

## TWISTING COCHAINS AND DUALITY BETWEEN MINIMAL ALGEBRAS AND MINIMAL LIE ALGEBRAS

BY

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**ABSTRACT.** An algebraic duality theory is developed between 1-connected minimal cochain algebras of finite type and connected minimal chain Lie algebras of finite type by means of twisting cochains. The duality theory gives a concrete relationship between Sullivan's minimal models, Chen's power series connections and the various Lie algebra models of a 1-connected topological space defined by Quillen, Allday, Baues-Lemaire and Neisendorfer. It can be used to compute the Lie algebra model of a space from the algebra model of the space and vice versa.

**1. Introduction.** Baues and Lemaire [2] have developed an algebraic duality theory between 1-connected minimal cochain algebras of finite type and connected minimal chain Lie algebras of finite type. The duality theory gives a one-to-one correspondence between isomorphism classes of 1-connected minimal cochain algebras of finite type and isomorphism classes of connected minimal chain Lie algebras of finite type. If the cochain algebra  $\mathfrak{M}$  corresponds to the chain Lie algebra  $L$  under the duality theory, then there are natural, nonsingular pairings

$$(1.1) \quad Q\mathfrak{M} \otimes sH_*(L) \rightarrow \mathbb{Q}$$

and

$$(1.2) \quad IH^*(\mathfrak{M}) \otimes sQL \rightarrow \mathbb{Q}$$

where  $Q\mathfrak{M}$  denotes the space of indecomposables of  $\mathfrak{M}$ ,  $QL$  denotes  $L/[L, L]$ ,  $s$  denotes the suspension operator and  $IH^*(\mathfrak{M}) = \bigoplus_{p>0} H^p(\mathfrak{M})$ . The duality theory reflects the adjointness of Quillen's  $\mathbb{L}$ - and  $\mathcal{C}$ -constructions.

For a 1-connected space  $K$  with finite Betti numbers, Sullivan's minimal model  $\mathfrak{M}_K$  of  $K$  is a 1-connected minimal cochain algebra of finite type which has the properties

$$Q\mathfrak{M}_K \approx \text{Hom}_{\mathbb{Z}}(\pi_*(K), \mathbb{Q}) \quad \text{and} \quad H^*(\mathfrak{M}_K) \approx H^*(K; \mathbb{Q})$$

(see [6, 15]). The primitive elements  $L_K$  of Chen's algebra model of  $K$  (see [5, 18]) is a connected minimal chain Lie algebra of finite type which has the properties

$$Q\mathfrak{M}_K \approx \text{Hom}_{\mathbb{Z}}(\pi_*(K), \mathbb{Q}) \quad \text{and} \quad H^*(\mathfrak{M}_K) \approx H^*(K; \mathbb{Q})$$

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(see [6, 15]). The Lie algebra model of  $K$  obtained by using K. T. Chen's method of power series connections (see [5, 18]) is a minimal chain Lie algebra which has the properties

$$sH_*(L_K) \approx \pi_*(K) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{and} \quad sQL_K \approx \tilde{H}_*(K; \mathbb{Q}).$$

(Similar Lie algebra models have been studied by Quillen [14], Allday [1], Baues and Lemaire [2] and Neisendorfer [13].) Under the duality theory,  $\mathfrak{M}_K$  and  $L_K$  correspond, and the pairings (1.1) and (1.2) correspond to the natural pairings

$$\text{Hom}_{\mathbb{Z}}(\pi_*(K), \mathbb{Q}) \otimes (\pi_*(K) \otimes_{\mathbb{Z}} \mathbb{Q}) \rightarrow \mathbb{Q}$$

and

$$\tilde{H}^*(K; \mathbb{Q}) \otimes \tilde{H}_*(K; \mathbb{Q}) \rightarrow \mathbb{Q}.$$

A direct consequence of the duality theory is that Chen's model of a 1-connected space  $K$  with finite Betti numbers is isomorphic to the minimal model of the  $\mathcal{L}$ -construction  $\mathcal{L}(\mathfrak{M}_K^*)$  on the dual of Sullivan's model  $\mathfrak{M}_K$  of  $K$ .

Given a 1-connected minimal cochain algebra  $\mathfrak{M}$  of finite type, the twisting cochain  $\omega$  and the corresponding Lie algebra  $L$  can be computed inductively from  $\mathfrak{M}$  and vice versa (Theorem (3.7)). Our approach eliminates Quillen's  $\mathcal{L}$ - and  $\mathcal{C}$ -constructions from the duality theory and, in the author's opinion, makes the computation of the correspondence and the pairings (1.1) and (1.2) simpler and more tractable than when using the Baues-Lemaire approach.

Independently, Tanré [19, 20] has obtained results that relate Chen's power series connections and Chen's Lie algebra model of a space to other algebraic models of the space. Specifically, he has shown that there is a natural one-to-one correspondence between power series connections on a cochain algebra  $\mathcal{Q}$  and Felix-Halperin-Stasheff (F.H.S.) models  $(\Delta Z, d) \rightarrow \mathcal{Q}$  of  $\mathcal{Q}$  (see [17]). In the case when  $\mathcal{Q}$  is the de Rham complex of a wedge of spheres, the F.H.S. model of  $\mathcal{Q}$  equals the Sullivan model of  $\mathcal{Q}$  and Tanré's result and ours coincide. It also follows from Tanré's result that Chen's algebra model of a space is isomorphic to the Adams-Hilton model of the space. Tanré's methods are similar to ours.

Let  $K$  be a 1-connected rational space with finite Betti numbers. In [18], it is proved that elementary extensions of Chen's Lie algebra model of  $K$  correspond to cofibrations  $K \rightarrow K \cup_f e_{(0)}^n$  where  $e_{(0)}^n$  is a rational cell of dimension  $n$  (that is, the cone over a rational  $(n-1)$  sphere  $S_{(0)}^{n-1}$ ) and where  $f_*: H_{n-1}(S_{(0)}^{n-1}) \rightarrow H_{n-1}(K)$  is trivial. Thus Chen's Lie algebra model of  $K$  determines a rational homology decomposition of  $K$ . Since Sullivan's model of  $K$  determines a Postnikov tower of  $K$ , the duality theory gives a method of computing the rational cell structure of  $K$  from a Postnikov tower of  $K$  and vice versa. This result is related to the Baues-Lemaire conjecture [2] which asserts that the minimal model of Quillen's Lie algebra model of  $K$  (see [1, 11]) is isomorphic to the minimal model of the  $\mathcal{L}$ -construction  $\mathcal{L}(\mathfrak{M}_K^*)$  on the dual of Sullivan's model  $\mathfrak{M}_K$  of  $K$ .

The results in this paper are excerpted from the author's Ph.D. thesis [8] which was supervised by Professor Kuo-Tsai Chen to whom the author wishes to express

his sincere gratitude for guidance and encouragement given during the preparation of the thesis.

The first four sections of this paper are elementary and self-contained. §2 is a list of notation and conventions used in this paper. In §3 connections on cochain algebras and chain Lie algebras are defined and their existence is proved. In §4 we state the main theorem of this paper and discuss its topological implications. §5 contains a proof of the main theorem.

**2. Notation and conventions.** A *graded vector space*  $V$  is a sequence  $\{V_p\}$  of  $\mathbf{Q}$ -vector spaces where  $p$  runs through the integers  $\mathbf{Z}$ . The degree of an element  $v$  of  $V$  will be denoted by  $\deg v$ . The graded vector space is said to be of *finite type* if each  $V_p$  has finite dimension. Let  $r \in \mathbf{Z}$ . We say that  $V$  is  $r$ -connected if  $V_p = 0$  whenever  $p \leq r$ . The  $r$ -fold suspension  $s^r V$  of  $V$  is the graded vector space defined by  $(s^r V)_p = V_{p-r}$ . The ground field  $\mathbf{Q}$  will be considered as a graded vector space concentrated in degree zero.

