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ON THE DISTRIBUTION OF THE PRINCIPAL SERIES IN $L^2(\Gamma\backslash G)$

BY

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ABSTRACT. Let G be a semisimple Lie group of split rank one with finite center. If $\Gamma \subset G$ is a discrete cocompact subgroup, then $L^2(\Gamma\backslash G) = \sum_{\omega \in \mathcal{E}(G)} n_\Gamma(\omega) \cdot \omega$. For fixed $\sigma \in \mathcal{E}(M)$, let $P(\sigma)$ denote the classes of irreducible unitary principal series $\pi_{\sigma,i\nu}$ ($\nu \in \mathfrak{A}^*$). Let, for $s > 0$, $\psi_\sigma(s) = \sum_{\omega \in P(\sigma)} n_\Gamma(\omega) \cdot e^{s\lambda_\omega}$, where λ_ω is the eigenvalue of Ω (the Casimir element of G) on the class ω. In this paper, we determine the singular part of the asymptotic expansion of $\psi_\sigma(s)$ as $s \to 0^+$ if Γ is torsion free, and the first term of the expansion for arbitrary Γ. As a consequence, if $N_\sigma(r) = \sum_{\omega \in P(\sigma), |\lambda_\omega| < r} n_\Gamma(\omega)$ and G is without connected compact normal subgroups, then

$$N_\sigma(r) \sim C_G \cdot |Z(G) \cap \Gamma| \cdot \operatorname{vol}(\Gamma\backslash G) \cdot \dim(\sigma) \cdot r^c \qquad \left(c = \tfrac{1}{2}\dim G/K\right),$$

as $r \to +\infty$. In the course of the proof, we determine the image and kernel of the restriction homomorphism $i^*: R(K) \to R(M)$ between representation rings.

Introduction. Let G be a connected, real semisimple Lie group with Lie algebra $\mathfrak{G}$. Let $G = K.A.N.$ (respectively $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{A} \oplus \mathfrak{N}$) be an Iwasawa decomposition of G (respectively $\mathfrak{G}$) and let M be the centralizer of A in K. We assume throughout this paper that G has finite center and split rank one. We do not assume that G is linear. Let $\mathcal{E}(G)$ denote the set of equivalence classes of irreducible unitary representations of G. If $\sigma \in \mathcal{E}(M)$, $\nu \in \mathfrak{A}^*_{\mathbf{C}}$ let $\pi_{\sigma,\nu}$ be the principal series representation of G, parametrized as in [**DW**, §3]. In this parametrization $\pi_{\sigma,\nu}$ is unitary if $\nu \in i\mathfrak{A}^*$. If $\omega \in \mathcal{E}(G)$ let λ_ω and θ_ω denote, respectively, the eigenvalue of the Casimir element of G on the class ω and the distributional character of ω. We will abbreviate by writing $\lambda_{\sigma,\nu} = \lambda_{\pi_{\sigma,\nu}}$. If $\omega \in \mathcal{E}_2(G)$, the discrete series of G, let $d(\omega)$ denote the formal degree of ω.

For fixed $\sigma \in \mathcal{E}(M)$ set

$$P(\sigma) = \{\pi_{\sigma,i\nu} \mid \nu \in \mathfrak{A}^* \text{ and } \pi_{\sigma,i\nu} \text{ is irreducible}\}.$$

Recall [**KS**, §12] that $\pi_{\sigma,i\nu}$ is reducible only if rank $G =$ rank K, $\nu = 0$ and in this case $\pi_{\sigma,0} = \pi^+_{\sigma,0} + \pi^-_{\sigma,0}$ where $\pi^\pm_{\sigma,0}$ are inequivalent irreducible representations. Let $R(\sigma) = \{\pi^\pm_{\sigma,0}\}$ if $\pi_{\sigma,0}$ is reducible and $R(\sigma) = \varnothing$, otherwise. Finally let $C(\sigma)$ denote the subset of $\mathcal{E}(G)$ of classes ω such that, if $(\pi_\omega, H_\omega) \in \omega$, then H_ω is infinitesimally equivalent to $J_{\sigma,\nu}$ for some ν s.t. $\operatorname{Re}\langle \nu, \lambda\rangle > 0$ ($J_{\sigma,\nu}$ is as in [**DW**, Theorem 4.1], and λ is the long positive restricted root).

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Let Γ be a discrete, cocompact subgroup of G. The right regular representation π_Γ of G in $\mathfrak{L}^2(\Gamma\backslash G)$ decomposes $\pi_\Gamma = \Sigma n_\Gamma(\omega)\cdot\omega$ and $n_\Gamma(\omega) < \infty$, for any $\omega \in \mathcal{E}(G)$. If $\tau \in \mathcal{E}(K)$,

$$\phi_\tau(s) = \sum_{\omega\in\mathcal{E}(G)} n_\Gamma(\omega)\cdot[\tau:\omega]\cdot e^{s\lambda_\omega}$$

defines a C^∞ function on $\mathbf{R}^+$, the series converging uniformly on compacta with all derivatives [**W**]. Hence, if $\sigma \in \mathcal{E}(M)$ is fixed, the series $\psi_\sigma(s) = \Sigma_{\omega\in P(\sigma)} n_\Gamma(\omega)\cdot e^{s\lambda_\omega}$ defines a C^∞ function for $s > 0$. The purpose of this paper is to study the asymptotic behavior of $\psi_\sigma(s)$, as $s \to 0^+$. By using the technique in [**M1**] we determine the singular part of the asymptotic expansion of $\psi_\sigma(s)$, as $s \to 0^+$, when Γ is torsion free.

THEOREM 1. *Let $\Gamma \subset G$ be a discrete, cocompact, torsion-free subgroup. Then*

$$\psi_\sigma(s) = \operatorname{vol}(\Gamma\backslash G)\cdot e^{-s(|\rho|^2+a\lambda_\sigma)}\left(\sum_{i=0}^{c+d-1} b_{2(i-d)+1}(\sigma)\cdot\Gamma(i+1-d)\cdot(4s)^{-i-1+d}\right) - g_\sigma(s)$$

where $g_\sigma(s)$ extends to $\tilde{g}_\sigma(s)$, a C^∞ function on $\mathbf{R}$, *such that*

(i) *if* rank G = rank K

$$\tilde{g}_\sigma(0) = \operatorname{vol}(\Gamma\backslash G)\left(\sum_{i=0}^{c-1} b_{2i+1}(\sigma)\frac{((\varepsilon+1)\cdot 2^{2i+1}-1)}{i+1}\cdot B_{2(i+1)}\right) + \sum_{\omega\in R(\sigma)\cup C(\sigma)} s(\omega)\cdot n_\Gamma(\omega) + \sum_{\omega\in\mathcal{E}_2(G)} s(\omega)(n_\Gamma(\omega) - d(\omega)\cdot\operatorname{vol}(\Gamma\backslash G)),$$

(ii) *if* rank G > rank K

$$\tilde{g}_\sigma(0) = \sum_{\omega\in C(\sigma)} s(\omega)\cdot n_\Gamma(\omega).$$

Here $c = \frac{1}{2}\dim(G/K)$, $d = c - [c]$, $a \in \mathbf{R}^+$, and λ_σ is the eigenvalue of the Casimir element of M on the class σ. Furthermore, $b_{2(i-d)+1}(\sigma)$ denotes, for $i = 1,\ldots,c+d-1$, the ith coefficient of the polynomial part of the Plancherel density associated to σ, B_{2j} is the jth Bernoulli number, and $\varepsilon = 1$ or -1 depending on σ. Finally, if $\omega \in R(\sigma)\cup C(\sigma)\cup\mathcal{E}_2(G)$, then

$$s(\omega) = [\eta:\omega] = \dim \operatorname{Hom}_K(\eta,\omega) \in \mathbf{Z}$$

where $\eta = \eta_\sigma$ is a virtual representation of K (in particular, $s(\omega)$ depends on σ but not on Γ).

