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ALL THREE-MANIFOLDS ARE PULLBACKS OF A BRANCHED COVERING S^3 TO S^3

BY

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ABSTRACT. There are two main results in this paper. First, we show that every closed orientable 3-manifold can be constructed by taking a pair of disjoint bounded orientable surfaces in S^3, call them F_1 and F_2; taking three copies of S^3; splitting the first along F_1, the second along F_1 and F_2, and the third along F_2; and then pasting in the natural way. Second, we show that given any closed orientable 3-manifold M^3 there is a 3-fold irregular branched covering space, $p: M^3 \to S^3$, such that $p: M^3 \to S^3$ is the pullback of the 3-fold irregular branched covering space $q: S^3 \to S^3$ branched over a pair of unknotted unlinked circles.

0. Introduction. The purpose of this paper is to continue the study of closed orientable 3-manifolds as 3-fold irregular branched coverings of the 3-sphere. There are two main results here. First, we show that every closed orientable 3-manifold can be obtained by a certain pasting and glueing method. Let F_1 and F_2 be a pair of disjoint bounded orientable surfaces in S^3. Take three copies of the triple (S^3_k, F_1^k, F_2^k), $k = 1, 2, 3$, then split and glue F_1^1 to F_1^2 and F_2^2 to F_2^3. (This answers a problem in Kirby's problem list. What is new is that both surfaces can be assumed orientable.)

Second, we show that *any* closed orientable 3-manifold is the pullback of *any* 3-*fold* irregular covering $p: S^3 \to S^3$ and some smooth map $g: S^3 \to S^3$, transversal to the knot or link that is the branch set of $p: S^3 \to S^3$.

1. Representation of three-manifolds by pairs of orientable surfaces. Let L be a link in S^3 and let $\omega: \pi_1(S^3 - L) \to \Sigma_3$ be a transitive representation on the permutation group of the indices $\{1, 2, 3\}$ such that meridians are sent to transpositions. Then (L, ω) determines a closed orientable 3-manifold $M(L, \omega)$ and a projection $p: M(L, \omega) \to S^3$ such that p is a branched covering space map with branch set the link L in S^3.

An interesting idea for defining a representation $\omega: \pi_1(S^3 - L) \to \Sigma_3$, given the link L, is due to Ralph Fox. Three colors (say G = green, R = red, B = blue) are used to color the bridges of the link. This is done in such a way that three colors that meet at an overcrossing are either all the same or all distinct. Then a representation is defined on the meridians $G \to (12)$, $B \to (13)$, $R \to (23)$. The color condition guarantees that the defining relations are sent to the identity in the Wirtinger

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presentation. If at least two colors are used, the representation is transitive. A representation defined in this way is called a colored knot or link. As an example, consider the trefoil knot in Figure 1.

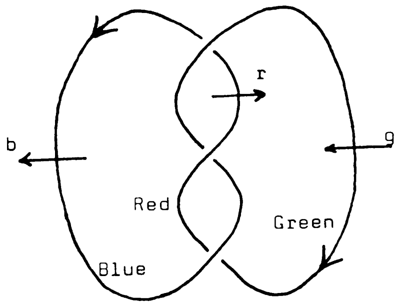

FIGURE 1

Of particular interest to us will be the following example. Let G^2 and R^2 be a pair of bounded disjoint surfaces in S^3 which can be thought of as disks with bands. We suppose that G^2 and R^2 are in "general position with respect to a plane" so that all the singularities of the projection are the type shown in Figure 2. If the boundary of G^2 is colored green except for those parts that are under R^2, and the boundary of R^2 is colord in red except for those parts that are under G^2 (Figure 2), we obtain a colored link in the sense of Fox. We will call this coloring of $\partial(R^2)$ and $\partial(G^2)$ the natural coloring.

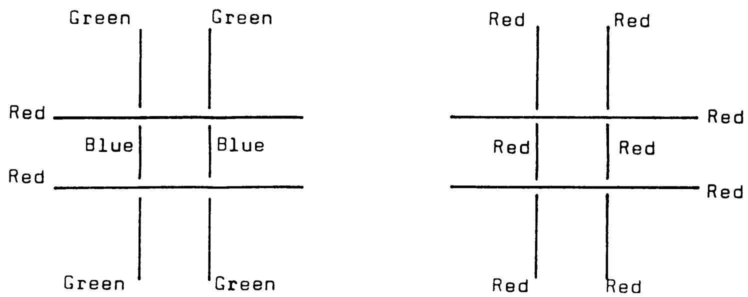

FIGURE 2

In **[HM]** it was demonstrated that every closed orientable 3-manifold could be represented as $M(\partial(G^2 \cup R^2), \omega)$ with the natural coloring. Also in **[HM]** (and in the problem list **[K]**) it was asked whether this could be done with G^2 and R^2 orientable. It is clear from the following theorem that the answer to this question is yes.

