) = [W_1]^{-1} \sum_{\sigma \in W_0} \sum_{v \in W} \det(\sigma v)$$
$$\times \int_{\mathfrak{a}^+} \sum_{\tau \in L_R} \overline{\xi_{\sigma\tau}(\sigma v j)}\, \varepsilon(\sigma\lambda) \kappa_{\sigma_\lambda}(\sigma v) \bar{c}(\sigma\tau : \sigma\Phi^+(\lambda)) \exp(\sigma\tau - i\mu)(H)\, dH$$
$$= \varepsilon(\lambda)[W_1]^{-1} \sum_{v \in W} \det v\, \kappa_\lambda(v) \sum_{\sigma \in W_0}$$
$$\times \int_{\sigma^{-1}\mathfrak{a}^+} \sum_{\tau \in L_R} \overline{\xi_\tau(vj)}\, \bar{c}(\tau : \Phi^+(\lambda)) \exp(\tau - i\sigma^{-1}\mu)(H)\, dH.$$

Let $\varphi \in \mathfrak{T}(\Phi(\lambda))$. Define $W_0(\varphi, \lambda) = \{s \in W(\Phi(\lambda)) \mid s\varphi^+ \subseteq \Phi^+(\lambda)\}$. Then as in **[5c, e]** $\mathfrak{T}(\Phi(\lambda)) = \{\sigma\varphi \mid \sigma \in W_0(\varphi, \lambda)\}$, $\varepsilon(\sigma\varphi : \Phi^+(\lambda)) = \det \sigma\, \varepsilon(\varphi : \Phi^+(\lambda))$, and (2.1) can be rewritten as

$$\bar{c}(\tau : \Phi^+(\lambda)) = [W_1(\Phi(\lambda) : \varphi^+)]^{-1} \varepsilon(\varphi : \Phi^+(\lambda)) \sum_{s \in W_0(\varphi, \lambda)} \det s\, \bar{c}(s^{-1}\tau : \varphi^+).$$

Write $W_1(\Phi(\lambda) : \varphi^+) = W_1'$, $W_0(\varphi, \lambda) = W_0'$, and $\varepsilon(\varphi) = \varepsilon(\varphi : \Phi^+(\lambda))$. Thus, using (3.1),

$$I(\lambda, \mu, j) = \varepsilon(\lambda)[W_1]^{-1}[W_1']^{-1}\varepsilon(\varphi) \sum_{s \in W_0'} \det s \sum_{v \in W} \det sv\, \kappa_\lambda(sv)$$
$$\times \sum_{\sigma \in W_0} \int_{s^{-1}\sigma^{-1}\mathfrak{a}^+} \sum_{\tau \in L_R} \overline{\xi_{s\tau}(svj)}\, \bar{c}(\tau : \varphi^+) \exp(\tau - is^{-1}\sigma^{-1}\mu)(H)\, dH$$
$$= \varepsilon(\lambda)[W_1]^{-1}[W_1']^{-1}\varepsilon(\varphi) \sum_{v \in W} \det v\, \kappa_\lambda(v)$$
$$\times \sum_{\sigma \in W_0} \sum_{s \in W_0'} I(vj, s^{-1}\sigma^{-1}\mu, s^{-1}\sigma^{-1}\mathfrak{a}^+)$$

where

$$I(j, \mu, \mathfrak{a}^+) = \int_{\mathfrak{a}^+} \sum_{\tau \in L_R} \overline{\xi_\tau(j)}\, \bar{c}(\tau : \varphi^+) \exp(\tau - i\mu)(H)\, dH,$$

for $j \in \exp(i\mathfrak{a}_j)$, $\mu \in \mathfrak{j}_p^*$, and $\mathfrak{a}^+$ a component of $\mathfrak{a}' = \{H \in \mathfrak{a} \mid \alpha(H) \neq 0$ for all $\alpha \in \Phi\}$.