Let $R(M)$ and $R(K)$ denote the representation rings of M and K. We make use of the following.

PROPOSITION 1. *Let i^*: $R(K) \to R(M)$ be the restriction homomorphism. Then* $\operatorname{Im}(i^*) = R(M)^W$, *where $W = W(\mathfrak{G}, \mathfrak{A})$. If* rank G = rank K, *then $R(M)^W = R(M)$ and i^* is surjective.*

Let $\Gamma \subset G$ be an arbitrary discrete cocompact subgroup. We assume for simplicity, that G has no nontrivial compact, connected, normal subgroups (see the remark below). Theorem 1.1 in [**W**], together with Proposition 1, imply

COROLLARY 1. *If* $Z(\Gamma) = Z(G) \cap \Gamma$ *and* $\sigma \in \mathcal{E}(M)$ *satisifies* $\sigma|_{Z(\Gamma)} = 1$, *then*

$$\lim_{s \to 0^+} s^c \cdot \psi_\sigma(s) = \frac{\dim(\sigma)\,|Z(\Gamma)|\,\mathrm{vol}(\Gamma\backslash G)}{(4\pi)^c}$$

(*if* $\sigma|_{Z(\Gamma)} \neq 1$ *then* $n_\Gamma(\pi_{\sigma,\nu}) = 0$, *for all* ν, *hence* $\psi_\sigma(s) = 0$).

Let $N_\sigma(r) = \sum_{\omega \in P(\sigma), |\lambda_\omega| < r} n_\Gamma(\omega)$ $(r > 0)$. Corollary 1 and the Tauberian theorem for the Laplace transform imply

$$\text{(1)} \qquad \lim_{r \to \infty} r^{-c} \cdot N_\sigma(r) = \frac{\dim(\sigma) \cdot |Z(\Gamma)|\,\mathrm{vol}(\Gamma\backslash G)}{\Gamma(c+1) \cdot (4\pi)^c}.$$

REMARK. When G has compact normal subgroups, (1) still holds with $\dim V_\sigma^{\Gamma \cap N} \cdot |\Gamma \cap N|$ substituting $\dim(\sigma) \cdot |Z(G) \cap \Gamma|$, where $N = \bigcap_{x \in G} xKx^{-1}$. This follows from [**W**, 1.1], with a correction factor as in [**BH**, §6], and Proposition 1. Indeed, if $\sigma = i^*(\eta)$, $\eta = \sum m_j \tau_j \in R(K)$, one can show that $\sum m_j \dim V_{\tau_j}^{N \cap \Gamma} = \dim V_\sigma^{N \cap \Gamma}$.

The asymptotic formula (1) for the spherical principal series (i.e. $\sigma = 1$) in $L^2(\Gamma\backslash G)$ was proposed by Gelfand ([**G**, p. 77], see also [**GGP**, pp. 82, 94]). It was proved by Gangolli for complex G, by Eaton for G of split rank one, and by Duistermaat-Kolk-Varadarajan, for general G ([**Ga**, **DKV**], see also [**GW**]). With the aid of Proposition 1, Theorem 1.1 in [**W**] implies the Gelfand type formula (1) for any $\sigma \in \mathcal{E}(M)$, when G is as above.

The outline of the paper is as follows. In §1, we prove Theorem 1.1 (assuming Proposition 1). The proof of Proposition 1 is given in §2. Finally, we show (Lemma 2.6) that if rank G = rank K, then $J = \ker i^* \neq 0$ and determine J explicitly. We recall that if Γ is torsion-free, each $\eta \in \ker i^*$ yields a finite alternating sum formula in the $n_\Gamma(\omega)$'s [**M3**, Theorem 1.2].

1. We first normalize Haar measures conveniently. If λ denotes the long positive restricted root of $(\mathfrak{G}, \mathfrak{A})$ let $H \in \mathfrak{A}$ be so that $\lambda(H) = 2$. Let $a = B(H, H)$, B the Killing form of $\mathfrak{G}$. Let $d\tilde{x}$, $d\tilde{k}$ denote respectively the invariant Riemannian measures on G and K induced by the inner product on $\mathfrak{P}$, $(X, Y) = a^{-1} \cdot B(X, Y)$. We will use on G and K the measures $dx = \mathrm{vol}(K)^{-1} \cdot d\tilde{x}$, $dk = \mathrm{vol}(K)^{-1} \cdot d\tilde{k}$. As usual, let dx on $\Gamma\backslash G$ be so that

$$\int_{\Gamma\backslash G} \Big(\sum_\gamma f(\gamma x) \Big)\, dx = \int_G f(x)\, dx, \quad \text{for } f \in C_c(G).$$

For fixed $\sigma \in \mathcal{E}(M)$, the Plancherel density associated to σ can be written $\mu_\sigma(x\lambda) = q_\sigma(x) \cdot \phi_\sigma(x)$, where $q_\sigma(x)$ is a polynomial of degree $2c - 1$ and $\phi_\sigma(x) = 1$, $\tanh \pi x$ or $\coth \pi x$, depending on σ [**O**]. Moreover, $\phi_\sigma = 1$ if and only if rank $G >$ rank K. Let $d = c - [c]$, that is, $d = 0$ if rank G = rank K and $d = \frac{1}{2}$, otherwise. If the Haar measure on G is normalized as above, then

$$q_\sigma(x) = \sum_{i=0}^{c+d-1} b_{2(i-d)+1}(\sigma) \cdot x^{2(i-d)+1}$$

and $b_{2c-1}(\sigma) = \dim(\sigma)/(\Gamma(c) \cdot \pi^c)$ [**M2**, §3].

For fixed τ set, if $x \in G$ and $s > 0$,

$$g_{\tau,s}(x) = \int_{\mathcal{E}(G)} \dim(\tau)^{-1} \cdot \phi_{\tau,\omega}(x^{-1}) \cdot e^{s\lambda_\omega} | \, d\mu(\omega)$$

where $\phi_{\tau,\omega}$ is the τ-spherical trace function associated to ω and $\mu(\omega)$ is the Plancherel measure on $\mathcal{E}(G)$. DeGeorge and Wallach (unpublished) have proved a general result which implies that $g_{\tau,s} \in \mathcal{C}^p(G)$ (the p-Schwartz space of G) for any $p > 0$ (G can be of arbitrary split rank). Using this fact, one shows [**M3**, 1.1] that $\theta_\omega(g_{\tau,s}) = [\tau : \omega]e^{s\lambda_\omega}$ for any $\omega \in \mathcal{E}(G)$, where $[\tau : \omega] = \dim \operatorname{Hom}_K(\tau, \omega)$.

Let $\Gamma \subset G$ be a discrete, cocompact, torsion-free subgroup. Fix $\sigma \in \mathcal{E}(M)$. We assume first that rank G = rank K. Then, by Proposition 1, there exists $\eta = \Sigma m_j \tau_j$, $m_j \in \mathbf{Z}$, $\tau_j \in \mathcal{E}(K)$ such that $i^*(\eta) = \sigma$. Set $g_{\eta,s} = \Sigma m_j g_{\tau_j,s}$. Since $g_{\eta,s} \in \mathcal{C}^p(G)$ for any $p > 0$, and $g_{\eta,s}$ is K-finite, the operator $\pi_\Gamma(g_{\eta,s})$ on $L^2(\Gamma \backslash G)$ is trace-class [**M1**, §2], and

$$\operatorname{tr} \pi_\Gamma(g_{\tau,s}) = \sum_{\omega \in \mathcal{E}(G)} n_\Gamma(\omega) \cdot [\eta : \omega] \cdot e^{s\lambda\omega},$$

where $[\eta : \omega] = \Sigma m_j \cdot [\tau_j : \omega]$.

