

A GENERALIZATION OF F -SPACES AND SOME TOPOLOGICAL CHARACTERIZATIONS OF GCH

BY

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ABSTRACT. Several topological characterizations involving F -spaces of the continuum hypothesis are due to R. G. Woods and E. K. van Douwen. We extend this work by defining a space X to be an F_α -space if the union of $< \alpha$ cozero-sets is C^* -embedded in X and by giving, for every infinite cardinal α , topological characterizations involving F_α -spaces of the cardinal equality $2^\alpha = \alpha^+$.

Topological characterizations of the continuum hypothesis (CH) abound in the theory of F -spaces. The two which are important for this paper are due to Woods ([**Wo**₁] and [**Wo**₂]) and to van Douwen [**vd**₁]. We state these as follows:

0.1. THEOREM (WOODS [**Wo**₁, 2.2 AND **Wo**₂, 1.1(a)] AND VAN DOUWEN [**vd**₁]). *The following are equivalent:*

- (a) CH.
- (b) Every small F -space is weakly Lindelöf.
- (c) Every small normal countably compact F -space is compact.

Topological characterizations of CH involving F -spaces have also been given by Dow [**Do**, 3.4], and van Douwen has pointed out a number of others in [**vd**₁]. (We give another such characterization of CH in this paper (see 3.19 and 5.3(b)).)

The present study, which comes from the author's Ph.D. thesis at Ohio University (done under the direction of R. L. Blair, to whom we are greatly indebted), and which is summarized in [**Sw**₃], has as its purpose an extension of the work of Woods and of van Douwen. We define " F_α -spaces" and give, for every infinite cardinal α , topological characterizations involving F_α -spaces of the segment $2^\alpha = \alpha^+$ of the generalized continuum hypothesis (GCH).

1. Definitions and conventions. By a *topological space* (or simply a *space*), we always mean a Tychonoff space.

Let X be a topological space. Then $C(X)$ (resp. $C^*(X)$) denotes the set of all real-valued (resp. bounded real-valued) continuous functions on X . For $f \in C(X)$, $Z(f) = \{x \in X: f(x) = 0\}$ is a *zero-set* of X . The complement in X of a zero-set of X is a *cozero-set* of X . A subset A of X is *C-embedded* (resp. *C*-embedded*) in X if

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every $f \in C(A)$ (resp. every $f \in C^*(A)$) has a continuous extension over X . A space X is an F -space if every cozero-set of X is C^* -embedded in X [GJ, 14.25].

What we intend to do in this paper is to add a cardinal parameter α to some of the properties mentioned in 0.1 in such a way that the Woods and van Douwen characterizations of CH become corollaries of our results in the case $\alpha = \omega$. To this end we make the following definitions: A subset A of a space X is a G_α -set in X if $A = \bigcap \mathcal{U}$, where $\mathcal{U}$ is a collection of open sets in X with $|\mathcal{U}| < \alpha$. A set $A \subset X$ is G_α -dense in X if every G_α -set in X meets A . A is α -open in X if $A = \bigcup \mathcal{U}$ where $\mathcal{U}$ is a collection of cozero-sets of X with $|\mathcal{U}| < \alpha$, and A is α -closed in X if $X - A$ is α -open in X . Clearly finite unions of α -open sets are α -open, and A is a α -closed if and only if $A = \bigcap \mathcal{Z}$ where $\mathcal{Z}$ is a collection of zero-sets of X with $|\mathcal{Z}| < \alpha$. It is obvious that an ω_1 -open set is a cozero-set, that an ω_1 -closed set is a zero-set and that, for $\alpha \geq \omega_1$, an α -closed set is a G_α -set.

A subset A of X is z -embedded in X if every zero-set of A is of the form $Z \cap A$ for some zero-set Z of X . Clearly, if A is z -embedded in X and if G is α -open (resp. α -closed) in A , then $G = G' \cap A$ for some α -open (resp. α -closed) subset G' of X .

A space X is an F_α -space if every α -open subset of X is C^* -embedded in X , and X is α -basically disconnected if every α -open set in X has open closure in X . X is *extremally disconnected* if every open set in X has open closure in X . Clearly X is an F_ω -space if and only if X is an F -space, and X is ω_1 -basically disconnected if and only if X is basically disconnected.

1.1. REMARKS. The terms “ F_α -space” and “ α -basically disconnected space” are used by other authors in senses different from the foregoing:

(a) In [CN, §14], Comfort and Negrepontis restrict attention to the class $\mathfrak{B}$ of spaces that have a base of closed-and-open sets. For $X \in \mathfrak{B}$ and G open in X , G is of type $< \alpha$ if G is the union of $< \alpha$ closed-and-open subsets of X . Then, in the sense of [CN], a space $X \in \mathfrak{B}$ is “ α -basically disconnected” if every open subset of X of type $< \alpha$ has open closure in X , and X is an “ F_α -space” if every open subset of X of type $< \alpha$ is C^* -embedded in X [CN, pp. 350, 343]. It is clear that if $X \in \mathfrak{B}$ and if X is an F_α -space in our sense, then X is an F_α -space in the sense of [CN]. Our definition of F_α -space is chosen to insure that our theorems reduce to results about F -spaces for the case $\alpha = \omega_1$.

(b) In [NL], Neville and Lloyd define a (compact) space X to be an “ F_α -space” if any two disjoint α -open subsets of X have disjoint closures in X . It can be shown that, in the presence of normality, the Neville and Lloyd definition of F_α -space coincides with ours, and that, if X is compact and $X \in \mathfrak{B}$, then all three definitions of an F_α -space (that of [CN], of [NL], and ours) coincide.

We shall need the following cardinal functions:

$L(X) = \min\{\kappa: \text{every open cover of } X \text{ has a subcover of cardinality } \leq \kappa\} + \omega$.

$wL(X) = \min\{\kappa: \text{if } \mathcal{U} \text{ is an open cover of } X, \text{ then there exists } \mathcal{V} \subset \mathcal{U} \text{ such that } |\mathcal{V}| \leq \kappa \text{ and } \bigcup \mathcal{V} \text{ is dense in } X\} + \omega$. We say that X is *weakly Lindelöf* if $wL(X) = \omega$.

$d(X) = \min\{|S|: S \text{ is a dense subset of } X\} + \omega$.

$w(X) = \min\{|\mathfrak{B}|: \mathfrak{B} \text{ is a base for } X\} + \omega$.

For $A \subset X$, define

$$\chi(A, X) = \min\{|\mathfrak{B}| : \mathfrak{B} \text{ is a base for the neighborhoods of } A \text{ in } X\} + \omega.$$

$$\psi(A, X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a collection of open subsets of } X \text{ and } A = \bigcap \mathcal{U}\} + \omega.$$

$$\chi_c(X) = \sup\{\chi(A, X) : A \text{ is closed in } X\}.$$

$$\chi_{nz}(X) = \sup\{\chi(A, X) : A \text{ is a nowhere dense zero-set of } X\}.$$

2. Properties of F_α -spaces and α -basically disconnected spaces. In proving some elementary results about F_α -spaces and α -basically disconnected spaces, we will frequently make use of the following proposition.