Because every element of $W = W_R(\gamma)$ centralizes γh_0, using (4.6),

$$\overline{b^*(\gamma h_0)}\, I(\gamma v h_1, s^{-1}\sigma^{-1}\mu, s^{-1}\sigma^{-1}\mathfrak{a}^+) \equiv \overline{b^*(\gamma h_0)}\, I(\gamma v h_1, \mu, s^{-1}\sigma^{-1}\mathfrak{a}^+)$$

for all $v \in W$, $s \in W_0'$, and $\sigma \in W_0$. Note that

$$\bigcup_{s \in W_0'} \bigcup_{\sigma \in W_0} s^{-1}\sigma^{-1}a^+ = \mathfrak{a}^+(\varphi) = \{H \in \mathfrak{j}_p \mid \alpha(H) > 0 \text{ for all } \alpha \in \varphi^+\}.$$

Thus

$$\overline{b^*(\gamma h_0)}\, I(\lambda, \mu, \gamma h_1) \equiv \varepsilon(\lambda)[W_1]^{-1}[W_1']^{-1}\varepsilon(\varphi)\, \overline{b^*(\gamma h_0)}$$

$$\times \sum_{v \in W} \det v\, \kappa_\lambda(v) \int_{\mathfrak{a}^+(\varphi)} \sum_{\tau \in L_R} \overline{\xi_\tau(\gamma v h_1)}\, \bar{c}(\tau : \varphi^+) \exp(\tau - i\mu)(H)\, dH.$$

Using the notation of (4.5), write $\mathfrak{a}_i^+ = \{H \in \mathfrak{a}_i \mid \alpha(H) > 0$ for all $\alpha \in \varphi_i^+\}$ and $\gamma v h_1 = \prod_{i=1}^s h_i(v)$ where $h_i(v) \in T_i$, $1 \leqslant i \leqslant s$. For any $\mu \in \mathfrak{a}_i^*$, $h \in T_i$, write

$$\bar{I}(h : \mu : \varphi_i^+) = \int_{\mathfrak{a}_i^+} \sum_{\tau \in L(\varphi_i)} \overline{\xi_\tau(h)}\, \bar{c}(\tau : \varphi_i^+) \exp(\tau - i\mu)(H)\, dH.$$

Then using (4.4),

$$\overline{b^*(\gamma h_0)}\, I(\lambda, \mu, \gamma h_1) \equiv 2^n c(\lambda)^{-1}\, \overline{b^*(\gamma h_0)} \sum_{v \in W} \det v\, \kappa_\lambda(v)$$

$$\times \sum_{(\eta_1, \dots, \eta_s) \in E} \prod_{i=1}^s \bar{I}(\eta_i h_i(v) : \mu_i : \varphi_i^+).$$

The factors $\bar{I}(h : \mu : \varphi_i^+)$ defined above are exactly the terms associated with the simple root system φ_i of types A_1 or B_2 which arise in the Fourier inversion formulas of stable orbital integrals. It follows from results of Sally and Warner [**10**] for the A_1 case and Chao [**2**] for the B_2 case, which are summarized in Lemma 4.6 [**5d**], that

$$\overline{b^*(\gamma h_0)} \sum_{u \in W(\varphi)} \det u \sum_{(\eta_1, \dots, \eta_s) \in E} \prod_{i=1}^s \bar{I}(\eta_i u h_i(v) : \mu_i : \varphi_i^+)$$

$$\equiv (\pi i)^n\, \overline{b^*(\gamma h_0)} \sum_{u \in W(\varphi)} \det u\, P(\varphi^+, \mu, u\gamma v h_1).$$

The lemma now follows from observing that $u h_i(v) = h_i(uv)$ for all $u \in W(\varphi)$, $v \in W$, and $1 \leqslant i \leqslant s$, and that $W(\varphi) \subseteq W(\Phi(\lambda))$ so that $\kappa_\lambda(uv) = \kappa_\lambda(v)$ for all $u \in W(\varphi)$, $v \in W$.

In (4.7) we have obtained an expression for I_f^H in terms of factors $S(\varphi^+, \mu, h)$, φ of types A_1 or B_2, which occur in the Fourier inversion formulas for stable orbital integrals. In Lemma 4.9 we will do more simplification to eliminate the sum over Λ_1 and the "characters" κ_λ.

Fix $\psi = \varphi_1 \cup \cdots \cup \varphi_s \in \mathfrak{T}(\Phi)$ for which every long root is of noncompact type. (We call α a long root of ψ if α belongs to any factor φ_i of type A_1 or if α is a long root of a factor φ_i of type B_2.) We use the notation of (4.5). Let $1 \leq i \leq s$. Let $\mu \in \mathfrak{a}_i^*$, $h \in T_i$. Define $T(\varphi_i^+, \mu, h) = S(\varphi_i^+, \mu, h)$ if φ_i is of type A_1. If φ_i is of type B_2, let $\varphi_{i,s}^+$ denote the short positive roots of φ_i, and define $S(\varphi_{i,s}^+, \mu, h)$ as in Theorem 1. Then define $T(\varphi_i^+, \mu, h) = S(\varphi_i^+, \mu, h) + \frac{1}{4}S(\varphi_{i,s}^+, \mu, h)$. Set

$$Q(\psi^+, \mu, h) = \sum_{(\eta_1, \ldots, \eta_s) \in E} \prod_{i=1}^{s} T(\varphi_i^+, \mu_i, \eta_i, h_i). \tag{4.8}$$

