

SUBCONTINUA WITH DEGENERATE TRANCHES IN HEREDITARILY DECOMPOSABLE CONTINUA¹

BY

LEX G. OVERSTEEGEN AND E. D. TYMCHATYN

ABSTRACT. A hereditarily decomposable, irreducible, metric continuum M admits a mapping f onto $[0, 1]$ such that each $f^{-1}(t)$ is a nowhere dense subcontinuum. The sets $f^{-1}(t)$ are the tranches of M and $f^{-1}(t)$ is a tranche of cohesion if $t \in \{0, 1\}$ or $f^{-1}(t) = \text{Cl}(f^{-1}([0, t))) \cap \text{Cl}(f^{-1}((t, 1]))$. The following answer a question of Mahavier and of E. S. Thomas, Jr.

THEOREM. *Every hereditarily decomposable continuum contains a subcontinuum with a degenerate tranche.*

COROLLARY. *If in an irreducible hereditarily decomposable continuum each tranche is nondegenerate then some tranche is not a tranche of cohesion.*

The theorem answers a question of Nadler concerning arcwise accessibility in hyperspaces.

1. Introduction. A continuum is a compact connected metric space. A continuum M is said to be *irreducible* between two points p and q if no proper subcontinuum of M contains both p and q .

A continuum M is said to be of *type λ* (see [8, p. 200]) if there exists a map ϕ of M onto $[0, 1]$ such that each point inverse under ϕ is a nowhere dense subcontinuum of M . The sets $\phi^{-1}(t)$ are called the *tranches* of M . The sets $\phi^{-1}(0)$ and $\phi^{-1}(1)$ are called *end-tranches* of M . The tranche $\phi^{-1}(t)$ is said to be a *tranche of cohesion* if $t \in \{0, 1\}$ or if

$$\phi^{-1}(t) = \text{Cl}(\phi^{-1}([0, t))) \cap \text{Cl}(\phi^{-1}((t, 1])).$$

We denote the closure of a set A by $\text{Cl}(A)$ and the boundary of A by $\text{Bd}(A)$.

Irreducible continua have been extensively studied, in particular, under the topic of continuous collections. For example, an irreducible continuum which admits a monotone open mapping onto $[0, 1]$ is a continuum of type λ and has the additional property that each tranche is a tranche of cohesion. Also, irreducible, hereditarily decomposable continua are of type λ .

Thomas in [14] and Mahavier in [9] proved that each hereditarily decomposable arc-like continuum contains a subcontinuum with a degenerate tranche. In the main result of this paper we extend the Thomas and Mahavier result to arbitrary hereditarily decomposable continua. This answers, in the affirmative, Problem 121 in

Received by the editors November 9, 1979 and, in revised form, August 3, 1982. Presented to the Society, at the AMS meeting in San Antonio, January 1980.

1980 *Mathematics Subject Classification*. Primary 54F20, 54F50; Secondary 54C10.

Key words and phrases. Hereditarily decomposable continua, monotone decomposition into tranches.

¹The research for this paper was supported in part by NSERC grant number A5616.

the *University of Houston problem book* (due to Mahavier). Our methods are patterned on those used by both Thomas and Mahavier. These methods are abstracted from a proof of Henderson [4].

To prove the existence of an indecomposable continuum, one constructs a sequence O_i of open covers such that O_{i+1} “folds” in O_i . The notion of folding in chain covers is intuitively clear. A large part of this paper is devoted to a definition of folding in covers whose nerves are arbitrary polyhedra.

In 1935 Knaster [6] constructed a monotone, open mapping of a certain irreducible continuum onto $[0, 1]$ such that each point inverse is nondegenerate. Dyer proved in [2] (see [7] for a simple proof) that each such mapping has a dense G_δ of indecomposable point inverses. As a corollary to our main theorem we complement Dyer’s theorem by proving that if M is a continuum of type λ , such that each tranche is nondegenerate and is a tranche of cohesion, then M contains indecomposable subcontinua of arbitrarily small diameters. Also, as a corollary to our main result we obtain an affirmative solution to a question of Nadler [12] concerning arcwise accessibility in hyperspaces.