For a differential graded (d.g.) vector space  $V$  and a cycle  $v$  in  $V$ , we shall denote the homology class of  $v$  by  $\{v\}$ .

A *cochain algebra* is a differential graded algebra whose differential has degree  $+1$ . That is,  $\mathcal{Q}$  is a cochain algebra with differential  $d$  and if  $a \in \mathcal{Q}^p$  and  $b \in \mathcal{Q}^q$ , then

$$a \wedge b = (-1)^{pq} b \wedge a,$$

$$d(a \wedge b) = (da) \wedge b + (-1)^p a \wedge (db)$$

and, in addition, there is a d.g. algebra map  $\varepsilon: \mathcal{Q} \rightarrow \mathbf{Q}$  of degree 0. The kernel of  $\varepsilon$  is called the *augmentation ideal* of  $\mathcal{Q}$  and will be denoted by  $I\mathcal{Q}$ . The  $r$ th power of  $I\mathcal{Q}$  will be denoted by  $I^r\mathcal{Q}$ . The *space of indecomposables* of  $\mathcal{Q}$  is the d.g. vector space  $I\mathcal{Q}/I^2\mathcal{Q}$  and will be denoted by  $Q\mathcal{Q}$ . Let  $r$  be a nonnegative integer. We say that  $\mathcal{Q}$  is  $r$ -connected if the graded vector space  $I\mathcal{Q}$  is  $r$ -connected.

We shall denote the *free graded commutative algebra* generated by the graded vector space  $V$  by  $\Lambda(V)$ . That is,  $\Lambda(V)$  is the tensor product of the polynomial algebra generated by the elements of  $V$  of even degree with the exterior algebra generated by the elements of  $V$  of odd degree. The free graded commutative algebra generated by the graded set  $\{x_1, x_2, \dots\}$  will be denoted by  $\Lambda(x_1, x_2, \dots)$ .

A *chain Lie algebra* is a graded Lie algebra whose differential has degree  $-1$ . That is, if  $L$  is a chain Lie algebra with differential  $\partial$  and if  $X \in L_p$ ,  $Y \in L_q$  and  $Z \in L_r$ , then

$$[X, Y] + (-1)^{pq}[Y, X] = 0,$$

$$(-1)^{pr}[X[Y, Z]] + (-1)^{qp}[Y[Z, X]] + (-1)^{rq}[Z[X, Y]] = 0,$$

$$\partial[X, Y] = [\partial X, Y] + (-1)^p[X, \partial Y].$$

The subspace  $[L, L]$  of  $L$  will be denoted by  $I^2L$ . For  $r > 2$  set  $I^rL = [L, I^{r-1}L]$ . The *space of indecomposables* of  $L$  is the d.g. vector space  $L/I^2L$  and will be denoted by  $QL$ . Let  $r$  be a nonnegative integer. A chain Lie algebra is said to be  $r$ -connected if it is  $r$ -connected as a graded vector space.

The *free graded Lie algebra* generated by the graded vector space  $V$  will be denoted by  $L(V)$ . The free graded Lie algebra generated by the graded set  $\{X_1, X_2, \dots\}$  will be denoted by  $L(X_1, X_2, \dots)$ .

**3. Connections.** Twisting cochains are the backbone of the duality theory. As they are viewed in this paper, they provide a concrete means for computing the minimal chain Lie algebra  $L$  that corresponds to a given minimal cochain algebra  $\mathfrak{M}$ , and vice versa, as well as the associated pairings

$$Q\mathfrak{M} \otimes sH_*(L) \rightarrow \mathbb{Q}, \quad IH^*(\mathfrak{M}) \otimes sQL \rightarrow \mathbb{Q}.$$

The twisting cochains that arise in the duality theory are called *connection forms*, the terminology used by Chen in the analogous topological situation [4, 5].

Let  $V, W$  be graded vector spaces. Denote by  $V \hat{\otimes} W$  the graded vector space  $\prod_{n=0}^{\infty} \bigoplus_{p+q=n} V_p \otimes W_q$ . A typical element of  $V \hat{\otimes} W$  being the infinite sum  $\sum_{j=0}^{\infty} v_j \otimes w_j$  where, for each  $n$ , there is only a finite number of terms  $v_j \otimes w_j$  of degree  $n$ .

For each cochain algebra  $\mathcal{Q}$ , define a linear automorphism  $J: \mathcal{Q} \rightarrow \mathcal{Q}$  by  $Ja = (-1)^{\deg a} a$  for each  $a \in \mathcal{Q}$ .

Let  $\mathcal{Q}$  be a cochain algebra with differential  $d$  and  $L$  a chain Lie algebra with differential  $\partial$ . We can extend the actions of  $d, \partial, J$  to  $\mathcal{Q} \hat{\otimes} L$  by setting  $d(\sum a_j \otimes X_j) = \sum(da_j) \otimes X_j$ ,  $\partial(\sum a_j \otimes X_j) = \sum a_j \otimes (\partial X_j)$ ,  $J(\sum a_j \otimes X_j) = \sum(Ja_j) \otimes X_j$  where  $a_j \in \mathcal{Q}$  and  $X_j \in L$ . Define a bracket  $[\cdot, \cdot]$  in  $\mathcal{Q} \hat{\otimes} L$  by  $[\sum a_i \otimes X_i, \sum b_j \otimes Y_j] = \sum(a_i \wedge b_j) \otimes [X_i, Y_j]$  where  $a_i, b_j \in \mathcal{Q}$  and  $X_i, Y_j \in L$ . Usually we will omit the tensor product sign and write  $\sum a_i \otimes X_i$  as  $\sum a_i X_i$ .

(3.1) DEFINITION. Let  $L$  be a chain Lie algebra (with differential  $\partial$ ) and  $\mathcal{Q}$  a cochain algebra (with differential  $d$ ). A twisting cochain in  $\mathcal{Q} \hat{\otimes} L$  is an element  $\omega$  of  $\prod_{p=0}^{\infty} \mathcal{Q}^{p+1} \otimes L_p$  satisfying the *twisting cochain condition*

$$\partial\omega + d\omega - \frac{1}{2}[J\omega, \omega] = 0.$$

If  $\mathcal{Q}$  is of finite type, then  $\text{Hom}(\mathcal{Q}, \mathbb{Q})$  is a d.g. coalgebra and the map  $\tau_\omega: \text{Hom}(\mathcal{Q}, \mathbb{Q}) \rightarrow L$  defined by  $\tau_\omega(\varphi) = \sum \varphi(a_i) A_i$ , where  $\omega = \sum a_i A_i$  and  $\varphi \in \text{Hom}(\mathcal{Q}, \mathbb{Q})$ , is a twisting cochain in the sense of Brown [3]. For our purposes there are two advantages in representing twisting cochains as elements of  $\mathcal{Q} \hat{\otimes} L$  rather than as functions. The first is that, in our context, it is more natural to deal with  $\mathcal{Q}$ , usually a minimal cochain algebra of a de Rham complex, than its dual. Second, twisting cochains take on a very explicit form when written as elements of  $\mathcal{Q} \hat{\otimes} L$ . This is an advantage, especially in computations.

The association between a minimal cochain algebra and a minimal chain Lie algebra is made by means of a connection.

(3.2) DEFINITIONS. (a) Let  $\mathcal{Q}$  be a cochain algebra with 1-connected homology and finite Betti numbers. A *connection on  $\mathcal{Q}$*  is a free chain Lie algebra  $L$  and twisting cochain  $\omega$  in  $\mathcal{Q} \hat{\otimes} L$  such that if  $(X_i)$  is a set that freely generates  $L$  and if

$$\omega \equiv \sum w_i X_i \pmod{\mathcal{Q} \hat{\otimes} I^2 L},$$

then each  $w_i$  is closed in  $\mathcal{Q}$  and their homology classes  $(\{w_i\})$  form a basis of  $IH^*(a)$ .