If $\omega \in \mathcal{E}(G)$, by Langlands' classification, either $\omega \in \mathcal{E}_2(G)$ or $\omega \in P(\xi) \cup R(\xi) \cup C(\xi)$, for some $\xi \in \mathcal{E}(M)$. If $\omega \in P(\xi)$, then

$$[\eta : \omega] = [i^*(\eta) : \xi] = \begin{cases} 0, & \xi \neq \sigma, \\ 1, & \xi = \sigma. \end{cases}$$

Hence $\operatorname{tr} \pi_\Gamma(g_{\tau,s}) = \psi_\sigma(s) + h_\sigma(s)$, where

$$h_\sigma(s) = \sum_{\omega \in \mathcal{E}_2(G) \cup R(\sigma) \cup C(\sigma)} n_\Gamma(\omega) \cdot [\eta : \omega] \cdot e^{s\lambda\omega}.$$

Note that the sets $\{\omega \in \mathcal{E}_2(G) | [\eta : \omega] \neq 0\}$, $\{\omega \in C(\sigma) | n_\Gamma(\omega) \neq 0\}$ are finite [**DW**, p. 489]. Hence $h_\sigma(s)$ is analytic.

On the other hand ([**M1**, 5.1], essentially)

$$\operatorname{tr} \pi_\Gamma(g_{\eta,s}) \sim \operatorname{vol}(\Gamma | G) \cdot g_{\eta,s}(1), \quad \text{as } s \to 0^+$$

(that is, $\operatorname{tr} \pi_\Gamma(g_{\eta,s}) - \operatorname{vol}(\Gamma | G) \cdot g_{\eta,s}(1) = o(s^n)$ for all $n \in \mathbf{N}$, as $s \to 0^+$).

Set $g^0_{\eta,s} = \Sigma m_j \cdot g^0_{\tau_j,s}$, where

$$g^0_{\tau_j,s} = \sum_{\omega \in \mathcal{E}_2(G)} d(\omega) \cdot [\eta : \omega] \cdot e^{s\lambda_\omega} \quad \text{(a finite sum)}.$$

By choice of η, if $h_{\eta,s} = g_{\eta,s} - g^0_{\eta,s}$, then

$$h_{\eta,s}(1) = \int_{-\infty}^{+\infty} e^{s\lambda_{\sigma,x\lambda}} \cdot \mu_\sigma(x\lambda) \, dx,$$

where $\lambda_{\sigma,x\lambda} = -(4x^2 + |\rho|^2 + a\lambda_\sigma)$ [**M1**, p. 17]. Here if $X_1,\ldots,X_r$ is a basis of $\mathfrak{M}$ such that $(X_i, X_j) = -\delta_{ij}$ and $\Delta_{\mathfrak{M}} = -\Sigma X_i^2$, λ_σ is so that $\sigma(\Delta_{\mathfrak{M}}) = \lambda_\sigma \cdot I$. On the other hand $\mu_\sigma(x\lambda) = q_\sigma(x)\cdot\phi_\sigma(x)$, $q_\sigma(x) = \Sigma_0^{c-1} b_{2i+1}(\sigma)\cdot x^{2i+1}$ and $\phi_\sigma(x) = \tanh\pi x$ or $\coth\pi x$. We may write (if $x \neq 0$) $1 - \phi_\sigma(x) = 2/(1+\varepsilon e^{2\pi x})$, where $\varepsilon = 1$ if $\phi_\sigma(x) = \tanh\pi x$ (respectively $\varepsilon = -1$, if $\phi_\sigma(x) = \coth\pi x$). Hence

$$h_{\eta,s}(1) = e^{-s(|\rho|^2+a\lambda_\sigma)}\cdot\left[2\int_0^{+\infty} e^{-4sx^2}\cdot q_\sigma(x)\,dx - 4\int_0^{+\infty}\frac{e^{-4sx^2}\cdot q_\sigma(x)}{1+\varepsilon e^{2\pi x}}dx\right]$$

$$= e^{-s(|\rho|^2+a\lambda_\sigma)}\cdot\left[\sum_0^{c-1} b_{2i+1}(\sigma)\cdot i!\,(4s)^{-i-1} - \sum_0^{c-1} b_{2i+1}(\sigma)\int_0^{+\infty}\frac{4\cdot e^{-4sx^2}\cdot x^{2i+1}}{1+\varepsilon e^{2\pi x}}dx\right].$$

Furthermore [**WW**, pp. 266–268]

$$\int_0^{+\infty}\frac{4x^{2i+1}}{1+e^{2\pi x}}dx = \frac{2^{2(i+1)}-1}{i+1}\cdot B_{2(i+1)},$$

$$\int_0^{+\infty}\frac{4x^{2i+1}}{1-e^{2\pi x}}dx = -\frac{B_{2(i+1)}}{i+1},$$

B_{2m} the mth Bernoulli number. Hence

$$\lim_{s\to 0^+}\sum_0^{c-1} b_{2i+1}(\sigma)\int_0^{+\infty}\frac{4\cdot e^{-4sx^2}\cdot x^{2i+1}}{1+\varepsilon e^{2\pi x}}dx = \sum_0^{c-1} b_{2i+1}(\sigma)\frac{[(\varepsilon+1)\cdot 2^{2i+1}-1]}{i+1}B_{2(i+1)}$$

(in fact, the full asymptotic expansion

$$\int_0^{+\infty}\frac{e^{-4sx^2}\cdot x^{2i+1}}{1+\varepsilon e^{2\pi x}}dx \sim \sum_{j=0}^{\infty} a_j\cdot s^j$$

can be written down explicitly).

Summing up

$$\psi_\sigma(s) \sim \mathrm{vol}(\Gamma\backslash G)\cdot\left(h_{\eta,s}(1) + g^0_{\eta,s}(1)\right) - h_\sigma(s),$$

$$\psi_\sigma(s) \sim \mathrm{vol}(\Gamma | G)e^{-s(|\rho|^2+2\lambda_\sigma)}\cdot\left(\sum_0^{c-1} b_{2i+1}(\sigma)\cdot i!\,(4s)^{-i-1}\right) - g_\sigma(s),$$

where

$$g_\sigma(s) = \mathrm{vol}(\Gamma\backslash G)\cdot e^{-s(|\rho|^2+a\lambda_\sigma)}\cdot\left(\sum_0^{c-1} b_{2i+1}(\sigma)\int_0^{+\infty}\frac{4\cdot e^{-4sx^2}\cdot x^{2i+1}}{1+\varepsilon e^{2\pi x}}dx\right)$$

$$-\mathrm{vol}(\Gamma\backslash G)\cdot g^0_{\eta,s}(1) + h_\sigma(s).$$

This concludes the proof of Theorem 1, in this case.

If rank $G >$ rank K, let $W = W(\mathfrak{G}, \mathfrak{A}) = \{1, u\}$. If $\sigma \in \mathcal{E}(M)$ is such that $\sigma = \sigma^u$, then by Proposition 1, $\sigma = i^*(\eta)$, $\eta \in R(K)$, and the above proof (with several simplifications) can be repeated. Moreover, in this case $\mu_\sigma(x\lambda) = q_\sigma(x)$, $g^0_{\eta,s} = 0$, hence $g_\sigma(s) = h_\sigma(s)$.

