TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 280, Number 2, December 1983

LENGTH DEPENDENCE OF SOLUTIONS OF FITZHUGH-NAGUMO EQUATIONS

BY

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ABSTRACT. We investigate the behavior of the solutions of the problem

$$u_t = u_{xx} - \alpha u - v + u^2(1 + \alpha - u), \qquad v_t = \eta v_{xx} + \sigma u - \gamma v,$$
$$u(0, t) = g(t), \quad v(0, t) = h(t), \qquad u_x(L, t) = v_x(L, t) = 0$$

where $t \geqslant 0$ and $0 < x < L \leqslant \infty$.

Solutions of the above equations are considered a qualitative model of conduction of nerve axon impulses. Using explicit constructions and semigroup methods, we obtain decay results on the norms of differences between the solution for L infinite and the solutions when L is large but finite. We conclude that nerve impulses for long finite nerves become uniformly close to those of the semi-infinite nerves away from the right endpoint of the finite nerve.

1. Introduction and notation. The FitzHugh-Nagumo equations [FN] are considered to be a qualitative model of nerve axon impulses. For background on nerve axon equations in general, see R. FitzHugh [3] or S. Hastings [5]. For background in neurophysiology, see Ochs [10] or R. D. Keynes [9]. For work specifically on FN, see Rauch and Smoller [12] or M. E. Schonbek [13]. The equations we consider are

$$\begin{aligned} u_t &= u_{xx} - \alpha u - v + f(u), \\ v_t &= \eta v_{xx} + \sigma u - \gamma v, \end{aligned} \qquad (x, t) \in (0, L) \times (0, \infty), \tag{1.1}$$

where η, α, σ and γ are positive constants and $0 < L \leqslant \infty$. The boundary conditions are

$$\begin{aligned} u(0, t) = g(t), \quad v(0, t) = h(t), &\qquad t \in [0, \infty), \\ u_x(L, t) = v_x(L, t) = 0, &\qquad t \in (0, \infty), \end{aligned} \tag{1.2}$$

where the second condition is omitted when $L = \infty$. The functions f and g are infinitely differentiable with compact support. Denote the initial conditions by $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$. The function $f(u) = u^2(1 + \alpha - u)$ is the remainder after linearization about zero of the usual cubic $u(1 - u)(u - \alpha)$ which is associated with FN; here $0 < \alpha < 1$. We use the boundary conditions (1.2) to model stimulation at one end of the axon, and usually have $u_0 = v_0 \equiv 0$ to model the initial rest state of the axon.

The central problem we address is the comparison of solutions of (1.1)–(1.2) where $L < \infty$ with the solution where $L = \infty$. To make sense of comparison of

Received by the editors January 24, 1983. Portions of this paper were presented at the 798th meeting of the A.M.S. on November 12, 1982, in Baton Rouge, Louisiana.

1980 *Mathematics Subject Classification*. Primary 35B30, 35K55; Secondary 35C15.

0002-9947/83 $1.00 + $.25 per page

solutions on different spaces, we extend the solutions on smaller domains by setting them equal to zero where not previously defined. We make the comparisons quantitative by examining norms of differences of solutions in the spaces $C^0(0, m; R^2)$ and $C^0(0, \xi L; R^2)$ for $m < L$ and $\xi \in (0, 1)$. The main result will be that these norms approach zero as $L \to \infty$. Consequently, two large L solutions should be close except near the right end of the solution for the smaller L. To paraphrase in physiological terms, the longer two finite nerves are, the more the impulses are alike except near the end of the shorter axon that is opposite the stimulation. This is to be expected since there are real complications near the end of the axon such as terminal branching.

Notation. Throughout we will denote partial and ordinary differential operators as D_t, D_x, D_{xx} and so on. Column vectors (usually 2-vectors) will sometimes be written as $(a, b)^{\mathrm{T}}$ instead of the vertical presentation. Several symbols will have fixed definitions in the sequel, unless otherwise specified. Let N, N_0, R and C be the sets of natural, nonnegative, real and complex numbers, respectively. $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ are the real and imaginary parts of the complex number s. Let $U = (u, v)^{\mathrm{T}}$; $W = W_1 = (w_1, w_2)^{\mathrm{T}}$; $W_3 = (w_3, w_4)^{\mathrm{T}}$. Let $F(U) = (f_1(U), f_2(U))^{\mathrm{T}}$. Let $D = \mathrm{diag}(1, \eta)$. Define the matrix B by $BU = (-\alpha u - v, \sigma u - \gamma v)^{\mathrm{T}}$. Let $\sigma(A)$ be the spectrum of an operator A; let $R(s, A) = (sI - A)^{-1}$ be the resolvent of an operator A at s. If $\phi \in (0, \pi/2)$, define

$$\Delta_\phi = \{s \in C - \{0\} : |\arg(s)| < \phi\},$$
$$S_\phi = \{s \in C - \{0\} : |\arg(s)| < \phi + \pi/2\}.$$

Here $\arg(s) = \arctan(\mathrm{Im}(s)/\mathrm{Re}(s))$. Let $(f, g) = \int_0^L f(x)\overline{g(x)}dx$, the inner product of the functions f and g. Let $L^p(0, L)$ be the usual Banach space of all real-valued Lebesgue measurable functions for which the (appropriate) norm below is finite:

$$\|u\|_p = \left(\int_0^L |u(x)|^p \, dx\right)^{1/p} \qquad \text{for } 1 \leq p < \infty,$$
$$\|u\|_p = \operatorname*{ess\,sup}_{[0, L]} |u(x)| \qquad \text{for } p = \infty.$$

We let

$$L^p = L^p(0, L; R^2) \simeq L^p(0, L) \times L^p(0, L), \qquad 0 < L \leq \infty, 1 \leq p \leq \infty.$$

For L finite, $C^0[0, L]$ is the set of all continuous real-valued functions on $[0, L]$; it is a Banach space under the norm

$$\|f\|_\infty = \max_{x \in [0, L]} |f(x)|.$$

For $L = \infty$, $C^0[0, \infty)$ is the set of all bounded continuous real-valued functions on $[0, \infty)$; it is a Banach space under the norm

$$\|f\|_\infty = \sup_{x \in [0, \infty)} |f(x)|.$$

We let

$$X = C^0(0, L; R^2) \simeq C^0(0, L) \times C^0(0, L), \qquad 0 < L \leq \infty.$$
$$X_0 = \{f \in X : f(0) = (0, 0)^{\mathrm{T}}\}.$$

Let

$$Y = C_0^\infty(0, \infty;\ R^2),$$

the usual space of functions having compact support in $(0, \infty)$ and having continuous derivatives of all orders. Along the same lines, let $C_0^k(\Omega)$ be the set of k-times continuously differentiable functions with compact support on the open set Ω.