LEMMA 4.9. *Let $\psi \in \mathfrak{T}(\Phi)$ for which every long root of ψ is of noncompact type. Then*

$$\sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}\, I_J^H(\gamma : h_1(w) : \mu)$$
$$\equiv (\pi i)^n [W_1(\Phi : \psi^+)]^{-1} [L_\psi : L_R]^{-1} \varepsilon(\psi : \Phi^+)$$
$$\times \sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}\, Q(\psi^+, \mu, \gamma h_1(w)).$$

PROOF. Suppose first that Φ^+ has been chosen so that Φ^+ and $\psi^+ = \psi \cap \Phi^+$ both satisfy (3.3). Then Λ_1 can be chosen so that for all $\lambda \in \Lambda_1$:

(i) $\varepsilon(\lambda : \Phi^+) = 1$;

(ii) $\varphi(\lambda) = \psi \cap \Phi(\lambda) \in \mathfrak{T}(\Phi(\lambda))$ and $\varepsilon(\varphi(\lambda) : \Phi^+(\lambda)) = \varepsilon(\psi : \Phi^+)$;

(iii) $W_1(\Phi : \Phi^+(\lambda)) \subseteq W_1(\Phi : \varphi^+)$;

(iv) λ is a sum of long roots of $\Phi(\lambda)$.

Fix $\varphi \subseteq \psi$ such that $\Lambda_1(\varphi) = \{\lambda \in \Lambda_1 \mid \varphi \in \mathfrak{T}(\Phi(\lambda))\} \neq \varnothing$. Let $U(\varphi) = S(\varphi)W_1(\Phi : \varphi^+)$ where $S(\varphi)$ is the subgroup of $W(\varphi)$ generated by reflections in long roots of φ. It is clear from the explicit formulas of the $S(\varphi_i^+, \mu_i, h_i)$ in Theorem 1 that for $s \in S(\varphi)$, $P(\varphi^+, \mu, sh) = \det s\, P(\varphi^+, \mu, h)$ for all $\mu \in \mathfrak{j}_p^*$ and $h \in \exp(i\mathfrak{j}_p)$. For $u \in W_1(\Phi : \varphi^+)$, $P(\varphi^+, \mu, uh) = P(\phi^+, u^{-1}\mu, h)$. Finally, $\nu^{-1}W_K \subseteq W(M^0, H_K^0)$. Thus

$$\sum_{\lambda \in \Lambda_1(\varphi)} c(\lambda)^{-1} [W_K]^{-1} \sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}$$
$$\times \sum_{v \in W} \det v\, \kappa_\lambda(v) P(\varphi^+, \mu, \gamma v h_1(w))$$
$$\equiv [U(\varphi) \cap W_K]^{-1} \sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}$$
$$\times \sum_{v \in U(\varphi) \backslash W / W_K} \det v\, P(\varphi^+, \mu, \gamma v h_1(w))$$
$$\times [S(\varphi)] \sum_{\lambda \in \Lambda_1(\varphi)} c(\lambda)^{-1} \sum_{u \in W_1(\Phi : \varphi^+)} \det u\, \kappa_\lambda(uv).$$

Using (3.2), $\kappa_\lambda(uv) = \kappa_\lambda(u)\kappa_{u^{-1}\lambda}(v)$. In the case that $\varphi = \psi$, $\psi \in \mathfrak{T}(\Phi)$ implies that $\det u = 1$ for all $u \in W_1(\Phi : \psi^+)$. Since all long roots of ψ are noncompact, and because of assumption (iv) on λ, $u^{-1}\lambda \equiv \lambda$ for all $u \in W_1(\Phi : \psi^+)$ so that $\kappa_\lambda(u) = 1$.

In the few cases that $\varphi \subsetneq \psi$ it is easy to check that $\det u = \kappa_\lambda(u)$ for all $u \in W_1(\Phi : \varphi^+)$. Thus $\det u\, \kappa_\lambda(uv) = \kappa_{u^{-1}\lambda}(v)$ for all $u \in W_1(\Phi : \varphi^+)$.

