ISOMORPHISM TYPES IN WREATH PRODUCTS AND EFFECTIVE EMBEDDINGS OF PERIODIC GROUPS

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ABSTRACT. For any finitely generated group Y, $\omega(Y)$ denotes the Turing degree of the word problem of Y. Let G be any non-Abelian 2-generator group and B an infinite group generated by $k \geq 1$ elements. We prove that if τ is any Turing degree with $\tau \geq \text{l.u.b.}\{\omega(G), \omega(B)\}$ then the unrestricted wreath product $W = G \operatorname{Wr} B$ has a $(k + 1)$-generator subgroup H with $\omega(H) = \tau$. If B is also periodic, then W has a k-generator subgroup H such that $\tau = \text{l.u.b.}\{\omega(B), \omega(H)\}$.

Easy consequences include: $G \operatorname{Wr} \mathbf{Z}$ has $2^{\aleph_0}$ pairwise nonembeddable 2-generator subgroups and if B is periodic then $G \operatorname{Wr} B$ has $2^{\aleph_0}$ pairwise nonembeddable k-generator subgroups.

Using similar methods, we prove an effective embedding theorem for embedding countable periodic groups in 2-generator periodic groups.

I. Introduction. Let G denote a non-Abelian 2-generator group and B an *infinite* group generated by $k \geq 1$ elements. $\mathbf{Z}$ is the infinite cyclic group. $G \operatorname{Wr} B = \Omega B =$ the unrestricted wreath product of G and $B =$ the split extension of the Cartesian power Ω of G over B, $\Omega = \{f \mid B \xrightarrow{f} G\} =$ the base group, by right B-translations of the domain, i.e. $f^b(c) = f(cb^{-1})$ for all $b, c \in B$ and $f \in \Omega$. $\omega(B)$ is the Turing degree of the word problem of a group B. If $S \leq B$, then $\omega(S, B)$ is the Turing degree of the decision problem of S in B, and the relation $\leq$ between Turing degrees is Turing reducibility.

Our first two theorems provide uniform means to obtain some old results and to prove some new ones on isomorphism types of finitely generated groups.

THEOREM 1. *Let τ be any Turing degree such that $\tau \geq \omega(G \times B)$. Then:*

(a) *$G \operatorname{Wr} B$ has a $(k + 1)$-generator subgroup H satisfying $\omega(H) = \tau$;*

(b) *if B is periodic, then $G \operatorname{Wr} B$ has a k-generator subgroup H such that $\tau = \omega(B \times H)$.*

Note that the hypothesis that G is non-Abelian is essential, at least if $B = \mathbf{Z}$, since there are only countably many isomorphism types of finitely generated metabelian groups (they are all recursively presented).

Note also that $\omega(G \times B)$ is the l.u.b. of $\omega(G)$ and $\omega(B)$ in the semilattice of Turing degrees. If τ and σ are Turing degrees we let $\tau \times \sigma$ denote their l.u.b.

While weaker than part (a), part (b) is strong enough to give the following general result.

COROLLARY 1. (a) $G \operatorname{Wr} \mathbf{Z}$ has $2^{\aleph_0}$ pairwise nonembeddable 2-generator subgroups.

(b) If B is periodic, then $G \operatorname{Wr} B$ has $2^{\aleph_0}$ pairwise nonembeddable k-generator subgroups.

PROOF. Let $\sigma = \omega(G \times B)$. By [11, p. 12] there is a set of $2^{\aleph_0}$ Turing degrees $\{\tau_\alpha \mid \alpha < 2^{\aleph_0}\}$ such that $\tau_\alpha \geqslant \sigma$ and $\tau_\alpha \nleqslant \tau_\beta$ for $\alpha \neq \beta$ (that is, $\{\tau_\alpha\}$ is an antichain of $2^{\aleph_0}$ Turing degrees above σ). We apply Theorem 1(a) and (b) to obtain 2- (resp. k-) generator subgroups $H_\alpha \leqslant G \operatorname{Wr} B$ with $B = \mathbf{Z}$ in part (a) such that $\tau_\alpha = \sigma \times \omega(H_\alpha)$. If, for some $\alpha \neq \beta < 2^{\aleph_0}$, H_α is isomorphic to a subgroup of H_β, then $\omega(H_\alpha) \leqslant \omega(H_\beta)$ implies $\tau_\alpha = \sigma \times \omega(H_\alpha) \leqslant \sigma \times \omega(H_\beta) = \tau_\beta$; a contradiction.

COROLLARY 2. Let p be a Novikov-Adyan prime and let B have exponent p and be 2-generated. Then, for all $n \geqslant 3$, there are $2^{\aleph_0}$ pairwise nonembeddable 2-generator groups of exponent pn. Each of these groups H has generators of orders p and pn and H has a class 2 nilpotent Hirsch-Plotkin radical N of exponent n with $H/N \cong B$.

PROOF. We use part (b) of Corollary 1, choosing G to be a class 2 nilpotent group of exponent n. The orders of the generators as well as the final properties will be evident in the proof of the theorem.

In the same way, we obtain

COROLLARY 3. For every prime p there are $2^{\aleph_0}$ pairwise nonembeddable 2-generator p-groups.

Other methods for obtaining $2^{\aleph_0}$ nonisomorphic 2-generator p-groups are known (e.g. [5, 6]). In [5], the method did not cover the cases $n = 3, 4$ or 5 of Corollary 2.

Let H_1 and H_2 be distinct groups as in Corollary 2. Thus $H_i/N_i \cong B$. Our construction does not yield $N_1 \cong N_2$; although a detailed analysis of the construction for $B = \mathbf{Z}$ would disclose the possibility of obtaining mutual embeddability of N_1 and N_2. We leave as open questions whether $2^{\aleph_0}$ H exist as in Corollary 2 with isomorphic Hirsch-Plotkin radicals N, and whether any of the results here can be obtained for some periodic B if G is Abelian.

All of the subgroups in Theorem 1 are carefully prescribed analogues of those first used in [8] to obtain embedding theorems. The key to controlling the word problem in such subgroups is our Lemma 2 (see §II).

To prove Theorem 1 we need to construct special infinite subsets of B, similar to those used in [10], to serve as support sets of functions of Ω. These are given in Lemma 1 (see §II) which also gives the computability conditions on these subsets needed in the proofs.