The Laplace transform of a function f is

$$L[f](s) = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s),$$

for those f's for which the above integral converges. The inverse Laplace transform is

$$L^{-1}[\bar{f}](t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s) e^{st}\,ds$$

for real c such that the poles of $\bar{f}$ lie to the left of the contour $\Gamma_c = \{c + wi : w \in R\}$.

Using vector-matrix notation, we write (1.1)–(1.2) in more compact form as

$$D_t U = DD_{xx}U + BU + F(U) \quad \text{for } (x, t) \in (0, L) \times (0, \infty), \tag{1.3}$$

$$\begin{aligned} &U(0, t) = G(t), && \text{where } G \in Y, \text{ and } G(0) = (0,0)^{\mathrm{T}},\\ &D_x U(L, t) = (0,0)^{\mathrm{T}}, && \text{for } t \in (0, \infty) \text{ provided } L < \infty,\\ &U(x, 0) = U_0(x) && \text{for } x \in [0, L]. \end{aligned} \tag{1.4}$$

To reduce (1.3)–(1.4) to an equivalent system with homogeneous boundary conditions, we employ the change of variable $W(x, t) = U(x, t) - G(t)$. Then $W(0, t) = (0,0)^{\mathrm{T}}$; when $L < \infty$, $D_x W(L, t) = (0,0)^{\mathrm{T}}$ as well. Since $G(0) = (0,0)^{\mathrm{T}}$, the initial value $U_0(x)$ is unchanged. Setting $U(x, t) = W(x, t) + G(t)$ in (1.3)–(1.4), we obtain a differential equation for W:

(1.5)

$$D_t W = (DD_{xx} + G)W + BG - D_t G + F(W + G), \qquad (x, t) \in (0, L) \times (0, \infty).$$

Let $F(t, W) = BG - D_t G + F(W + G)$; that is, use the same symbol F, now with two arguments. The full system in the new variable is:

$$D_t W = (DD_{xx} + B)W + F(t, W), \qquad (x, t) \in (0, L) \times (0, \infty), \tag{1.6}$$

$$\begin{cases} W(0, t) = (0,0)^{\mathrm{T}}, & t \in (0, \infty),\\ D_x W(L, t) = (0,0)^{\mathrm{T}}, & \text{provided } L < \infty, t \in (0, \infty),\\ W(x, 0) = U_0(x), & x \in [0, L]. \end{cases} \tag{1.7}$$

We shall refer to (1.1)–(1.2), (1.3)–(1.4) or (1.6)–(1.7) as the FitzHugh-Nagumo equations, abbreviated FN. We shall work with the system

$$D_t U = (D_{xx} + B)U, \qquad (x, t) \in (0, L) \times (0, \infty), \tag{1.8}$$

$$\begin{cases} U(0, t) = (0,0)^{\mathrm{T}}, & t \in (0, \infty),\\ D_x U(L, t) = (0,0)^{\mathrm{T}}, & \text{provided } L < \infty, t \in (0, \infty),\\ U(x, 0) = U_0(x), & x \in [0, L], \end{cases} \tag{1.9}$$

for the purpose of studying semigroups. We shall call (1.8)–(1.9) the homogeneous linearized FitzHugh-Nagumo equations, or HLFN.

2. Linear results for $L < \infty$. Define the operator $A^L: D(A^L) \to X$ by

$$A^L U(x) = \begin{bmatrix} D_{xx} & 0 \\ 0 & \eta D_{xx} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(x) = DD_{xx}U(x), \tag{2.1}$$

with $D(A^L) = Z \times Z$, where Z is given by

$$Z = \{u \in C^2(0, L) \cap C^0[0, L] : u(0) = 0 = D_x u(L) \text{ and } D_{xx}u(0+) \text{ and } D_{xx}u(L-) \text{ exist}\}.$$

A^L is well defined on this domain. Note [**8**, Theorem 5.5.6] gives a criterion for an operator T to be closable. That is, T is closable if and only if

$$u_n \in D(T),\ u_n \to 0 \text{ and } Tu_n \to v \text{ imply } v = 0. \tag{2.2}$$

Let $T = A^L$; note that the test functions $[C_0^\infty(0, L)]^2$ are contained in $D(A^L)$. Considering A^L as a distribution, let q be an arbitrary test function. Let $u_n \in D(A^L)$ be such that $u \to 0$ and $A^L u_n \to v$. Let $\langle \cdot, \cdot \rangle$ be the usual distributional pairing. Then $u_n \to 0$ implies

$$\langle q, A^L u_n \rangle = \langle A^L q, u_n \rangle \to 0.$$

But $A^L u_n \to v$ implies $\langle q, A^L u_n \rangle \to \langle q, v \rangle$. Uniqueness of limits implies $\langle q, v \rangle = 0$. Since q is an arbitrary test function, we have $v = 0$. Hence A^L is closable. Henceforth we shall identify A^L with its closure. Given the work above, we may write HLFN as the operator equation

$$D_t U = (A^L + B)U, \qquad U(0) = U_0. \tag{2.3}$$

From [**2**] we have $\sigma(A^L + B) = \{a_{1k}, a_{2k} : k \in N_0\}$, where

$$a_{1k} = \frac{1}{2}\left[-\left(\alpha + \gamma + (1 + \eta)b_k^2 L^{-2}\right) + \left(\left[\alpha - \gamma + (1 - \eta)b_k^2 L^{-2}\right]^2 - 4\sigma\right)^{1/2}\right],$$

with $b_k = (k + \frac{1}{2})\pi$ and a_{2k} equal to the conjugate of a_{1k}. Hence

$$\operatorname{Re}\sigma(A^L + B) < -\delta = \min(\alpha, \gamma) < 0, \qquad |\operatorname{Im}\sigma(A^L + B)| \leqslant \sqrt{\sigma}.$$