If $\sigma \neq \sigma^u$ then $\sigma + \sigma^u = i^*(\eta)$, $\eta \in R(K)$. Define $g_{\eta,s}$ as before. In this case $g_{\eta,s} = h_{\eta,s}$. Arguing as above, one obtains

$$\sum_{\omega \in \mathcal{E}(G)} n_\Gamma(\omega) \cdot [\eta : \omega] \cdot e^{s\lambda_\omega} \sim \operatorname{vol}(\Gamma \backslash G) \cdot h_{\eta,s}(1), \quad \text{as } s \to 0^+ .$$

The left-hand side equals

$$2\psi_\sigma(s) + 2 \sum_{\omega \in C(\sigma)} n_\Gamma(\omega) \cdot e^{s\lambda_\omega}$$

since $\mathcal{E}_2(G) = R(\sigma) = \phi$, $\pi_{\sigma,\nu} = \pi_{\sigma^u,-\nu}$ ($\nu \in \mathfrak{A}^*_{\mathbf{C}}$) and $[\eta : \omega] = 1$ if $\omega \in C(\sigma)$, in this case. Similarly,

$$\begin{aligned} h_{\eta,s}(1) &= 2 \cdot e^{-s(|\rho|^2 + a\lambda_\sigma)} \cdot \left(\sum_0^{c-1/2} b_{2i}(\sigma) \int_{-\infty}^{+\infty} e^{-4sx^2} \cdot x^{2i}\, dx \right) \\ &= 2e^{-s(|\rho|^2 + a\lambda_\sigma)} \cdot \left(\sum_0^{c-1/2} b_{2i}(\sigma) \cdot \Gamma(i + \tfrac{1}{2}) \cdot (4s)^{-i-1/2} \right). \end{aligned}$$

This concludes the proof. We observe that, if $\sigma \in \operatorname{Im}(i^*)$, Corollary 1 is an immediate consequence of Theorem 1.1 in [**W**] and Proposition 1 (with our normalization of measures $C_G = 1$, C_G as in [**W**, 1.1]). If $\sigma \notin \operatorname{Im}(i^*)$, then $\sigma + \sigma^u = i^*(\eta)$ and (essentially) the above argument yields the result.

2. This section is mainly devoted to the proof of Proposition 1. Assume first that rank $G >$ rank K. Then rank K = rank M. Let $T_1 \subset M$ be a maximal torus. There is a commutative diagram

$$\begin{array}{ccc} R(K) & \overset{i^*}{\to} & R(M) \\ & {\scriptstyle j_K^*} \searrow & \downarrow {\scriptstyle j_M^*} \\ & & R(T_1)^{W_m} \end{array}$$

where j_K^* is an isomorphism onto $R(T_1)^{W_K}$. If $M^* = N_K(A)$ (the normalizer of A in K), there is $u \in M^* \cap N_K(T)$, $u \notin M$. Therefore, W_K is generated by W_M and $u(|W_K/W_M| = 2)$. Thus $\operatorname{Im}(j_K^*) = R(T)^{W_K} = (R(T)^{W_M})^W$ and Proposition 1 is clear, in this case.

From now on, we assume that rank G = rank K. Fix $\mathfrak{H} \subset \mathfrak{K}$, a Cartan subalgebra, and let $\Delta = \Delta(\mathfrak{G}_{\mathbf{C}}, \mathfrak{H}_{\mathbf{C}})$. Then $\Delta = \Delta_c \cup \Delta_n$, where Δ_c (Δ_n) is the set of compact (noncompact) roots. Fix $\Delta^+ \subset \Delta$ a system of positive roots, $\Delta^+ = \Delta_c^+ \cup \Delta_n^+$. Let $\{X_\alpha\}_{\alpha \in \Delta}$, $\{H_\alpha\}_{\alpha \in \Delta}$ be a Weyl basis of $\mathfrak{G}_{\mathbf{C}}$ adapted to the compact form $\mathfrak{G}_u = \mathfrak{K} \oplus i\mathfrak{P}$ [**H**, p. 421]. Then, if σ denotes the conjugation of $\mathfrak{G}_{\mathbf{C}}$ with respect to $\mathfrak{G}$, $\sigma X_\alpha = -X_{-\alpha}$ ($\alpha \in \Delta_c$) and $\sigma X_\alpha = X_{-\alpha}$ ($\alpha \in \Delta_n$). From now on, we fix $\beta \in \Delta_n^+$ and choose $\mathfrak{A} = \mathbf{R}(X_\beta + X_{-\beta})$. The following lemma is not difficult.

2.1. LEMMA.

$$\mathfrak{M}_{\mathbf{C}} = \ker\beta \oplus \sum_{\substack{\alpha\in\Delta_c \\ \alpha\pm\beta\notin\Delta}} \mathbf{C}\cdot X_\alpha \oplus \sum_{\substack{\alpha\in\Delta_c \\ \alpha+2\beta\in\Delta}} \mathbf{C}(X_\alpha + c_\alpha X_{\alpha+2\beta})$$

where $c_\alpha = -N_{\alpha,\beta}/N_{\alpha+2\beta,-\beta}$ *and* $N_{\alpha,\beta}$ *is such that* $[X_\alpha, X_\beta] = N_{\alpha,\beta}\cdot X_{\alpha,\beta}$. *Furthermore*, $\ker\beta$ *is a Cartan subalgebra of* $\mathfrak{M}_{\mathbf{C}}$ *and*

$$\Delta_{\mathfrak{M}} = \Delta(\mathfrak{M}_{\mathbf{C}}, \ker\beta) = \{\alpha' = \alpha|_{\ker\beta} \mid \alpha\pm\beta\notin\Delta\} \cup \{\alpha' = \alpha|_{\ker\beta} \mid \alpha + 2\beta\in\Delta\}.$$

The root spaces are $\mathfrak{G}_{\alpha'} = \mathbf{C}X_\alpha$, *if* $\alpha\pm\beta\notin\Delta$ *and* $\mathfrak{G}_{\alpha'} = \mathbf{C}(X_\alpha + c_\alpha X_{\alpha+\beta})$, *if* $\alpha + 2\beta \in \Delta$.

Let $\Delta^+_{\mathfrak{M}} \subset \Delta_{\mathfrak{M}}$ be the positive system induced by Δ^+. Let also $T_1 = \exp(\ker\beta \cap \mathfrak{H})$, a maximal torus of M^0 (the connected component of 1 in M).

2.2. LEMMA. *Let G be semisimple, of split rank one, and such that* rank G = rank K. *Let* $W = W(\mathfrak{G}, \mathfrak{A}) = \{1, u\}$. *Then* $\sigma = \sigma^u$, *for any* $\sigma \in \mathcal{E}(M)$.

PROOF. In [**KS**, §16] the lemma is verified for $G = \mathrm{Spin}(2n, 1)$, $G = SU(n, 1)$ and $G = Sp(n, 1)$. We give a different proof. It is well known that $M = Z(G)\cdot M_0$. Moreover, W is generated by $u = \exp(\pi i H_\beta/\langle\beta,\beta\rangle)$. If $\sigma\in\mathcal{E}(M)$, then $\chi_{\sigma^u}(x) = \chi_\sigma(x)$ for any $x\in M$, since this holds for $x\in T_1$ and $x\in Z(G)$. Hence $\sigma^u = \sigma$.

REMARK. In [**KS**, Theorem 12.5], Knapp and Stein prove that if G is a linear group of split rank one, $\pi_{\sigma,\nu}$ is reducible only if $\nu = 0$. Moreover, $\pi_{\sigma,0}$ is reducible if and only if (i) $\sigma = \sigma^u$, (ii) $\mu_\sigma(0) > 0$. Lemma 2.2 says that if rank G = rank K, (i) is automatic. If rank $G >$ rank K it is no longer true that $\sigma = \sigma^u$. In fact, $\sigma = \sigma^u$ forces $\mu_\sigma(0) = 0$, hence $\pi_{\sigma,0}$ is irreducible.

We next prove a lemma. Let K_1 be a Lie group with finitely many components, such that $\mathrm{Ad}(K_1)$ is compact. Let $K_2 \subset K_1$ be a closed subgroup. As usual, let $R(K_i)$ and $\mathcal{E}(K_i)$ denote, respectively, the representation ring and the unitary dual of K_i $(i = 1, 2)$. Let S be a closed subgroup of $Z(K_1)$ (the center of K_1) such that $S\subset K_2$. Then $R(K_i/S)$ can be identified with the subring of $R(K_i)$ generated by those representations τ of K_i such that $S\subset\ker\tau$. Let i^*_S: $R(K_1/S)\to R(K_2/S)$, i^*: $R(K_1)\to R(K_2)$ denote the restriction homomorphisms.