We shall denote a connection on  $\mathcal{Q}$  by  $(\omega, L)$  and shall call  $\omega$  the *connection form* of the connection  $(\omega, L)$ .

(b) Let  $L$  be a chain Lie algebra with connected homology and finite Betti numbers. A *connection on  $L$*  is a free cochain algebra  $\mathfrak{N}$  and a twisting cochain  $\omega$  in  $\mathfrak{N} \hat{\otimes} L$  such that if  $(x_i)$  is a set that freely generates  $\mathfrak{N}$  and if

$$\omega \equiv \sum x_i W_i \pmod{I^2 \mathfrak{N} \hat{\otimes} L},$$

then each  $W_i$  is closed in  $L$  and their homology classes  $(\{W_i\})$  form a basis of  $H_*(L)$ .

We shall denote a connection on  $L$  by  $(\mathfrak{N}, \omega)$  and shall call  $\omega$  the *connection form* of the connection  $(\mathfrak{N}, \omega)$ .

Note that since  $\omega$  is a twisting cochain then  $\deg w_i = 1 + \deg X_i$  and  $\deg x_i = -1 + \deg W_i$ .

(3.3) EXAMPLES. (a) If  $\mathcal{Q} = \Lambda(a, b: \deg a = 2, \deg b = 3, da = 0, db = a^2)$ , then a connection  $(\omega, L)$  on  $\mathcal{Q}$  is given by

$$L = \mathbf{L}(X: \deg X = 1, \partial X = 0), \quad \omega = aX + \frac{1}{2}b[X, X].$$

(b) If  $L = \mathbf{L}(X, Y, Z: \deg X = \deg Y = 2, \deg Z = 5, \partial X = \partial Y = 0, \partial Z = [X, Y])$ , then a connection  $(\mathfrak{N}, \omega)$  on  $L$  is given by

$$\mathfrak{N} = \Lambda(a, b: \deg a = \deg b = 3, da = db = 0), \quad \omega = aX + bY + Ja \wedge bZ.$$

Observe that if  $\mathcal{Q}$  is a 1-connected cochain algebra with finite Betti numbers and  $(\omega, L)$  is a connection on  $\mathcal{Q}$ , then there is a nonsingular pairing

$$(3.4) \quad IH^*(\mathcal{Q}) \otimes sQL \rightarrow \mathbf{Q}$$

defined as follows: if  $\omega \equiv \sum w_i X_i \pmod{\mathcal{Q} \hat{\otimes} I^2 L}$ , then  $\{w_i\} \otimes sX_j \mapsto \delta_{ij}$ . Since the pairing is nonsingular, it follows that as a graded Lie algebra

$$L \approx \mathbf{L}(s^{-1}\text{Hom}(IH^*(\mathcal{Q}), \mathbf{Q})).$$

In the dual situation, a connection  $(\mathfrak{N}, \omega)$  on a connected chain Lie algebra  $L$  defines a nonsingular pairing

$$(3.5) \quad Q\mathfrak{N} \otimes sH_*(L) \rightarrow \mathbf{Q}.$$

Because the pairing is nonsingular, it follows that as a graded algebra  $\mathfrak{N} \approx \Lambda(s\text{Hom}(H_*(L), \mathbf{Q}))$ .

To prove the existence of a connection on a cochain algebra  $\mathcal{Q}$ , we have to define a suitable differential on  $L = \mathbf{L}(s^{-1}\text{Hom}(IH^*(\mathcal{Q}), \mathbf{Q}))$  and a connection form in  $\mathcal{Q} \hat{\otimes} L$ . Dually, to prove the existence of a connection on a chain Lie algebra  $L$  we have to find a suitable differential on  $\mathfrak{N} = \Lambda(s\text{Hom}(H^*(L), \mathbf{Q}))$  and a connection form in  $\mathfrak{N} \hat{\otimes} L$ . The following theorem asserts that connections always exist. Its proof is constructive. First we recall the definitions of minimal cochain algebras and minimal chain Lie algebras.

(3.6) DEFINITIONS. (a) A connected cochain algebra  $\mathfrak{N}$  is said to be a *minimal cochain algebra* if:

- (i)  $\mathfrak{N}$  is a free graded commutative algebra,
- (ii) its differential  $d$  is decomposable; that is,  $d(I\mathfrak{N}) \subseteq I^2\mathfrak{N}$ ,
- (iii) there is a chain

$$\mathbf{Q} = \mathfrak{N}_0 \subseteq \mathfrak{N}_1 \subseteq \mathfrak{N}_2 \subseteq \cdots \subseteq \mathfrak{N}$$

of sub-d.g.-algebras of  $\mathfrak{N}$  such that  $d\mathfrak{N}_n \subseteq \mathfrak{N}_{n-1}$  and  $\bigcup_{n=1}^{\infty} \mathfrak{N}_n = \mathfrak{N}$ .

When  $\mathfrak{N}$  is 1-connected (iii) follows from (ii).

(b) A chain Lie algebra  $L$  is said to be a *minimal chain Lie algebra* if:

- (i)  $L_p = 0$  whenever  $p < 0$ ,
- (ii)  $L$  is a free graded Lie algebra,
- (iii) its differential  $\partial$  is decomposable; that is,  $\partial L \subseteq [L, L]$ .

(3.7) THEOREM. (a) *Every cochain algebra with 1-connected homology of finite type has a connection. The chain Lie algebra associated to such a cochain algebra by a connection is a connected minimal chain Lie algebra.*

(b) *Every chain Lie algebra with connected homology of finite type has a connection. The cochain algebra associated to such a chain Lie algebra by a connection is a 1-connected minimal cochain algebra.*

PROOF. We prove only (b). The proof of (a) is an algebraic analogue of Chen's proof of Theorem 1.3.1 in [4] and is dual to the proof of (b). Gugenheim has proved an assertion similar to (a) in [7].

Let  $L$  be a chain Lie algebra with connected homology of finite type. Set  $\mathfrak{N} = \Lambda(s \operatorname{Hom}(H_*(L), \mathbb{Q}))$ . Choose a basis  $(x_i)_{i \in S}$  of  $s \operatorname{Hom}(H_*(L), \mathbb{Q})$ . Choose closed elements  $(W_i)_{i \in S}$  of  $L$  such that their homology classes  $(\{W_i\})_{i \in S}$  form a basis of  $H_*(L)$  dual to the basis  $(s^{-1}x_i)$  of  $\operatorname{Hom}(H_*(L), \mathbb{Q})$ . We have to find a decomposable differential  $d$  on  $\mathfrak{N}$  and a twisting cochain  $\omega$  in  $\mathfrak{N} \hat{\otimes} L$  such that

$$\omega \equiv \sum x_i W_i \pmod{I^2 \mathfrak{N} \hat{\otimes} L} \quad \text{and} \quad \partial \omega + d\omega - \frac{1}{2}[J\omega, \omega] = 0.$$

For this we need the following lemma.

(3.8) LEMMA. *Let  $n$  be a positive integer. If  $d$  is a derivation of  $\mathfrak{N}$  of degree 1 such that  $d(I\mathfrak{N}) \subseteq I^2 \mathfrak{N}$  and if  $\omega$  is an element of  $\prod_{p=0}^{\infty} \mathfrak{N}^{p+1} \otimes L_p$  such that:*

- (a)  $\omega \equiv \sum x_i W_i \pmod{I^2 \mathfrak{N} \hat{\otimes} L}$ ,
- (b)  $\partial \omega + d\omega - \frac{1}{2}[J\omega, \omega] \equiv 0 \pmod{I^n \mathfrak{N} \hat{\otimes} L}$ ,

*then*

- (c)  $\partial(\partial \omega + d\omega - \frac{1}{2}[J\omega, \omega]) \equiv 0 \pmod{I^{n+1} \mathfrak{N} \hat{\otimes} L}$ ,
- (d)  $d^2 \equiv 0 \pmod{I^{n+1} \mathfrak{N}}$ .