The differential equation

$$\begin{aligned} &\kappa D_{xx}u + zu = 0, && x \in (0, L), \\ &u(0) = D_x u(L) = 0, && \kappa \text{ a positive constant,} \end{aligned} \tag{2.4}$$

has eigenvalues $z = z_k = -b_k^2 L^{-2}$ and normalized eigenfunctions

$$e_k(x) = (2/L)^{1/2}\sin\left(b_k L^{-1}x\right) \quad \text{for } k \in N_0.$$

Since (2.4) is a regular Sturm-Liouville problem, the set of eigenfunctions is complete in L^2. See [**1**] for a discussion of Sturm-Liouville problems and (L^2) completeness. The evolution equation

$$\begin{cases} D_t U = DD_{xx}U, & (x, t) \in (0, L) \times (0, \infty), \\ U(0, t) = D_x U(L, t) = (0, 0)^{\mathrm{T}} & \text{for } t \in (0, \infty), \\ U(x, 0) = U_0(x) & \text{for } x \in (0, L), \end{cases} \tag{2.5}$$

has the associated operators $f \to T_0^L(t)f$ on L^2 defined by

$$(2.6)\quad \begin{cases} T_0^L(0)f = f, \\ T_0^L(t)f = \sum_{k=0}^{\infty} \operatorname{diag}\left[\exp(z_k t), \exp(\eta z_k t)\right] \cdot (f, e_k)e_k(x) \quad \text{for } t > 0. \end{cases}$$

Since X is contained in L^2, we may restrict the operator to X. For $k \in N_0$ we note that

$$\|e_k\|_\infty |(f, e_k)| \leq 2b_k^{-1}\|f\|_X.$$

This inequality, together with integral estimates on sums, tells us that for t positive,

$$(2.7)\quad \begin{aligned} \|T_0^L(t)f\|_X &\leq 2\|f\|_X \cdot \sum_{k=0}^{\infty} \left[\exp(z_k t) + \exp(\eta z_k t)\right] b_k^{-1} \\ &\leq 4\|f\|_X \pi^{-1}\Big[\left(1 + L^2\pi^{-2}t^{-1}\right)\exp\left(-t\pi^2 L^{-2}/4\right) \\ &\qquad + \left(1 + L^2(\pi^2\eta t)^{-1}\right)\exp\left(-t\pi^2\eta L^{-2}/4\right)\Big]. \end{aligned}$$

A standard maximum principle argument shows that

$$(2.8)\quad \|T_0^L(t)\| \leq 1 \quad \text{for all } t \in [0, \infty),$$

so $T_0^L(t)$ is always nonexpansive. Estimate (2.8) implies $T_0^L(t)$ is a bounded operator on X for all $t \geq 0$. In view of (2.7),

$$(2.9)\quad \|T_0^L(t)\| \to 0 \quad \text{as } t \to \infty.$$

Let t and s be in $[0, \infty)$. Then for f in X, the orthonormality of the e_k's and the usual properties of exponentials show that

$$(2.10)\quad T_0^L(t)T_0^L(s)f = T_0^L(t + s)f.$$

Suppose $f \in C^1(0, L; R^2)$. Then the Fourier development, $\Sigma_0^M(f, e_k)e_k(x)$, converges to f in X as M becomes arbitrarily large. For $\varepsilon > 0$ given and M sufficiently large,

$$\|T_0^L(t)f - f\|_X \leq \left\| \sum_{k=0}^{M} \operatorname{diag}\left[\exp(z_k t) - 1, \exp(\eta z_k t) - 1\right](f, e_k)e_k \right\|_X + \frac{\varepsilon}{3} + \frac{\varepsilon}{3},$$

since the relevant series converge in X. Since $\exp(z_k t) - 1$ and $\exp(\eta z_k t) - 1$ go to zero uniformly as $t \downarrow 0$, there exists $t_0 > 0$ such that

$$(2.11)\quad 0 < t < t_0 \text{ implies } \|T_0^L(t)f - f\|_X < \varepsilon \quad \text{for } f \text{ in } C^1.$$

Let f be arbitrary in X. Select $\{f_n\} \subset C^1(0, L; R^2)$ such that $f_n \to f$ in X. Let $\varepsilon > 0$ be given. Fix n large enough that $\|f_n - f\|_X < \varepsilon/4$. Choose t_0 small enough that $\|T_0^L(t)f_n - f_n\|_X < \varepsilon/2$ for $0 < t < t_0$. Then triangularization shows

$$\begin{aligned} \|T_0^L(t)f - f\|_X &\leq \left(1 + \|T_0^L(t)\|\right)\|f_n - f\|_X + \|T_0^L(t)f_n - f_n\|_X \\ &\leq 2\|f_n - f\|_X + \|T_0^L(t)f_n - f_n\|_X < \varepsilon. \end{aligned}$$

Hence (2.11) holds for f arbitrary in X. That is, T_0^L is strongly continuous from the right at $t = 0$. From [**14**, Theorem 9.1.1], we have $T_0^L(t)$ is strongly continuous as

$t \to t_0$ for $t_0 \in [0, \infty)$. This fact, plus (2.8) and (2.10) show that $T_0^L(t)$ is a strongly continuous semigroup on X.