2.3. LEMMA. $\mathrm{Im}(i^*_S) = \mathrm{Im}(i^*)\cap R(K_2/S)$.

PROOF. Let $\tau\in\mathcal{E}(K_1)$. If $i^*(\tau) = \sum_1^l r_j\cdot\xi_j$ $(r_j\neq 0)$ we say that ξ_j is a K_2-type of τ. We note that if τ has a K_2-type ξ such that $\xi|_S = 1$, then $\tau|_S = 1$. Indeed, since S is central in K_1, then $\mathrm{Ind}_{K_2}^{K_1}(\xi)|_S = 1$. Thus $\tau|_S = 1$, too. As a consequence, if $\tau, \gamma\in\mathcal{E}(K_1)$ have a common K_2-type and $\tau|_S = 1$, then $\gamma|_S = 1$. We now prove the lemma. Let $\eta\in R(K_1)$ be such that $i^*(\eta)\in R(K_2/S)$. If $\eta = \sum_1^k r_j\cdot t_j$, $i^*\eta = \sum_1^l s_i\cdot\sigma_i$ set $\mathcal{E}_\eta(K_1) = \{\tau_1,\ldots,\tau_k\}$, $\mathcal{E}_{i^*(\eta)}(K_2) = \{\sigma_1,\ldots,\sigma_l\}$. By assumption $\sigma_j|_S = 1$, $j = 1,\ldots,l$.

Define inductively

$$\mathcal{S}_1 = \{\gamma\in\mathcal{E}_\eta(K_1) \mid \gamma \text{ contains a } K_2\text{-type in } \mathcal{E}_{i^*(\eta)}(K_2)\},$$

$$\mathcal{S}_{i+1} = \{\gamma\in\mathcal{E}_\eta(K_1) \mid \gamma \text{ has a common } K_2\text{-type with some } \tau\in\mathcal{S}_i\}.$$

Then $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{E}_\eta(K_1)$. By the above observation, if $\tau \in \mathcal{S}_j$ for some j, then $\tau|_S = 1$. Thus, if $\mathcal{S}_n = \mathcal{E}_\eta(K_1)$ for some $n \in \mathbf{N}$, then $\eta \in R(K_1/S)$ and the lemma is proved. Otherwise, there exists n such that $\mathcal{S}_n = \mathcal{S}_{n+1} \neq \mathcal{E}_\eta(K_1)$. It is then easy to see that if $\eta' = \sum_{\tau_j \in \mathcal{S} - \mathcal{S}_n} m_j \tau_j$, then $i^*(\eta') = 0$. Thus $i^*(n) = i^*(\eta - \eta')$ and $\eta - \eta' \in R(K_1/S)$. We note that in general it is not true that $\ker i^* \subset R(K_1/S)$, as the example $K_1 = S^1$, $K_2 = S = \{\pm 1\}$ already shows.

2.4. Lemma. *Let G be a simply connected Lie group of split rank one. Assume that* $\operatorname{rank} G = \operatorname{rank} K$ *and* $\mathfrak{G} \neq \mathfrak{S}l(2, \mathbf{R})$. *Then M is simply connected.*

Proof. By applying the long exact sequence in homotopy to the fibration $M \to K \to K/M$, one readily obtains $\pi_0(M) = \pi_1(M) = \{1\}$ (K/M is diffeomorphic to the unit sphere in $\mathfrak{P}$ and $\dim \mathfrak{P} \geqslant 4$).

2.5. Proof of Proposition 1. By Lemma 2.2, in order to prove Proposition 1, we must show that $i^*\colon R(K) \to R(M)$ is surjective, if $\operatorname{rank} G = \operatorname{rank} K$. By Lemma 2.3 (applied to $(K_1, K_2) = (K, M)$), it is enough to verify this under the assumption that K (hence G) be simply connected. Now, since G has split rank one, we may assume that G is simple and, on the other hand, if $\mathfrak{G} = \mathfrak{S}l(2, \mathbf{R})$, it is clear that i^* is surjective. We thus assume that G is simple, simply connected, $\operatorname{rank} G = \operatorname{rank} K$ and $\mathfrak{G} \neq \mathfrak{S}l(2, \mathbf{R})$.

It will be enough to show, by Lemma 2.4, that the fundamental representations of $\mathfrak{M}_{\mathbf{C}}$ are restrictions of virtual representations of $\mathfrak{K}_{\mathbf{C}}$. We give a proof by case-by-case verification. Though a direct proof would be desirable, by this method, one finds explicitly $\eta \in R(K)$ with $i^*(\eta) = \sigma$, for each fundamental representation σ of $\mathfrak{M}_{\mathbf{C}}$. Since, by Theorem 1, the coefficients a_i ($i \geqslant 0$) of the asymptotic expansion of $\psi_\sigma(s)$ involve the numbers $[\eta : \omega]$ ($\omega \in \mathcal{E}(G)$), the knowledge of η may be of some use.

From now on we identify, via the Killing form, the imaginary dual of $\mathfrak{H}$ with a convenient subspace of $\mathbf{R}^n$, so that the usual inner product of $\mathbf{R}^n$ corresponds to a multiple of the Killing form. Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the canonical basis of $\mathbf{R}^n$. We often denote by 1 the trivial representation (of any Lie algebra). We will make use of the well-known branching formulas (see [**Z**, pp. 128–132 or **B**, 10]).

(i) $\mathfrak{G} = \mathfrak{S}\mathfrak{U}(n, 1)$ ($n \geqslant 2$).

$$i\mathfrak{H}^* = \left\{ \sum_1^{n+1} t_i \varepsilon_i \,|\, t_1 + \cdots + t_{n+1} = 0 \right\}, \quad \mathfrak{K}_{\mathbf{C}} \simeq \mathfrak{U}(n), \quad \mathfrak{M}_{\mathbf{C}} \simeq \mathfrak{U}(n-1),$$

$$\Delta_c^+ = \{\varepsilon_i - \varepsilon_j \,|\, 2 \leqslant i < j \leqslant n+1\}, \quad \Delta_n^+ = \{\varepsilon_1 - \varepsilon_i \,|\, 2 \leqslant i \leqslant n+1\}, \quad \beta = \varepsilon_1 - \varepsilon_2.$$

The centers of $\mathfrak{K}$ and $\mathfrak{M}$ correspond, respectively, to $\mathbf{R}(\varepsilon_1 - \frac{1}{n}(\varepsilon_2 + \cdots + \varepsilon_{n+1}))$ and $\mathbf{R}(\varepsilon_1 + \varepsilon_2 - 2(\varepsilon_3 + \cdots + \varepsilon_{n+1})/(n-1))$. Any $a \in \mathbf{R}$ defines a character ϕ_a (ϕ'_a) on $\mathfrak{Z}(\mathfrak{K})$ ($\mathfrak{Z}(\mathfrak{M})$) by the rule

$$\phi_a\left(\varepsilon_1 - \frac{1}{n}\left(\sum_2^{n+1} \varepsilon_i\right)\right) = ia\left(\phi'_a\left((\varepsilon_1 + \varepsilon_2) - \frac{2\left(\sum_3^{n+1} \varepsilon_i\right)}{n-1}\right) = ia\right).$$

Hence ϕ_a (ϕ'_a) defines a one-dimensional representation of $\mathfrak{K}_{\mathbf{C}}$ ($\mathfrak{M}_{\mathbf{C}}$) and it is easy to verify that $i^*(\phi_a) = \phi'_a$.