Philip Hall used a pigeonhole argument to produce $2^{\aleph_0}$ 2-generator center-by-metabelian groups with prescribed countable centers [2, p. 433]. Part (a) of Theorem 1 with $B = \mathbf{Z}$ does not yield center-by-metabelian groups, but certain 2-generator class 2 nilpotent-by-infinite cyclic groups

$$(1.1) \qquad\qquad\qquad H = \langle f, x \rangle \leqslant G \operatorname{Wr} \mathbf{Z}$$

where G is class 2 *nilpotent,* $\mathbf{Z} = \langle x \rangle$, *and* $f \in \Omega$. As we should expect, the isomorphism-types of those H as in (1.1) can be distinguished much more dramatically than Theorem 1 suggests. Indeed, it is possible to determine $2^{\aleph_0}$ H's as in (1.1) which are torsion-free and satisfy: the $2^{\aleph_0}$ classes $\mathfrak{M}_H =$ the nonmetabelian members of $QS(H)$ ($=$ sections of H) are pairwise disjoint. Further, an identical result can be obtained by elaborating Hall's construction to obtain center-by-metabelian groups with the above QS-incomparability. These results will be presented in the forthcoming papers [4 and 13].

Hall's construction is the easiest and most elegant construction of $2^{\aleph_0}$ nonisomorphic 2-generator torsion-free groups of which we know. The standard method of proving this would be to use Britton's Lemma to embed every countable torsion-free group into a 2-generator torsion-free group having the same word problem degree (using the quick *HNN* method [7, p. 188]). Though elementary, the use of Turing degrees is less so than Hall's construction. The use of Turing degrees does seem very appropriate to obtain nonembeddable groups.

The types of wreath embeddings used in Theorem 1 preserve word problems about as nicely as *HNN* extensions do (see Lemma 2 in §II), and so can be used toward similar ends while producing groups of small exponent, etc. We will illustrate this by proving an effective embedding theorem similar to that using Britton's Lemma mentioned above, but for periodic groups.

DEFINITION 1. Let $P = \langle X \rangle$ be a countable group where $X = \{x_n \mid n \geq 1\}$ is a recursive set of generators. Let τ be a Turing degree such that $\tau \geq \omega_X(P) = $ *the Turing degree of the word problem of P in the generators X* (if P is finitely generated, then we need not specify X and we write $\omega(P)$). Suppose $f: P \to G$ is an embedding of P into a group $G = \langle Y \rangle$. We say that f is τ-*effective* provided:

(i) $\tau \geq \omega_Y(G)$;

(ii) $\tau \geq \omega(f(P), G)$ (where G is given in the generators Y); and

(iii) if F is the free group with basis Y, then there is a subset $\{w_n\} \leq F$ which is recursive in τ such that *in G we have* $f(x_n) = w_n$ for all $n \geq 1$.

If $\tau = \omega_X(P)$ we simply say that f is an *effective embedding* of $P = \langle X \rangle$ into $G = \langle Y \rangle$. If G is finitely generated, then we need not specify Y since any finite generating set can be used.

It might be considered more natural to replace the condition "recursive in τ" in (iii) by the stronger condition "recursive" since the $\{w_n\}$ are merely a way to *label* the generators of P in G, and not a decision procedure for them in G. However, the stronger condition that $\{w_n\} \leq F$ be recursive would present a rather serious roadblock to Theorem 2 below; hence we are forced to consider condition (iii) to be an appropriate one.

Note the following transitivity property of τ-effectiveness: if $A \leq B \leq C$ are groups (with prescribed generating sets) and the embeddings $A \leq B$ and $B \leq C$ are τ-effective, then $A \leq C$ is also τ-effective.

It is well known that the groups of Golod [1] have recursively enumerable presentations. We do not know if finitely generated infinite p-groups exist which have solvable word problems, and hence we are forced to formulate Theorem 2 somewhat relativistically.

THEOREM 2. *Suppose $P = \langle X \rangle$ is a countably infinite, periodic π-group where π is some set of primes and $\tau \geqslant \omega_X(P)$ is a Turing degree. Then, P is τ-effectively contained in a 2-generator periodic π-group if and only if there exists an infinite finitely generated periodic π-group whose word problem has Turing degree $\leqslant \tau$.*

The forward implication is, of course, trivial. The construction of the embedding is a streamlined version of [10] where P was embedded into a 2-generator π-group but no attention was paid to recursiveness. The embedding has two major steps. The first is to embed P into the derived group of a π-group Q. This can be done effectively in a standard manner via an unrestricted wreath product with the diagonal embedding, and we must also arrange in this step that P is generated in a certain way by elements of bounded orders in Q. The second step embeds Q' in the 1-coordinate of a finitely generated subgroup of an unrestricted wreath product $Q \operatorname{Wr} B$ where B is a certain finitely generated periodic π-group whose word problem has degree $\leqslant \tau$. The effectiveness of this embedding is similar to the method used in Theorem 1, the trick being to choose support sets effectively (see Lemma 2). Theorem 2 is proved in §V.

The extension of this effective embedding technique to residually finite groups will be undertaken in a forthcoming paper [3] where a careful analysis will be made of John Wilson's recent clever adaption of wreath embeddings to residually finite groups [12] in order to prove similar effective embedding theorems.

II. Preliminaries.

2.1. Here we state Lemma 1 which is needed for the proof of Theorem 1. The proof will be given in §2.2.

LEMMA 1. *Let $B = \langle D \rangle$ be an infinite group where $|D| = k$ and let F be the free group with free basis D. Further, let $(w_i \,|\, i \geqslant 1)$ be some recursive enumeration of F, and X be a fixed finite subgroup of B. Put $\beta = \omega(B)$. Then B has an infinite subset $T = \{t_n \,|\, n \geqslant 1\}$ such that:*

(i) *$t_1 = 1$ and $(t_n \,|\, n \geqslant 1)$ is a nonredundant enumeration of T which is effective in β;*

(ii) *$T \cap X = \{1\}$;*

(iii) *if $x \neq y \in X$, then $Tx \cap Ty = \varnothing$;*

(iv) *if $1 \in M \leqslant T$, then $T \cap (XM^{-1}MX) = M$;*

(v) *for all $1 \neq b \in B$, $|T \cap Tb| \leqslant 2$.*

If $b \in B$, let $\lambda(b)$ denote the least j such that in B we have $b = w_j$. Note that λ is a β-computable function.