From (2.7) and (2.8), we see there exist $C > 1$ and $\rho > 0$ such that $\|T_0^L(t)\| \leqslant C \cdot \exp(-\rho t)$ for $t > 0$. Let $\phi = \operatorname{arccot}(\sqrt{\sigma}/\delta) \in (0, \pi/2)$. For $t \in \Delta_\phi$, $|t|\cos(\phi) \leqslant \operatorname{Re}(t)$. Hence $(\operatorname{Re}(t))^{-1} \leqslant (\sec(\phi))|t|^{-1}$. The above estimates go through with "t" replaced by "$\operatorname{Re}(t)$" in the exponents, and "t^{-1}" replaced by "$\sec(\phi)|t|^{-1}$" otherwise. In particular,

$$\|T_0^L(t)\| \leqslant C \cdot \exp(-\rho \operatorname{Re}(t)) \quad \text{for } t \in \Delta_\phi. \tag{2.12}$$

Since $T_0^L(t)$ is strongly continuous, we have from [**4**, Theorem II.1.1] that

$$A^L T_0(t) f = D_t T_0^L(t) f \quad \text{for } f \in X.$$

Again using integral estimates of series, we obtain for $t \in \Delta_\phi$:

$$\begin{aligned} \|A^L T_0^L(t) f\|_X &= \left\| \sum_{k=0}^{\infty} \operatorname{diag}[z_k \exp(z_k t), \eta z_k \exp(\eta z_k t)](f, e_k) e_k(\cdot) \right\|_X \\ &\leqslant 2L^{-1} \sum_{k=0}^{\infty} b_k^{-1}[|z_k \exp(z_k t)| + \eta |z_k \exp(\eta z_k t)|] \cdot \|f\|_X \\ &\leqslant \|f\|_X \cdot \left[(\pi L^{-2} + \sec(\phi)\pi^{-1}|t|^{-1}) \exp(-L^{-2}\pi^2 \operatorname{Re}(t)/4) \right. \\ &\qquad \left. + (\eta\pi L^{-2} + \sec(\phi)\pi^{-1}|t|^{-1}) \exp(-L^{-2}\pi^2 \operatorname{Re}(t)/4) \right]. \end{aligned} \tag{2.13}$$

Since

$$\tilde{f}(t) = |t| \exp(-L^{-2} 8^{-1} \eta \pi^2 \operatorname{Re}(t)) \to 0 \quad \text{as } |t| \to \infty \text{ in } \Delta_\phi,$$

$\tilde{f}(t)$ is bounded in $\Delta_\phi - \theta$, where θ is a neighborhood of the origin. Hence

$$\|A^L T_0^L(t)\| \leqslant C \cdot |t|^{-1} \exp(-L^{-2} 8^{-1} \eta \pi^2 \operatorname{Re}(t)) \quad \text{for } t \in \Delta_\phi. \tag{2.14}$$

LEMMA 2.1. *$T_0^L(t)$ is a holomorphic semigroup on X_0 for $t \in \Delta_\phi$.*

PROOF. The strong continuity of $T_0^L(t)$ was shown above. This, coupled with (2.12) and (2.14), implies $T_0^L(t)$ is holomorphic in Δ_ϕ. Since $e_k(0) = 0$ for $k \in N_0$, we have $[T_0^L(t) f](0) = (0, 0)^{\mathrm{T}}$ for $t \geqslant 0$. ■

THEOREM 2.2. *$A^L + B$ generates a holomorphic semigroup $T^L(t)$ on X_0 for $t \in \Delta_\phi$.*

PROOF. B is a bounded perturbation of A^L. By [**6**, Theorem 1.3.2], $A^L + B$ generates a holomorphic semigroup $T^L(t)$. The bounds given above on $\sigma(A^L + B)$ show that the vertex of Δ_ϕ is 0 and the sector angle is $\phi = \operatorname{arccot}(\sqrt{\sigma}/\delta)$. ■

LEMMA 2.3. *There exist $C, \delta > 0$ such that*

$$\|T^L(t) f\|_X \leqslant C \cdot \exp(-\delta t) \|f\|_X \quad \text{for } f \in X. \tag{2.15}$$

PROOF. We consider the lemma in the light of [**6**, §1.5]. Define $\sigma(A) = \sigma(A) \cup \{\infty\}$. Then a subset $Y \subset \sigma(A)$ is a *spectral set* provided both Y and $\sigma(A) - Y$ are closed in the extended plane. Hence the empty set and $\sigma(A)$ are spectral sets for a given operator A. Note that $-(A^L + B)$ is sectorial in the sense of Henry, and $0 < \delta < \operatorname{Re}(\sigma(-(A^L + B)) \cup \{\infty\})$, where $\delta = \frac{1}{2}\min(\alpha, \gamma) > 0$. By [**6**, Theorem 1.5.3], there exists $C > 0$ such that (2.15) holds. ■

Next we shall develop explicit formulas for a representative of the resolvent $R(s, A^L + B)$. First we shall mention properties of the functions $\lambda_i(s)$; for proofs, see [**2**, Appendix]. Second, we shall construct Green's functions for certain second order ordinary differential equations. Third, we shall relate Green's functions to the resolvents.