The fundamental representations are $\lambda_i = \varepsilon_2 + \cdots + \varepsilon_i$ $(2 \leq i \leq n)$, for $\mathfrak{K}_{\mathbf{C}}$, and $\lambda'_j = \varepsilon_3 + \cdots + \varepsilon_j$ $(3 \leq j \leq n)$, for $\mathfrak{M}_{\mathbf{C}}$. The branching formulas imply

$$\begin{aligned} i^*(\lambda_2) &= \phi_1 \oplus \phi_2 \otimes \lambda'_3, \\ i^*(\lambda_i) &= \phi'_{2i-3} \otimes \lambda'_i \oplus \phi'_{2i-2} \otimes \lambda'_{i+1}, \qquad 3 \leq i \leq n-1, \\ i^*(\lambda_n) &= \phi'_{2n-3} \otimes \lambda'_n \oplus \phi_{2n-2}, \end{aligned}$$

where $\phi'_j = \phi'_{a_j}$ (a_j can be easily computed). Since $\mathrm{Im}(i^*)$ contains ϕ'_a for any a, this clearly implies that $\lambda'_j \in \mathrm{Im}(i^*)$ for $3 \leq j \leq n$.

(ii) $\mathfrak{G} = \mathfrak{SO}(2n, 1)$.

$$i\mathfrak{H}^* = \left\{ \sum_1^n t_i \varepsilon_i \,|\, t_i \in \mathbf{R} \right\}, \qquad \Delta^+ = \{\varepsilon_i \,|\, 1 \leq i \leq n,\, \varepsilon_i \pm \varepsilon_j \,|\, 1 \leq i < j \leq n\},$$

$$\Delta_c^+ = \{\varepsilon_i \pm \varepsilon_j \,|\, 1 \leq i < j \leq n\}, \qquad \Delta_n^+ = \{\varepsilon_i,\, 1 \leq i \leq n\},$$

$$\beta = \varepsilon_1, \qquad \Delta_{\mathfrak{M}}^+ = \{\varepsilon_i \,|\, 2 \leq i \leq n,\, \varepsilon_i \pm \varepsilon_j \,|\, 2 \leq i < j \leq n\}.$$

Fundamental weights:

$$\mathfrak{K}_{\mathbf{C}}\colon \lambda_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (i = 1, \ldots, n-2), \qquad \lambda_\pm = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} \pm \varepsilon_n),$$

$$\mathfrak{M}_{\mathbf{C}}\colon \lambda'_i = \varepsilon_2 + \cdots + \varepsilon_i \quad (i = 2, \ldots, n-1), \qquad \lambda'_+ = \tfrac{1}{2}(\varepsilon_2 + \cdots + \varepsilon_n).$$

By the branching formulas

$$i^*(\lambda_i) = \lambda'_i \oplus \lambda'_{i+1}, \qquad i = 1, \ldots, n-2,\ (\lambda'_1 = 1),\ i^*(\lambda_\pm) = \lambda'_+ .$$

Hence $\lambda'_{i+1} = i^*(\lambda_i - \lambda_{i-1} + \lambda_{1-2} - \cdots \pm 1)$, $\lambda'_+ = i^*(\lambda_\pm)$. We include the case $\mathfrak{G} = \mathfrak{S}\mathfrak{O}(2n+1, 1)$, for completeness.

(iii) $\mathfrak{G} = \mathfrak{SO}(2n+1, 1)$.

$$i\mathfrak{H}^* = \left\{ \sum_1^{n+1} t_i \varepsilon_i \,|\, t_i \in \mathbf{R} \right\}, \qquad \Delta^+ = \{\varepsilon_i \pm \varepsilon_j \,|\, 1 \leq i < j \leq n+1\},$$

$$\Delta_c^+ = \{\varepsilon_i \pm \varepsilon_j \,|\, 2 \leq i < j \leq n-1,\, \varepsilon_i \,|\, 2 \leq i \leq n+1\},$$

$$\Delta_{\mathfrak{M}}^+ = \{\varepsilon_i \pm \varepsilon_j \,|\, 2 \leq i < j \leq n+1\}.$$

Fundamental weights:

$$\mathfrak{K}_{\mathbf{C}}\colon \lambda_i = \varepsilon_2 + \cdots + \varepsilon_i \quad (2 \leq i \leq n), \qquad \lambda_+ = \tfrac{1}{2}(\varepsilon_2 + \cdots + \varepsilon_{n+1}),$$

$$\mathfrak{M}_{\mathbf{C}}\colon \lambda'_j = \varepsilon_2 + \cdots + \varepsilon_j \quad (2 \leq j \leq n+1), \qquad \lambda'_\pm = \tfrac{1}{2}(\varepsilon_2 + \cdots + \varepsilon_n \pm \varepsilon_{n+1}).$$

Moreover, $i^*(\lambda_i) = \lambda'_i \oplus \lambda'_{i-1}$ $(2 \leq i \leq n)$, $i^*(\lambda_+) = \lambda'_+ \oplus \lambda'_-$.

Hence $\lambda'_i = i^*(\lambda_i - \lambda_{i-1} + \lambda_{i-2} - \cdots \pm 1) \in \mathrm{Im}(i^*)$ $(i = 2, \ldots, n)$.

Recall [**Hu**, p. 188] that $\lambda'_+ \otimes \lambda'_- = \lambda'_n \oplus \lambda'_{n-2} \oplus \cdots$.

Thus $\lambda'_+ \oplus \lambda'_-$ and $\lambda'_+ \otimes \lambda'_- \in \mathrm{Im}(i^*)$. On the other hand, if $W = \{1, u\}$ one knows that $(\lambda'_i)^u = \lambda'_i$ $(i = 2, \ldots, n-1)$, $(\lambda'_\pm)^u = \lambda'_\mp$.

Hence $R(M)^W = \mathbf{Z}[\lambda'_2, \ldots, \lambda'_{n-1}][\lambda'_+, \lambda'_-]^W$ is a polynomial ring over $\mathbf{Z}[\lambda'_2, \ldots, \lambda'_{n-1}]$ in the symmetric functions $\lambda'_+ \oplus \lambda'_-$, $\lambda'_+ \otimes \lambda'_-$. Hence, if M is simply connected (i.e. $G = \mathrm{Spin}(2n+1, 1)$) $\mathrm{Im}(i^*) = R(M)^W$.

The case $G = SO(2n,1)$ follows from Lemma 2.3.

(iv) $\mathfrak{G} = \mathfrak{S}p(n,1)$ $(n \geqslant 2)$.

$$\mathfrak{K} \simeq \mathfrak{S}p(1) \times \mathfrak{S}p(n), \qquad i\mathfrak{H}^* = \left\{ \sum_1^{n+1} t_i \varepsilon_i \,|\, t_i \in \mathbf{R} \right\},$$

$$\Delta^+ = \{2\varepsilon_i, 1 \leqslant i \leqslant n+1;\ \varepsilon_i \pm \varepsilon_j, 1 \leqslant i < j \leqslant n+1\},$$

$$\Delta_c^+ = \{2\varepsilon_i, 1 \leqslant i \leqslant n+1;\ \varepsilon_i \pm \varepsilon_j, 1 \leqslant i < j \leqslant n\}, \qquad \Delta_n^+ = \{\varepsilon_i \pm \varepsilon_{n+1}, 1 \leqslant i \leqslant n\},$$

$$\beta = \varepsilon_1 - \varepsilon_{n+1}, \qquad \ker\beta = \mathbf{C}(\varepsilon_1 + \varepsilon_{n+1}) + \sum_2^n \mathbf{C} \cdot \varepsilon_i,$$

$$\Delta_{\mathfrak{M}}^+ = \{\varepsilon_1 + \varepsilon_{n+1}|_{\ker\beta}\} \cup \{2\varepsilon_i|_{\ker\beta} \,|\, 2 \leqslant i \leqslant n\} \cup \{\varepsilon_i \pm \varepsilon_j|_{\ker\beta} \,|\, 2 \leqslant i < j \leqslant n\}.$$

It will be understood from now on that roots and weights of $\mathfrak{M}_{\mathbf{C}}$ are restricted to $\ker\beta$.