2. Definitions and preliminaries. We let M be a continuum with a fixed but arbitrary metric d . If $\mathcal{U}$ is a collection of subsets of M and $A \subset M$ we set

$$S^1(A, \mathcal{U}) = S(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U} \mid U \cap A \neq \emptyset\}$$

and, inductively,

$$S^n(A, \mathcal{U}) = S(S^{n-1}(A, \mathcal{U}), \mathcal{U}).$$

We let

$$\mathcal{U}^* = \{S(U, \mathcal{U}) \mid U \in \mathcal{U}\} \quad \text{and} \quad \mathcal{U}^{**} = \{S^2(U, \mathcal{U}) \mid U \in \mathcal{U}\}.$$

If $\mathcal{U}$ and $\mathcal{V}$ are two collections of subsets of X we say $\mathcal{U}$ *refines* $\mathcal{V}$ if for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ with $U \subset V$. If $\mathcal{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ and $\mathcal{V} = \{V_\gamma \mid \gamma \in \Gamma\}$ and $U_\gamma \subset V_\gamma$ for each $\gamma \in \Gamma$, then $\mathcal{U}$ is said to be a *precise refinement* of $\mathcal{V}$.

A collection $\mathcal{U}$ of sets is said to be *taut* if $U, V \in \mathcal{U}$ with $C1(U) \cap C1(V) \neq \emptyset$ implies $U \cap V \neq \emptyset$. The collection $\mathcal{U}$ is said to be *coherent* if $U, V \in \mathcal{U}$ implies there exists $U_1 = U, U_2, \dots, U_n = V$ in $\mathcal{U}$ with $U_i \cap U_{i+1} \neq \emptyset$ for each $i = 1, \dots, n - 1$. If $\mathcal{U}$ is a collection of open sets in a set M and $U \in \mathcal{U}$ let

$$i(U, \mathcal{U}) = U \setminus C1(\bigcup \{V \mid U \neq V \in \mathcal{U}\}).$$

If $K \subset M$ is a continuum we say a collection $\mathcal{U}$ of subsets of M *strongly irreducibly covers* K if $\{C1(U) \mid U \in \mathcal{U}\}$ is an irreducible cover of K . Notice that if $\mathcal{U}$ is an irreducible open cover of K then there exists an open, taut, precise refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ covers K and $\mathcal{V}$ is a strongly irreducible cover of K .

We shall need the following well-known result (see [8, p. 172]):

BOUNDARY BUMPING THEOREM. *If K is a component of a proper open subset U of a continuum X then $\text{Bd}(U) \cap C1(K) \neq \emptyset$.*

REMARK. If M is a continuum which contains no indecomposable continuum of diameter less than ε for some $\varepsilon > 0$ then M is one dimensional. To see this, let

$p \in M$ and let U be a closed neighbourhood of p of diameter less than ε . Then every component of U has dimension ≤ 1 by the theorem of Mazurkiewicz [10] that every continuum of dimension ≥ 2 contains an indecomposable continuum. Let $f: U \rightarrow Y$ be the map that identifies components of U to points. Then $\dim Y = 0$ and f is a closed map. Then $\dim U = 1$ by the Hurewicz Theorem [5, VI, 7]. For the sake of geometric intuition the reader may suppose, therefore, that every open cover $\mathcal{U}$ that will be needed for the proofs of the main results of this paper has nerve $\mathcal{N}(\mathcal{U})$ which is a finite graph.

If $K \subset M$ are continua then by a M - K -cover A we mean a taut collection of open sets in K which covers and strongly irreducibly covers K . If $N \subset M$ are continua and A is a N - K -cover then A is a M - K -cover.

Let A be an irreducible open cover of a continuum M and let $U, V \in A$ such that $U \not\subset S^6(V, A)$. Let $\langle U, V \rangle$ be a subcollection of A such that

- (1) $U, V \subset \langle U, V \rangle$,
- (2) $\langle U, V \rangle$ is a cover of some subcontinuum K of M such that $U \cap K \neq \emptyset \neq V \cap K$, and
- (3) if $W \subset \langle U, V \rangle$ is a cover of a subcontinuum L of M such that $L \cap U \neq \emptyset \neq L \cap V$ and $U, V \in W$ then $W = \langle U, V \rangle$.

If $B = \langle U, V \rangle$ we call U the *first link* of B and write $U = FB$. Similarly, we call V the *last link* of B and write $V = LB$. We call a M - K -cover A a M - K -cover from FA to LA if $FA, LA \in A$ and $A = \langle FA, LA \rangle$.