(vi) *T can be chosen so that for all $n \geqslant 1$ and $i < n$ we have*

$$\lambda\big(t_n^{-1} t_i\big) \geqslant n \quad and \quad \lambda\big(t_i^{-1} t_n\big) \geqslant n;$$

(vii) *$\omega(T^{-1}T, B) \leqslant \beta$ and $\omega(T, B) \leqslant \beta$;*

(viii) *if $M \leqslant T$ and $\mu = \omega(M, B)$, then $\omega(M^{-1}M, B) \leqslant \beta \times \mu$;*

(ix) *there is a β-effective algorithm to compute the function $\sigma(b) = |T \cap Tb|$ from B to $\{0, 1, 2\}$ and (with M as in (viii)) there is a $\beta \times \mu$-effective algorithm to compute the function $\sigma(b) = |M \cap Mb|$ from B to $\{0, 1, 2\}$.*

We first note that property (vii) is a consequence of (i) and (vi). To see this, let $b \in B$; to decide whether $b \in T^{-1}T$ we need only check, according to (vi), whether $b = t_i^{-1}t_j$ for some $1 \leq i, j \leq \lambda(b)$ and this can be done effectively in β by the enumeration (i). Since $t_1 = 1$, this procedure also determines whether or not $b \in T$.

Likewise, (viii) follows from (i) and (vi); to specify an algorithm effective in $\mu \times \beta$ to decide membership in $M^{-1}M$ we may assume that M is infinite. Let $b \in B$ be given; we first compute $\lambda(b)$ and begin generating the enumeration $\{t_{k(i)}\}$ of M. If $b \in M^{-1}M$, then by (vii) we must find that $t_{k(i)}^{-1}t_{k(j)} = b$ for some $\max(k(i), k(j)) \leq \lambda(b)$; otherwise we will conclude that $b \notin M^{-1}M$.

Also (ix) follows from (i), (v), (vi) and (vii); for the members of $T \cap Tb$ can be determined effectively in β using (vi) and (vii) as follows. We first determine whether $b \in T^{-1}T$ and we compute $\lambda(b)$. If $b \notin T^{-1}T$, then $T \cap Tb = \varnothing$; if $b \in T^{-1}T$, then by (vi) we need only check the values $t_1, \ldots, t_{\lambda(b)} \in T$ to find all pairs $s, t \in T$ such that $b = s^{-1}t$. Each such pair gives an element of $T \cap Tb$ which we thus determine.

Hence, all parts of Lemma 1 follow if we establish parts (i)–(vi) and we take up the proof of these in the next section.

2.2. PROOF OF LEMMA 1. The set T will be constructed by induction. Let $t_1 = 1 = w_1$; then obviously $T = \{t_1\}$ satisfies the conditions (i)–(vi). Suppose we have β-effectively (with a uniform algorithm) constructed $T_k = \{t_1, t_2, \ldots, t_k\}$ such that for $1 \leq n \leq k$, T_n satisfies (i)–(vi) (in place of T). We will exhibit a uniform procedure effective in β for selecting $t_{k+1} \in B$ such that T_{k+1} satisfies (i)–(vi) and then define $T = \cup \{T_k \mid k \geq 1\}$.

For $1 \leq n \leq k$, let

$$F_n = \left\{ \prod_{i=1}^{4} a_i \mid a_i \in T_n \cup T_n^{-1} \cup X \right\}.$$

Note that F_n is finite and that $F_i \leq F_j$ when $1 \leq i \leq j \leq k$. We assume that during the construction we choose the t's so that

$$(2.2.1) \qquad\qquad t_n \notin F_{n-1} \quad \text{for } 2 \leq n \leq k.$$

Let z be any element of B with

$$(2.2.2) \qquad\qquad z \notin F_k.$$

We now show

$$(2.2.3) \qquad\qquad T_z = T_k \cup \{z\}$$

satisfies the properties (ii)–(v) of the lemma.

Parts (ii) and (iii) of (2.2.3) are proved by assuming that T_z fails to satisfy the stated condition and then obtaining a contradiction of the form $z \in F_k$.

For the property (iv), let $1 \in M \leq T_z$ and note that $M \leq T_z \cap (XM^{-1}MX)$ follows from $1 \in T_z \cap M$. Now let $u \in T_z \cap (XM^{-1}MX)$; then there are elements $\alpha, \beta \in X$ and $v, w \in M$ such that

$$(2.2.4) \qquad\qquad u = \alpha v^{-1}w\beta.$$

If z does not occur in (2.2.4) the inductive hypothesis

$$T_k \cap \left(\left(X(T_k \cap M)(T_k \cap M)^{-1} X \right) \right) = T_k \cap M$$

gives $u \in M$. Thus, we assume that z occurs in (2.2.4); further, since $z \notin F_k$, z must occur exactly twice in (2.2.4). The cases $u = v = z$ or $u = w = z$ give the desired conclusion and we are left with the case $u = \alpha z^{-1} z \beta = \alpha\beta$ where $u \in T_k$. We now use (2.2.1) to deduce that $u = t_1 = 1 \in M$ and this gives property (iv).

To verify (v) for the set T_z we suppose that for some $1 \neq d \in B$ we have $|T_z \cap T_z d| \geq 3$. Then there are three distinct elements a, b, c of T_z and elements u, v, w of T_z such that

$$d = u^{-1}a = v^{-1}b = w^{-1}c.$$

Evidently, u, v, w are also distinct and our choice of z forces z to occur in each of the three products $u^{-1}a$, $v^{-1}b$, $w^{-1}c$; i.e., $z \in \{u, a\} \cap \{v, b\} \cap \{w, c\}$. This, in turn, contradicts the fact that u, v and w are distinct.

At this stage we have verified all parts of (2.2.3). Note that the property (2.2.2) is β-decidable since membership in F_k is β-decidable.

Now let $U_k = \{w_1, \ldots, w_{k-1}\} \leq B$. Thus $R_k = T_k U_k \cup T_k U_k^{-1}$ is a finite subset of B. It is easy to show that $z \notin R_k$ implies (vi) of the lemma.