Simple roots:

$$\mathfrak{K}_{\mathbf{C}}: 2\varepsilon_{n+1}, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n,$$

$$\mathfrak{M}_{\mathbf{C}}: \tfrac{1}{2}(\varepsilon_1 + \varepsilon_{n+1}), \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n.$$

Fundamental weights: $\mathfrak{K}_{\mathbf{C}}$: $\lambda = \varepsilon_{n+1}$, $\lambda_j = \varepsilon_1 + \cdots + \varepsilon_j$ $(1 \leqslant j \leqslant n)$ (with dimensions $d_\lambda = 2$, $d_{\lambda_j} = \binom{2n}{j} - \binom{2n}{j-2}$ respectively)

$$\mathfrak{M}_{\mathbf{C}}: \lambda' = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_{n+1}), \quad \lambda'_j = \varepsilon_2 + \cdots + \varepsilon_j \quad (2 \leqslant j \leqslant n).$$

It can be shown directly or by using the branching formulas in [**B**, 10.7] that

$$i^*(\lambda) = \lambda' \otimes 1, \quad i^*(\lambda_1) = \lambda' \otimes 1 \oplus 1 \otimes \lambda'_2, \quad i^*(\lambda_2) = \lambda' \otimes \lambda'_2 \oplus 1 \otimes \lambda'_3 \oplus 1,$$

$$i^*(\lambda_j) = \lambda' \otimes \lambda'_j \oplus 1 \otimes \lambda'_{j+1} \oplus 1 \otimes \lambda'_{j-1} \quad \text{if } 3 \leqslant j \leqslant n-1,$$

$$i^*(\lambda_n) = \lambda' \otimes \lambda'_n \oplus 1 \otimes \lambda'_{n-1}.$$

It then follows by induction that i^* is surjective. Note that making some conventions these formulas can be written in a closed form.

(v) $\mathfrak{G} = \mathfrak{f}_4(-20)$.

$$i\mathfrak{H}^* = \mathbf{R}^4, \qquad \Delta^+ = \{\tfrac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4);\ \varepsilon_i \pm \varepsilon_j, 1 \leqslant i < j \leqslant 4;\ \varepsilon_i, 1 \leqslant i \leqslant 4\},$$

$$\Delta_c^+ = \{\varepsilon_i \pm \varepsilon_j, 1 \leqslant i < j \leqslant 4;\ \varepsilon_i, 1 \leqslant i \leqslant 4\}, \qquad \Delta_n^+ = \{\tfrac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\},$$

$$\beta = \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4),$$

$$\Delta_{\mathfrak{M}}^+ = \{\alpha|_{\ker\beta} \,|\, \alpha = \varepsilon_i \pm \varepsilon_j, 2 \leqslant i < j \leqslant 4 \text{ or } \alpha = \varepsilon_1 + \varepsilon_i, 2 \leqslant i \leqslant 4\}$$

$$= \{\alpha|_{\ker\beta} \,|\, \alpha = \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \text{ (even number of } +\text{)},$$

$$\alpha = \varepsilon_1 + \varepsilon_i, 2 \leqslant i \leqslant 4, \text{ or } \alpha = \varepsilon_i - \varepsilon_j, 2 \leqslant i < j \leqslant 4\}.$$

We have: $\mathfrak{K}_{\mathbf{C}} \simeq \mathfrak{SO}(9)$, $\mathfrak{M}_{\mathbf{C}} \simeq \mathfrak{SO}(7)$. The simple roots for $\Delta_{\mathfrak{M}}^+$ are $\varepsilon_3 - \varepsilon_4$, $\varepsilon_2 - \varepsilon_3$, $\frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$. The fundamental representations for $\mathfrak{K}_{\mathbf{C}}$: $\lambda_1 = \varepsilon_1$, $\lambda_2 = \varepsilon_1 + \varepsilon_2$, $\lambda_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $\lambda_+ = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$ of dimensions 9, 36, 84 and 16, respectively,

$$\mathfrak{M}_{\mathbf{C}}: \lambda'_1 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)|_{\ker\beta'},$$

$$\lambda'_2 = (\varepsilon_1 + \varepsilon_2)|_{\ker\beta'} \lambda'_+ = \tfrac{1}{4}(3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)|_{\ker\beta}$$

(dimensions 7, 21, and 8). The branching formulas are

$$i^*(\lambda_1) = \lambda'_+ \oplus 1, \quad i^*(\lambda_2) = \lambda'_1 \oplus \lambda'_2 \oplus \lambda'_+, \quad i^*(\lambda_+) = \lambda'_1 \oplus \lambda'_+ \oplus 1$$

and

$$i^*(\lambda_3) = (\lambda'_1 + \lambda'_+) \oplus \lambda'_1 \oplus \lambda'_2 \oplus \lambda'_+ .$$

Therefore, $\lambda'_+ = i^*(\lambda_1 - 1)$, $\lambda'_1 = i^*(\lambda_+ - \lambda_1)$, $\lambda'_2 = i^*(\lambda_2 - \lambda_+ + 1)$ and i^* is surjective. We sketch the proof of the branching formulas.

A basis for the unipotent radical of the Borel subalgebra of $\mathfrak{M}_{\mathbf{C}}$ defined by $\Delta^+_{\mathfrak{M}}$ is $X_{\varepsilon_1+\varepsilon_i}$, $2 \leqslant i \leqslant 4$, $X_{\varepsilon_i-\varepsilon_j}$, $2 \leqslant i < j \leqslant 4$, $X_{\varepsilon_3+\varepsilon_4} + c_1 \cdot X_{\varepsilon_1-\varepsilon_2}$, $X_{\varepsilon_2+\varepsilon_4} + c_2 \cdot X_{\varepsilon_1-\varepsilon_3}$ and $X_{\varepsilon_2+\varepsilon_3} + c_3 \cdot X_{\varepsilon_1-\varepsilon_4}$, where the constants c_i are as in Lemma 2.1.

Since $\lambda_1 = \frac{1}{2}\beta + \lambda'_+$, the restriction of λ_1 contains λ'_+. Since $\dim(\lambda_1) = 9$, $\dim(\lambda'_+) = 8$, the first identity is clear.

Now $\lambda_+ = (-\frac{1}{2})\beta + \lambda'_+$. Hence $i^*(\lambda_+)$ contains λ'_+. Since any weight of λ_+ is of the form $\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ one checks that any vector of weight $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)$ is $\mathfrak{M}_{\mathbf{C}}$-dominant. Thus, λ_+ restricted to $\mathfrak{M}_{\mathbf{C}}$ contains λ'_1. Since $\dim(\lambda') = 7$, the third identity follows. Now we study $\lambda_2 = \varepsilon_1 + \varepsilon_2$, restricted to $\mathfrak{M}_{\mathbf{C}}$. This is the adjoint representation of $\mathfrak{K}_{\mathbf{C}}$ with weights $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leqslant i < j \leqslant 4$, $\pm\varepsilon_i$, $1 \leqslant i \leqslant 4$, and 0, with multiplicity 4. Clearly, $i^*(\lambda_2)$ contains the $\mathfrak{M}_{\mathbf{C}}$-module with highest weight λ'_2. On the other hand, it is easily checked that any vector of weight ε_1 is $\mathfrak{M}_{\mathbf{C}}$-dominant. Since $\varepsilon_1|_{\ker\beta} = \lambda'_+$, then $i^*(\lambda_2) = \lambda'_2 \oplus \lambda'_+ \oplus \mu$, μ a representation of dimension 7. Now if $v_1 \neq 0$ is of weight $\varepsilon_2 + \varepsilon_3$ and $v_2 \neq 0$ is of weight $\varepsilon_1 - \varepsilon_4$, then $X_{\varepsilon_1-\varepsilon_2}(v_1)$ and $X_{\varepsilon_3+\varepsilon_4}(v_2)$ are nonzero vectors of weight $\varepsilon_1 + \varepsilon_2$. Hence, we can choose v_1 and v_2 so that $c_1 X_{\varepsilon_1-\varepsilon_2}(v_1) + X_{\varepsilon_3+\varepsilon_4}(v_2) = 0$. It is easy to verify that with this choice $v_1 + v_2$ is $\mathfrak{M}_{\mathbf{C}}$-dominant. Since

$$\varepsilon_2 + \varepsilon_3 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4) - \beta, \qquad \varepsilon_1 - \varepsilon_4 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4) + \beta,$$

the $\mathfrak{M}_{\mathbf{C}}$-submodule spanned by $v_1 + v_2$ has highest weight λ'_1. This proves the third identity, since $\dim(\lambda'_1) = 7$. We omit the proof of the last one, since from the first three one already concludes that i^* is surjective.