First, the functions $\lambda_i(s)$. Let

$$E = \begin{bmatrix} a & 1 \\ -b & c \end{bmatrix}.$$

The eigenvalues of E are given by

$$\lambda^2_{\pm} = \frac{1}{2}\left[c + a \pm \{(c-a)^2 - 4b\}^{1/2}\right].$$

Define λ_1 and λ_3 to be the principal square roots of λ^2_+ and λ^2_-, respectively. Define $\lambda_2 = -\lambda_1$, $\lambda_4 = -\lambda_3$. Given $a = s + \alpha$, $b = \sigma/\eta$, and $c = (s + \gamma)/\eta$, λ_1 and λ_3 are functions of the complex variable s. For $\mathrm{Re}(s) > 0$ and $\mathrm{Im}(s)$ arbitrary, we have the estimates:

$$\mathrm{Re}(\lambda_1) \geqslant [(\mathrm{Re}(s) + \gamma)/2\eta]^{1/2} > [\gamma/2\eta]^{1/2} > 0, \tag{2.16}$$

$$\mathrm{Re}(\lambda_3) \geqslant [\mathrm{Re}(s) + \alpha]^{1/2} > \alpha^{1/2} > 0, \tag{2.17}$$

$$\mathrm{Re}(\lambda_1) \geqslant [(1+\eta)/4\eta]^{1/2} |w|^{1/2}, \tag{2.18}$$

$$\mathrm{Re}(\lambda_3) \geqslant \tfrac{1}{2}|w|^{1/2} \quad \text{for } |w| > R > 0, \tag{2.19}$$

where R is a fixed real number, and $w = \mathrm{Im}(s)$. Further.

$$|\lambda_1^2 - \lambda_3^2|^{-1} \leqslant \text{constant} < \infty \quad \text{for fixed } \mathrm{Re}(s) > 0, \tag{2.20}$$

$$|\lambda_1^2 - \lambda_3^2|^{-1} \to 0 \quad \text{as } |\mathrm{Im}(s)| = |w| \to \infty, \tag{2.21}$$

$$|(\lambda_i^2 - a)/(\lambda_1^2 - \lambda_3^2)| < C_i < \infty, \qquad i = 1, 3,\ C_i \text{ constant}. \tag{2.22}$$

The following identities are derived by algebraic manipulation:

$$(\lambda_1^2 - a)(\lambda_3^2 - a) = b; \tag{2.23}$$

$$b - (\lambda_1^2 - a)^2 = -(\lambda_1^2 - a)(\lambda_1^2 - \lambda_3^2); \tag{2.24}$$

$$b - (\lambda_3^2 - a)^2 = (\lambda_3^2 - a)(\lambda_1^2 - \lambda_3^2);$$

$$\left[b - (\lambda_1^2 - a)^2\right] + \left[b - (\lambda_3^2 - a)^2\right] = -(\lambda_1^2 - \lambda_3^2)^2; \tag{2.25}$$

$$\begin{aligned} \left[b - (\lambda_3^2 - a)^2\right](\lambda_1^2 - a) &= b(\lambda_1^2 - \lambda_3^2); \\ \left[b - (\lambda_1^2 - a)^2\right](\lambda_3^2 - a) &= -b(\lambda_1^2 - \lambda_3^2). \end{aligned} \tag{2.26}$$

The eigenvectors of E are $(1, \lambda_i^2 - a)^{\mathrm{T}}$ for $i = 1, 3$.

Second, we indicate the main steps in finding certain Green's functions. Let $F : [0, L] \to R^2$ in this section. Consider

$$D_{xx}U = EU + F(x), \qquad 0 < x < L, \tag{2.27}$$

$$U(0) = (0,0)^{\mathrm{T}} = D_x U(L), \tag{2.28}$$

$$U(0) = (0,0)^{\mathrm{T}} = D_x U(0). \tag{2.29}$$

We wish to find Green's function for (2.27)–(2.28). To do this, we alter Green's function for (2.27)–(2.29). Note that the vectors $\exp(\lambda_i x)(1, \lambda_i^2 - a)^{\mathrm{T}}$, $i = 1, 2, 3, 4$, are solutions of (2.27) when $F \equiv (0,0)^{\mathrm{T}}$. Convert (2.27) to the first order system

$$D_x \begin{bmatrix} u \\ v \\ u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 1 & 0 & 0 \\ -b & c & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ u_x \\ v_x \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_1(x) \\ F_2(x) \end{bmatrix} \quad \text{for } 0 < x < L,$$

or, in more compact form, $V_x = HV + \tilde{F}(x)$, where V, H and $\tilde{F}$ are defined by the previous equation. Note that λ_i, $i = 1, 2, 3, 4$, are the eigenvalues of H with the corresponding eigenvectors $(1, \lambda_i^2 - a, \lambda_i, \lambda_i(\lambda_i^2 - a))^{\mathrm{T}}$. The kernel for the inhomogeneous initial value problem

$$V_x = HV + \tilde{F}(x), \qquad V(0) = (0,0,0,0)^{\mathrm{T}},$$

is given by

$$V(x) = \int_0^x e^{H(x-y)} \tilde{F}(y)\, dy.$$

If we restrict to the problem (2.27)–(2.29), we have the solution

$$U(x) = \int_0^x K(x-y) F(y)\, dy,$$

where $K(x)$ is the upper right-hand 2×2 block of the matrix e^{Hx}. To find Green's function for the boundary value problem (2.27)–(2.28), we shall find constants c_i, $i = 1, 2, 3, 4$, such that

$$\begin{aligned} U(x) &= \int_0^x K(x-y) F(y)\, dy + \sum_{i=1}^{4} c_i \left(1, \lambda_i^2 - a\right)^{\mathrm{T}} \cdot \exp(\lambda_i x) \\ &= \int_0^L G^L(x, y) F(y)\, dy, \end{aligned} \tag{2.30}$$

for some kernel $G^L(x, y)$. We use (2.28) to determine the constants c_i. At $x = 0$,

$$U(0) = (0,0)^{\mathrm{T}} = \sum_{i=1}^{4} c_i \left(1, \lambda_i^2 - a\right)^{\mathrm{T}}. \tag{2.31}$$

A computation shows that

$$K(x) = K_1(x) + K_3(x) = \Psi(\lambda_1, \lambda_3)\left[\sinh(\lambda_1 x) Z_1 - \sinh(\lambda_3 x) Z_3\right], \tag{2.32}$$

where $\Psi(\lambda_1, \lambda_3) = -(\lambda_1 \lambda_3 [\lambda_1^2 - \lambda_3^2])^{-1}$, and

$$Z_i = \begin{bmatrix} \lambda_j \left(\lambda_j^2 - a\right) & -\lambda_j \\ \lambda_j & -\lambda_j \left(\lambda_i^2 - a\right) \end{bmatrix}$$

for $i = 1, 3$ and $j = 3, 1$, respectively. The form of (2.32) shows that $K(x)$ is odd; in particular, $K(0)$ is the 2×2 zero matrix. Let $'$ denote D_x. Differentiate (2.30), then evaluate at $x = L$ to obtain

$$U'(L) = \int_0^L K'(L-y)F(y)\,dy + \sum_{i=1}^{4} \lambda_i c_i \left(1, \lambda_i^2 - a\right)^{\mathrm{T}} \exp(\lambda_i L).$$