2.1. PROPOSITION. *If G is α -open in X and if Z_1 and Z_2 are disjoint zero-sets of G , then Z_1 and Z_2 are contained in disjoint α -open subsets of X .*

PROOF. Let $G = \bigcup_{\xi < \kappa} P_\xi$, where each P_ξ is a cozero-set of X and $\kappa < \alpha$, and let Z_1 and Z_2 be disjoint zero-sets of G . There exist disjoint cozero-sets Q_1 and Q_2 of G with $Z_1 \subset Q_1$ and $Z_2 \subset Q_2$. For each $\xi < \kappa$, $Q_1 \cap P_\xi$ and $Q_2 \cap P_\xi$ are disjoint cozero-sets of P_ξ and hence of X [En, 2.1.B(c)], and thus $\bigcup_{\xi < \kappa} (Q_1 \cap P_\xi)$ and $\bigcup_{\xi < \kappa} (Q_2 \cap P_\xi)$ are disjoint α -open subsets of X containing Z_1 and Z_2 respectively. $\square$

2.2. PROPOSITION. *If X is a space, then the following are equivalent:*

- (a) X is extremally disconnected.
- (b) X is α -basically disconnected for every infinite cardinal α .
- (c) X is an F_α -space for every infinite cardinal α .

PROOF. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c) Let X be α -basically disconnected, let G be an α -open subset of X , and let Z_1 and Z_2 be disjoint zero-sets of G . It suffices to show that Z_1 and Z_2 are completely separated in X .

By 2.1, Z_1 and Z_2 are contained in disjoint α -open subsets P_1 and P_2 of X respectively. Then $\text{cl}_X P_1$ and $\text{cl}_X P_2$ are disjoint closed-and-open subsets of X and hence Z_1 and Z_2 are completely separated in X .

(c) $\Rightarrow$ (a) By [GJ, 1H.6] it suffices to show that every open subset of X is C^* -embedded in X , and this follows from (c) and the fact that the cozero-sets form a base for X . $\square$

The following is a generalization of [GJ, 14N.4].

2.3. THEOREM. *A space X is an F_α -space if and only if any two disjoint α -open subsets of X are completely separated in X .*

PROOF. If G and H are disjoint α -open subsets of the F_α -space X , then the function $f \in C^*(G \cup H)$ that is 0 on G and 1 on H has an extension $g \in C(X)$, and g completely separates G and H . Conversely, let G be α -open in X and let Z_1 and Z_2 be disjoint zero-sets of G . By 2.1, Z_1 and Z_2 are contained in disjoint α -open subsets of X , which are completely separated in X by hypothesis, and hence G is C^* -embedded in X . $\square$

The well-known fact that the F -space property is hereditary with respect to C^* -embedded subsets does not generalize to F_α -spaces (see 2.6), but we do have the following result:

2.4. THEOREM. *Let S be dense and C^* -embedded in the space T . Then S is an F_α -space if and only if T is an F_α -space.*

PROOF. Let S be an F_α -space and let G be α -open in T . Let $f \in C^*(G)$. Since $S \cap G$ is α -open in S , $S \cap G$ is C^* -embedded in S and hence in T , and thus $f|(S \cap G)$ has an extension $g \in C(T)$. Since $S \cap G$ is dense in G , $g|G = f$. Thus G is C^* -embedded in T , and we conclude that T is an F_α -space.

Conversely, assume that T is an F_α -space and let G be α -open in S . Let Z_1 and Z_2 be disjoint zero-sets of G . By 2.1 there exist disjoint α -open subsets H_1 and H_2 of S such that $Z_1 \subset H_1$ and $Z_2 \subset H_2$. Since S is C^* -embedded in T , there exist α -open subsets L_1 and L_2 of T such that $H_1 = L_1 \cap S$ and $H_2 = L_2 \cap S$, and since S is dense in T , L_1 and L_2 are disjoint. By 2.3, L_1 and L_2 are completely separated in S . We conclude that G is C^* -embedded in S . $\square$

The following corollary generalizes an important property of F -spaces [GJ, 14.25].

2.5. COROLLARY. *A space X is an F_α -space if and only if βX is an F_α -space.*

2.6. REMARK. For $\alpha > \omega_1$, a C^* -embedded subset (in fact, a compact subset) of an F_α -space need not be an F_α -space: $\beta\omega$ is extremally disconnected, hence an F_α -space for all α , but $\beta\omega - \omega$ is not basically disconnected [GJ, 6W.3], and so there exists a cardinal α such that $\beta\omega - \omega$ is not an F_α -space. (The referee of this paper has noted that $\beta\omega - \omega$ is not an F_{ω_2} -space since Hausdorff has shown that there is an (ω_1, ω_1) -gap in $\beta\omega - \omega$ [Ha, §1].)

The following result generalizes [GJ, 6M.1].

2.7. PROPOSITION. *A space X is α -basically disconnected if and only if βX is α -basically disconnected.*

PROOF. Let X be α -basically disconnected and let G be α -open in βX . Then $G \cap X$ is α -open in X and hence $\text{cl}_{\beta X} G = \text{cl}_{\beta X}(\text{cl}_X(G \cap X))$ is open in βX . Conversely, assume that βX is α -basically disconnected and let H be α -open in X . Then $H = H' \cap X$, where H' is α -open in βX . Since $\text{cl}_{\beta X} H'$ is open in βX , $\text{cl}_X H = X \cap \text{cl}_{\beta X} H'$ is open in X . $\square$

A collection of pairwise disjoint open subsets of a space X is called a *cellular family* in X . A subset A of X is *cellularly embedded* in X if every cellular family in A is the restriction to A of a cellular family in X . We will use the next proposition in §5.

2.8. PROPOSITION. *If A is cellularly embedded in the F_α -space X , and if $wL(A) < \alpha$, then A is C^* -embedded in X .*

PROOF. Let Z_1 and Z_2 be disjoint zero-sets of A . There exist disjoint cozero-sets P_1 and P_2 of A and disjoint open sets U_1 and U_2 of X such that $Z_1 \subset P_1 \subset U_1$ and $Z_2 \subset P_2 \subset U_2$. For each $x \in Z_1$ and each $y \in Z_2$ pick cozero-sets P_x and Q_y of X

with $x \in P_x \subset U_1$ and $y \in Q_y \subset U_2$. There exist $I \subset Z_1$ and $J \subset Z_2$ with $|I|, |J| < \alpha$ such that $Z_1 \subset \text{cl}_X \bigcup_{x \in I} P_x$ and $Z_2 \subset \text{cl}_X \bigcup_{y \in J} Q_y$. Since $\bigcup_{x \in I} P_x$ and $\bigcup_{y \in J} Q_y$ are disjoint α -open subsets of X , Z_1 and Z_2 are completely separated in X and hence A is C^* -embedded in X . $\square$

Corollary 2.9 is well known (see [GJ, 14N.1]).

2.9. COROLLARY. *No point of an F -space is the limit of a sequence of distinct points.*

3. F_α -spaces and χ_α -remote points. We call a space X α -small if $|C^*(X)| \leq 2^\alpha$ and we call X small if X is ω -small. We denote the Stone-Čech remainder $\beta X - X$ of X by X^* .

The theorem of Fine and Gillman below (3.1) plays a crucial role in the proof that (a) $\Rightarrow$ (c) in 0.1.

3.1. THEOREM (FINE AND GILLMAN [FG, 4.6]) [CH]. *If X is a small locally compact σ -compact space, then X^* has no proper dense C^* -embedded subspace.*

Before stating a consequence of 3.1, we need the following result.