PROOF. We prove the lemma by induction on  $n$ . When  $n = 1$  the lemma is easily verified. Suppose that  $n > 1$  and that the lemma is true for  $n - 1$ . By induction  $d^2 \equiv 0 \pmod{I^n \mathfrak{N}}$ . Observe that

$$d^2 \omega \equiv \sum (d^2 x_i) W_i \pmod{I^{n+1} \mathfrak{N} \hat{\otimes} L}.$$

Denote the universal enveloping algebra of  $L$  by  $UL$  and define the element  $J\omega \wedge \omega$  of  $\mathfrak{N} \hat{\otimes} UL$  by  $J\omega \wedge \omega = \sum (Ju_i \wedge u_j) U_i U_j$ , where  $\omega = \sum u_i U_i$ . It is straightforward to verify that  $\frac{1}{2}[J\omega, \omega] = J\omega \wedge \omega$  in  $\mathfrak{N} \hat{\otimes} UL$  so that in  $\mathfrak{N} \hat{\otimes} UL$  we have

$$\begin{aligned}
\partial(\partial\omega + d\omega - \tfrac{1}{2}[J\omega, \omega]) &= \partial(\partial\omega + d\omega - J\omega \wedge \omega) \\
&= d\partial\omega - J(\partial\omega) \wedge \omega + \omega \wedge \partial\omega \\
&\equiv d\partial\omega + J(d\omega - J\omega \wedge \omega) \wedge \omega \\
&\quad - \omega \wedge (d\omega - J\omega \wedge \omega) \mod I^{n+1}\mathfrak{N} \hat{\otimes} UL \\
&\equiv d(\partial\omega - J\omega \wedge \omega) \mod I^{n+1}\mathfrak{N} \hat{\otimes} UL \\
&\equiv -d^2\omega \mod d(I^n\mathfrak{N}) \hat{\otimes} UL \\
&\equiv -d^2\omega \mod I^{n+1}\mathfrak{N} \hat{\otimes} UL.
\end{aligned}$$

Therefore

$$\partial(\partial\omega + d\omega - \tfrac{1}{2}[J\omega, \omega]) \equiv \sum (d^2x_i)W_i \mod I^{n+1}\mathfrak{N} \hat{\otimes} L.$$

Since the  $W_i$  represent linearly independent homology classes of  $L$ , and since the left-hand side of the previous equation is in  $I^{n+1}\mathfrak{N} \hat{\otimes} (\partial L)$ , it follows that  $d^2x_i \in I^{n+1}\mathfrak{N}$  so that  $d^2 \equiv 0 \mod I^{n+1}\mathfrak{N}$  and

$$\partial(\partial\omega + d\omega - \tfrac{1}{2}[J\omega, \omega]) \equiv 0 \mod I^{n+1}\mathfrak{N} \hat{\otimes} L. \quad \square$$

To complete the proof of (3.7) we construct a sequence  $(\omega_n)_{n \geq 1}$  of elements of  $\prod_{p=0}^{\infty} \mathfrak{N}^{p+1} \otimes L_p$  and a sequence of derivations  $(d_n)_{n \geq 1}$  of  $\mathfrak{N}$ , each of degree 1, that satisfy:

- (a)  $\omega_1 = \sum x_i W_i$  and  $d_i = 0$ ,
- (b)  $\partial\omega_n + d_n\omega_n - \tfrac{1}{2}[J\omega_n, \omega_n] \equiv 0 \mod I^{n+1}\mathfrak{N} \hat{\otimes} L$ ,
- (c)  $\omega_{n+1} \equiv \omega_n \mod I^{n+1}\mathfrak{N} \hat{\otimes} L$ ,
- (d)  $d_{n+1}x_i \equiv d_nx_i \mod I^{n+1}\mathfrak{N}$ .

Before giving the proof of the existence of  $(\omega_n)$  and  $(d_n)$  we need to introduce some notation. For each finite sequence  $I = (i_1, i_2, \dots, i_p)$  of elements of  $S$ , the set indexing the  $x_i$ 's, denote by  $x_I$  the element  $x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_p}$  of  $\mathfrak{N}$ . For each positive integer  $p$ , choose a subset  $\mathcal{G}(p)$  of  $S \times S \times \dots \times S$  ( $p$  times) such that  $\{x_I : I \in \mathcal{G}(p), p \geq 1\}$  is a basis of  $I\mathfrak{N}$ . Set  $\mathcal{G} = \bigcup_{p=1}^{\infty} \mathcal{G}(p)$ .

Each element  $\mu$  of  $I\mathfrak{N} \hat{\otimes} L$  can be written uniquely in the form  $\sum_{I \in \mathcal{G}} x_I U_I$  where  $U_I \in L$ . We shall denote by  $\mu(n)$  the "truncation"  $\sum_{p \leq n} \sum_{I \in \mathcal{G}(p)} x_I U_I$  of  $\mu$ .

Set  $\omega_1 = \sum x_i W_i$  and  $d_1 = 0$ . Conditions (a) through (d) are trivially satisfied. Suppose that  $n \geq 2$  and that  $\omega_1, \omega_2, \dots, \omega_{n-1}$  and  $d_1, d_2, \dots, d_{n-1}$  have already been constructed satisfying conditions (a) through (d). To construct  $\omega_n$  and  $d_n$  we need to find, for each  $I \in \mathcal{G}(n)$ , rational numbers  $(a_i^I)_{i \in S}$  and an element  $W_I$  of  $L$  such that if we set

$$\omega_n = \omega_{n-1} + \sum_{I \in \mathcal{G}(n)} x_I W_I, \quad d_n x_i = d_{n-1} x_i + \sum_{I \in \mathcal{G}(n)} a_i^I x_I,$$

then

$$\partial\omega_n + d_n\omega_n - \tfrac{1}{2}[J\omega_n, \omega_n] \equiv 0 \mod I^{n+1}\mathfrak{N} \hat{\otimes} L.$$

Since

$$\partial\omega_{n-1} + d_{n-1}\omega_{n-1} - \frac{1}{2}[J\omega_{n-1}, \omega_{n-1}] \equiv 0 \pmod{I^n\mathfrak{N} \hat{\otimes} L}$$

and since  $d_{n-1}(I^2\mathfrak{N}) \subseteq I^{n+1}\mathfrak{N}$ , it follows from a short calculation that if  $(a_i^I)_{i \in S}$  and  $W_i$  satisfy the equation

$$(*) \quad \sum_{I \in \mathfrak{g}(n)} \left( x_I \partial W_I + \sum_{i \in S} a_i^I x_I W_i \right) = -(\partial\omega_{n-1} + d_{n-1}\omega_{n-1} - \frac{1}{2}[J\omega_{n-1}, \omega_{n-1}])(n),$$

then  $\omega_n$  and  $d_n$  satisfy conditions (a) through (d).

It follows from (3.8) that

$$\partial(\partial\omega_{n-1} + d_{n-1}\omega_{n-1} - \frac{1}{2}[J\omega_{n-1}, \omega_{n-1}])(n) = 0.$$

Since the homology classes of the  $W_i$  span  $H_*(L)$ , we can find numbers  $a_i^I$  and elements  $W_i$  of  $L$  such that equation (\*) is satisfied. Thus  $\omega_n$  and  $d_n$  can be constructed.

Define  $\omega$  and  $d$  by insisting that

$$\omega \equiv \omega_n \pmod{I^{n+1}\mathfrak{N} \hat{\otimes} L} \quad \text{and} \quad dx_i \equiv d_n x_i \pmod{I^{n+1}\mathfrak{N}}. \quad \square$$

**4. The main theorem.** In this section we state the main theorem of this paper, give examples of its applications and outline some topological applications of the theorem.