We now display the uniform β-effective algorithm for the choice of t_{k+1}. Using a β-oracle we check the elements w_i of B (with increasing subscripts) for

(1) membership in $B - F_k$, and

(2) membership in $B - R_k$.

The element t_{k+1} is the first w_i which meets these requirements; the existence of t_{k+1} follows from the fact that both R_k and F_k are finite. We have already shown that $T_{k+1} = \{t_1, \ldots, t_k, t_{k+1}\}$ satisfies (ii)–(vi) and property (i) now also holds. This completes the proof of Lemma 1.

2.3. The next lemma is the key for controlling Turing degrees. Necessary symbols and concepts are as in the introduction.

LEMMA 2. *Let $G = \langle X \rangle$ and B be groups with B finitely generated, let $f \in \Omega =$ the base group of $W = G \operatorname{Wr} B$ with $M = \operatorname{supp}(f) = \{b \in B \mid f(b) \neq 1\}$, and let τ be a Turing degree. Assume the following three conditions.*

(i) $\omega(B) \leq \tau$ and $\omega_X(G) \leq \tau$;

(ii) there is a τ-effective algorithm for computing the values $f(b) \in \langle X \rangle$ ($b \in B$), and

(iii) for all $1 \neq c \in B$, $\sigma(c) = |M \cap Mc|$ is finite and $\sigma(c)$ is computable by a τ-effective algorithm.

Then, there is a τ-effective algorithm to decide, given $g \in f^B \leq \langle f, B \rangle \cap \Omega \leq W$, whether $\operatorname{supp}(g) \leq \{1\}$ and whether $g = 1$.

PROOF. Write $g = (f^{\varepsilon_1})^{b_1} \cdots (f^{\varepsilon_n})^{b_n}$ where, for $1 \leq i \leq n$, $\varepsilon_i = \pm 1$ and $b_i \in B$. For $1 \leq i \leq n$, compute the exponent sum

$$\mu_i = \sum_{1 \leq j \leq n} \{\varepsilon_j \mid b_j = b_i\}$$

using an oracle for $\tau \geqslant \omega(B)$. Define

$$F_1 = \cup \ \{Mb_i \cap Mb_j \mid 1 \leqslant i, \ j \leqslant n \text{ and } b_i \neq b_j\}$$

and note that F_1 is finite by (iii).

Suppose first that, for all finite subsets $M_0 \leqslant M$, $\{|f(s)| \mid s \in M - M_0\}$ is unbounded. If $\mu_i \neq 0$ for some i, then we claim that $\mathrm{supp}(g)$ is infinite because we can choose infinitely many $x \in Mb_i - F_1$ such that $|f(xb_i^{-1})|$ does not divide μ_i and we have $g(x) = [f^{b_i}(x)]^{\mu_i} = f(xb_i^{-1})^{\mu_i} \neq 1$.

On the other hand suppose that there is a bound on almost all of the numbers $|f(x)|$, $s \in M$. Define for $d > 1$ $U_d = \{b \in M \mid |f(b)| = d\}$ and let $r = $ least common multiple of all those d's for which U_d is infinite; thus, r is finite by the hypothesis of this case. If, for some i, r does not divide μ_i, then there is some $d > 1$ such that U_d is infinite and d does not divide μ_i and an argument identical to the first case above shows that $\mathrm{supp}(g)$ is infinite.

Thus, in these two cases we would decide that $\mathrm{supp}(g) \nleqslant \{1\}$ and so we may suppose either:

Case A. $\mu_i = 0$ for all $1 \leqslant i \leqslant n$, or

Case B. There is a finite subset $V \leqslant M$ and for all $s \in M - V$, $|f(s)|$ divides r (as defined above) and r divides μ_i for all $1 \leqslant i \leqslant n$.

Note that since f is given we determine whether Case A or B holds after computing the μ_i. Define $F_2 = \varnothing$ if Case A holds; and $F_2 = Vb_1 \cup \cdots \cup Vb_n$ if Case A does not hold, but Case B holds. Thus $F_1 \cup F_2$ is finite and an easy argument shows that $\mathrm{supp}(g) \leqslant F_1 \cup F_2$. From the hypotheses (i) and (ii), the lemma will be proved if we can display a τ-effective algorithm for generating (in a definite finite time) a complete list of $F_1 \cup F_2$. Since V can be specified as soon as f is known, there is no problem in constructing a list of F_2. To list F_1 we observe

(2.3.1)

$z \in F_1$ if and only if there exist $u, v \in M$ and $b_i \neq b_j$ such that $z = ub_i = vb_j$.

Thus, to effectively list (with τ-oracle) all $z \in F_1$, we first decide for each pair $b_i \neq b_j$ whether $\sigma(b_i b_j^{-1}) \neq 0$, i.e. whether $Mb_i \cap Mb_j \neq \varnothing$, by computing this value of σ in accordance with (iii). If we find $\sigma(b_i b_j^{-1}) \neq 0$, then we begin generating a τ-effective enumeration of $M = \mathrm{supp}(f)$ (this can be done by (i) and (ii)) and listing all pairs $u, v \in M$ (as we find them) for which $b_i b_j^{-1} = u^{-1}v$. Now each such distinct pair corresponds to a distinct element of $Mb_i \cap Mb_j$, and hence we must be able to list exactly $\sigma(b_i b_j^{-1}) = |Mb_i \cap Mb_j|$ such pairs. Since σ is a τ-computable function on B, we will know to stop the enumeration when exactly $\sigma(b_i b_j^{-1})$ distinct pairs (u, v) have been found. These pairs (u, v) (for all possible $b_i \neq b_j$) give us an exhaustive list of F_1 by (2.3.1).

We will also need a simple lemma on relative recursiveness.

LEMMA 3. *If T is an infinite set of Turing degree $\leqslant \beta$ and $\tau \geqslant \beta$ is a Turing degree, then there is an infinite subset M of T whose Turing degree μ satisfies $\tau = \mu \times \beta$.*

PROOF. Let $T = \{t_k \mid k \geqslant 1\}$ and let $S = \{k_i \mid i \geqslant 1\}$ be an infinite set of natural numbers of Turing degree τ. Define $M = \{t_{k_i} \mid i \geqslant 1\}$. One checks easily that the Turing degree μ of M satisfies $\tau = \mu \times \beta$.