THEOREM 1. *Every closed orientable 3-manifold is homeomorphic to an $M(\partial(G^2 \cup R^2), \omega)$ in which G^2 and R^2 are orientable and the presentation is derived from the natural coloring.*

PROOF. Our point of departure is the representation of closed orientable 3-manifolds by surgery on "special" links due to Birman and Powell [**BP**]. They prove in Theorem 3.1 that every 3-manifold can be obtained by surgery on a link of "special" type. (It will help the reader to refer to Figure 3 as we describe what "special" means.) There are three types of components in a "special" link.

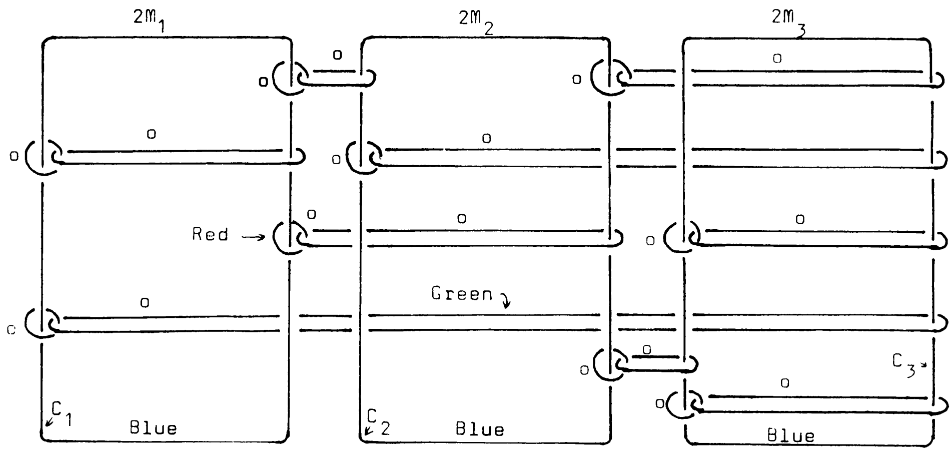

FIGURE 3
All small meridional circles are red.
All elongated horizontally stretched circles are green.

1. There is a set of disjoint squares of side length 1, lying in the x-y plane with centers on the x-axis and sides parallel to the x- and y-axes. These components all have even framing.

2. There is a set of pairs of unknotted once simply linked circles each with framing zero. The projection on the x-y plane of one pair is disjoint from another pair, and each component of a pair projects as a topological circle. The left components project as small meridional circles for some square and the right components as "ribbons" which link only the left components and some of the squares. (It is not stated explicitly in Theorem 3.1 of [**BP**] that the framed link is like this, but it is clear from the proof that we can obtain exactly this representation.)

We wish to isotope this link somewhat. Without moving the squares, we push all the left components of the paired unknotted circles so that they fall into sets of concentric circles centered at the intersection of sides of the squares with the x-axis and all lying in a plane making, say a 45° angle with the x-y plane. We do this in such a way that the images of the right components all lie in the "slab" $\{0 < y < 1\}$ and bound disjoint disks in $\{0 < y < 1\}$. At this point the set of squares and the set of left components are invariant under a 180° rotation about the x-axis (see Figure 4). The link of Figure 4 is symmetric in the sense defined in [**M**, p. 321] so that the proof of the theorem follows easily using the algorithm in [**M**, Theorem 3]. To see this it is only necessary to note that, since the link cuts the x-y plane only in the

x-axis, the surfaces obtained in **[M]** are embedded, not just immersed, in S^3. Also, since the framings are all even, there are an even number of twists in the bands, and the surfaces obtained in **[M]** are orientable.

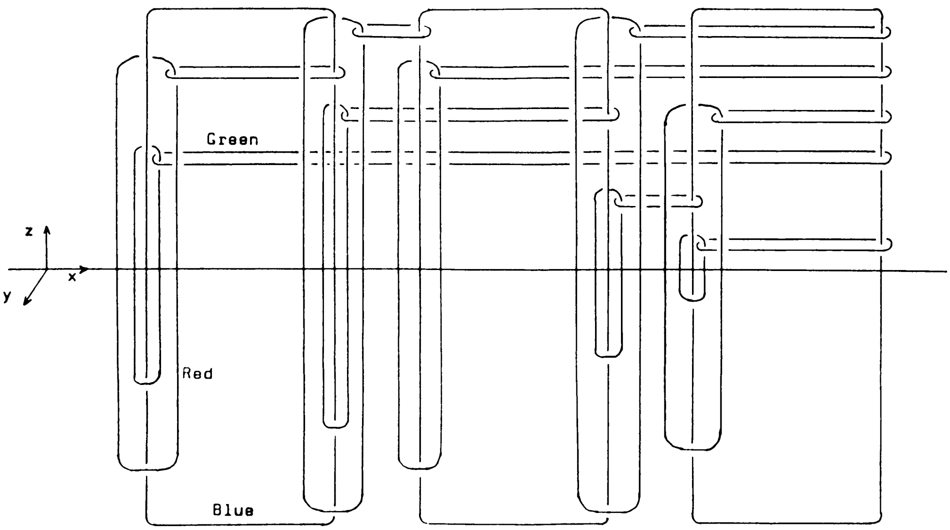

FIGURE 4

2. Lifting representations. If K or L is a knot or link, then $E(K)$ or $E(L)$ represents the exterior. Let F_2 be the free group on the two generators x and y.