1. *Note.* Let A be an irreducible open cover of a continuum M and suppose $U, V \in A$ such that $U \not\subset S^6(V, A)$. Since A is finite there exists a $\langle U, V \rangle$ which need not, however, be unique. If N is a continuum in M such that $N \subset \bigcup \langle U, V \rangle$ and $N \cap U \neq \emptyset \neq N \cap V$, let $\langle \langle U, V \rangle \rangle$ be a M - N -cover which is a precise refinement of $\langle U, V \rangle$ (we can show as before that $\langle \langle U, V \rangle \rangle$ exists). For each $W \in \langle U, V \rangle$ let W' be the element of $\langle \langle U, V \rangle \rangle$ which corresponds to W . Then $\langle \langle U, V \rangle \rangle$ is a M - N -cover from U' to V' . It suffices to show that if K is a continuum in $\bigcup \langle \langle U, V \rangle \rangle$ such that $K \cap U' \neq \emptyset \neq K \cap V'$ and $O \in \langle U, V \rangle \setminus \{U, V\}$, then $K \not\subset \bigcup (\langle \langle U, V \rangle \rangle \setminus \{O'\})$. If $K \subset \bigcup (\langle \langle U, V \rangle \rangle \setminus \{O'\})$, then $K \subset \bigcup (\langle U, V \rangle \setminus \{O\})$ and $K \cap U \neq \emptyset \neq K \cap V$, which contradicts the definition of $\langle U, V \rangle$.

We say that a N - K -cover B is *embedded* in a M - N -cover A if $\{S^3(U, B) \mid U \in B\}$ refines A . If A is a M - N -cover from FA to LA , then we say a N - K -cover B is *embedded in A from FA to LA* if B is embedded in A , B is a N - K -cover from FB to LB , $C1(FB) \subset i(FA, A)$ and $C1(LB) \subset i(LA, A)$ for some $U \in B$.

REMARK. If B is a N - K -cover embedded in a M - N -cover A from FA to LA then for each $U \in A$ there exists $W \in B$ such that $C1(W) \subset U$.

PROOF. Without loss of generality $FA \neq U \neq LA$. Let $V \in B$ such that $V \subset i(LA, A)$. Since B is an irreducible cover of $K \subset N$, $FB \subset i(FA, A)$, $V \subset i(LA, A)$ and A is a M - N -cover from FA to LA , there exists $x \in K \cap U \setminus \bigcup \{T \in A \mid T \neq U\}$. Let $W \in B$ such that $x \in W$. Then $C1(W) \subset U$.

If it follows from the above Remark that if B is a N - K -cover embedded in the M - N -cover A from FA to LA and C is a K - L -cover embedded in B from FB to LB , then C is embedded in A from FA to LA .

2. *Note.* If A is a M - N -cover from FA to LA , then there exists for each $\varepsilon > 0$, by Note 1, a N - K -cover B of mesh less than ε embedded in A from FA to LA and $C1(LB) \subset i(LA, A)$.

If A is a M - N -cover from FA to LA then by an *endpiece* T of A we mean a coherent subcollection of A which contains $\{W \in A \mid W \subset S^3(LA, A)\}$ and such that $FA \cap \bigcup T = \emptyset$. By fT we denote

$$\{W \in T \mid W \cap Z \neq \emptyset \text{ for some } Z \in A \setminus T\},$$

and we call fT the *first links* of T .

Let B be a N - K -cover embedded in a M - N -cover A from FA to LA . Let S be an endpiece of A and let T be an endpiece of B . We say that T *folds in* S if $C1(LB) \subset \bigcup fS$, $C1(\bigcup fT) \subset \bigcup fS$, $C1(\bigcup T) \subset \bigcup S$, and no coherent subcollection of $\{W \in T \mid W \not\subset i(LA, A)\}$ contains both LB and an element of fT .

3. **LEMMA.** *Let B be a N - K -cover embedded in a M - N -cover A from FA to LA , let S be an endpiece of A and suppose $\{W \in B \mid C1(W) \not\subset i(LA, A)\}$ contains at least two maximal distinct coherent subcollections P and Q and elements $U \in P$ and $V \in Q$ such that $C1(U \cup V) \subset \bigcup fS$. Then there exists a K - L -cover C embedded in A from FA to LA and an endpiece T of C such that T folds in S .*

PROOF. Let $\{U \in B \mid C1(U) \not\subset i(LA, A)\} = R \cup R'$, where R is the maximal coherent subcollection of $\{U \in B \mid C1(U) \not\subset i(LA, A)\}$ which contains FB and $R \cap R' = \emptyset$. Let $R'' = \{U \in R' \mid C1(U) \subset \bigcup S\}$.