III. Proof of Theorem 1(a). We first define elements in the base group of $W = G \operatorname{Wr} B = \Omega B$ which have special support sets and which are used in the proofs of both parts of Theorem 1.

Let M be an infinite subset of T with $1 \in M$ where $T \leqslant B$ is the subset given in Lemma 1. Let $G = \langle x, y \rangle$. Define $f = f_M \in \Omega$ as follows.

(3.1) $f_M(1) = x$, $f_M(t) = y$ if $t \in M - \{1\}$ and $f_M(u) = 1$ if $u \notin M$.

For the proof of part (a) of Theorem 1, we define

(3.2) $H = H_M = \langle f, B \rangle \leqslant W;$

and note that H has $k + 1$ generators.

Let $\tau \geqslant \omega(G \times B)$ be a Turing degree. From Lemma 3 we obtain $M \leqslant T$ satisfying

(3.3) $\tau = \omega(M, B) \times \omega(B).$

Since the addition of 1 to M will leave (3.3) unchanged, we can assume $1 \in M$.

We must show that $\omega(H) = \tau$.

To show that $\omega(H) \leqslant \tau$ suppose that w is a word in the generators of H. We first write w in the form bh where $h \in \Omega$ is a product of conjugates of $f^{\pm 1}$ and $b \in B$. Since $\omega(B) \leqslant \tau$ we decide τ-effectively whether $b = 1$. If $b \neq 1$, then $w \neq 1$; so assume we find $b = 1$. We now use Lemma 2 to decide τ-effectively whether $w = h = 1$. Hypothesis (i) of Lemma 2 holds, i.e. $\tau \geqslant \omega(G \times B)$, (ii) holds by (3.1) and (3.3), and (iii) holds according to (ix) of Lemma 1.

To establish $\tau \leqslant \omega(H)$, assume that a decision procedure for $\omega(H)$ is available. Since the k generators of the top group B of W are among the generators of H, we can decide $\omega(B)$. In view of (3.3) we need only display a decision procedure for $\omega(M, B)$ to complete our proof. Let

$$\Delta_M = \left\{ b \in B \mid [f^b, f] \neq 1 \right\}.$$

It is not difficult to show, using (3.1) that $\Delta_M = (M - \{1\}) \cup (M^{-1} - \{1\})$. Hence this set is $\omega(H)$-decidable.

Since we have a decision procedure for $\omega(B)$ available, we can also decide the set $M \cup M^{-1}$. Since $M \cup M^{-1} \leqslant M^{-1}M$ (because $1 \in M$), we have $T \cap (M \cup M^{-1}) = M$ by (iv) of Lemma 1. Since $\omega(T, B) \leqslant \omega(B)$, the above intersection property provides a decision procedure for $\omega(M, B)$ which is effective in $\omega(H)$, thus finishing the proof.

IV. Proof of Theorem 1(b). We will denote any sequence $(b_1, \ldots, b_{k-1})$ having $k - 1$ terms by $\vec{b}$. Let F be a free group of rank k with free generators $\{s, \vec{t}\}$ and $B = \langle a, \vec{b} \rangle$. Let $F = (w_n \mid n \geqslant 1)$ be a recursive list of F where *$i \leqslant j$ implies that the length of w_i is less than or equal to the length of w_j.*

Let θ be the homomorphism from F onto B defined by $\theta(s) = a$ and $\theta(\vec{t}) = \vec{b}$. Then with λ defined as in Lemma 1 and $b \in B$ we have

(4.1) *if $\lambda(b) = j$, then w_j is a word of minimal length in the coset $\theta^{-1}(b)$.*

Thus,

(4.2) $\mathscr{B} = \{w_j \,|\, j = \lambda(b)$ *for some $b \in B\}$ is a transversal of* $\ker\theta$ *in F with* $1 \in \mathscr{B}$.

We also adopt the notation

(4.3) *if $U = \langle r, \vec{v} \rangle$ is a k-generator group and $w \in F$, then $w(r, \vec{v})$ is the image of w in U under the unique homomorphism of F to U that maps s to r and $\vec{t}$ to $\vec{v}$. In the special case $U = B = \langle a, \vec{b} \rangle$ we use the notation $w(a, \vec{b}) = w^0$.*

Recalling that B is periodic, let $n = |a|$, $X = \langle a \rangle$, and let $T \leqslant B$ be the subset given in Lemma 1 (for our particular choice of $X \leqslant B$ and λ). Let $\tau \geqslant \omega(B \times G)$ be a Turing degree and choose $M \leqslant T$ as in (3.3) and so that $1 \in M$. We define $f = f_M \in \Omega$ as in (3.1). Define

$$\text{(4.4)} \qquad\qquad H = H_M = \langle af, b_1, \ldots, b_{k-1} \rangle,$$

and

$$\text{(4.5)} \qquad g = g_M = (af)^n = f^{a^{n-1}} \cdots f^a f \quad (\text{since } n = |a|).$$

Thus, $g \in H \cap \Omega$ and property (iii) of Lemma 1 together with (3.1) gives

$$\text{(4.6)} \qquad \text{supp}(g) = MX \text{ and } g(c) = x \text{ if } c \in X \quad \text{while } g(c) = y \text{ if } c \in MX - X.$$

Note that if $h \in \Omega$ and $w \in B$, then

(4.7) $w(ah, \vec{b}) = w^0(h^{\varepsilon_1})^{w_1^0} \cdots (h^{\varepsilon_n})^{w_n^0}$, *where for $1 \leqslant i \leqslant n$, $\varepsilon_i = \pm 1$ and w_i is a proper final segment of w.*

An easy consequence of (4.1) and (4.2) is the crucial fact

(4.8) if $1 \neq w \in \mathscr{B}$ and $w_1, \ldots, w_n$ are proper final segments of w, then

$$\left(f_M^{\varepsilon_1}\right)^{w_1^0} \cdots \left(f_M^{\varepsilon_n}\right)^{w_n^0}(w^0) \in \langle y \rangle.$$

To prove (4.8) note that the choice of our set $\mathscr{B}$ ((4.1) and (4.2)) forces $w^0(w_i^0)^{-1} \neq 1$ for $1 \leqslant i \leqslant n$.