3.2. PROPOSITION [Sw₁, 5.3]. *If F is closed in the normal space X , then $\text{cl}_{\beta X} F$ is α^+ -closed in βX if and only if $\chi(F, X) \leq \alpha$.*

3.3. PROPOSITION [CH]. *If X is a small normal F -space and if $p \in X^*$, then $p \notin \text{cl}_{\beta X} F$ for every closed nowhere dense subset F of X with $\chi(F, X) = \omega$.*

PROOF. Suppose $p \in X^*$ and $p \in \text{cl}_{\beta X} F$ where F is a closed nowhere dense subset of X with $\chi(F, X) = \omega$. By 3.2, $\text{cl}_{\beta X} F$ is a nowhere dense zero-set of βX and hence $\beta X - \text{cl}_{\beta X} F$ is a dense cozero-set of the F -space βX . Then $\text{cl}_{\beta X} F$ is the Stone-Čech remainder of the small locally compact σ -compact space $\beta X - \text{cl}_{\beta X} F$. By 3.1, then, $\text{cl}_{\beta X} F$ has no proper dense C^* -embedded subset, which is, since $p \in (\text{cl}_{\beta X} F) - F$, a contradiction. $\square$

What we have, then, is that, under the hypothesis of 3.3, all points of X^* are “remote” in some sense. We make this notion precise by defining a point $p \in X^*$ to be a χ_α -remote point of X if $p \notin \text{cl}_{\beta X} F$ for every closed nowhere dense subset F of X with $\chi(F, X) \leq \alpha$. A point $p \in X^*$, then, is remote in the sense of [vD₂] if and only if p is χ_α -remote for every cardinal α ; and if $\chi_c(X) \leq \alpha$, then for every $p \in X^*$, p is χ_α -remote if and only if p is remote.

What we will establish in this paper is that, under a suitable generalization of the hypothesis of 0.1(c), no point of X^* is χ_α -remote. In the case $\alpha = \omega$, then, by 3.3, $X^* = \emptyset$.

If $\mathcal{F}$ is a collection of sets, we say that $\mathcal{F}$ has the α -intersection property if $\bigcap \mathcal{F}' \neq \emptyset$ for all $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| < \alpha$. We set $\beta_\alpha X = \{p \in \beta X : p \text{ has the } \alpha\text{-intersection property}\}$ (where points of βX are regarded as z -ultrafilters on X). A space X is α -pseudocompact if $\beta_{\alpha^+} X = \beta X$ [Ke]. Note that the Hewitt realcompactification νX of X is $\beta_{\omega_1} X$ and that X is pseudocompact if and only if X is ω -pseudocompact.

A space X is $[\alpha, \beta]$ -compact if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq \beta$ has a subcover $\mathcal{U}'$ such that $|\mathcal{U}'| < \alpha$. It can be shown (see [Ke, 2.2 and §3]) that if X is $[\omega, \alpha]$ -compact, then X is α -pseudocompact.

For our purposes, it turns out that “ α -pseudocompact” is the appropriate generalization of “countably compact” in 0.1(c). The important fact about an α -pseudocompact space X for this study is that, since X is G_α -dense in $\beta_\alpha X$, if X is α -pseudocompact, then every G_α -set in βX meets X .

We are now ready to state the main theorem of this section.

3.4. THEOREM [$2^\alpha = \alpha^+$]. *If X is an α -small, α -pseudocompact, normal F_α -space, then X has no χ_α -remote point.*

Before proceeding with the proof of 3.4, let us note that 0.1(a) $\Rightarrow$ (c) is a corollary.

3.5. COROLLARY (WOODS [$\mathbf{Wo}_1$, 1.1(a)]) [CH]. *If X is a small normal countably compact F -space, then X is compact.*

PROOF. By 3.3, every point of X^* is χ_ω -remote. Hence by 3.4, $X^* = \emptyset$. $\square$

We postpone the proof of 3.4 until after we have established a generalization of 0.1(a) $\Rightarrow$ (b). This we do in a sequence of lemmas in order to show precisely where the hypothesis $2^\alpha = \alpha^+$ is needed.

The easy proof of the first lemma is omitted.

3.6. LEMMA. *If $L(X) \leq \alpha$, and if G is α^+ -open in X , then $L(G) \leq \alpha$.*

The next lemma is a generalization of [FG, 4.1]. The lemma is essentially [CN, 14.1] (except that the latter has an unnecessary regularity hypothesis on α), and we are therefore omitting the proof. A detailed proof can be found in [Sw₂, 16.10].

3.7. LEMMA. *Let X be an F_α -space and let $V = \bigcup_{\xi < \alpha} V_\xi$ be an α^+ -open subset of X . If $G \subset V$ and if $G \cap V_\xi$ is α -open in V for each $\xi < \alpha$, then G is C^* -embedded in V .*

The proof of the following lemma uses techniques similar to those of [$\mathbf{Wo}_1$, 2.2].

3.8. LEMMA. *If $wL(X) = L(X) = \alpha^+$, then there exists an α^{++} -open subset $V = \bigcup_{\xi < \alpha^+} V_\xi$ of βX such that $X \subset V$ and there exists a pairwise disjoint collection $\{P_\xi: \xi < \alpha^+\}$ of cozero-sets of βX with each $P_\xi \subset V$ and such that for all $\beta < \alpha^+$, $V_\beta \cap \bigcup_{\xi < \alpha^+} P_\xi$ is α^+ -open in V .*

PROOF. Since $wL(X) = \alpha^+$, there is an open cover $\mathcal{Q}$ of cozero-sets of X such that no subcover of cardinality α has dense union in X . Since $L(X) = \alpha^+$, $\mathcal{Q}$ has a subcover $\{U_\xi: \xi < \alpha^+\}$ of cardinality α^+ . Then each $U_\xi = V_\xi \cap X$, where V_ξ is a cozero-set of βX . Let $V = \bigcup_{\xi < \alpha^+} V_\xi$ and note that $X \subset V$. We will construct, by recursion, a sequence $\langle P_\xi: \xi < \alpha^+ \rangle$ satisfying the following two conditions:

- (1) For each $\xi < \alpha^+$, P_ξ is a nonempty cozero-set in βX and $P_\xi \subset V$.
- (2) For each $\beta, \xi < \alpha^+$, if $\beta < \xi$, then $P_\xi \cap (P_\beta \cup V_\beta) = \emptyset$.

Let $\xi < \alpha^+$ and assume that for all $\beta < \xi$, P_β has been selected subject to conditions (1) and (2).

We will now show that $V - \text{cl}_{\beta X} \bigcup_{\beta < \xi} P_\beta \neq \emptyset$. Assume the contrary. Since $\bigcup_{\beta < \xi} (P_\beta \cup V_\beta)$ is α^+ -open in βX , since $\{V_\xi: \xi < \alpha^+\}$ covers $\bigcup_{\beta < \xi} (P_\beta \cup V_\beta)$ and since $L(\beta X) = \omega \leq \alpha$, by 3.6 there exists $I \subset \alpha^+$ such that $|I| \leq \alpha$ and such that

$\{V_\xi: \xi \in I\}$ covers $\bigcup_{\beta < \xi} (P_\beta \cup V_\beta)$. Then $\bigcup_{\beta < \xi} (P_\beta \cup V_\beta) \subset \bigcup_{\xi \in I} V_\xi \subset V \subset \text{cl}_{\beta X} \bigcup_{\beta < \xi} (P_\beta \cup V_\beta)$, and hence $\bigcup_{\xi \in I} V_\xi$ is dense in V . Therefore $(\bigcup_{\xi \in I} V_\xi) \cap X = \bigcup_{\xi \in I} U_\xi$ is dense in $V \cap X = X$, contradicting the assumption that no subcollection of $\mathcal{U}$ of cardinality $\leq \alpha$ has dense union in X .