We conclude the paper by computing $\ker(i^*\colon R(K) \to R(M))$ explicitly. Recall that each $\eta \in \ker(i^*)$ yields an alternating sum formula in the multiplicities $n_\Gamma(\omega)$, if Γ is torsion-free [**M3**, 1.2]. We assume from now on that G is a connected, semisimple Lie group of split rank one, with finite center. If K is compact and $\tilde{K} \xrightarrow{p} K$, a finite covering, we identify $\mathcal{E}(K)$ with $\{\tau \in \mathcal{E}(\tilde{K}) \mid \ker p \subset \ker \tau\}$ and $R(K)$ with the corresponding subring of $R(\tilde{K})$. Let T be a maximal torus of K and $\tilde{T} = p^{-1}(T)$.

2.6. Lemma. (i) *If* rank $G >$ rank K, *then* $\ker i^* = 0$.

(ii) *If* rank $G =$ rank K, *let* $\tilde{G} \xrightarrow{p} G$ *be a finite covering so that* $\delta_n = \frac{1}{2}(\sum_{\Delta_n^+}\alpha)$ *is a weight of* $\tilde{T} = p^{-1}(T)$. *Then* $\ker i^* = R(K) \cap R(\tilde{K}) \cdot \eta_1$, *where* $\eta_1 \in R(\tilde{K})$ *is such that* $\eta_1(t) = t^{-\delta_n}\prod_{\Delta_n^+}(t^\gamma - 1)$, $t \in \tilde{T}$.

Proof. As noted at the beginning of the section, if rank $G >$ rank K, $i^*\colon R(K) \to R(M)^W$ is an isomorphism.

We thus assume that rank G = rank K. We also assume that δ_n is a weight of T. The lemma is obvious once it is proved in this case.

Let $\beta \in \Delta_n^+$ and $\mathfrak{A} = \mathbf{R}(X_\beta + X_{-\beta})$, as above.

If $\eta \in \ker i^*$, then $\eta(t) = 0$ for $t \in T_\beta$, since $T_\beta \subset M$. Therefore ([**A**, 6.4], essentially), there is $\eta' \in R(T)$ so that

$$\eta(t) = (t^\beta - 1) \cdot \eta'(t), \qquad t \in T.$$

Since $\eta^s = \eta$ $(s \in W_K)$, then $\eta(t) = 0$, for $t \in sT_\beta = T_{s\beta}$. If $s\beta \neq \pm\beta$, then $\dim T_\beta \cap T_{s\beta} < \dim T_\beta$. Thus, by continuity, $\eta'(t) = 0$, $t \in T_{s\beta}$. Hence, $\eta'(t) = (t^{s\beta} - 1) \cdot \eta''(t)$, for some $\eta'' \in R(T)$.

We may thus write

$$(*) \qquad \eta = \prod_{\gamma \in \Psi} (t^\gamma - 1) \cdot \eta' \qquad (\eta' = \eta'(\Psi) \in R(T)),$$

where Ψ is any subset of $W_K \cdot \beta$ such that $\Psi \cap -\Psi = \varnothing$.

Since $\mathfrak{G}$ is of split rank one, then either $\mathfrak{K}_\mathbf{C}$ acts irreducibly on $\mathfrak{P}_\mathbf{C}$, or $\mathfrak{P}_\mathbf{C} = \mathfrak{P}^+ \oplus \mathfrak{P}^-$, where $\mathfrak{P}^+ = \Sigma_{\Delta_n^+}\mathfrak{G}_\alpha, \mathfrak{P}^- = \Sigma_{\Delta_n^+}\mathfrak{G}_\alpha$ and $\mathfrak{P}^\pm$ are irreducible subspaces. Furthermore, all noncompact roots have the same length [**KW**, 12.1]. Thus $W_K \cdot \beta = \Delta_n^+$ or $W_K \cdot \beta = \Delta_n$, since W_K acts transitively on weights of a fixed length.

Then, if $\Psi = \Delta_n^+$ in $(*)$, we may write

$$\eta = \eta_0 \cdot \eta'' \qquad \text{with } \eta_0(t) = \prod_{\gamma \in \Delta_n^+} (t^\gamma - 1), \eta'' \in R(T),$$

or

$$\eta = \eta_1 \cdot \eta', \quad \text{where } \eta_1(t) = t^{-\delta_n} \cdot \eta_0(t) \in R(T)^{W_K} \text{ and } \eta' \in R(T)^{W_K}.$$

On the other hand, $M = Z(G) \cdot M^0$ (M^0, the connected component of 1 in M) and $T_\beta = Z(G) \cdot T_\beta^0$ ($T_\beta^0 = \exp(\ker\beta \cap \mathfrak{H})$, a maximal torus of M^0). Hence, $M = \bigcup\{x \cdot T_\beta \cdot x^{-1} \mid x \in M\}$ and $\eta_1 \in \ker i^*$, since $\eta_1(t) = 0$ for $t \in T_\beta$. Thus $\ker i^* = R(K) \cdot \eta_1$, as asserted.

EXAMPLES. (i) G simply connected. Then $\ker i^* = R(K) \cdot \eta_1$.

(ii) $G = Sl(2, \mathbf{R})$.

Then

$$K = T = \left\{ k(\theta) = \begin{vmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{vmatrix} \right\},$$

$\mathcal{E}(K) = \{\tau_n \mid \tau_n(k(\theta)) = e^{in\theta}\}$, $\Delta = \Delta_n = \{\pm\alpha\}$, $k(\theta)^\alpha = e^{2i\theta} = \tau_2(k(\theta))$. Hence $\ker i^* = R(K) \cdot (\tau_1 - \tau_{-1})$, as in [**M3**, Lemma 2.1].

(iii) $\mathfrak{G} = \mathfrak{SU}(n, 1)$.

Then $W_K \cdot \beta = \Delta_n^+$ and $\eta_0(t) = \prod_{\Delta_n^+}(t^\gamma - 1) \in R(T)^{W_K}$. Hence $\ker i^* = R(K) \cdot \eta_0$ (if δ_n is a weight of K, η_0 and η_1 differ by a unit in $R(T)^{W_K} \simeq R(K)$).

(iv) $G = SO(2n, 1)$.

In the notation of 2.5(ii), by Lemma 2.6,

$$\ker i^* = \{\eta = \eta' \otimes \eta_1 \mid \eta' \in R\,\mathrm{Spin}(2n), \eta \in RSO(2n)\},$$

where $\eta_1 = \lambda_+ - \lambda_- \in R\,\mathrm{Spin}(2n)$. Now

$R\,\mathrm{Spin}(2n) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n][\lambda_+, \lambda_-] \subset RSO(2n)[\lambda_+, \lambda_-]\}$ [**Hu**, Chapter 13].

It is then easy to check that $\eta' \otimes \eta_1 \in RSO(2n)$ if and only if $\eta' = \eta^+ \otimes \lambda_+ + \eta^- \otimes \lambda_-$, $\eta^\pm \in RSO(2n)$. That is,

$$\ker i^* = \left\{ (\eta^+ \otimes \lambda_+ + \eta^- \otimes \lambda_-) \otimes (\lambda_+ - \lambda_-) \mid \eta^\pm \in RSO(2n) \right\}.$$

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