Clearly $\varphi \in \mathfrak{T}(\Phi(u\lambda))$ for all $u \in W_1(\Phi : \varphi^+)$, $\lambda \in \Lambda_1(\varphi)$. Conversely, if $\varphi \in \mathfrak{T}(\Phi(\lambda))$ for any $\lambda \in \Lambda$, then $\lambda + \Lambda_0 = w\lambda_1 + \Lambda_0$ for a unique $\lambda_1 \in \Lambda_1(\varphi)$. Since $\varphi, w^{-1}\varphi \in \mathfrak{T}(\Phi(\lambda_1))$, there is $\sigma \in W(\Phi(\lambda_1))$ with $w^{-1}\varphi^+ = \sigma\varphi^+$ so that $u = w\sigma \in W_1(\Phi : \varphi^+)$ and $\Phi(u\lambda_1) = \Phi(\lambda)$. Further, for $u_1, u_2 \in W_1(\Phi : \varphi^+)$, $\Phi(u_1\lambda) = \Phi(u_2\lambda)$ if and only if $u_1^{-1}u_2 \in \{u \in W(\Phi) \mid u\Phi(\lambda) = \Phi(\lambda),\ u\varphi^+ = \varphi^+\} = W_1(\Phi : \Phi^+(\lambda))W_1(\Phi(\lambda) : \varphi^+)$.

Thus

$$[S(\varphi)] \sum_{\lambda \in \Lambda_1(\varphi)} c(\lambda)^{-1} \sum_{u \in W_1(\Phi : \varphi^+)} \det u\, \kappa_\lambda(uv)$$
$$= \varepsilon(\psi : \Phi^+)\left[L_\varphi : L_R\right]^{-1} \sum_{\lambda + \Lambda_0 \ni \varphi \in \mathfrak{T}(\Phi(\lambda))} \kappa_\lambda(v).$$

Case I. Suppose that $\varphi = \psi$. Then $\{\lambda \in \Lambda \mid \psi \in \mathfrak{T}(\Phi(\lambda))\} = \{\lambda \in \Lambda \mid \psi \subseteq \Phi(\lambda)\}$ is a sublattice of Λ containing Λ_0 which we will denote by Λ_ψ. Further, for fixed $v \in W$, $\lambda \to \kappa_\lambda(v)$ is a character Λ_ψ/Λ_0. Thus

$$\sum_{\lambda \in \Lambda_\psi/\Lambda_0} \kappa_\lambda(v) = \begin{cases} \left[\Lambda_\psi : \Lambda_0\right] & \text{if } \kappa_\lambda(v) = 1 \text{ for all } \lambda \in \Lambda_\psi, \\ 0 & \text{otherwise.} \end{cases}$$

Note $\Lambda_\psi = \{\lambda \in \Lambda \mid \langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle \in Z$ for all $\alpha \in \psi\}$ and $\Lambda_0 = \{\lambda \in \Lambda \mid \langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle \in Z$ for all $\alpha \in \Phi\}$. Thus $[\Lambda_\psi : \Lambda_0] = [L_\psi : L]$ and $[E]^{-1}[\Lambda_\psi : \Lambda_0] = [L : L_R]^{-1}$. It can be checked that $\Lambda_\psi/\Lambda_0 = \{\sum_{i=1}^k \alpha_i + \Lambda_0 \mid \alpha_i$ is a long root of ψ, $1 \leqslant i \leqslant k\}$. The set ψ_l^+ of positive long roots of ψ is a set of strongly orthogonal roots of noncompact type and $\lambda \to \kappa_\lambda(v)$ is the trivial character of Λ_ψ/Λ_0 if and only if $v^{-1}\alpha \equiv \alpha$ for all $\alpha \in \psi_l^+$. This is the case if $v \in U(\psi)W_K$. Conversely, if $v^{-1}\alpha \equiv \alpha$ for all $\alpha \in \psi_l^+$, then ψ_l^+ and $v^{-1}\psi_l \cap \Phi^+$ are two sets of strongly orthogonal roots of noncompact type in Φ, and hence conjugate by an element of W_K. Thus $wv^{-1}\psi_l = \psi_l$ for some $w \in W_K$ and so $v \in UW_K$ where $U = \{u \in W \mid u\psi_l = \psi_l\}$. In all cases, $UW_K = U(\psi)W_K$. Finally, note that $W_1(\Phi : \psi^+) \subseteq W_K$ so that $[U(\psi) \cap W_K] = [S(\psi) \cap W_K][W_1(\Phi : \psi^+)]$ and $[S(\psi) \cap W_K] = [L_\psi : L]$. Thus

$$\sum_{\lambda \in \Lambda_1(\psi)} c(\lambda)^{-1}[W_K]^{-1} \sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}$$
$$\times \sum_{v \in W} \det v\, \kappa_\lambda(v) P(\psi^+, \mu, \gamma v h_1(w))$$
$$\equiv \varepsilon(\psi : \Phi^+)\left[W_1(\Phi : \psi^+)\right]^{-1}\left[L_\psi : L_R\right]^{-1}$$
$$\times \sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}\, P(\psi^+, \mu, \gamma h_1(w)).$$

If ψ is of type A_1^n, then $\Lambda_1(\psi) = \Lambda_1$ and $P(\psi^+, \mu, \gamma h_1(w)) = Q(\psi^+, \mu, \gamma h_1(w))$ so the proof is complete.