We may therefore select a nonempty cozero-set P_ξ of βX such that $P_\xi \subset V - \text{cl}_{\beta X} \bigcup_{\beta < \xi} (P_\beta \cup V_\beta)$. This completes the recursion.

By conditions (1) and (2), the collection $\{P_\xi: \xi < \alpha^+\}$ satisfies the requirements of the lemma. $\square$

3.9. LEMMA. *If $wL(X) = L(X) = \alpha^+$ and if X is an F_{α^+} -space, then $|C^*(X)| \geq 2^{\alpha^+}$.*

PROOF. By 3.8, there exists an α^{++} -open subset V of βX with $X \subset V$, and there exists $P = \bigcup_{\xi < \alpha^+} P_\xi$, a pairwise disjoint collection of cozero-sets of βX with $P \subset V$ and such that for all $\xi < \alpha^+$, $V_\xi \cap P$ is α^+ -open in V . Since βX is an F_{α^+} -space, P is C^* -embedded in V by 3.7, and, since $X \subset V$, V is C^* -embedded in βX . Then $2^{\alpha^+} \leq |C^*(P)| \leq |C^*(V)| \leq |C^*(\beta X)| = |C^*(X)|$. $\square$

3.10. THEOREM. *If $|C^*(X)| \leq \alpha^+$ and if X is an F_{α^+} -space, then $wL(X) \leq \alpha$.*

PROOF. If $wL(X) \geq \alpha^+$, then $wL(X) = L(X) = w(X) = |C^*(X)| = \alpha^+$, contradicting 3.9. $\square$

3.11. COROLLARY [$2^\alpha = \alpha^+$]. *If X is an α -small F_{α^+} -space, then $wL(X) \leq \alpha$.*

We see in 5.2 that Corollary 3.11 is, in fact, equivalent to $2^\alpha = \alpha^+$.

We still have some more work to do before we can proceed with the proof of 3.4. First we need a couple of lemmas.

3.12. LEMMA. *Let X be a space with $wL(X) \leq \alpha$. If $p \in X^*$, then there exists a dense α^+ -open subset V of βX with $p \notin V$.*

PROOF. $\beta X - \{p\}$ is open in βX , and hence $\beta X - \{p\} = \bigcup \mathcal{U}$, where $\mathcal{U}$ is a collection of cozero-sets of βX . Since $X \subset \bigcup \mathcal{U}$ and $wL(X) \leq \alpha$, there exists $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| \leq \alpha$ and such that $(\bigcup \mathcal{U}') \cap X$ is dense in X . Let $V = \bigcup \mathcal{U}'$. V is a dense α^+ -open subset of βX and $p \notin V$. $\square$

3.13. LEMMA. *If X is an α -pseudocompact space and if V is a dense α^+ -open subset of βX , then $\text{cl}_{\beta X}(X - V) = \beta X - V$.*

PROOF. Let $p \in \beta X - V$ and let U be an open neighborhood of p in βX . Since V is α^+ -open in βX , $\beta X - V$ is α^+ -closed in βX and is thus a G_{α^+} -set in βX . Hence $U - V = U \cap (\beta X - V)$ is a G_{α^+} -set in $\beta X = \beta_{\alpha^+} X$, and hence $U - V$ meets X . Then clearly U meets $X - V$, which implies that $p \in \text{cl}_{\beta X}(X - V)$. The reverse inclusion is obvious. $\square$

The following result is fundamental to the proof of 3.4.

3.14. THEOREM. *If X is a normal α -pseudocompact space with $wL(X) \leq \alpha$, then no point of X^* is a χ_α -remote point of X .*

PROOF. Let $p \in X^*$. By 3.12 there exists a dense α^+ -open subset V of βX such that $p \notin V$. By 3.13, $\text{cl}_{\beta X}(X - V) = \beta X - V$, and hence $p \in \text{cl}_{\beta X}(X - V)$ and $\text{cl}_{\beta X}(X - V)$ is α^+ -closed in βX . By 3.2, $\chi(X - V, X) \leq \alpha$, which implies that p is not χ_α -remote. $\square$

We digress in order to give two corollaries of 3.14. (It is known, of course, that Lindelöf pseudocompact spaces are compact. The second corollary of 3.14 answers the question: What property can one add to weakly Lindelöf pseudocompact normal spaces to insure that they are compact?)

3.15. COROLLARY. *Let α be an infinite cardinal. If X is a normal space, then the following are equivalent:*

- (a) X is compact.
- (b) $wL(X) \leq \alpha$, X is α -pseudocompact, and every nowhere dense closed subset F of X with $\chi(F, X) \leq \alpha$ is compact.

PROOF. (a) $\Rightarrow$ (b) is trivial. Assume (b) and suppose that (a) is false. Then there exists $p \in X^*$ and, by 3.14, p is not χ_α -remote. Therefore some nowhere dense closed subset F of X with $\chi(F, X) \leq \alpha$ is not compact, a contradiction. $\square$

3.16. COROLLARY. *If X is a normal space, then the following are equivalent:*

- (a) X is compact.
- (b) X is weakly Lindelöf and pseudocompact, and every closed nowhere dense subset F of X with $\chi(F, X) = \omega$ is compact.

3.17. REMARKS. (a) To show that the hypothesis of normality is necessary in 3.16, let $\phi(\mathbf{R}) = \{p \in \beta \mathbf{R} : p \notin \text{cl}_{\beta \mathbf{R}} A \text{ for every closed discrete subset } A \text{ of } \mathbf{R}\}$. ($\phi(\mathbf{R})$ is the set of far points of $\mathbf{R}$. See [vD₂] for a discussion of $\phi(\mathbf{R})$.) Let $Y = \mathbf{R} \cup \phi(\mathbf{R})$. Y is separable, hence weakly Lindelöf, and pseudocompact [vD₂, §17]. We will show that every closed nowhere dense subset of Y with countable character is compact (and is, in fact, contained in $\mathbf{R}$).

Let $F \subset Y$ be closed and nowhere dense and let $\{U_n : n \in \omega\}$ be a base for the neighborhoods of F in Y . We show first that $F \cap \mathbf{R}$ is compact.

Assume the contrary. Thus $F \cap \mathbf{R}$ is not bounded and hence contains $\{x_n : n \in \omega\}$, an infinite closed discrete set. There exists a collection $\{H_n : n \in \omega\}$ of open subsets of $\mathbf{R}$ such that for all $n \in \omega$, $x_n \in H_n \subset U_n$ and such that $\{H_n : n \in \omega\}$ is discrete in $\mathbf{R}$. Since H_n is open in Y and since F is nowhere dense, we can pick $y_n \in H_n - F$ for all $n \in \omega$. Then $A = \{y_n : n \in \omega\}$ is closed discrete in $\mathbf{R}$ and, since no point of $\phi(\mathbf{R})$ is a limit point of A , A is also closed in Y . Then $Y - A$ is a neighborhood of F in Y but no U_n is contained in $Y - A$, a contradiction.

Next we show that $F \cap \phi(\mathbf{R})$ has countable character in Y . Since $F \cap \mathbf{R}$ is compact and $\phi(\mathbf{R})$ is closed in Y , $F \cap \mathbf{R}$ has a closed neighborhood H in Y such that $H \cap (F \cap \phi(\mathbf{R})) = \emptyset$. Then $\{U_n - H : n \in \omega\}$ is a base for the neighborhoods of $F \cap \phi(\mathbf{R})$ in Y .