If $R'' = R'$ let $U \in R''$ such that $C1(U) \subset \bigcup fS$. By Note 1 choose $\langle FB, U \rangle \subset B$, a continuum $L \subset K \cap \bigcup \langle FB, U \rangle$ such that $L \cap FB \neq \emptyset \neq L \cap U$, and a K - L -cover $C = \langle \langle FB, U \rangle \rangle$ from FC to LC with $FC \subset FB$ and $LC \subset U$. Then C is embedded in A from FA to LA . Let T be the maximal coherent subcollection of C which contains LC and such that $C1(\bigcup T) \subset \bigcup S$. Then $C1(\bigcup fT) \subset \bigcup fS \cap \bigcup R$. To see this, let $W \in fT$. There exists $Z \in C \setminus T$ such that $Z \cap W \neq \emptyset$. Let $V \in A$ such that $S^3(Z, C) \subset V$. Since $C1(Z) \subset V$, $V \notin S$. Let $V_1, \dots, V_n \in S$ such that $C1(W) \subset V_1 \cup \dots \cup V_n$. Then $V_i \cap V \neq \emptyset$ for each i and, hence, $V_i \in fS$. Hence, T folds in S .

If $R'' \neq R'$, let $x_0 \in K \cap i(FB, B)$. Then by the Boundary Bumping Theorem there exists a continuum $K' \subset K \cap (\bigcup R \cup LA \cup \bigcup R'')$ such that $x_0 \in K'$ and

$$C1(S(K', B)) \cap \bigcup R' \not\subset \bigcup S.$$

Let $B' \subset R'' \cup (A \setminus R')$ be an irreducible cover of K' . Let $U \in B' \cap R''$ such that $C1(S(U, B)) \not\subset \bigcup S$. Notice $U \subset V$ for some $V \in A \setminus S$. Thus, $C1(U) \subset \bigcup fS$. Choose $\langle FB, U \rangle \subset B'$, a continuum $L \subset K' \cap \bigcup \langle FB, U \rangle \subset K$ such that $L \cap FB \neq \emptyset \neq L \cap U$, and a K - L -cover $C = \langle \langle FB, U \rangle \rangle$ from FC to LC such that $FC \subset FB$ and $LC \subset U$. Then C is embedded in A from FA to LA . Let T be the maximal coherent collection of C which contains LC and such that $C1(\bigcup T) \subset \bigcup S$. As above T folds in S .

The following is a variation on a theorem of Rogers [13] and Bellamy [1]:

4. LEMMA. Let M_0 be a continuum. Suppose $B_1, B_2, \dots$ is a sequence such that B_{i+1} is a M_i - M_{i+1} -cover embedded in the M_{i-1} - M_i -cover B_i from FB_i to LB_i , and $T_1, T_2, \dots$ is a sequence such that T_i is an endpiece of B_i and T_{i+1} folds in T_i . Then $\bigcap_{i=1}^{\infty} (\bigcup \{U \in T_i\})$ contains an indecomposable continuum.

PROOF. By the Boundary Bumping Theorem there exists a continuum K_i in $\bigcup T_i$ such that K_i meets fT_i and LB_i . Then K_i also meets fT_j and LB_j for $j < i$. Without loss of generality $\text{Lim } K_i = K$. Then K is a continuum,

$$K \subset \bigcap_{i=1}^{\infty} \text{Cl}(\bigcup \{U \in T_i\}) = \bigcap_{i=1}^{\infty} (\bigcup \{U \in T_i\})$$

and K meets $\text{Cl}(\bigcup fT_i)$ and $\text{Cl}(LB_i)$ for each i . So $K \cap \bigcup fT_i \neq \emptyset$ for each i . Let $h: [0, 1] \rightarrow [0, 1]$ be defined by