(4.1) THEOREM. (a) *Let  $\mathfrak{N}$  be a 1-connected minimal cochain algebra of finite type and  $L$  a connected minimal cochain algebra of finite type. If  $\omega$  is a twisting cochain in  $\mathfrak{N} \hat{\otimes} L$ , then  $(\omega, L)$  is a connection on  $\mathfrak{N}$  if and only if  $(\mathfrak{N}, \omega)$  is a connection on  $L$ .*

(b) *The minimal chain Lie algebra associated to a 1-connected minimal cochain algebra of finite type by a connection is unique up to an isomorphism that preserves the pairings (3.4) and (3.5) defined by the connection form. Dually, the minimal cochain algebra associated to a connected minimal chain Lie algebra of finite type is unique up to an isomorphism that preserves the pairings (3.4) and (3.5) defined by the connection form.*  $\square$

In the case when the minimal chain Lie algebra  $L$  is associated to the minimal cochain algebra  $\mathfrak{N}$  by a connection (equivalently,  $\mathfrak{N}$  is associated to  $L$  via a connection) we shall say that there is a *connection between*  $\mathfrak{N}$  and  $L$ .

(4.2) EXAMPLES. (a) Let  $n$  be an odd integer,  $n > 1$ . Let

$$\mathfrak{N} = \Lambda(a: \deg a = n, da = 0) \quad \text{and} \quad L = \mathbf{L}(X: \deg X = n - 1, \partial X = 0).$$

A connection between  $\mathfrak{N}$  and  $L$  is given by the connection form  $\omega = aX$ . In this case  $I\mathfrak{N}$  is spanned by  $a$  and  $L$  by  $X$  and both pairings associated to  $\omega$  are given by  $\langle a, sX \rangle = 1$ .

(b) Let  $n$  be an even integer,  $n \geq 2$ . For

$$\begin{aligned} \mathfrak{N} &= \Lambda(a, b: \deg a = n, \deg b = 2n - 1, da = 0, db = a^2), \\ L &= \mathbf{L}(X: \deg X = n - 1, \partial X = 0), \end{aligned}$$



a connection between  $\mathfrak{M}$  and  $L$  is given by the connection form  $aX + \frac{1}{2}b[X, X]$ . Here the pairing  $IH^*(\mathfrak{M}) \otimes sQL \rightarrow \mathbf{Q}$  is given by  $\langle a, sX \rangle = 1$  and the pairing  $QL \otimes sH_*(L)$  is given by  $\langle a, X \rangle = 1, \langle b, X \rangle = 0, \langle a, [X, X] \rangle = 0, \langle b, [X, X] \rangle = 2$ .

(c) Theorem (4.1) can be used to compute the homology of connected minimal chain Lie algebras. For example, we can easily compute the homology of the chain Lie algebra  $(\mathbf{L}(X_1, X_2, \dots, X_n), \partial)$  where  $\deg X_k = 2k - 1$ ,  $\partial X_1 = 0$  and  $\partial X_k = \frac{1}{2}\sum_{i+j=k}[X_i, X_j]$ . Denote this Lie algebra by  $L$ . Let

$$\mathfrak{M} = \Lambda(a, b: \deg a = 2, \deg b = 2n + 1, da = 0, db = a^{n+1}).$$

The element  $\omega$  of  $\mathfrak{M} \hat{\otimes} L$  given by

$$\omega = \sum_{k=1}^n a^k X_k + \frac{1}{2} \sum_{k=1}^n a^{k-1} b \sum_{i+j=n+k} [X_i, X_j]$$

is a twisting cochain in  $\mathfrak{M} \hat{\otimes} L$ . Since  $\omega \equiv \sum_{k=1}^n a^k X_k \bmod \mathfrak{M} \hat{\otimes} I^2 L$ , it follows that  $(\omega, L)$  is a connection on  $\mathfrak{M}$ . By Theorem (4.1),  $(\mathfrak{M}, \omega)$  is a connection on  $L$ . Since  $\omega \equiv aX_1 + \frac{1}{2}b\sum_{i+j=n+1}[X_i, X_j] \bmod I^2 \mathfrak{M} \hat{\otimes} L$ , it follows that the homology classes  $\{X_1\}$  and  $\{\sum_{i+j=n+1}[X_i, X_j]\}$  are nontrivial in, and span  $H_*(L)$ .

(d) Often we want to find the chain Lie algebra associated to the minimal model  $\mathfrak{M}$  of a 1-connected cochain algebra  $\mathcal{Q}$  with finite Betti numbers and compute the pairings (1.1) and (1.2) in low dimensions (a typical example being when  $\mathcal{Q}$  is the de Rham complex of a 1-connected manifold with finite Betti numbers). This is most easily achieved by first finding a connection  $(\omega, L)$  on  $\mathcal{Q}$ . If  $(\tilde{\omega}, \tilde{L})$  is a connection on  $\mathfrak{M}$ , then it can be shown that  $L \approx \tilde{L}$  (cf. [8, Proposition (6.2)]). The following example illustrates this and other techniques for dealing with not so trivial examples.

Let  $\mathcal{Q}$  be the truncated polynomial algebra

$$\Lambda(c_1, c_2: \deg c_1 = 2, \deg c_2 = 4)/\mathcal{G}$$

where  $\mathcal{G}$  is the ideal of  $\Lambda(c_1, c_2)$  generated by  $c_1^3, c_1 \wedge c_2$  and  $c_2^2$ . We will compute a connection on  $\mathcal{Q}$  and the pairing  $Q\mathfrak{M} \otimes H_*(L) \rightarrow \mathbf{Q}$  in low dimensions, where  $\mathfrak{M}$  is the minimal model of  $\mathcal{Q}$  and where  $L$  is the Lie algebra associated to  $\mathcal{Q}$  by a connection.

A connection  $(\omega, L)$  on  $\mathcal{Q}$  is given by

$$L = \mathbf{L}(X_1, X_{11}, X_2: \deg X_1 = 1, \deg X_{11} = \deg X_2 = 3, \partial X_2 = 0, \\ \partial X_{11} = \frac{1}{2}[X_1, X_1]),$$

$$\omega = c_1 X_1 + c_1^2 X_{11} + c_2 X_2.$$

Set  $\mathfrak{M}' = \Lambda(x_1, x_2, u_{1,11}, u_{1,2}, u_{12,11}, u_{12,2})$  where  $\deg x_1 = 2, \deg x_2 = 4, \deg u_{1,11} = \deg u_{1,2} = 5, \deg u_{2,2} = 7, \deg u_{12,11} = \deg u_{12,2} = 8$  and where  $dx_1 = dx_2 = 0, du_{1,11} = x_1^3, du_{1,2} = x_1 \wedge x_2, du_{2,2} = x_2^2, du_{12,11} = x_1^2 \wedge u_{1,2} - x_2 \wedge u_{1,11}, du_{12,2} = u_{1,2} \wedge x_2 - x_1 \wedge u_{2,2}$ . Define a d.g. algebra map  $\rho: \mathfrak{M}' \rightarrow \mathcal{Q}$  by  $\rho(x_1) = c_1, \rho(x_2) = c_2$  and  $\rho(u_{I,J}) = 0$  for all  $I, J$ . It is easy to check that  $\rho$  induces an isomorphism on cohomology in degrees  $\leq 9$ . Consequently, there is an inclusion of  $\mathfrak{M}'$  into the minimal model  $\mathfrak{M}$  of  $\mathcal{Q}$  that is an isomorphism in degrees  $\leq 8$ .