Thus, for each i, $f_M^{\varepsilon}(w^0(w_i^0)^{-1})$ is either 1, y or y^{-1} and (4.8) follows easily.

Now define

$$\text{(4.9)} \qquad \Delta_M = \left\{ w \in \mathscr{B} \,\Big|\, \left[g^{w(af, \vec{b})}, g \right] \neq 1 \right\} \cup \{1\}$$

and note the slight technical difference from the way Δ_M was defined in §III. The key observation is

LEMMA 4. $M \leqslant \Delta_M^0 \leqslant XM^{-1}MX$.

PROOF. It follows from (4.6) and (iii) of Lemma 1 that $g_M(1) = x$ and that $g_M(w^0) = y$ if $1 \neq w^0 \in M$. Let $1 \neq w^0 \in M$; then, as in (4.7),

$$w\left(af_M, \vec{b}\right) = w^0 h, \quad \text{where } h = \left(f_M^{\varepsilon_1}\right)^{w_1^0} \cdots \left(f_M^{\varepsilon_n}\right)^{w_n^0}.$$

Thus (4.8) implies that $h(w^0) \in \langle y \rangle$, and so

$$\left[g_M^{w(af_M, \vec{b})}, g_M \right](w^0) = \left[g_M^{w^0 h}, g_M \right](w^0) = \left[\left(g_M\left(w^0(w^0)^{-1}\right) \right)^{h(w^0)}, g_M(w^0) \right]$$

$$= \left[g_M(1)^{h(w^0)}, g_M(w^0) \right] = \left[x^{y^i}, y \right],$$

which is not 1 since $[x, y] \neq 1$. Hence $w^0 \in \Delta_M^0$.

Suppose now that $w \in \Delta_M$; then $[g_M^{w(af_M, \vec{b})}, g_M] \neq 1$ and using the notation above, $[g_M^{w^0 h}, g_M] \neq 1$. Now by (3.9) and standard facts about wreath products, $\operatorname{supp}[g_M^{w^0 h}, g_M] \leq MXw^0 \cap MX$ and so $MXw^0 \cap MX \neq 1$. Thus, $w^0 \in XM^{-1}MX$ and this completes the proof of Lemma 4.

PROOF THAT $\omega(B \times H) \leq \tau$. We may assume that we have available decision procedures for M, for T, and for the word problem of B, and we want to construct a decision procedure for the word problem of H. Let

$$d = w\left(af, \vec{b} \right) = w^0 (f^{\epsilon_n})^{w_1^0} \cdots (f^{\epsilon_n})^{w_n^0}$$

be an element of H as expressed in (4.7). We first decide whether $w^0 = w(a, \vec{b}) = 1$ in B. We may assume that we find $w^0 = 1$ for otherwise $d \neq 1$. Thus $d = (f^{\epsilon_1})^{w_1^0} \cdots (f^{\epsilon_n})^{w_n^0}$.

Since $\tau \geq \omega(G \times B) \times \omega(M, B)$ and in view of (3.1) and (ix) of Lemma 1, all hypotheses of Lemma 2 are satisfied and Lemma 2 gives a τ-effective algorithm to decide whether $d = 1$.

PROOF THAT $\tau \leq \omega(B \times H)$. We assume that we have available oracles for $\omega(B)$ and $\omega(H)$. We will first observe that this enables us to decide $\omega(\Delta_M^0, B)$. Suppose $b \in B$. Using our $\omega(B)$-oracle we can determine the unique $w \in \mathfrak{B}$ such that $b = w^0$. Thus, to decide whether $b \in \Delta_M^0$, consulting (4.9), it suffices to decide whether either $b = 1$ or $[g^{w(af,b)}, g] \neq 1 \in H$ which we can do with our oracles. Now, in view of the equation $T \cap XM^{-1}MX = M$ ((iv) of Lemma 1) and Lemma 4, we have $T \cap \Delta_M^0 = M$ and hence $\omega(M, B)$ can be decided with our oracles for $\omega(B)$ and $\omega(H)$; and in view of (3.3) we can also decide τ.

V. Proof of Theorem 2.

5.1. We first note some elementary facts regarding restricted wreath products.

(5.1.1) *Let $H = \langle h_1, \ldots, h_n \rangle$ be a finitely generated group and put $B = H \operatorname{wr} H$. Then B has generating set $\{h_1, \ldots, h_n, \bar{h}_1, \ldots, \bar{h}_n\}$ where for $1 \leq i \leq n$, $\bar{h}_i$ is the function in the base group of B with $\operatorname{supp}(\bar{h}_i) = \{1\}$ and $\bar{h}_i(1) = h_i$. Further, $\omega(B) = \omega(H)$.*

The equality $\omega(B) = \omega(H)$ follows from the fact that elements $x \in B$ have the unique representation $x = hc$ where $h \in H$ and c is in the base group of B and both h and c are $\omega(H)$-computable.

(5.1.2) *Let f be an element in the base group of B with $f^2 \neq 1$ and $\operatorname{supp}(f) = \{1\}$. Then the set $M = \{f^h \mid h \in H\}$ has the properties*

(i) *for all $1 \neq c \in B$, the equation $c = u^{-1}v$ can hold for at most one pair $(u, v) \in M \times M$; this is called sparseness property [5, p. 181]; and is equivalent to $|M \cap Mc| \leq 1$ if $1 \neq c \in B$;*

(ii) $\omega(M, B) \times \omega(M^{-1}M, B) \leqslant \omega(H)$;

(iii) *for all* $1 \neq c \in B$, $\sigma(c) = |M \cap Mc|$ *is computable by an* $\omega(H)$-*effective algorithm.*

In (5.1.2), the verification of (i) is straightforward while (ii) follows as in the proof of (5.1.1). For (iii), we see from (i) that $|M \cap Mc| \leqslant 1$. Note that $|M \cap Mc| = 1$ if and only if $c \in M^{-1}M$ and this can be decided by (ii). Thus (iii) holds, and we note that

(5.1.3) *if* $\tau \geqslant \omega(H)$, M *satisfies the condition* (iii) *of Lemma* 2.