THEOREM 2. *Let G^2 and R^2 be a pair of orientable disjoint bounded surfaces in S^3 with a normal projection on a plane. Let ω be the natural coloring of $\partial(G^2 \cup R^2)$. Then there is a homomorphism*

$$\mu: \pi_1(E(\partial(G^2 \cup R^2))) \to F_2 = \langle x, y \rangle$$

such that:

(i) $\tau\mu = \omega$ (*where* τ: $F_2 \to \Sigma_3$ *is defined by* $x \to (12)$ *and* $y \to (23)$).

(ii) μ *sends green and red meridians to elements of* $\{x, y\}$.

(iii) *Each component of the boundary of* $E(\partial(G^2 \cup R^2))$ *has a longitude whose image under* μ *is the identity.*

PROOF. We orient $G^2 \cup R^2$ and give $\partial(G^2 \cup R^2)$ the induced orientation. Then we arrange that all the crossings of bands of G^2 with bands of R^2 or of bands of G^2 with G^2 or R^2 with R^2 are as in Figure 5.

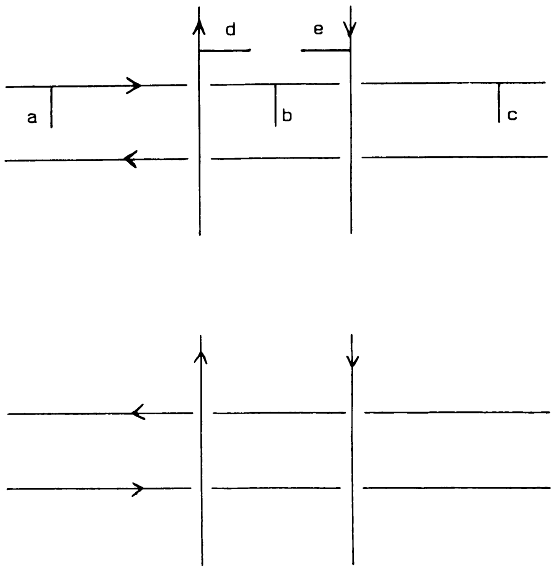

FIGURE 5

In the Wirtinger presentation for $\pi_1(E(\partial(G^2 \cup R^2)))$ there is a method for deleting one relator and one generator every time we have an undercrossing of a band. For example, referring to Figure 5, we see that the five generators a, b, c, d, e and two relators $a^{-1}d^{-1}bd$ and $b^{-1}ece^{-1}$ may be replaced by the four generators a, d, e, c, and single relation $a^{-1}d^{-1}ece^{-1}d$.

If we now map green meridians into x and red meridians into y, we can see that we have a homomorphism μ: $\pi_1(E(\partial(G^2 \cup R^2))) \to F_2$ just by checking the relations in the new presentation. This completes the proof of (i) and (ii). To prove (iii) we just take the longitude lying in the plane of the projection.

Let 8 be the figure eight space obtained by joining a pair of circles C_1 and C_2 by a boundary point P as in Figure 6. We let $\hat{8}$ be the union of the two corresponding discs whose centers we denote by A and B.

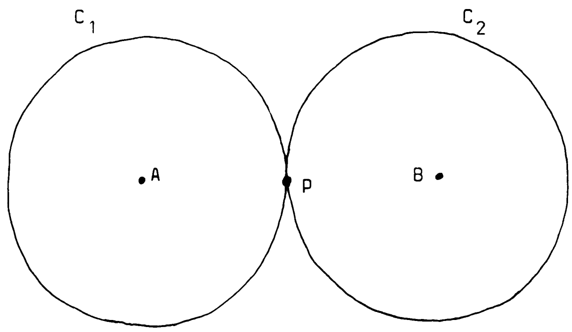

FIGURE 6

COROLLARY. *Let G^2 and R^2 be a pair of disjoint orientable surfaces in S^3 in normal projection. Let ω be the natural coloring. Then there is a PL function f: $(E(\partial(G^2 \cup R^2))$; boundary corresponding to G^2, boundary corresponding to $R^2) \to (8; C_1, C_2)$ such that*

(i) $\tau f_* = \omega$, *where f_* is the induced homomorphism on fundamental groups.*

(ii) $f|$ *boundary corresponding to G^2 (resp. R^2) is an S^1 fibration with base C_1 (resp. C_2).*

PROOF. One starts by mapping the longitude of Theorem 2 into the point P, then extending to $\partial(E(\partial(G^2 \cup R^2)))$ and then extending to $E(\partial(G^2 \cup R^2))$. Using Theorem 2 and the facts that $\pi_k(8) = \pi_k(C_1) = \pi_k(C_2) = 0$, $k \geqslant z$, and $\pi_1(8) = F_2 = \langle x, y \rangle$, there are no obstructions to extending.