Finally, we show that no nonempty subset of $\phi(\mathbf{R})$ has countable character in Y and conclude that $F = F \cap \mathbf{R}$ and is therefore compact. Let $\emptyset \neq A \subset \phi(\mathbf{R})$ and let $\{H_n : n \in \omega\}$ be a decreasing neighborhood base for A in Y . For all $n \in \omega$, pick

$y_n \in H_n \cap \mathbf{R}$ and consider $K = \text{cl}_{\mathbf{R}}\{y_n: n \in \omega\}$. If K is compact, then $Y - K$ is open, $A \subset Y - K$, but for all $n \in \omega$, $H_n \not\subset Y - K$, a contradiction. If K is not compact, $\{x_n: n \in \omega\}$ is not bounded and therefore contains an infinite subset $D = \{x_{nk}: k \in \omega\}$ which is closed discrete in $\mathbf{R}$. Again no point of $\phi(\mathbf{R})$ is a limit point of D and therefore $Y - D$ is a neighborhood of A in Y . Since $\{H_n: n \in \omega\}$ is a decreasing neighborhood base for A in Y , $\{H_{nk}: k \in \omega\}$ is also a neighborhood base for A in Y , but no H_{nk} is contained in $Y - D$. This contradiction completes the proof.

(b) We show next that, in the implication (b) $\Rightarrow$ (a) of 3.16, none of the hypotheses of (b) can be omitted.

The ordinal space ω_1 is normal and pseudocompact and every closed nowhere dense subset of ω_1 with countable character is compact. Thus the hypothesis “ X is weakly Lindelöf” cannot be omitted. To see that the hypothesis of pseudocompactness cannot be omitted, we note that the ordinal space ω is normal and Lindelöf and contains no nonempty nowhere dense closed set. The last example shows that the hypothesis on closed nowhere dense sets cannot be omitted from 3.16(b): Let κ be an uncountable cardinal, and for each $\xi < \kappa$, let $X_\xi = \{0, 1\}$. Let $X = \prod_{\xi < \kappa} X_\xi$ and let $Y = \{x \in X: x_\xi = 1 \text{ for at most countably many } \xi < \kappa\}$. (Y is a “ Σ -product” of the family $\langle X_\xi: \xi < \kappa \rangle$. See [Co] and [En, 2.7.13] for further details.) By [Co, Theorem 1], Y is normal, and by [Co, Theorem 2], $\nu Y = X = \beta Y$, and hence Y is pseudocompact but not compact. Moreover, by [Co, Proposition 3], Y has a dense Lindelöf subspace and is therefore weakly Lindelöf. (We are grateful to E. K. van Douwen for calling [Co] to our attention.)

We are now ready to prove 3.4.

PROOF OF 3.4. Let x be an α -small α -pseudocompact normal F_{α^+} -space. By 3.11, $wL(X) \leq \alpha$ and hence by 3.14, X has no χ_α -remote point. $\square$

The statement of the following theorem is also equivalent to $2^\alpha = \alpha^+$ (see 5.2).

3.18. THEOREM [$2^\alpha = \alpha^+$]. *If X is an α -small F_{α^+} -space with $\chi(F, X) \leq \alpha$ for every α^+ -closed nowhere dense subset F of X , then $\beta_{\alpha^+} X - X$ contains no χ_α -remote point of X .*

PROOF. Let $p \in \beta_{\alpha^+} X - X$. By 3.11 and 3.12 there exists a dense α^+ -open subset V of βX with $p \notin V$. Then $\beta X - V$ is α^+ -closed in βX and hence is a G_{α^+} -set in βX . Since $p \in \beta_{\alpha^+} X$, $(\beta X - V) \cap U$ meets X for every neighborhood U of p in βX , and hence $p \in \text{cl}_{\beta X}(X - V)$. Since $\chi(X - V, X) \leq \alpha$, p is not χ_α -remote. $\square$

3.19. COROLLARY [CH]. *If X is a small normal F -space with $\chi_{nz}(X) = \omega$, then X is realcompact.*

PROOF. Let $p \in \nu X - X$. By 3.18, $p \in \text{cl}_{\beta X} F$ for some closed nowhere dense set F of X with $\chi(F, X) = \omega$, contradicting 3.3. $\square$

3.20. REMARKS. (a) 3.19 is equivalent to CH (see 5.3(b)).

(b) In [Sw₁] we show that if X is countably compact, then $\chi(Z, X) = \omega$ for every zero-set Z of X , and hence $\chi_{nz}(X) = \omega$. Note then that 0.1(a) $\Rightarrow$ (c) is a corollary of 3.19. Note also that for $p \in \beta\omega - \omega$, 3.19 shows that, under CH, $\beta\omega - \{p\}$ is not

normal. (See [Wa, 7.3 and 7.4] for a proof that CH implies that $\omega^* - \{p\}$ (and hence $\beta\omega - \{p\}$) is not normal.)

(c) In [Te], Terada shows that no point of $\nu X - X$ is remote, provided only that every collection of disjoint open subsets of X has Ulam-nonmeasurable cardinality. We will see later, however, that for every α , there is a space Φ_α such that $\nu\Phi_\alpha - \Phi_\alpha$ contains a χ_α -remote point (see 4.26).

4. Generalization of van Douwen's space Φ . In this section we define, for every infinite cardinal α , a space Φ_α that is normal, almost compact, noncompact, α^+ -basically disconnected (hence an F_{α^+} -space), $[\omega, \alpha^+]$ -compact (hence α -pseudo-compact) and such that $\chi(F, \Phi_\alpha) \leq \alpha$ for every α^+ -closed nowhere dense subset F of Φ_α . We shall also show that $wL(\Phi_\alpha) > \alpha$, that there exists a point in Φ_α^* , hence in $\beta_{\alpha^+}\Phi_\alpha - \Phi_\alpha$, that is χ_α -remote, and that, if $2^\alpha \neq \alpha^+$, then Φ_α is α -small. Hence if $2^\alpha \neq \alpha^+$, then Φ_α is an example of a space for which the statements of 3.4, 3.11 and 3.18 fail. The conclusion then is that the statements of 3.4, 3.11 and 3.18 are each equivalent to the segment $2^\alpha = \alpha^+$ of GCH (see 5.2).

Φ_α is a straightforward cardinal generalization of van Douwen's space Φ of $[\nu\mathbf{D}_1]$ (in fact, $\Phi_\omega = \Phi$), and many of the techniques used in proving that Φ_α has the properties listed above are those of $[\nu\mathbf{D}_1]$.

The space Φ_α is defined as a subspace of βB_α , where B_α is a cardinal generalization of the space P of $[\nu\mathbf{D}_1]$ (P is also described in [GJ, 9L]). B_α is a subspace of $\alpha^{++} + 1$, and is defined as follows:

$$B_\alpha = \{\xi \leq \alpha^{++} : \xi \text{ is a successor ordinal or } \text{cf}(\xi) \in \{\alpha^+, \alpha^{++}\}\}.$$

We now turn to the properties of B_α that will be used in discussing Φ_α .