Case II. Suppose $\varphi \subsetneq \psi$ and Φ is simple of type B_{2s}, $s \geqslant 1$, or F_{2s}, $s = 2$. Then ψ is of type B_2^s and φ is of type $B_2^{s-1} \times B_1^2$. Then $\{\lambda + \Lambda_0 \mid \varphi \subseteq \Phi(\lambda)\}$ is the disjoint union of $\{\lambda + \Lambda_0 \mid \varphi \in \mathfrak{T}(\Phi(\lambda))\}$ and $\{\lambda + \Lambda_0 \mid \psi \subseteq \Phi(\lambda)\}$. Thus

$$\sum_{\lambda+\Lambda_0 \not\ni \varphi \in \mathfrak{T}(\Phi(\lambda))} \kappa_\lambda(v) = \sum_{\lambda \in \Lambda_\varphi/\Lambda_0} \kappa_\lambda(v) - \sum_{\lambda \in \Lambda_\psi/\Lambda_0} \kappa_\lambda(v),$$

where, as before, $\Lambda_\varphi = \{\lambda \in \Lambda \mid \varphi \subseteq \Phi(\lambda)\}$. Now $U(\varphi)W_K = U(\psi)W_K$ so that if $v \notin U(\psi)W_K$, $\lambda \to \kappa_\lambda(v)$ is nontrivial on Λ_ψ and hence also on $\Lambda_\varphi \supseteq \Lambda_\psi$ and so $\sum_{\lambda+\Lambda_0 \not\ni \varphi \in \mathfrak{T}(\Phi(\lambda))} \kappa_\lambda(v) = 0$. The double coset $U(\varphi)W_K$ of $U(\varphi)\backslash W/W_K$ can be represented by $v = 1$ so that

$$\sum_{\lambda+\Lambda_0 \not\ni \varphi \in \mathfrak{T}(\Phi(\lambda))} \kappa_\lambda(1) = [L_\varphi : L] - [L_\psi : L] = [L_\psi : L].$$

Further, $[L_\psi : L]/[U(\varphi) \cap W_K][L_\varphi : L_R] = s/[W_1(\Phi : \psi^+)]4[L_\psi : L_R]$ so that

$$\begin{aligned}\sum_{\lambda \in \Lambda_1(\varphi)} c(\lambda)^{-1}[W_K]^{-1} \sum_{w \in W(M^0, H_K^0)} & \det w \, \overline{b^*(\gamma h_0(w))} \\ & \times \sum_{v \in W} \det v \, \kappa_\lambda(v) P(\varphi^+, \mu, \gamma v h_1(w)) \\ & \equiv s/4[W_1(\Phi : \psi^+)]^{-1}[L_\psi : L_R]^{-1} \varepsilon(\psi : \Phi^+) \\ & \times \sum_{w \in W(M^0, H_K^0)} \det w \, \overline{b^*(\gamma h_0(w))} \, P(\varphi^+, \mu, \gamma h_1(w)).\end{aligned}$$

In this case $\Lambda_1 = \Lambda_1(\psi) \cup \Lambda_1(\varphi)$. If E is defined as in (4.5) with respect to $\psi = \varphi_1 \cup \cdots \cup \varphi_s$, then there is $1 \leqslant j \leqslant s$ so that

$$P(\varphi^+, \mu, h) = \sum_{(\eta_1, \ldots, \eta_s) \in E} \prod_{\substack{i=1 \\ i \neq j}}^{s} S(\varphi_i^+, \mu_i, \eta_i h_i) \times S(\varphi_{j,s}^+, \mu_j, \eta_j h_j)$$

for all $\mu \in \mathfrak{j}_p^*$, $h \in \exp(i\mathfrak{j}_p)$. Further, the s possible φ's which could have been chosen, corresponding to $1 \leqslant j \leqslant s$, are all conjugate by elements of determinant one in $W_K \subseteq \nu W(M^0, H_K^0)$. Thus