$$h(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}], \\ 2 - 2x & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Define $f_1: \text{Cl}(\bigcup T_1) \rightarrow [0, 1]$ to be a continuous function such that $f_1^{-1}(0) = \text{Cl}(\bigcup fT_1)$ and $f_1^{-1}(1) = \text{Cl}(LB_1)$. Define $f_2: \text{Cl}(\bigcup T_2) \rightarrow [0, 1]$ so that if R is the union of the maximal coherent subcollections of $\{U \in B_2 \mid U \not\subset i(LB_1, B_1)\}$ which meets $\bigcup fT_2$, then

$$f_2(x) = \begin{cases} \frac{1}{2}f_1(x) & \text{if } x \in \text{Cl}(\bigcup R) \cap \text{Cl}(\bigcup T_2), \\ \frac{1}{2} & \text{if } x \in f_1^{-1}(1) \cap \text{Cl}(\bigcup T_2), \\ 1 - \frac{1}{2}f_1(x) & \text{if } x \in \text{Cl}(\bigcup T_2) \setminus (\text{Cl}(\bigcup R) \cup f_1^{-1}(1)). \end{cases}$$

Notice that $LB_2 \cap \text{Cl}(\bigcup R) = \emptyset$ and $K \cap \bigcup fT_2 \neq \emptyset \neq K \cap \text{Cl}(LB_2)$, so $f_2(K) = [0, 1]$. Then $f_1(x) = h \circ f_2(x)$ for each $x \in \text{Cl}(\bigcup T_2)$. By induction we define continuous functions $f_i: \text{Cl}(\bigcup T_i) \rightarrow [0, 1]$ such that $f_i(x) = h \circ f_{i+1}(x)$ for each positive integer i and for each $x \in \text{Cl}(\bigcup T_{i+1})$ and $f_i(K) = [0, 1]$. Then $f = \varprojlim f_i|_K$ is a mapping of $K \subset \bigcap_{i=1}^{\infty} (\bigcup \{U \in T_i\})$ onto Knaster's indecomposable continuum $Y = \varprojlim (I_i, h_i)$, where $I_i = [0, 1]$ and $h_i^j = h$ for all i and j (as in the proof of Bellamy [1 Theorem, p. 305]). Since f maps K onto Y , K contains an indecomposable continuum.

5. LEMMA. Let M be a continuum which contains no indecomposable subcontinuum of diameter less than ϵ for some $\epsilon > 0$. Let A be a M - N -cover of mesh less than ϵ from FA to LA . Then there exists a N - N_1 -cover B embedded in A from FA to LA , and an endpiece T of B such that $\text{Cl}(\bigcup T) \subset i(LA, A)$, and such that if C is a N_1 - K -cover embedded in B from FB to LB then no endpiece of C folds in T . Moreover, B can be chosen to have arbitrarily small mesh.

PROOF. As in Note 2 there is a N - L -cover B_1 of arbitrarily small mesh embedded in A from FA to LA and an endpiece T_1 of B_1 such that $\text{Cl}(\bigcup T_1) \subset LA$. The lemma now follows by contradiction from Lemma 4.

3. The main results. The first three results in this section were proved by Thomas in [14] and Mahavier in [9] for the special case of arc-like continua.

6. THEOREM. *If M is a continuum which contains no indecomposable subcontinuum of diameter less than ϵ for some $\epsilon > 0$ and $x, y \in M$, then there exists a subcontinuum K of M irreducible from p to q such that K is locally connected at q , $d(x, p) < \epsilon$ and $d(y, q) < \epsilon$. In particular, if K is irreducible from p to q' then $q' = q$.*

PROOF. Let $N_0 = M$. Let $\mathcal{O}$ be a M - N_0 -cover of mesh less than $\min\{\epsilon, 1\}$ such that $y \notin S^7(x, 0)$. Let $x \in U \in \mathcal{O}$ and $y \in V \in \mathcal{O}$. By Note 1 choose $\langle U, V \rangle \subset \mathcal{O}$, a continuum $N_1 \subset \bigcup \langle U, V \rangle$ such that $N_1 \cap U \neq \emptyset \neq N_1 \cap V$ and $B_1 = \langle \langle U, V \rangle \rangle$, a N_0 - N_1 -cover from $FB_1 \subset U$ to $LB_1 \subset V$. By Lemma 5 there exists a N_1 - N_2 -cover B_2 of mesh less than $\frac{1}{2}$ embedded in B_1 from FB_1 to LB_1 and an endpiece T_2 of B_2 such that $C1(\bigcup T_2) \subset i(LB_1, B_1)$, and such that if D is a N_2 - K -cover embedded in B_2 from FB_2 to LB_2 , then no endpiece of D folds in T_2 .