4.1. PROPOSITION. *If A is an initial segment of B_α and if $f \in C(A)$, then f is constant on some neighborhood of each point in A .*

PROOF. The result follows from the fact that countable intersections of open sets are open in B_α . $\square$

4.2. PROPOSITION. *Every α^+ -open subset of B_α is closed in B_α .*

PROOF. Let $G = \bigcup_{\xi < \alpha} P_\xi$, where each P_ξ is the cozero-set of $f_\xi \in C(B_\alpha)$ and suppose there exists $\beta \in (\text{cl}_{B_\alpha} G) - G$. Then $f_\xi(\beta) = 0$ for all $\xi < \alpha$. β is not a successor ordinal since $\beta \notin G$, and hence $\text{cf}(\beta) \geq \alpha^+$. By 4.1, for each $\xi < \alpha$, there exists $\nu_\xi < \beta$ such that $(\nu_\xi, \beta] \cap B_\alpha \subset f_\xi^{-1}(0)$. Let $\nu = \sup\{\nu_\xi : \xi < \alpha\}$. Then $\nu < \beta$ and $(\nu, \beta] \cap B_\alpha \subset f_\xi^{-1}(0)$ for all $\xi < \alpha$. Hence $(\nu, \beta] \cap G = \emptyset$, which contradicts the assumption that $\beta \in \text{cl}_{B_\alpha} G$. $\square$

4.3. COROLLARY. *B_α is α^+ -basically disconnected, and hence an F_{α^+} -space.*

4.4. PROPOSITION. $L(B_\alpha) = \alpha$.

PROOF. Let $\mathcal{U}$ be an open cover of B_α . There exists $U \in \mathcal{U}$ and there exists $\beta < \alpha^{++}$ such that $(\beta, \alpha^{++}] \cap B_\alpha \subset U \in \mathcal{U}$. Hence it suffices to show that

$$(*) \quad L([0, \beta] \cap B_\alpha) \leq \alpha \quad \text{if } \beta < \alpha^{++}.$$

Assume then that $\mathcal{U}$ is an open cover of $[0, \beta] \cap B_\alpha$. We may assume that $\mathcal{U} = \{\mathcal{U}_\xi: \xi < \alpha^+\}$ and that each $U_\xi \subset [0, \beta] \cap B_\alpha$. For all $\xi < \alpha^+$, let $V_\xi = U_\xi - \bigcup_{\delta < \xi} U_\delta$, and let $I = \{\xi < \alpha^+: V_\xi \neq \emptyset\}$. It suffices to show that $|I| \leq \alpha$.

Assume that $|I| = \alpha^+$ and pick $v_\xi \in V_\xi$ for all $\xi \in I$. Let $\nu = \sup\{v_\xi: \xi \in I\}$. Then $\nu \leq \beta$ and $\text{cf}(\nu) = \alpha^+$ and hence there exists $\delta < \alpha^+$ and there exists $\lambda < \nu$ such that $(\lambda, \nu] \cap B_\alpha \subset U_\delta$. Then for all $\eta > \delta$, $(\lambda, \nu] \cap B_\alpha \cap V_\eta = \emptyset$, a contradiction. Thus $(*)$ holds, which completes the proof. $\square$

4.5. PROPOSITION. $w(B_\alpha) = \alpha^{++}$.

PROOF. $\alpha^{++} = \chi(B_\alpha) \leq w(B_\alpha) \leq |B_\alpha| \cdot \chi(B_\alpha) = \alpha^{++} \cdot \alpha^{++} = \alpha^{++}$. $\square$

For a space X , we set $\text{CO}(X) = \{A: A \text{ is closed-and-open in } X\}$.

We call a space X *strongly zero-dimensional* if every pair of disjoint zero-sets of X can be separated by disjoint closed-and-open subsets of X [En, 6.2.4].

We turn now to a sequence of propositions leading to a computation of $w(\beta B_\alpha)$.

4.6. PROPOSITION. B_α is strongly zero-dimensional.

PROOF. This is immediate from 4.2. $\square$

The next two propositions are no doubt well known, but no references are known to the author.

4.7. PROPOSITION. For any space X , $|\text{CO}(X)| \leq w(X)^{L(X)}$.

PROOF. Let $\mathfrak{B}$ be a base for X with $|\mathfrak{B}| \leq w(X)$. Define $\Psi: \text{CO}(X) \rightarrow \{\mathfrak{B}' \subset \mathfrak{B}: |\mathfrak{B}'| \leq L(X)\}$ by $\Psi(A) = \mathcal{U}_A$ if $A = \bigcup \mathcal{U}_A$. Then Ψ is injective and $|\{\mathfrak{B}' \subset \mathfrak{B}: |\mathfrak{B}'| \leq L(X)\}| \leq w(X)^{L(X)}$. $\square$

4.8. PROPOSITION. For any space X , $|\text{CO}(X)| = |\text{CO}(\beta X)|$.

PROOF. Define $\Psi: \text{CO}(X) \rightarrow \text{CO}(\beta X)$ by $\Psi(U) = \text{cl}_{\beta X} U$. Then Ψ is bijective. $\square$

The following two cardinal equalities will be used in subsequent calculations. Proofs can easily be given using [Mo, 22.5, 22.6, 22.13 and 22.14].

4.9. PROPOSITION. (a) $(\alpha^{++})^\alpha = \alpha^{++} \cdot 2^\alpha$.

(b) $(\alpha^{++} \cdot 2^\alpha)^{\alpha^{++}} = 2^{\alpha^{++}}$.

4.10. PROPOSITION. $|\text{CO}(B_\alpha)| = \alpha^{++} \cdot 2^\alpha$, and hence $|\text{CO}(\beta B_\alpha)| = \alpha^{++} \cdot 2^\alpha$.

PROOF. $|\text{CO}(B_\alpha)| \leq (\alpha^{++})^\alpha = \alpha^{++} \cdot 2^\alpha$ by 4.4, 4.5, 4.7 and 4.9(a). Let $\mathcal{Q} = \{\{\xi\}: \xi \text{ is a successor ordinal and } \xi > \alpha\}$ and let $\mathfrak{B} = \{A: A \subset [0, \alpha) \cap B_\alpha\}$. Then $|\text{CO}(B_\alpha)| \geq |\mathcal{Q}| \cdot |\mathfrak{B}| = \alpha^{++} \cdot 2^\alpha$, and hence $|\text{CO}(\beta B_\alpha)| = \alpha^{++} \cdot 2^\alpha$ by 4.8. $\square$

4.11. PROPOSITION. $w(\beta B_\alpha) = \alpha^{++} \cdot 2^\alpha$.

PROOF. Since B_α is strongly zero-dimensional, so is βB_α [En, 6.2.12] and therefore $w(\beta B_\alpha) = |\text{CO}(\beta B_\alpha)|$ by [CN, 2.24]. The result then follows from 4.10. $\square$

We prove two more results about B_α that will be needed later.

4.12. PROPOSITION. If G is α^+ -open and dense in βB_α , then $B_\alpha \subset G$.

PROOF. If G is α^+ -open and dense in βB_α , then $G \cap B_\alpha$ is α^+ -open and dense in B_α . By 4.2, $G \cap B_\alpha$ is closed in B_α , and hence $G \cap B_\alpha = B_\alpha$. $\square$

4.13. PROPOSITION. *Every real-valued continuous function on $B_\alpha - \{\alpha^{++}\}$ is constant on a tail of $B_\alpha - \{\alpha^{++}\}$, and hence $B_\alpha - \{\alpha^{++}\}$ is C -embedded in B_α .*

PROOF. Let $f \in C(B_\alpha - \{\alpha^{++}\})$. Suppose f is not constant on a tail of $B_\alpha - \{\alpha^{++}\}$. We can then pick recursively two sequences $\langle \beta_\xi: \xi < \alpha^+ \rangle$ and $\langle \nu_\xi: \xi < \alpha^+ \rangle$ such that $f(\beta_\xi) \neq f(\nu_\xi)$ for all $\xi < \alpha^+$ and such that if $\delta < \xi$, then $\beta_\delta < \nu_\delta < \beta_\xi$.