$$\begin{aligned}\sum_{w \in W(M^0, H_K^0)} & \det w \, \overline{b^*(\gamma h_0(w))} \, I_J^H(\gamma : h_1(w) : \mu) \\ & \equiv (\pi i)^n [W_1(\Phi : \psi^+)]^{-1}[L_\psi : L_R]^{-1} \varepsilon(\psi : \Phi^+) \sum_{w \in W(M^0, H_K^0)} \det w \, \overline{b^*(\gamma h_0(w))} \\ & \times \sum_{(\eta_1, \ldots, \eta_s) \in E} \sum_{j=1}^{s} \prod_{\substack{i=1 \\ i \neq j}}^{s} S(\varphi_i^+, \mu_i, \eta_i h_i) \times T(\varphi_j^+, \mu_j, \eta_j h_j).\end{aligned}$$

The lemma follows because any term in $Q(\psi^+, \mu, h)$ which contains a factor of the form $S(\varphi_{i,s}^+, \mu_i, h_i) S(\varphi_{j,s}^+, \mu_j, h_j)$ for some $1 \leqslant i \neq j \leqslant s$ is stable under $h \to vh$, where v is an element of W_K of determinant -1 and thus will cancel out in the sum over $W(M^0, H_K^0)$.

Case III. Suppose Φ is simple of type B_{2s+1}. Then $\psi = \varphi_1 \cup \cdots \cup \varphi_{s+1}$ is of type $B_2^s \times B_1$ and $\Lambda_1(\psi) = \Lambda_1$. The lemma follows from Case I together with the observation that any term in $Q(\psi^+, \mu, h)$ which contains a factor of the form

$S(\varphi_{i,s}^+, \mu_i, h_i)S(\varphi_j^+, \mu_j, h_j)$ for i, j with φ_i of type B_2 and φ_j of type B_1 is stable under $h \to vh$ for an element $v \in W_K$ with $\det v = -1$.

Case IV. Suppose $\varphi \subsetneq \psi$ and Φ is simple of type C_n. Then ψ is of type C_2^k or $C_2^k \times C_1$ depending on whether $n = 2k$ or $2k+1$, and φ is of type $A_1^{2p} \times C_2^{k-p}$ or $A_1^{2p} \times C_2^{k-p} \times C_1$ for some $0 < p \leqslant k$ where the A_1 roots are short. In any case, $U(\varphi)W_K = U(\varphi)W_K = W$, so the only double coset of $U(\varphi)\backslash W/W_K$ can be represented by $v = 1$. Also in this case, for a given φ, there is only one coset of Λ/Λ_0 with $\varphi \in \mathfrak{T}(\Phi(\lambda))$. Finally, $[U(\psi) \cap W_K] = 2^{-p}\binom{k}{p}[U(\varphi) \cap W_K]$ and $[L_\varphi : L_\psi] = 2^p$ so that

$$\sum_{\lambda \in \Lambda_1(\varphi)} c(\lambda)^{-1}[W_K]^{-1} \sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}$$
$$\times \sum_{v \in W} \det v\, \kappa_\lambda(v) P(\varphi^+, \mu, \gamma v h_1(w))$$
$$\equiv \binom{k}{p} 4^{-p}[W_1(\Phi : \psi^+)]^{-1}[L_\psi : L_R]^{-1}\varepsilon(\psi : \Phi^+)$$
$$\times \sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}\, P(\varphi^+, \mu, \gamma h_1(w)).$$

Now, if E is defined as in (4.5) with respect to $\psi = \varphi_1 \cup \cdots \cup \varphi_s$, $s = k$ or $k+1$ depending on whether Φ is of type C_{2k} or C_{2k+1}, there is a subset P of $\{1, \ldots, s\}$ with p elements so that

$$P(\varphi^+, \mu, h) = \sum_{(\eta_1, \ldots, \eta_s) \in E} \prod_{i \notin P} S(\varphi_i^+, \mu_i, \eta_i h_i) \prod_{i \in P} S(\varphi_{i,s}^+, \mu_i, \eta_i h_i).$$

For fixed p, $0 \leqslant p \leqslant k$, there are $\binom{k}{p}$ such φ which could have been chosen all conjugate by elements of determinant one in W_K. Thus

$$\sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}\, I_J^H(\gamma : h_1(w) : \mu)$$
$$\equiv (\pi i)^n [W_1(\Phi : \psi^+)]^{-1}[L_\psi : L_R]^{-1}\varepsilon(\psi : \Phi^+)$$
$$\times \sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}\, Q(\psi^+, \mu, \gamma h_1, (w)).$$

This concludes the proof of the lemma for Φ^+ chosen as above.