This equation, coupled with (2.31), yields a system of four linear equations in four unknowns, namely the c_i. After solving for the c_i and substituting back into (2.30), we obtain

(2.33)

$$U(x) = \int_0^x K(x-y)F(y)\,dy + \frac{\left[(\lambda_3^2 - a)k_1 - k_2\right]\sinh(\lambda_1 x)}{\lambda_1(\lambda_1^2 - \lambda_3^2)\cosh(\lambda_1 L)}\left(1, \lambda_1^2 - a\right)^{\mathrm{T}}$$
$$- \frac{\left[(\lambda_1^2 - a)k_1 - k_2\right]\sinh(\lambda_3 x)}{\lambda_3(\lambda_1^2 - \lambda_3^2)\cosh(\lambda_3 L)}\left(1, \lambda_3^2 - a\right)^{\mathrm{T}},$$

where

$$k_i = \int_0^L r_i(K)'(L-y)F(y)\,dy, \qquad i = 1, 2,$$

for $r_i(K) = i$th row of K. Upon the examination of the rows of K in (2.32), we see that

$$U(x) = \int_0^x K(x-y)F(y)\,dy + \left(\int_0^x + \int_x^L\right)\left[\tilde{G}_1^L(x, y) + \tilde{G}_3^L(x, y)\right]F(y)\,dy,$$

where

$$\tilde{G}_1^L(x, y) = \frac{\sinh(\lambda_1 x)\cosh(\lambda_1(L-y))}{\lambda_1(\lambda_1^2 - \lambda_3^2)\cosh(\lambda_1 L)}\begin{bmatrix} \dfrac{b - (\lambda_3^2 - a)^2}{\lambda_1^2 - \lambda_3^2} & -1 \\ \dfrac{\left[b - (\lambda_3^2 - a)^2\right](\lambda_1^2 - a)}{\lambda_1^2 - \lambda_3^2} & -(\lambda_1^2 - a) \end{bmatrix}$$

and

$$\tilde{G}_3^L(x, y) = \frac{\sinh(\lambda_3 x)\cosh(\lambda_3(L-y))}{\lambda_3(\lambda_1^2 - \lambda_3^2)\cosh(\lambda_3 L)}\begin{bmatrix} \dfrac{b - (\lambda_1^2 - a)}{\lambda_1^2 - \lambda_3^2} & 1 \\ \dfrac{\left[b - (\lambda_1^2 - a)^2\right](\lambda_3^2 - a)}{\lambda_1^2 - \lambda_3^2} & (\lambda_3^2 - a) \end{bmatrix}.$$

Note the hyperbolic identity

(2.34) $$\sinh(\lambda_i(x-y))\cosh(\lambda_i L) + \sinh(\lambda_i x)\cosh(\lambda_i(L-y)) = \sinh(\lambda_i y)\cosh(\lambda_i(L-x)),$$

for $i = 1, 3$. Comparing $K_1(x-y)$ with $\tilde{G}_1^L(x, y)$ and $K_3(x-y)$ with $\tilde{G}_3^L(x, y)$ in light of (2.34), we see that

$$K_1(x, y) + \tilde{G}_i^L(x, y) = \tilde{G}_i^L(y, x)$$

for $i = 1, 3$. Hence we may legitimately define the kernel $G^L(x, y)$ in (2.30) by

(2.35)
$$G^L(x, y) = G_1^L(x, y) + G_3^L(x, y) = \begin{cases} \tilde{G}^L(y, x) + \tilde{G}_3^L(y, x) & \text{for } y < x, \\ \tilde{G}_1^L(x, y) + \tilde{G}_3^L(x, y) & \text{for } x < y. \end{cases}$$

Then

$$S^L(x) = \int_0^L G^L(x, y) F(y)\, dy$$

solves (2.28) by construction. That $S^L(x)$ solves (2.27) can be seen by noting that the λ_i^2 are eigenvalues of E, and by using the identities (2.24)–(2.26) along with the identity

$$-\sinh(\lambda_i x)\sinh(\lambda_i(L - x)) + \cosh(\lambda_i x)\cosh(\lambda_i(L - x)) \equiv \cosh(\lambda_i L).$$

Hence $G^L(x, y)$ is the desired kernel for (2.27)–(2.28).

Third, and last for this section, we relate the kernel $G^L(x, y)$ to the resolvent $R(s, A^L + B)$. Consider the initial boundary value problem (1.8)–(1.9). Apply the Laplace transform to these equations in the time variable t, introducing the transform variable s and the new equations

(2.36)
$$s\bar{U}(x, s) - U_0(x) = (DD_{xx} + B)\bar{U}(x, s), \qquad (x, s) \in (0, L) \times (0, \infty),$$
$$\bar{U}(0, x) = (0, 0)^{\mathrm{T}} = D_x\bar{U}(L, s).$$

Recalling the definitions of D and B from §1, we see that we may rearrange this system in transform space to the following:

$$D_{xx}\bar{U}(x, s) = E\bar{U} - D^{-1}U_0(x), \qquad (x, s) \in (0, L) \times (0, \infty),$$
$$\bar{U}(0, s) = (0, 0)^{\mathrm{T}} = D_x\bar{U}(L, s).$$

The work just done shows that

$$\bar{U}(x, s) = -\int_0^L G^L(x, y, s) D^{-1} U_0(y)\, dy$$

solves the inhomogeneous boundary value problem in Laplace transform space.