Let $\nu = \sup\{\beta_\xi: \xi < \alpha^+\} = \sup\{\nu_\xi: \xi < \alpha^+\}$. Then $\nu \in B_\alpha - \{\alpha^{++}\}$, but f is not constant on a neighborhood of ν , contradicting 4.1. $\square$

We now define the space Φ_α as follows: $\Phi_\alpha = \beta B_\alpha - \{\alpha^{++}\}$.

The remainder of this section is devoted to showing that Φ_α has the desired properties.

4.14. PROPOSITION. $\beta \Phi_\alpha = \beta B_\alpha$, and hence Φ_α is almost compact but not compact.

PROOF. Φ_α is clearly dense in βB_α , and since, by 4.13, $B_\alpha - \{\alpha^{++}\}$ is C -embedded in B_α , Φ_α is C^* -embedded in βB_α . $\square$

4.15. PROPOSITION. Φ_α is collectionwise normal.

PROOF. We show that every pair of noncompact closed subsets of Φ_α meet. Let F and K be closed subsets of Φ_α with $\alpha^{++} \in \text{cl}_{\beta \Phi_\alpha} F \cap \text{cl}_{\beta \Phi_\alpha} K$. We will pick, recursively, four sequences $\langle x_\xi: \xi < \alpha^+ \rangle$, $\langle y_\xi: \xi < \alpha^+ \rangle$, $\langle \nu_\xi: \xi < \alpha^+ \rangle$ and $\langle \beta_\xi: \xi < \alpha^+ \rangle$ subject to the following conditions:

(1) For all $\xi < \alpha^+$,

$$x_\xi \in F \cap \text{cl}_{\beta \Phi_\alpha}((\beta_\xi, \nu_\xi] \cap B_\alpha) \quad \text{and} \quad y_\xi \in K \cap \text{cl}_{\beta \Phi_\alpha}((\beta_\xi, \nu_\xi] \cap B_\alpha).$$

(2) If $\delta < \xi < \alpha^+$, then $\beta_\delta < \nu_\delta \leq \beta_\xi < \alpha^{++}$.

Let $\xi < \alpha^+$ and assume that for all $\delta < \xi$, x_δ , y_δ , ν_δ and β_δ have been selected satisfying (1) and (2). Let $\beta_\xi = \sup\{\nu_\delta: \delta < \xi\}$. Pick

$$x_\xi \in F \cap \text{cl}_{\beta \Phi_\alpha}((\beta_\xi, \alpha^{++}] \cap B_\alpha) \quad \text{and} \quad y_\xi \in K \cap \text{cl}_{\beta \Phi_\alpha}((\beta_\xi, \alpha^{++}] \cap B_\alpha).$$

Pick ν_ξ so that $x_\xi, y_\xi \notin \text{cl}_{\beta \Phi_\alpha}((\nu_\xi, \alpha^{++}] \cap B_\alpha)$. This completes the recursion.

Let $\nu = \sup\{\beta_\xi: \xi < \alpha^+\} = \sup\{\nu_\xi: \xi < \alpha^+\}$. Since $\text{cf}(\nu) = \alpha^+$, $\nu \in \Phi_\alpha$. Also

$$\nu \in \text{cl}_{\Phi_\alpha}\{x_\xi: \xi < \alpha^+\} \subset F \quad \text{and} \quad \nu \in \text{cl}_{\Phi_\alpha}\{y_\xi: \xi < \alpha^+\} \subset K.$$

Thus F and K meet which proves that Φ_α is normal. Since Φ_α is almost compact, Φ_α is collectionwise normal. $\square$

4.16. PROPOSITION. Φ_α is α^+ -basically disconnected and hence an F_{α^+} -space.

PROOF. By 4.3, B_α is α^+ -basically disconnected. The result then follows from 2.7 and 4.14. $\square$

4.17. PROPOSITION. $wL(\Phi_\alpha) = \alpha^{++}$.

PROOF. Since $B_\alpha - \{\alpha^{++}\}$ is dense in Φ_α , $wL(\Phi_\alpha) \leq |B_\alpha - \{\alpha^{++}\}| = \alpha^{++}$. To see that $wL(\Phi_\alpha) \geq \alpha^{++}$, we note that $\mathcal{U} = \{\beta B_\alpha - \text{cl}_{\beta B_\alpha}((\xi, \alpha^{++}) \cap B_\alpha) : \xi < \alpha^{++}\}$ is an open cover of Φ_α of cardinality α^{++} . No subcover of $\mathcal{U}$ of smaller cardinality has dense union in Φ_α . $\square$

4.18. PROPOSITION. *If F is α^+ -closed and nowhere dense in Φ_α , then F is α^+ -closed in βB_α and hence F is compact.*

PROOF. We may write $F = \bigcap_{\xi < \alpha} (\Phi_\alpha - P_\xi)$, where each P_ξ is a cozero-set in $\beta \Phi_\alpha = \beta B_\alpha$. Then $\bigcup_{\xi < \alpha} P_\xi$ is dense and α^+ -open in βB_α , which implies by 4.12, that $B_\alpha \subset \bigcup_{\xi < \alpha} P_\xi$. Hence $\alpha^{++} \in \bigcup_{\xi < \alpha} P_\xi$, and so $F = \bigcap_{\xi < \alpha} (\beta B_\alpha - P_\xi)$, an α^+ -closed subset of βB_α . $\square$

4.19. PROPOSITION. $\chi(F, \Phi_\alpha) \leq \alpha$ for every nowhere dense α^+ -closed subset F of Φ_α .

PROOF. By 4.18, F is α^+ -closed in βB_α and hence $\chi(F, \Phi_\alpha) \leq \chi(F, \beta B_\alpha) = \psi(F, \beta B_\alpha) \leq \alpha$. $\square$

The next group of propositions leads to the conclusion that Φ_α is α^+ -pseudocompact. We will show, in fact, that Φ_α is $[\omega, \alpha^+]$ -compact.

A point $p \in X$ is a *complete accumulation point* of a subset A of X if for every neighborhood U of p in X , $|U \cap A| = |A|$.

4.20. LEMMA [En, 3.12.1]. *A space X is compact if and only if every infinite subset of X has a complete accumulation point in X .*

4.21. PROPOSITION. *If $A \subset \Phi_\alpha$ and if $|A| \leq \alpha^+$, then α^{++} is not a limit point of A in βB_α .*

PROOF. Since $\text{cf}(\alpha^{++}) > \alpha^+$, $\alpha^{++} \notin \text{cl}_{\beta B_\alpha} A$ if $|A| \leq \alpha^+$. $\square$

4.22. PROPOSITION. Φ_α is $[\omega, \alpha^+]$ -compact.

PROOF. Let $\mathcal{U}$ be an open cover of Φ_α such that $|\mathcal{U}| \leq \alpha^+$, and assume that $\mathcal{U}$ has no finite subcover. Choose $\mathcal{U}' \subset \mathcal{U}$ of minimal cardinality such that $\mathcal{U}'$ covers Φ_α . Write $\mathcal{U}' = \{U_\xi : \xi < \kappa\}$ with $\kappa \leq \alpha^+$, and for each $\xi < \kappa$, let $V_\xi = U_\xi - \bigcup_{\beta < \xi} U_\beta$. Let $I = \{\xi < \kappa : V_\xi \neq \emptyset\}$, and note that $\Phi_\alpha = \bigcup_{\xi \in I} U_\xi$. By the minimality of κ , $|I| = \kappa$. For each $\xi \in I$, choose $x_\xi \in V_\xi$, and let $A = \{x_\xi : \xi \in I\}$. Clearly $|A| = \kappa \geq \omega$, and hence A has a complete accumulation point $p \in \beta B_\alpha$. By 4.21, $p \neq \alpha^{++}$, and hence $p \in \Phi_\alpha$. Then $p \in U_\beta$ for some $\beta \in I$, and therefore $|U_\beta \cap A| = |A| = \kappa$. But for all $\xi \in I$ with $\xi > \beta$, we have $x_\xi \notin U_\beta$ and hence $|U_\beta \cap A| \leq |\beta| < \kappa$, a contradiction. $\square$

4.23. COROLLARY. Φ_α is α^+ -pseudocompact, hence α -pseudocompact.

In order to calculate $|C^*(\Phi_\alpha)|$, we need the following result from elsewhere. (We discuss this result in more detail in §5.)