Now suppose Φ^+ is replaced by $v\Phi^+$, $v \in W(\Phi)$. The the left-hand side of (4.9) changes by $\det v$. Write $v = u\sigma^{-1}$ where $u \in W(\psi)$ and $\sigma\psi^+ \subseteq \Phi^+$. Then

$$\varepsilon(\psi : v\Phi^+) = \varepsilon(v^{-1}\psi : \Phi^+) = \det \sigma\, \varepsilon(\psi : \Phi^+).$$

Further, $\psi \cap v\Phi^+ = u\psi^+$ and

$$\sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}\, Q(u\psi^+, \mu, \gamma h_1(w))$$
$$\equiv \det u \sum_{w \in W(M^0, H_K^0)} \det w\, \overline{b^*(\gamma h_0(w))}\, Q(\psi^+, \mu, \gamma h_1(w)).$$

Thus (4.9) is valid for any choice of positive root system for Φ.

We now have, using (4.9), a complete description of the terms $I_J^H(\gamma : j : \mu)$ which appear in Theorem 4.3. To complete the proof of Theorem 1, it is only necessary to use this information to compute the terms

$$\sum_{S \in \mathbb{S}(H, J)} (-1)^l I(S : b^* : \mu : h_k)$$

appearing in (4.3). This stage of the proof can be accomplished in the same way as the corresponding analysis in **[5d, e]**. It is slightly more complicated for three reasons.

First, comparing the formulas for I_J^H to those for $\bar{I}_J^H$ in **[5d, e]**, we see that the I_J^H may have extra terms. These will be carried along at each stage and cause no extra difficulties. Second, in computing the Fourier inversion formula for $F_f^H(h_k h_p)$, many sums of the form $\sum_{w \in W(M^0, H_K^0)} \det w g(b^*, \mu, wh_k)$ appear, where g is some function of $b^* \in \hat{J}_K$, $\mu \in \mathfrak{j}_p^*$, and $h_k \in H_K$ where $J \in \mathrm{Car}'(M)$. In **[5d]**, the corresponding terms would be averaged over the larger Weyl group $W_I = W_I(\mathfrak{g}, \mathfrak{h}) = W(\mathfrak{m}_{\mathbf{C}}, \mathfrak{h}_{k\mathbf{C}})$. Frequently $\sum_{w \in W_I} \det wg(b^*, \mu, wh_k) \equiv 0$, while $\sum_{w \in W(M^0, H_K^0)} \det wg(b^*, \mu, wh_k)$ makes a nonzero contribution to the Fourier inversion formula for F_f^H. Thus in passing from $I_J^H(\gamma : j : \mu)$ to $\sum_{\gamma \in \Gamma(\mathfrak{j}_p)} b^*(\gamma) I_J^H(\gamma : j : \mu)$ and on to

$$\sum_{S \in \mathbb{S}(H, J)} (-1)^l I(S : b^* : \mu : h_k),$$

terms which would make no contribution to the Fourier inversion formula for an averaged invariant integral, need to be retained in the formula for F_f^H. However, they can be handled by the same techniques used in **[5d]**.

Finally, in **[5d, e]** we restricted ourselves to the case that $H = T$ is a compact Cartan subgroup of G. In this paper, we drop that simplifying assumption. For the most part, this causes no problems. For example, for $J \in \mathrm{Car}'(M)$, $\mathbb{S}(H, J)$ can be described in the same way as $\mathbb{S}(J)$ in **[5d]**, where M^0 takes the role of G and H_K^0 that of T. For a fixed $S \in \mathbb{S}(H, J)$, the constants picked up in evaluating

$$\prod_{i=1}^{l} w_i \sum_{\gamma_i \in \Gamma(\mathfrak{a}_i)} b^*(\gamma_i) I_{J_i}^{\tilde{J}_{i-1}}(\gamma_i : h_i : \mu_i)$$

are the same as those occurring in **[5d, e]**, except that if $Z(\mathfrak{h}_p) \cap Z(\mathfrak{j}_p)$ is not trivial, the extra term $[Z(\mathfrak{h}_p) \cap Z(\mathfrak{j}_p)]^{-1}$ factors out. In this way, we obtain the following.