3. Representation of 3-manifolds as pullbacks of branched coverings. Let (L, ω) be any colored link such that $M(L, \omega) = S^3$ and let p: $M(L, \omega) \to S^3$ be the corresponding projection.

LEMMA 3. *Let f: $S^3 \to S^3$ be a smooth map transversal to the link L. Let $M(f)$ be the pullback*

$$\begin{array}{ccc} M(f) & \xrightarrow{\tilde{f}} & M(L, \omega) = S^3 \\ p(f) \downarrow & & \downarrow p \\ S^3 & \xrightarrow{f} & S^3 \end{array}$$

Then $M(f)$ is $M(f^{-1}L, \omega\hat{f}_*)$, where $\hat{f} = f \mid S^3 - f^{-1}L$.

PROOF. Since f is transversal to L, $f^{-1}L$ is a link in S^3 and $\omega\hat{f}_*$ is a coloring. Moreover, $M(f)$ is the branched covering corresponding to the subgroup $\hat{f}_*^{-1}(H)$, where H is the subgroup that defines $M(L, \omega)$. Since H is the preimage by ω of the stabilizer of an index, the proof of the Lemma follows easily.

Now we are in position to prove our main theorem.

THEOREM 4. *Let p: $S^3 \to S^3$ be the covering projection associated to any 3-fold irregular branched covering of S^3 by itself branched over a knot or link L, and let M^3 be any closed orientable 3-manifold. Then there is a smooth map g: $S^3 \to S^3$ transversal to L such that M^3 is the pullback of p: $S^3 \to S^3$ by the map g.*

PROOF. By Theorem 1 we can assume M^3 is $M(\partial(G^2 \cup R^2), \omega)$ in which G^2 and R^2 are orientable and ω is the natural coloring. By the corollary to Theorem 2 there exists f: $(E(\partial(G^2 \cup R^2))$; part of boundary corresponding to G^2, part corresponding to $R^2)$ into $(8; C_1, C_2)$ such that f restricted to either part of the boundary is an S^1 fibration. We extend f by coning to a map F: $S^3 \to \hat{8}$. We can easily find an embedding e: $\hat{8} \to S^3$ such that $e(\{A\} \cup \{B\}) = e(\hat{8}) \cap L$, eF is transversal to L so that $(eF)^{-1}(L)$ is $\partial(G^2 \cup R^2)$, and $\omega^1(e\hat{F})_* = \omega$, where ω^1 is the coloring for L, and $e\hat{F}$ is the restriction to $S^3 - \partial(G^2 \cup R^2)$.

4. Notes. 1. Theorem 4 can be considered as generalizing the representations of cyclic branched coverings of S^3 by functions g: $S^3 \to S^3$ transversal to the trivial knot.

2. An interesting open question is what relationship exists between functions g, g^1: $S^3 \to S^3$, transversal to T of Figure 1, such that $M(g) \cong M(g^1)$. In particular suppose $M(g) = S^3$.

3. The theorem of Hilden [**Hi**] on embeddings of M^3 in $S^3 \times S^2$, such that $M^3 \to S^3 \times S^2 \overset{\pi_1}{\to} S^3$ is a branched covering, can be deduced directly from Theorem 4. Let (T, τ) be the coloring of the trefoil illustrated in Figure 1. It is enough to observe that $M(T, \tau)$ is the Seifert variety $(0oo \mid -1; (2, 1))$ and that

$$t\colon \left(0oo \mid -1; (2,1)\right) = S^3 \to S^3 = \left(0oo \mid -1; (3,1), (2,1)\right)$$

preserves fibres and satisfies

$$\#\pi t^{-1}x = \#t^{-1}x,$$

where π is the projection of $(0oo \mid -1; (2,1))$ over the base S^2. The embedding $M^3 \to S^3 \times S^2$ is then $(p(f), \pi\tilde{f})$ where $M^3 = M(f)$. In fact, by studying the embedding of $\hat{8}$ in S^3 used in the proof of Theorem 4, one sees that we actually get an embedding $M^3 \to S^3 \times D^2$. Since $S^3 \times D^2$ embeds in E^5, we also get another proof of Morris Hirsh's theorem that every closed orientable 3-manifold embeds in E^5.

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