By repeated application of Lemma 5 there exist sequences of continua $N_1, N_2, \dots$, covers $B_1, B_2, \dots$ and endpieces $T_2, T_3, \dots$ such that for each $i = 1, 2, \dots$:

- (i) B_{i+1} is a N_i - N_{i+1} -cover embedded in B_i from FB_i to LB_i ;
- (ii) $\text{mesh } B_i < 1/i$;
- (iii) T_{i+1} is an endpiece of B_{i+1} with $C1(\bigcup T_{i+1}) \subset i(LB_i, B_i)$;
- (iv) if D is a N_{i+1} - K -cover embedded in B_{i+1} from FB_{i+1} to LB_{i+1} , then no endpiece of D folds in T_{i+1} .

Let $K = \bigcap N_i$, $\{p\} = \bigcap FB_i$ and $\{q\} = \bigcap LB_i$. Then K is a continuum. Since $B_i = \langle FB_i, LB_i \rangle$ for each i , B_i is an irreducible cover of every subcontinuum of K which contains both p and q . Since $\text{mesh } B_i < 1/i$ it follows that K is an irreducible continuum from p to q .

Suppose K is not connected im kleinen at q . There exists $\delta > 0$ such that $\delta < \epsilon$ and such that no subcontinuum of K of diameter less than δ contains a neighbourhood of q in K . Let Q be the component of q in $K \cap C1(B(q, \delta/4))$, where $B(q, \delta/4)$ denotes the open $\delta/4$ ball centered at q . Let $r \in Q \setminus B(q, \delta/4)$. Let n be an integer so that $LB_{n-1} \subset B(q, \delta/4)$. If for each sufficiently large integer i the maximal coherent subcollection of $\{U \in B_i \mid C1(U) \not\subset i(LB_n, B_n)\}$ which contains FB_i also contains r in its union, then there exists a component N of $K \setminus LB_{n+1}$ which contains both p and r . By the irreducibility of K from p to q this would imply that $K = N \cup Q$ since $N \cup Q$ is a continuum in K which contains p and q . Thus, q is in the interior of Q in K which contradicts the choice of δ . Thus, for some sufficiently large $m > n$ the maximal coherent subcollection of $\{U \in B_m \mid C1(U) \not\subset i(LB_n, B_n)\}$ which contains FB_m does not contain r in its union. By Lemma 3 there exists a N_n - K -cover D embedded in B_n from FB_n to LB_n , and an endpiece T of D such that T folds in T_n . This is a contradiction and the connectedness im kleinen of K at q is proved.

Finally, we show that K is locally connected at q . Let V be any closed connected neighbourhood of q of diameter less than ϵ such that $p \notin V$. Let L be the closure of the component of $K \setminus V$ which contains p . Then $U = K \setminus L$ is an open set containing q . By the irreducibility of K from p to q , $U \subset V$ since V is connected. Let N be the

component of U which contains q . Then $C1(N) \cap \text{Bd}(U) \neq \emptyset$ and, since $\text{Bd}(U) \subset L$, $C1(N) \cap L \neq \emptyset$. Moreover, $C1(N) \setminus N \subset L$ and, by the irreducibility of K , $K = N \cup L$. Hence $N = K \setminus L = U \subset V$ is a connected open set containing q . This completes the proof of the theorem.

The next corollary gives an affirmative answer to a question of Mahavier (cf. Problem 121, *University of Houston problem book*).

7. COROLLARY. *Let M be a hereditarily decomposable continuum and $x, y \in M$. Then for each $\epsilon > 0$, there exists a subcontinuum K' of M such that K' is irreducible from x to q for some q with $d(y, q) < \epsilon$ and such that $\{q\}$ is an end-tranche of K .*

PROOF. Assume M is irreducible from x to y . Let $f: M \rightarrow [0, 1]$ be a finest monotone map with $f(x) = 0$ and $f(y) = 1$. Choose K and $p \in f^{-1}([0, \frac{1}{2}))$ and $q \in B(y, \epsilon) \cap f((\frac{1}{2}, 1])$ as in Theorem 6. Let $K' = K \cup f^{-1}([0, f(p)))$.