LEMMA 4.10. *Let $J \in \mathrm{Car}(M)$. Then*

$$\frac{(-1)^{r_I(H)}}{(4\pi)^{\dim \mathfrak{j}_p}[W(M_J, J_K)]} \sum_{w \in W(M^0, H_K^0)} \det w \sum_{S \in \mathbb{S}(H, J)} (-1)^l I(S : b^* : \mu : wh_k)$$

$$\equiv \frac{(-1)^{r_I(J)}}{[W(M, J)][Z(\mathfrak{h}_p) \cap Z(\mathfrak{j}_p)]} \left(\frac{i}{2}\right)^{\dim \mathfrak{j}_p} \times \sum_{w \in W(M^0, H_K^0)} \det w K(M, J, b^*, \mu, wh_k).$$

To complete the proof of Theorem 1 it is necessary to make two observations. First, representatives w of $W(M, H_K)/W(M^0, H_K^0)$ can be chosen so that for all $h_k \in H_K$, $K(M, J, b^*, \mu, wh_k) \equiv \det w K(M, J, b^*, \mu, h_k)$. Thus

$$[W(M, H_K) : W(M^0, H_K^0)] \sum_{w \in W(M^0, H_K^0)} \det w K(M, J, b^*, \mu, wh_k) \equiv \sum_{w \in W(M, H_K)} \det w K(M, J, b^*, \mu, wh_k).$$

Second, although Theorem 4.3 is valid only for h in the dense open subset H^* of H', both sides of the equation in Theorem 1 are continuous functions of $h \in H'$. Thus the formula is true for all $h \in H'$.

REFERENCES

1. D. Barbasch, *Fourier inversion for unipotent invariant integrals*, Trans. Amer. Math. Soc. **249** (1979), 51–83.
2. W. Chao, *Fourier inversion and the Plancherel formula for semisimple Lie groups of real rank two*, Ph. D. Thesis, Univ. of Chicago, Chicago, Ill., 1977.
3. Harish-Chandra, (a) *Discrete series for semisimple Lie groups*. I, Acta Math. **113** (1965), 241–318. (b) *Two theorems on semi-simple Lie groups*, Ann. of Math. (2) **83** (1966), 74–128. (c) *Supertempered distributions on real reductive groups*, 1979 (preprint).
4. H. Hecht, *The characters of Harish-Chandra representations*, Math. Ann. **219** (1976), 213–226.
5. R. Herb, (a) *Character formulas for discrete series on semisimple Lie groups*, Nagoya Math. J. **64** (1976), 47–61. (b) *Fourier inversion of invariant integrals on semisimple real Lie groups*, Trans. Amer. Math. Soc. **249** (1979), 281–302. (c) *Characters of averaged discrete series on semisimple real Lie groups*, Pacific J. Math. **80** (1979), 169–177. (d) *Fourier inversion and the Plancherel theorem for semisimple real Lie groups*, Amer. J. Math. **104** (1982), 9–58. (e) *Fourier inversion and the Plancherel theorem*, (Proc. Marseille Conf., 1980), Lecture Notes in Math., vol. 880, Springer-Verlag, Berlin and New York, 1982, pp. 197–210.
6. R. Herb and P. Sally, *Singular invariant eigendistributions as characters in the Fourier transform of invariant distributions*, J. Funct. Anal. **33** (1979), 195–210.
7. T. Hirai, *Invariant eigendistributions of Laplace operators on real simple Lie groups*. IV: *Characters of discrete series for* $\mathrm{Sp}(n, \mathbf{R})$, Japan J. Math. **3** (1977), 1–48.
8. S. Martens, *The characters of the holomorphic discrete series*, Proc. Nat. Acad. Sci. U.S.A. **72** (1975), 3275–3276.
9. H. Midorikawa, *On the explicit formulae of characters for discrete series representations*, Japan J. Math (N.S.) **3** (1977), 313–368.
10. P. Sally and G. Warner, *The Fourier transform on semisimple Lie groups of real rank one*, Acta Math. **131** (1973), 1–26.
11. W. Schmid, *On the characters of discrete series*, Invent. Math. **30** (1975), 47–144.
12. D. Shelstad, (a) *Orbital integrals and a family of groups attached to a real reductive group*, Ann. Sci. École Norm. Sup. **12** (1979), 1–31. (b) *Embeddings of L-groups* (preprint). (c) *L-indistinguishability for real groups* (preprint).
13. J. Vargas, *A character formula for the discrete series of a semisimple Lie group*, Bull. Amer. Math. Soc. **2** (N.S.) (1980), 465–467.
14. G. Warner, *Harmonic analysis on semisimple Lie groups*, Vols. I, II, Springer-Verlag, Berlin and New York, 1972.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742