Definition 2.4. Let Q be a function of x and y. Define

$$\|Q\|_{1,\infty} = \sup_{x \in (0, L)} \int_0^L |Q(x, y)|\, dy,$$

provided Q is defined in $(0, L)^2$.

Note that the following inequality holds:

$$\|S^L(\cdot, s)\|_X \leq \|G^L(\cdot, \cdot, s)\|_{1,\infty} \cdot \|U_0\|_X.$$

In order to estimate the $1, \infty$ norm of Green's function G^L, we consider the functions

$$h_i(x, y) = \cosh(\lambda_i L) g_i^L(x, y) = \begin{cases} \lambda_i^{-1}\cosh[\lambda_i(L - x)]\sinh(\lambda_i y), & y < x, \\ \lambda_i^{-1}\cosh[\lambda_i(L - y)]\sinh(\lambda_i x), & x < y. \end{cases}$$

Noting that $|\cosh(\lambda_i\xi)|$, $|\sinh(\lambda_i\xi)| \leq \cosh(\xi \operatorname{Re}(\lambda_i))$ for $\xi \in R$, we get the estimate

$$\|h_i(x, \cdot)\|_1 \leq (|\lambda_i| \operatorname{Re}(\lambda_i))^{-1} \sinh(L \operatorname{Re}(\lambda_i)).$$

Since $|\cosh(L\lambda_i)| \geq \sinh[L \operatorname{Re}(\lambda_i)]$, we combine inequalities to get

$$\|g_i^L(x, \cdot)\|_1 \leq (|\lambda_i| \operatorname{Re}(\lambda_i))^{-1} \leq (\operatorname{Re}(\lambda_i))^{-2} \leq C_i < \infty \tag{2.37}$$

by (2.16)–(2.17) for all $\operatorname{Re}(s) > 0$. Recalling the definition of the kernel $G^L(x, y, s)$, we see that all the (x, y)-dependence resides in the scalar multipliers g_i^L. Hence (2.37) proves that $\|G^L(\cdot, \cdot, s)\|_{1,\infty}$ is globally bounded in s.

We say that a kernel represents an operator if the action of the operator on a function is given by integration against the kernel.

THEOREM 2.5. *The resolvent* $R(s, A^L + B)$ *is a bounded operator on* X_0 *for* $\operatorname{Re}(s) > 0$. *Further, the resolvent is represented by the kernel* $-G^L(x, y, s)D^{-1}$.

PROOF. The solution of the HLFN system or, equivalently, (2.3), is given by $U(x, t) = [T^L(t)U_0](x)$, or $U(t) = T^L(t)U_0$, respectively. Applying the Laplace transform, we obtain

$$\bar{U}(s) = L[T^L(t)U_0](s) = R(s, A^L + B)U_0 = (sI - (A^L + B))^{-1}U_0$$

from [**4**, Corollary II.1.3]. As remarked above, direct calculations show that

$$\bar{U}(x, s) = -\int_0^L G^L(x, y, s)D^{-1}U_0(y)\, dy.$$

Hence $-G^L(x, y, s)D^{-1}$ represents $R(s, A^L + B)$. From (2.37) and below, we see that the operator norm

$$\|R(s, A^L + B)\| \leq \|G^L(\cdot, \cdot, s)D^{-1}\|_{1,\infty} \leq C < \infty \quad \text{for } \operatorname{Re}(s) > 0,$$

uniformly in s. Our construction shows that the resolvent sends X_0 into itself. Hence the resolvent is a bounded operator on X_0 for $\operatorname{Re}(s) > 0$. ■

3. Results for $L = \infty$. Define the operator $A : D(A) \to X$ by

$$AU(x) = \begin{bmatrix} D_{xx} & 0 \\ 0 & \eta D_{xx} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(x) = DD_{xx}U(x), \tag{3.1}$$

with $D(A) = Z_1 \times Z_1$, where $Z_1 = \{u \in X \cap C^2(0, \infty) : u(0) = 0$ and $D_{xx}u(0+)$ exists$\}$. The definition of Z_1 makes A well defined. A is closable by an argument similar to the one advanced in §2. Given the above, we may write HLFN as the operator equation:

$$D_tU = (A + B)U, \quad t \in (0, \infty), \qquad U(0) = U_0. \tag{3.2}$$

Next we consider the spectrum of $A + B$. Consider the elliptic problem

$$(DD_{xx} + B)U = zU + Q, \tag{3.3}$$

where $Q \in L^2$. Extend Q by $Q(-x)$, so that $Q \in J = L^2(R^1, R^2)$. Let $\Phi : J \to J$ be the usual Fourier transformation

$$\Phi(Q)(s) = \hat{Q}(s) = (2\pi)^{-1/2}\int_R \exp(-isx)Q(x)\, dx.$$

Applying Φ to (3.3), we obtain

$$\Phi[(DD_{xx} + B - zI)U] = \begin{bmatrix} -(s^2+\alpha+z) & -1 \\ \sigma & -(\eta s^2+\gamma+z) \end{bmatrix} \hat{U} \equiv E\hat{U} = \hat{Q},$$

where E is defined by the equation. When the inverse exists,

$$\hat{U} = E^{-1}\hat{Q} = (\det E)^{-1}\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}\hat{Q}$$

$$\equiv (\det E)^{-1}\begin{bmatrix} -(\eta s^2+\gamma+z) & 1 \\ -\sigma & -(s^2+\alpha+z) \end{bmatrix}\hat{Q}.$$

Provided that E^{-1} has bounded entries, $U = \Phi^{-1}(E^{-1}\hat{Q})$ is the unique solution of (3.3). Routine computations show that for $|\operatorname{Im}(z)| \geqslant \sqrt{\sigma+\varepsilon}$, $|\det E| \geqslant \varepsilon^2 > 0$, and for $\operatorname{Re}(z) \geqslant -(1-\varepsilon)\delta$, $\delta = \min(\alpha,\gamma) > 0$, we have $|\det E| \geqslant (\varepsilon\delta)^4 > 0$. Both these estimates are true for arbitrary $s \in R$. Hence $|\det E|^{-1}$ is bounded on each set

$$\Omega_\varepsilon = \left\{ t \in C : |\operatorname{Im}(t)| \,|\sqrt{\sigma+\varepsilon} \text{ or } \operatorname{Re}(t) > -(1-\varepsilon)\delta \right\}.$$