The method we use to compute the pairing  $Q\mathfrak{N} \otimes H_*(L) \rightarrow \mathbf{Q}$  in degrees  $\leq 8$  is to find an element  $\omega'$  of  $\mathfrak{N}' \otimes L$  that is an "approximation" to a connection form in  $\mathfrak{N} \hat{\otimes} L$ . Specifically, we will find an element  $\omega'$  of  $\mathfrak{N} \hat{\otimes} L$  such that there exists a connection form  $\tilde{\omega}$  in  $\mathfrak{N} \hat{\otimes} L$  such that

$$\tilde{\omega} \equiv \omega' \pmod{\bigoplus_{p \geq 9} (\mathfrak{N}^p \otimes L_{p-1})}.$$

The procedure for doing this is to find a sequence  $(\omega'_r)_{r \geq 1}$  of elements of  $\mathfrak{N} \hat{\otimes} L$  such that:

$$(a) \omega'_1 = x_1 X_1 + x_1^2 X_{11} + x_2 X_2,$$

$$(b) \partial \omega'_r + d\omega'_r - \frac{1}{2}[J\omega'_r, \omega'_r] \equiv 0 \pmod{\mathfrak{N} \hat{\otimes} I^{r+1}L},$$

$$(c) \omega'_{r+1} \equiv \omega'_r \pmod{I^{r+1}\mathfrak{N} \hat{\otimes} L}.$$

To find  $\omega'_{r+1}$  given  $\omega'_r$ , set  $\omega'_{r+1} = \omega'_r + \mu$  where  $\mu$  is an element of  $\mathfrak{N} \hat{\otimes} I^{r+1}L$  which satisfies the equation

$$-d\mu \equiv \partial \omega'_r + d\omega'_r - \frac{1}{2}[J\omega'_r, \omega'_r] \pmod{\mathfrak{N} \hat{\otimes} I^{r+2}L}.$$

We begin with  $\omega'_1 = x_1 X_1 + x_1^2 X_{11} + x_2 X_2$ . Now

$$\begin{aligned} \partial \omega'_1 + d\omega'_1 - \frac{1}{2}[J\omega'_1, \omega'_1] &= -\frac{1}{2}x_1^4[X_{11}, X_{11}] - \frac{1}{2}x_2^2[X_2, X_2] - x_1^3[X_1, X_{11}] \\ &\quad - x_1 \wedge x_2[X_1, X_2] - x_1^2 \wedge x_2[X_{11}, X_2] \\ &= -d\left(\frac{1}{2}x_1 \wedge u_{1,11}[X_{11}, X_{11}] + \frac{1}{2}u_{2,2}[X_2, X_2]\right) \\ &\quad + u_{1,11}[X_1, X_{11}] + u_{1,2}[X_1, X_2] + x_1 \wedge u_{1,2}[X_{11}, X_2]. \end{aligned}$$

Thus

$$\begin{aligned} \omega'_2 &= \omega'_1 + \frac{1}{2}x_1 \wedge u_{1,11}[X_{11}, X_{11}] + \frac{1}{2}u_{2,2}[X_2, X_2] \\ &\quad + u_{1,11}[X_1, X_{11}] + u_{1,2}[X_1, X_2] + x_1 \wedge u_{1,2}[X_{11}, X_2]. \end{aligned}$$

After a similar, but more tedious calculation, one can show that

$$\begin{aligned} \omega'_3 &= \omega'_2 + u_{12,2}[[X_1, X_2]X_2] + u_{12,11}[X_2[X_1, X_{11}]] \\ &\quad + x_1 \wedge u_{12,2}[[X_{11}, X_2]X_2] + x_1 \wedge u_{12,11}[[X_{11}, X_2]X_{11}]. \end{aligned}$$

If we continued the procedure above and found a sequence  $(\omega'_r)$  satisfying (a), (b) and (c) above, then we could define a connection form  $\tilde{\omega}$  in  $\mathfrak{N} \hat{\otimes} L$  by insisting that  $\tilde{\omega} \equiv \omega'_r \pmod{\mathfrak{N} \hat{\otimes} I^{r+1}L}$ .

Set  $\omega' = \omega'_3$ . A short computation shows that

$$\partial \omega' + d\omega' - \frac{1}{2}[J\omega', \omega'] \equiv 0 \pmod{\bigoplus_{p \geq 9} (\mathfrak{N}^{p+1} \otimes L_{p-1})}.$$

Thus we have found our approximation to a connection  $\tilde{\omega}$  on  $\mathfrak{N}$ . Observe that

$$\begin{aligned} \omega' &\equiv x_1 X_1 + x_2 X_2 + \frac{1}{2}u_{2,2}[X_2, X_2] + u_{1,11}[X_1, X_{11}] + u_{1,2}[X_1, X_2] \\ &\quad + u_{12,2}[[X_1, X_2]X_2] + u_{12,11}[X_2[X_1, X_{11}]] \pmod{I^2\mathfrak{N} \hat{\otimes} L}. \end{aligned}$$

The following table gives dual bases of  $Q\mathfrak{M}$  and  $H_*(L)$  in low dimensions:

Basis of $\bigoplus_{p \leq 8} (Q\mathfrak{M})^p$	Dual basis of $\bigoplus_{p \leq 7} H_p(L)$
$x_1$	$\{X_1\}$
$x_2$	$\{X_2\}$
$u_{1,11}$	$\{[X_1, X_{11}]\}$
$u_{1,2}$	$\{[X_1, X_2]\}$
$u_{2,2}$	$\frac{1}{2}\{[X_2, X_2]\}$
$u_{12,11}$	$\{[X_2[X_1, X_{11}]]\}$
$u_{12,2}$	$\{[[X_1, X_2]X_2]\}$

As remarked in the introduction, if  $K$  is a 1-connected space with finite Betti numbers, if  $\mathfrak{M}_K$  is Sullivan's minimal model of  $K$  and if  $L_K$  is the Lie algebra model of  $K$ , then  $\mathfrak{M}_K$  and  $L_K$  correspond under the duality theory,  $H^*(\mathfrak{M}_K) \approx H^*(K; \mathbf{Q})$ ,  $Q\mathfrak{M} \approx \text{Hom}_{\mathbf{Z}}(\pi_*(K), \mathbf{Q})$ ,  $H_*(L) \approx s^{-1}\pi_*(K) \otimes_{\mathbf{Z}} \mathbf{Q}$  and  $QL \approx s^{-1}\tilde{H}_*(K; \mathbf{Q})$ . In (3.3), Example (a) corresponds to  $S^2$ , and (b) to  $S^3 \times S^3$ . In (4.2), Examples (a) and (b) correspond to  $S^n$  where  $n$  is odd and  $n$  is even, respectively. Example (c) corresponds to  $CP^n$  and (d) to the 4-skeleton of  $BU$ .

If  $K$  is a 1-connected rational space with finite Betti numbers, then  $\mathfrak{M}_K$  determines the Postnikov tower of  $K$  [6, 15] and  $L_K$  a homology decomposition of  $K$  [13, 18]. The topological implication of Theorems (3.7) and (4.1) is that we can inductively compute the  $k$ -invariants of  $K$  from the characteristic maps of a homology decomposition of  $K$  and vice versa (see [8 or 10] for example).

**5. A proof of Theorem (4.1).** The proof of (4.1) is a composite of results of Quillen [14, Appendix B], Sullivan [16], Lehmann [9] and Baues and Lemaire [2]. We begin by reviewing the basic properties of Quillen's  $\mathcal{L}$  and  $\mathcal{C}$ -constructions. Good references for this material include [2] and [13] as well as Appendix B of Quillen's paper [14].

A *chain coalgebra* is a d.g. coalgebra  $C$  with a differential  $\partial$  of degree  $-1$  and a coaugmentation  $\eta: \mathbf{Q} \rightarrow C$ . We shall denote the cokernel of  $\eta$  by  $\bar{C}$  and the image in  $\bar{C}$  of an element  $c$  of  $C$  by  $\bar{c}$ . The set of *primitive* elements  $\{c \in C: \Delta c = c \otimes 1 + 1 \otimes c\}$  of  $C$  will be denoted by  $PC$ . We say that  $C$  is  $r$ -connected if the graded vector space  $\bar{C}$  is  $r$ -connected. A chain coalgebra  $C$  is *commutative* if

$$\Delta c = \sum a_i \otimes b_i = \sum (-1)^{\deg a_i \deg b_i} b_i \otimes a_i$$

for all  $c \in C$ .