It is also convenient to record here a restricted version of Theorem 2.

LEMMA 5. *If R is a finitely generated periodic π-group, then R is effectively contained in a 2-generator periodic π-group S.*

PROOF. The embedding is the one given first by Neumann and Neumann [8] (see also Phillips [10, p. 251]) and the reader is referred to these proofs. More complicated variants of this procedure will appear in the proof of Theorem 2; here we only give a very brief sketch of the embedding. The essential features are given in

(5.1.4) *there are finite cyclic groups C_n and C_k and a function f in the base group of* $(R \, \mathrm{wr} \, C_n) \, \mathrm{wr} \, C_k$ *such that R is embedded in $f^C k \leqslant \langle f, C_k \rangle$ as the diagonal of the* 1-*coordinate subgroup.*

The effectiveness of this embedding follows easily from the fact that both C_n and C_k are finite.

5.2. *The construction.* Let $P = \langle X \rangle$ be an infinite periodic π-group and p be a fixed prime involved in P. Put

$$n = \begin{cases} p & \text{if } p \text{ is odd,} \\ 4 & \text{if } p = 2. \end{cases}$$

There are two steps in the embedding of Theorem 2.

Step 1. We will embed P effectively into a periodic π-group $Q = \langle Y \rangle$ in such a way as to satisfy:

(5.2.1) *For each $x \in X$ there are elements c_1, c_2, c_3, c_4 in Q such that $x = [c_1 c_2, c_3 c_4]$ and the c_i's may be chosen so that $c_i = [d_i, e_i]$ where d_i and e_i are elements of Q of order n which are $\omega_X(P)$-computable in the generators Y of Q.*

For the proof of (5.2.1) we require a group $J = \langle y_1, y_2 \rangle$ which is nilpotent of class 2 in which $|y_1| = |y_2| = |[y_1, y_2]| = n$. Thus $|J| = n^3$.

The generators Y of Q will be denoted (and specified later) by

(5.2.2) $\qquad\qquad Y = \{w_x \mid x \in X\} \cup \{z_x \mid x \in X\} \cup J.$

Put $A = \mathrm{Dr}\{\langle z_x \rangle \mid x \in X\}$ (direct product); Q will be a subgroup of $W = (P \, \mathrm{Wr} \, A) \, \mathrm{wr} \, J$ where P is embedded in the base group of $P \, \mathrm{Wr} \, A$ diagonally and $P \, \mathrm{Wr} \, A$ is embedded in $(P \, \mathrm{Wr} \, A)^J =$ the base group of W, as the 1-coordinate subgroup.

We now specify the generators of Q introduced in (5.2.2).

(5.2.3) (i) $|z_x| = |x|$ *for all* $x \in X$; *thus* $|z_x|$ *is* $\omega_X(P)$-*computable since P is periodic. A will be identified with the top group of the* 1-*coordinate of W.*

(ii) *The element w_x in the base group of $P \operatorname{Wr} A$ is defined by $w_x(t) = x^{-i}$ where $t \equiv z_x^i (\operatorname{mod} \operatorname{Dr}\{\langle z_\alpha \rangle \mid x \neq \alpha \in X\})$. The element $[w_x, z_x]$ is a diagonal element with value x; we henceforth write $[w_x, z_x] = x$ and this describes the diagonal embedding of P in $P \operatorname{Wr} A$.*

(iii) *J is the top group of W.*

Note that $\langle w_x \mid x \in X \rangle$ is contained in the set Λ of all elements in the base group of $P \operatorname{Wr} A$ which have finite range. The set Λ is obviously a normal subgroup of $P \operatorname{Wr} A$ and as P is periodic, an element of Λ must have finite order. Thus, $\langle w_x \mid x \in X \rangle^A$ is a periodic π-group and it now follows easily that $Q = \langle Y \rangle$ is a periodic π-group.

Also, $\omega_X(P) \leq \omega_Y(Q)$ since $x = [w_x, z_x]$. For the next part of the proof of Step 1 we verify

$$(5.2.4) \qquad\qquad \omega(P, Q) \leq \omega_Y(Q) = \omega_X(P).$$

Let $h \in Q = \langle Y \rangle$; since we already know that $\omega_X(P) \leq \omega_Y(Q)$, (5.2.4) will follow if we specify an algorithm effective in $\omega_X(P)$ to decide whether or not $h \in P$ and, if so whether or not $h = 1$. First reduce h to the form $h = \sigma(\prod_{g \in J} f_g^g)$ where $f_g \in \langle w_x, z_x \mid x \in X \rangle \leq P \operatorname{Wr} A$ and $\sigma \in J$. If $\sigma \neq 1$ we conclude that $h \notin P$, so assume $\sigma = 1$. Note that the factors $\{f_g^g \mid g \in J\}$ commute pairwise since $P \operatorname{Wr} A$ is embedded as the 1-coordinate subgroup of the base group of W. Next we reduce every f_g, $g \in J$:

$$(5.2.5) \qquad\qquad f_g = a_g \left[\prod_{i=1}^{n} \left((w_{x_i})^{k_i} \right)^{u_i} \right]$$

where $a_g, u_i \in A$, $x_i \in X$ and k_i is an integer (all of which depend on g). Using a $\omega_X(P)$-oracle to compute the orders of the z_x we can determine for all $g \in J$ whether or not $a_g = 1$. If $a_g \neq 1$ for some $g \in J$ then we conclude $h \notin P$. So, we can assume that $a_g = 1$ and so f_g is in the base group of $P \operatorname{Wr} A$ for all $g \in J$.

We will now show how to decide, for every $g \in J$, (i) whether f_g lies in the diagonal of P^A and (ii) if so, whether $f_g = 1$. Having decided (i) and (ii) we decide whether $h \in P$ as follows. If $f_g \neq 1$ for some $g \neq 1$ then $h \notin P$ since P is embedded in the 1-coordinate of W; if $f_g = 1$ for all $g \neq 1$ then $h \in P$ if and only if f_1 is in the diagonal of $P \operatorname{Wr} A$ and $h = 1$ if and only if $f_1 = 1$.