In [14 and 3] are given examples of hereditarily decomposable arc-like continua X so that if K is a subcontinuum of X with a degenerate tranche L then L is an end-tranche of K .

8. COROLLARY. *If M is a continuum of type λ which contains no indecomposable continuum of diameter $< \epsilon$ for some $\epsilon > 0$ and such that each tranche of M is a tranche of cohesion, then a dense G_δ -set of tranches of M are degenerate.*

PROOF. Let $\Phi: M \rightarrow [0, 1]$ be a map such that for each $t \in [0, 1]$, $\Phi^{-1}(t)$ is a nowhere dense subcontinuum of M . Let $[a, b] \subset [0, 1]$ such that $a < b$. By Theorem 6, there exists a continuum $K \subset \Phi^{-1}([a, b])$ irreducible from p to q such that $\Phi(p) < \Phi(q)$ and $\{q\}$ is a degenerate end-tranche of M . It also follows that if $t \in D = \{s \in [0, 1] \mid \Phi^{-1}(s) \text{ is a tranche of continuity}\}$, then $\Phi^{-1}(t)$ is degenerate, and it is known (see [8, p. 202]) that D is a dense G_δ in $[0, 1]$.

9. COROLLARY. *If M is an irreducible hereditarily decomposable continuum such that each tranche of M is a tranche of cohesion, then a dense G_δ -set of tranches of M is degenerate.*

For any compact metric space M we denote by 2^M (respectively, $C(M)$) the space of all nonempty, compact subsets (respectively, subcontinua) of M with the topology induced by the Hausdorff metric.

Let M be a continuum and let $x \in M$. Then $\{x\}$ is said to be *arcwise accessible* from $2^M \setminus C(M)$ (see [11 and 12]) provided there exists an arc A in 2^M such that $A \cap C(M) = \{x\}$. The next corollary follows from Theorem 6 and Theorem 4.1 of [3]. It gives a positive solution to a question of Nadler (see [11, 12.19 and 12, 8.1]).

10. COROLLARY. *Let M be a hereditarily decomposable continuum. There exists a point $x \in M$ such that $\{x\}$ is arcwise accessible from $2^M \setminus C(M)$.*

In view of Corollary 8 the following question is interesting:

11. Question. If X is an irreducible continuum which admits a continuous monotone decomposition onto an arc, does X contain hereditarily indecomposable tranches? In particular, does Knaster's continuum in [6] contain tranches which are pseudoarcs?

REFERENCES

1. D. P. Bellamy, *Composants of Hausdorff indecomposable continua; a mapping approach*, Pacific J. Math. **47** (1973), 303–309.
2. E. Dyer, *Irreducibility of the sum of the elements of a continuous collection*, Duke Math. J. **20** (1953), 589–592.
3. J. Grispolakis and E. D. Tymchatyn, *Irreducible continua with degenerate end-tranches and arcwise accessibility in hyperspaces*, Fund. Math. **110** (1980), 117–130.
4. G. W. Henderson, *Proof that every compact decomposable continuum which is topologically equivalent to each of its non-degenerate subcontinua is an arc*, Ann. of Math. (2) **72** (1960), 421–428.
5. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Univ. Press., Princeton, N. J., 1941.
6. B. Knaster, *Un continu irréductible à décomposition continue en tranches*, Fund. Math. **25** (1935), 568–577.
7. J. Krasinkiewicz, *On two theorems of Dyer* (to appear).
8. K. Kuratowski, *Topology*, Vol. 2, Academic Press, New York, 1968.
9. W. S. Mahavier, *Upper semi-continuous decompositions of irreducible continua*, Fund. Math. **60** (1967), 53–57.
10. S. Mazurkiewicz, *Sur l'existence des continus indecomposables*, Fund. Math. **26** (1935), 327–328.
11. S. B. Nadler, *Hyperspaces of sets*, Dekker, New York, 1968.
12. ———, *Arcwise accessibility in hyperspaces*, Dissertationes Math. **138** (1976), 1–33.
13. J. W. Rogers, Jr., *On mapping indecomposable continua onto certain chainable indecomposable continua*, Proc. Amer. Math. Soc. **25** (1970), 449–456.
14. E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Rozprawy Mat. **50** (1966).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, BIRMINGHAM, ALABAMA 35294

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SASKATCHEWAN, SASKATOON, CANADA S7N 0W0