Let  $C$  be a commutative chain coalgebra. The  $\mathcal{L}$ -construction on  $C$  is a chain Lie algebra  $\mathcal{L}(C)$  that, as a Lie algebra, is isomorphic to  $\mathbf{L}(s^{-1}\bar{C})$ . As a d.g. vector space,  $Q\mathcal{L}(C) \approx s^{-1}\bar{C}$ .

Let  $L$  be a chain Lie algebra. The  $\mathcal{C}$ -construction on  $L$  is a cocommutative chain coalgebra  $\mathcal{C}(L)$  that, as a coalgebra, is isomorphic to the symmetric coalgebra  $S(sL)$  on  $sL$ . As a d.g. vector space,  $P\mathcal{C}(L)$  is isomorphic to  $sL$ .

There is a canonical inclusion  $i: s^{-1}\bar{C} \rightarrow \mathcal{L}(C)$  that corresponds to the canonical inclusion  $s^{-1}\bar{C} \rightarrow \mathbf{L}(s^{-1}\bar{C})$ . There is a canonical projection  $p: \mathcal{C}(L) \rightarrow sL$  that

corresponds to the canonical projection of  $S(sL)$  onto  $sL$ . In general, neither  $i$  nor  $p$  is a d.g. vector space map.

The basic properties of the  $\mathcal{L}$ - and  $\mathcal{C}$ -constructions are summarized by the following theorem.

(5.1) THEOREM. *Let  $C$  be a cocommutative chain coalgebra and  $L$  a chain Lie algebra. There is a natural one-to-one correspondence between d.g. coalgebra maps  $C \rightarrow \mathcal{C}(L)$  and d.g. Lie algebra maps  $\mathcal{L}(C) \rightarrow L$ : the map  $\varphi: C \rightarrow \mathcal{C}(L)$  corresponds to the map  $\psi: \mathcal{L}(C) \rightarrow L$  if and only if the linear maps  $\psi \circ i \circ s^{-1}: \bar{C} \rightarrow L$  and  $s^{-1} \circ p \circ \varphi: \bar{C} \rightarrow L$  are equal. Moreover, if  $H_*(C)$  is 1-connected and  $H_*(L)$  is connected, and if the maps  $\varphi: C \rightarrow \mathcal{C}(L)$  and  $\psi: \mathcal{L}(C) \rightarrow L$  correspond as above, then  $\varphi$  induces an isomorphism on homology if and only if  $\psi$  induces an isomorphism on homology.  $\square$*

Let  $\mathcal{Q}$  be a cochain algebra of finite type and  $L$  a chain Lie algebra. Denote the dual coalgebra  $\text{Hom}(\mathcal{Q}, \mathbf{Q})$  of  $\mathcal{Q}$  by  $\mathcal{Q}^*$ . Each twisting cochain  $\omega$  in  $\mathcal{Q} \hat{\otimes} L$  determines a chain Lie algebra map  $\theta: \mathcal{L}(\mathcal{Q}^*) \rightarrow L$  that we shall call the *holonomy map* associated to  $\omega$ . If  $\omega = \sum u_i U_i$  and  $\varphi \in \mathcal{Q}^*$ , define  $\langle \omega, \bar{\varphi} \rangle = \sum \varphi(u_i) U_i$ . Observe that this is a finite sum so that  $\langle \omega, \bar{\varphi} \rangle \in L$ . The holonomy map  $\theta: \mathcal{L}(\mathcal{Q}^*) \rightarrow L$  is then defined by

$$\theta(s^{-1}\bar{\varphi}) = \langle \omega, \bar{\varphi} \rangle \quad \text{for all } \varphi \in \mathcal{Q}^*.$$

Using the fact that  $\omega$  is a twisting cochain, one can show that  $\theta$  is a d.g. Lie algebra map (see [14, Appendix B]).

We would like to characterize the connections on  $\mathcal{Q}$  as the pairs  $(\omega, L)$  where the associated holonomy map  $\theta: \mathcal{L}(\mathcal{Q}^*) \rightarrow L$  induces an isomorphism on homology. To do this we need the following lemma. A proof of it can be found in [2, p. 226].

(5.2) LEMMA. *Let  $L_1$  and  $L_2$  be free chain Lie algebras with connected homology. A d.g. Lie algebra map  $\varphi: L_1 \rightarrow L_2$  induces an isomorphism on homology if and only if  $Q\varphi: QL_1 \rightarrow QL_2$  induces an isomorphism on homology.  $\square$*

We can now prove the desired characterization of connections on a cochain algebra.

(5.3) LEMMA. *Let  $\mathcal{Q}$  be a 1-connected cochain algebra of finite type and  $L$  a connected minimal chain Lie algebra. If  $\omega$  is a twisting cochain in  $\mathcal{Q} \hat{\otimes} L$ , then  $(\omega, L)$  is a connection on  $\mathcal{Q}$  if and only if the associated holonomy map  $\theta: \mathcal{L}(\mathcal{Q}^*) \rightarrow L$  induces an isomorphism on homology.*

PROOF. Let  $(X_i)$  be a set that freely generates  $L$ . Suppose that  $\omega \equiv \sum w_i X_i \pmod{\mathcal{Q} \hat{\otimes} I^2 L}$ . Since  $\partial$  (the differential in  $L$ ) is decomposable, the twisting cochain condition implies that  $d\omega \equiv 0 \pmod{\mathcal{Q} \hat{\otimes} I^2 L}$ . That is, each  $w_i$  is closed in  $\mathcal{Q}$ .

The holonomy map  $\theta: \mathcal{L}(\mathcal{Q}^*) \rightarrow L$  induces the map  $s^{-1}\bar{\varphi} \mapsto \sum \varphi(w_i) X_i$  on indecomposables ( $\varphi \in \mathcal{Q}^*$ ). Observe that  $(\omega, L)$  is a connection on  $\mathcal{Q}$  if and only if this map induces an isomorphism on homology. The result now follows from (5.2).  $\square$

To prove the analogous result for connections on Lie algebras, we need an analogue of (5.2) for free commutative cochain algebras. The following proposition is proved in Lehmann [9, Proposition V.7] and is implicit in Theorem (2.2) of [16].

(5.4) PROPOSITION. *Let  $\mathcal{Q}$  be a free, connected cochain algebra with 1-connected homology. Let  $\mathfrak{N}$  be a minimal cochain algebra. A d.g. algebra map  $\rho: \mathfrak{N} \rightarrow \mathcal{Q}$  induces an isomorphism on homology if and only if  $Q\rho: Q\mathfrak{N} \rightarrow Q\mathcal{Q}$  induces an isomorphism on homology.*  $\square$

An easy consequence of the above proposition is the natural analogue of (5.2) for free commutative cochain algebras.

(5.5) LEMMA. *Let  $\mathcal{Q}$  and  $\mathcal{Q}'$  be connected, free cochain algebras with 1-connected homology. A d.g. algebra map  $\varphi: \mathcal{Q} \rightarrow \mathcal{Q}'$  induces an isomorphism on homology if and only if  $Q\varphi: Q\mathcal{Q} \rightarrow Q\mathcal{Q}'$  induces an isomorphism on homology.*  $\square$

Let  $\mathcal{Q}$  be a cochain algebra of finite type and  $L$  a chain Lie algebra. For a twisting cochain  $\omega \in \mathcal{Q} \hat{\otimes} L$ , we shall denote by  $\theta': \mathcal{Q}^* \rightarrow \mathcal{C}(L)$  the d.g. coalgebra map that corresponds to the holonomy map  $\theta: \mathcal{L}(\mathcal{Q}^*) \rightarrow L$  associated to  $\omega$ .

We can now characterize connections on chain Lie algebras in terms of their holonomy maps.