To decide (i) and (ii) let

$$f_g = f = \prod_{i=1}^{n} \left((w_{x_i})^{k_i} \right)^{u_i}$$

as in (5.2.5) with $a_g = 1$. Let $z_1, \ldots, z_j$ denote *all* of the generators of A which occur in the elements $u_1, \ldots, u_n$ and let $A_0 = \langle z_1, \ldots, z_j, z_{x_1}, \ldots, z_{x_n} \rangle$. Thus, A_0 is a finite group and the computation of the values of f is identical on the cosets of A_0 in A; i.e., if $a \in A_0$ and $t \in \operatorname{Dr}\{\langle z_x \rangle \mid z_x$ is not one of the generators of $A_0\}$, then $f(t + a) = f(a)$. Thus, to check whether f is diagonal (resp., nontrivial) we need only compute its values on A_0 using a decision procedure for $\omega_X(P)$ to get the orders of the z_x's and to compare the values of f that we obtain. These values are obtained from conjugates of the w_x, and so are computed as words in the generators X of P. This completes the proof of (5.2.4).

The final part of Step 1 is the proof of (5.2.1) which we now take up.

In the group J we let $y = [y_1, y_2]$. Then $|y| = n > 2$. If $x \in X$, then

$$x = [w_x, z_x] = \left[\left[w_x^{-1}, y\right], \left[z_x^{-1}, y^2\right]\right]$$

because $y^2 \neq 1$ and the embedding of $P \operatorname{Wr} A$ into W is on the 1-coordinate. Since y is the commutator $[y_1, y_2]$ whose entries have order n we have

$$\left[w_x^{-1}, y\right] = \left(y^{-1}\right)^{w_x^{-1}} y = c_1 c_2,$$

a product of two such commutators. Similarly, noting that $y^2 = [y_1^2, y_2]$ we have $[z_x^{-1}, y^2] = c_3 c_4$. The computation from x of the entries of the commutators c_i (the d_i's and e_i's) in the generators Y of Q is clearly effective. This concludes the proof of Step 1.

Step 2. We now embed $P = \langle X \rangle$ τ-effectively into a $(2n + 1)$-generator periodic π-group R.

Let $H = \langle h_1, \ldots, h_n \rangle$ be an infinite finitely generated π-group with $\omega(H) \leqslant \tau$ as given in the hypothesis of Theorem 2. Thus, H does not have exponent 2 and $B = H \operatorname{wr} H$ has $2n$ generators as in (5.1.1) and $\omega(B) = \omega(H)$.

Let M be the sparse subset of B as given in (5.1.2); from (5.1.2)(ii), M is recursive in $\omega(B) \leqslant \tau$. Let $x \in X$ and for $1 \leqslant i \leqslant 8$ define $\alpha_i(x)$ by

$$\alpha_i(x) = \begin{cases} d_i & \text{if } 1 \leqslant i \leqslant 4, \\ e_i & \text{if } 5 \leqslant i \leqslant 8, \end{cases}$$

where the d_i and e_i's are as in (5.2.1). As noted earlier $\alpha_i(x)$ is effectively computable as a word in the generators Y of Q from x and i. Now let $\{x_j \,|\, j \geqslant 0\}$ be an effective enumeration of the recursive set X (if X is finite, put $x_j = 1$ after a point). We now define an element v in the base group of $V = Q \operatorname{Wr} B$ as follows:

$$v(c) = \begin{cases} 1 & \text{if } c \notin M, \\ \alpha_i(x_j) & \text{if } c = t_{8j+i} \ (1 \leqslant i \leqslant 8). \end{cases}$$

Thus given $c \in B$ we can compute the value $v(c) \in Q$ τ-effectively. Now put $R = v^B B \leqslant V$ and note that R has $(2n + 1)$-generators, namely v and the $2n$ generators of B. We now show that R has the properties asserted in Step 2.

To prove that R is a periodic π-group, note first that $|v| = n$ (since each of the d_i and e_i has order n); this together with the "sparseness" of M implies that v^B is a periodic π-group [10, p. 247], and it is now immediate that R is a periodic π-group. We now take up the embedding aspects of Step 2.

Define $\delta_i \in v^B$ by

$$\delta_i = \left[v^{(t_{8j+i})^{-1}}, v^{(t_{8j+i+4})^{-1}}\right] \qquad (j \geqslant 0, 1 \leqslant i \leqslant 4)$$

and notice that δ_i has support $M(t_{8j+i})^{-1} \cap M(t_{8j+i+4})^{-1} = \{1\}$ by the sparseness of M. Further, for $1 \leqslant i \leqslant 4$ we have

$$\delta_i(1) = \left[v\left(t_{8j+i}\right), v\left(t_{8j+i+4}\right)\right] = \left[\alpha_i(x_j), \alpha_{i+4}(x_j)\right]$$

$$= [d_i, e_i] = c_i \quad \left(\text{corresponding to } x_j \text{ in (5.2.1)}\right).$$

Hence $[\delta_1\delta_2, \delta_3\delta_4]$ (1) $= x_j$, displaying the embedding of P into the 1-coordinate of v^B. Since M is τ-recursive and v is τ-computable, the above embedding displays a τ-effective procedure for computing the generators $x \in X$ in terms of the generators of R. Thus $\omega_X(P) \leqslant \omega(R)$.

Also since $\omega_y(Q) = \omega_x(P) \leqslant \tau$ and v is τ-computable, (5.1.3) together with Lemma 2 imply that $\omega(R) \leqslant \tau$ and that there is a τ-effective procedure for deciding when an element in v^B has support $\{1\}$. Now we use the effectiveness of P in Q ($=$ the 1-coordinate of V) to conclude that $\tau \geqslant \omega(P, R)$.

This concludes the proof of Step 2 and Theorem 2 now follows directly from Lemma 4 because the relation "is τ-effectively embedded in" is transitive.

ADDED IN PROOF. We have recently seen the constructions of Grigorcuk for $p = 2$ and of Gupta and Said for $p > 2$ of Burnside groups which have solvable word problems. It follows that our Theorem 2 can be improved to read: *Every countable π-group can be effectively embedded in a 2-generator π-group*. In the proof we can choose the sparse subset M to be recursive and it follows that in the definition of an effective embedding (Definition 1) we can strengthen condition (iii) by replacing "recursive in τ" by "recursive", which is a more natural concept.

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