4.24. THEOREM [CoHa, 2.2]. *For any space X , $|C(X)| \leq w(X)^{wL(X)}$.*

4.25. PROPOSITION. $|C(\Phi_\alpha)| = \alpha^{++} \cdot 2^\alpha$.

PROOF. It suffices to calculate $|C^*(\beta B_\alpha)|$. By 4.11, 4.24 and 4.9, $|C^*(\beta B_\alpha)| \leq (\alpha^{++} \cdot 2^\alpha)^\omega \leq (\alpha^{++} \cdot 2^\alpha)^\alpha = \alpha^{++} \cdot 2^\alpha$, and, since the characteristic function of a closed-and-open set is continuous, $|C^*(\beta B_\alpha)| \geq |\text{CO}(\beta B_\alpha)| = \alpha^{++} \cdot 2^\alpha$ by 4.10. $\square$

We need one last result about Φ_α .

4.26. PROPOSITION. α^{++} is a χ_α -remote point of Φ_α .

PROOF. Let F be a closed nowhere dense subset of Φ_α with $\chi(F, \Phi_\alpha) \leq \alpha$. By 3.2, $\text{cl}_{\beta\Phi_\alpha} F$ is α^+ -closed in $\beta\Phi_\alpha$, which implies that $F = \Phi_\alpha \cap \text{cl}_{\beta\Phi_\alpha} F$ is α^+ -closed in Φ_α . By 4.18, F is compact, and hence $\alpha^{++} \notin \text{cl}_{\beta\Phi_\alpha} F$. $\square$

5. Topological characterizations of cardinal equalities. As noted in 4.24, Comfort and Hager prove in [CoHa, 2.2] that, for every space X , $|C(X)| \leq w(X)^{wL(X)}$, and in [CoHa, 5.5], they give an example of a space X such that $|C(X)| < w(X)^{wL(X)}$. In [vDZ], van Douwen and Zhou observe that, for every cardinal λ such that $\lambda^\omega = \lambda$, the ordinal space λ has the property that $|C(\lambda)| = \lambda$ while $w(\lambda)^{wL(\lambda)} > \lambda$.

What we now show is that, for every infinite cardinal α , $|C(\Phi_\alpha)| < w(\Phi_\alpha)^{wL(\Phi_\alpha)}$ if and only if $2^\alpha < 2^{\alpha^{++}}$. In fact, we have the following topological characterizations of the cardinal equality $2^\alpha = 2^{\alpha^{++}}$.

5.1. THEOREM. If α is an infinite cardinal, then the following are equivalent:

- (a) $2^\alpha = 2^{\alpha^{++}}$.
- (b) If X is any space with $d(X) \leq \alpha^{++}$ and if X has either a discrete C^* -embedded subspace of cardinality α or at least 2^α closed-and-open subsets, then $|C(X)| = w(X)^{wL(X)}$.
- (c) If X is an F_{α^+} -space with $d(X) \leq \alpha^{++}$, and if X has a discrete cellularly embedded subset of cardinality α , then $|C(X)| = w(X)^{wL(X)}$.
- (d) $|C(\Phi_\alpha)| = w(\Phi_\alpha)^{wL(\Phi_\alpha)}$.

PROOF. (a) $\Rightarrow$ (b) Under either hypothesis on X , $|C^*(X)| \geq 2^\alpha$. Then, by [CoHa, 2.2] and [En, 1.5.6], $2^\alpha \leq |C^*(X)| = |C(X)| \leq w(X)^{wL(X)} \leq (2^{d(X)})^{d(X)} \leq 2^{\alpha^{++}} = 2^\alpha$.

(b) $\Rightarrow$ (c) This is immediate from 2.8.

(c) $\Rightarrow$ (d) Since $B_\alpha - \{\alpha^{++}\}$ is dense in Φ_α , $d(\Phi_\alpha) \leq \alpha^{++}$. Moreover, the set $S = [0, \alpha) \cap B_\alpha$ is cellularly embedded in Φ_α and $|S| = \alpha$.

(d) $\Rightarrow$ (a) If $2^\alpha < 2^{\alpha^{++}}$, then $|C(\Phi_\alpha)| = \alpha^{++} \cdot 2^\alpha < 2^{\alpha^{++}} = (\alpha^{++} \cdot 2^\alpha)^{\alpha^{++}} = w(\Phi_\alpha)^{wL(\Phi_\alpha)}$. $\square$

As a summary of §§3 and 4, we now present several topological characterizations of the segment $2^\alpha = \alpha^+$ of GCH. It should again be noted that 5.2, in the case $\alpha = \omega$, is due partly to Woods [Wo₁, 2.2] and [Wo₂, 1.1(a)] and partly to van Douwen [vD₁].

5.2. THEOREM. If α is an infinite cardinal, then the following are equivalent:

- (a) $2^\alpha = \alpha^+$.
- (b) If X is an α -small F_{α^+} -space, then $wL(X) \leq \alpha$.
- (c) If X is an α -small, α -pseudocompact, normal F_{α^+} -space, then X has no χ_α -remote point.

(d) If X is an α -small F_{α^+} -space such that $\chi(F, X) \leq \alpha$ for every α^+ -closed nowhere dense subset F of X , then $\beta_{\alpha^+} X - X$ contains no χ_{α} -remote point.

PROOF. (a) $\Rightarrow$ (b) is 3.11; (a) $\Rightarrow$ (c) is 3.4; and (a) $\Rightarrow$ (d) is 3.18.

Now assume that (a) is false. Then $|C^*(\Phi_{\alpha})| = \alpha^{++} \cdot 2^{\alpha} = 2^{\alpha}$ and hence Φ_{α} is α -small. By 4.16, Φ_{α} is an F_{α^+} -space; by 4.15, Φ_{α} is normal; by 4.23, Φ_{α} is α -pseudocompact; and by 4.19, $\chi(F, \Phi_{\alpha}) \leq \alpha$ for every α^+ -closed nowhere dense subset F of X . Then (b) is false by 4.17 while (c) and (d) are false by 4.26 (since $\alpha^{++} \in \beta_{\alpha^+} \Phi_{\alpha} - \Phi_{\alpha}$).

5.3. REMARKS. (a) Since by 4.22, Φ_{α} is $[\omega, \alpha]$ -compact, the hypothesis " α -pseudo-compact" can be replaced by " $[\omega, \alpha]$ -compact" in 5.2(c).

(b) If $\neg\text{CH}$, then Φ_{ω} is a small normal F -space and $\chi_{nz}(\Phi_{\omega}) = \omega$ by 4.19. Since Φ_{ω} is countably compact but not compact, Φ_{ω} is not realcompact and therefore the statement of 3.19 is equivalent to CH.

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