(5.6) LEMMA. *Let  $L$  be a connected chain Lie algebra of finite type and let  $\mathfrak{N}$  be a 1-connected minimal cochain algebra of finite type. If  $\omega$  is a twisting cochain in  $\mathfrak{N} \hat{\otimes} L$ , then  $(\mathfrak{N}, \omega)$  is a connection on  $L$  if and only if the associated holonomy map  $\theta: \mathcal{L}(\mathfrak{N}^*) \rightarrow L$  induces an isomorphism on homology.*

PROOF. Let  $(x_i)$  be a set that freely generates  $\mathfrak{N}$ . Suppose that  $\omega \equiv \sum x_i W_i \pmod{I^2 \mathfrak{N} \hat{\otimes} L}$ . Since  $d$  (the differential in  $\mathfrak{N}$ ) is decomposable, the twisting cochain condition implies that  $\partial \omega \equiv 0 \pmod{I^2 \mathfrak{N} \hat{\otimes} L}$ . That is, each  $W_i$  is closed in  $L$ .

The holonomy map  $\theta: \mathcal{L}(\mathfrak{N}^*) \rightarrow L$  associated to  $\omega$  induces an isomorphism on homology if and only if the coalgebra map  $\theta': \mathfrak{N}^* \rightarrow \mathcal{C}(L)$  induces an isomorphism on homology. Since  $\mathfrak{N}$  is of finite type,  $\text{Hom}(\mathfrak{N}^*, \mathbf{Q}) \approx \mathfrak{N}$ , and since  $L$  is connected and of finite type,  $\text{Hom}(\mathcal{C}(L), \mathbf{Q})$  is a 1-connected, free cochain algebra. Thus  $\theta'$  induces an isomorphism on homology if and only if its dual  $\theta'^*: \text{Hom}(\mathcal{C}(L), \mathbf{Q}) \rightarrow \mathfrak{N}$  induces an isomorphism on homology.

Observe that there is a natural isomorphism  $Q \text{Hom}(\mathcal{C}(L), \mathbf{Q}) \approx \text{Hom}(sL, \mathbf{Q})$  and that, with respect to this isomorphism,  $Q\theta'^*$  takes the functional  $\psi$  on  $sL$  to  $\sum x_i \psi(sW_i)$  (a finite sum). This follows from (5.1). The result now follows from (5.5).  $\square$

The first assertion of (4.1) now follows directly from (5.3) and (5.6).

Let  $L$  be a connected chain Lie algebra of finite type. Suppose that  $(\mathfrak{N}_1, \omega_1)$  and  $(\mathfrak{N}_2, \omega_2)$  are connections on  $L$  with associated holonomy maps  $\theta_j: \mathcal{L}(\mathfrak{N}_j^*) \rightarrow L$  ( $j = 1, 2$ ). As in the proof of (5.6), the dual of the adjoint of  $\theta_j$ ,

$$\theta_j'^*: \text{Hom}(\mathcal{C}(L), \mathbf{Q}) \rightarrow \mathfrak{N}_j, \quad j = 1, 2,$$

induces an isomorphism on homology. In fact,  $\theta_j'^*$  splits as the following argument shows. Let  $\rho: \mathfrak{N} \rightarrow \text{Hom}(\mathcal{C}(L), \mathbf{Q})$  be a minimal model of  $\text{Hom}(\mathcal{C}(L), \mathbf{Q})$ . (For a definition of minimal model, see [6, 15].) The composite  $\theta_j'^* \circ \rho: \mathfrak{N} \rightarrow \mathfrak{N}_j$  induces an isomorphism on homology. Because  $\mathfrak{N}$  and  $\mathfrak{N}_j$  are 1-connected and minimal,  $\theta_j'^* \circ \rho$  is an isomorphism. Define

$$\Psi_j: \mathfrak{N}_j \rightarrow \text{Hom}(\mathcal{C}(L), \mathbf{Q})$$

by  $\Psi_j = \rho \circ (\theta_j'^* \circ \rho)^{-1}$ . Observe that  $\theta_j'^* \circ \Psi_j$  equals  $\text{id}: \mathfrak{N}_j \rightarrow \mathfrak{N}_j$ .

Define a d.g. algebra map  $\Phi: \mathfrak{N}_1 \rightarrow \mathfrak{N}_2$  by  $\Phi = \theta_2'^* \circ \Psi_1$ . Since  $\Phi$  induces an isomorphism on homology and since both  $\mathfrak{N}_1$  and  $\mathfrak{N}_2$  are 1-connected and minimal,  $\Phi$  is an isomorphism.

To see that the diagram

$$\begin{array}{ccc} Q\mathfrak{N}_1 \otimes sH_*(L) & & \\ & \searrow & \\ Q\Phi \otimes \text{id} \downarrow & & \mathbf{Q} \\ & \nearrow & \\ Q\mathfrak{N}_2 \otimes sH_*(L) & & \end{array}$$

commutes, observe that the diagram

$$\begin{array}{ccc} H_*(Q\text{Hom}(\mathcal{C}(L), \mathbf{Q})) & \xrightarrow{(\theta_1'^*)^*} & Q\mathfrak{N}_1 \\ & \searrow (\theta_2'^*)^* & \downarrow Q\Phi \\ & & Q\mathfrak{N}_2 \end{array}$$

commutes and that with respect to the identification

$$\text{Hom}(sH_*(L), \mathbf{Q}) \approx H_*(Q\text{Hom}(\mathcal{C}(L), \mathbf{Q})),$$

the map  $\theta_j'^*$  is the polarity of the pairing  $Q\mathfrak{N}_j \otimes sH_*(L) \rightarrow \mathbf{Q}$  given by  $\omega_j$ .

If, in addition,  $L$  is minimal, then it follows from (4.1)(a) that the homology of  $\text{Hom}(\mathcal{C}(L), \mathbf{Q})$  is isomorphic to  $\text{Hom}(sQL, \mathbf{Q})$ . Using this and the fact that the diagram

$$\begin{array}{ccc} H^*(\text{Hom}(\mathcal{C}(L), \mathbf{Q})) & \xrightarrow{\theta_1'^*} & H^*(\mathfrak{N}_1) \\ & \searrow \theta_2'^* & \downarrow \Phi^* \\ & & H^*(\mathfrak{N}_2) \end{array}$$

commutes, it follows that the diagram

$$\begin{array}{ccc} IH^*(\mathfrak{N}_1) \otimes sQL & & \\ & \searrow & \\ I\Phi^* \otimes \text{id} \downarrow & & \mathbf{Q} \\ & \nearrow & \\ IH^*(\mathfrak{N}_2) \otimes sQL & & \end{array}$$

commutes up to homotopy.

The homotopy class of  $\Phi$  is uniquely determined, for if

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{C}(L), \mathbf{Q}) & \xrightarrow{\theta_1^*} & \mathfrak{M}_1 \\ & \theta_2^* \searrow & \downarrow \Phi' \\ & & \mathfrak{M}_2 \end{array}$$

commutes up to homotopy, then  $\Phi' = \Phi' \circ \theta_1^* \circ \Psi_1 \simeq \theta_2^* \circ \Psi_1 = \Phi$ .

The uniqueness of the Lie algebra in the dual case is proved by a similar argument.  $\square$

(5.7) COROLLARY. *If  $L$  and  $\mathfrak{M}$  correspond under the duality theory, then  $L$  is isomorphic to the minimal model of  $\mathcal{L}(\mathfrak{M}^*)$  and  $\mathfrak{M}$  is isomorphic to the minimal model of  $\mathrm{Hom}(\mathcal{C}(L), \mathbf{Q})$ .*  $\square$

It follows from Theorem 3.1.1 in [5] in the real case, and Theorem (6.9) in [18] in the rational case, that the universal enveloping algebra of Chen's Lie algebra model of a space is isomorphic to the Adams-Hilton model of the space (defined in [2]). This fact, together with (5.7) provides a correct proof of Proposition (3.6) in [2].

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