Since the entries of E^{-1} are rational functions in s, $\operatorname{Im}(z)$, and $\operatorname{Re}(z)$ with the degrees of the numerators less than or equal to the degrees of the denominators, the entries are bounded for $(z,s) \in \Omega_\varepsilon \times R^1$ for any $\varepsilon > 0$. Hence $\Omega_\varepsilon \cap \sigma(A+B) = \varnothing$ for all $\varepsilon > 0$. This shows we have the same bounds on $\sigma(A+B)$ as on $\sigma(A^L+B)$, since $\sigma(A+B)$ is contained in the complement of S_ϕ for the same ϕ as mentioned previously.

Next we get semigroup results similar to those in §2. First, consider the uncoupled system

$$\text{(3.4)} \qquad \begin{cases} D_tU = DD_{xx}U, & (x,t) \in (0,\infty)^2, \\ U(0,t) = (0,0), & t \in (0,\infty), \\ U(x,0) = U_0(x), & x \in (0,\infty). \end{cases}$$

Let

$$H(x,y,t) = (4\pi t)^{-1/2}\left[\exp\left(-(x-y)^2/4t\right) - \exp\left(-(x+y)^2/4t\right)\right].$$

Classical techniques yield

$$\text{(3.5)} \qquad \begin{cases} U(x,0) = T_0(0)U_0(x) = U_0(x), \\ U(x,t) = T_0(t)U_0(x) = \int_0^\infty \operatorname{diag}[H(x,y,t), H(x,y,\eta t)]\,U_0(y)\,dy, \end{cases}$$

and hence

$$\text{(3.6)} \qquad \|U(x,t)\|_X \leqslant \|T_0(t)U_0\|_X \leqslant \|U_0\|_X.$$

Estimate (3.6) shows $U(x,t)$ is bounded and uniformly continuous. Taking $x = 0$ in (3.5), we see that $U(0,t) = (0,0)^T$. Hence $T_0(t)\colon X_0 \to X_0$ for $t \geqslant 0$; (3.6) yields $\|T_0(t)\| \leqslant 1$ for $t \geqslant 0$. That is, T_0 is nonexpansive. The usual integration techniques show

$$\text{(3.7)} \qquad T_0(t)T_0(s) = T_0(t+s) \quad \text{for all } t,s \geqslant 0.$$

We use a technique of John [7]. Note that

$$\int_0^\infty H(x, y, \kappa t)\, dy = 2\pi^{-1/2}\int_0^{x/(4\kappa t)^{1/2}} \exp(-v)\, dv \uparrow 1 \quad \text{as } t \downarrow 0.$$

Let $\varepsilon > 0$ be given; choose t_0 small enough that the integral differs from 1 by less than $\varepsilon/(3\|U_0\|_X)$ for both $\kappa = 1$ and $\kappa = \eta$. For $m(x, t)$ equal to one of $u(x, t)$ or $v(x, t)$,

$$\begin{aligned}
|m(x,t) - m_0(x)| &\leqslant \int_0^\infty |m_0(y) - m_0(x)| H(x, y, \kappa t)\, dy + \frac{\varepsilon}{3} \\
&\leqslant \frac{\varepsilon}{3} + \sup_{|x-y|<2\delta} |m_0(y) - m_0(x)| \int_0^\infty H(x, y, \kappa t)\, dy \\
&\quad + 2\|m_0\|_\infty \int_{|x-y|>\delta} (4\pi\kappa t)^{1/2} \exp\left(\frac{-(x-y)^2}{4\kappa t}\right) dy \\
&\leqslant \frac{\varepsilon}{3} + \sup_{|x-y|<2\delta} |m_0(y) - m_0(x)| \\
&\quad + 2\pi^{-1/2}\|m_0\|_\infty \int_{|x-y|>\delta(4\kappa t)^{-1/2}} \exp(-v^2)\, dv.
\end{aligned}$$

Choose $\delta > 0$ small enough that the supremum above is less than $\varepsilon/3$; fix such a δ. Let t_1 be small enough that the term involving the integrals is also less than $\varepsilon/3$ for $0 < t < t_1$. Then for positive t less than $\min(t_0, t_1)$, we have $|m_0(x, t) - m_0(x)| < \varepsilon$, regardless of x. From our choice of m we see that

$$\|U(\cdot, t) - U_0\|_X = \|T_0(t)U_0 - U_0\|_X < 2\varepsilon$$

for t in the interval $(0, \min(t_0, t_1))$. Hence $T_0(t)$ is strongly continuous from the right at $t = 0$; [**14**, Theorem 9.1.1] shows that $T_0(t)$ is strongly continuous at $t \to t_0 \in [0, \infty)$. Hence $T_0(t)$ is a strongly continuous semigroup on X_0.

Let $\rho > 0$ be given. Define a new semigroup $T_0^{(\rho)}(t)$ by

(3.8) $$T_0^{(\rho)}(t) = \exp(-\rho t)T_0(t) \quad \text{for } t \geqslant 0.$$

Then $T_0^{(\rho)}(t)$ is a strongly continuous semigroup on X_0, with

(3.9) $$\left\|T_0^{(\rho)}(t)\right\| \leqslant \exp(-\rho \operatorname{Re}(t))$$

for $t \in \Delta_\phi$. We wish to show that $T_0^{(\rho)}(t)$ extends to a holomorphic semigroup. First, we note that

(3.10) $$\|AT_0(t)f\|_X = \|D_t T_0(t)f\|_X \leqslant 2|t|^{-1}\|f\|_X,$$

which follows from direct computation and natural estimates. Let $t \in \Delta_\phi$. Using (3.10), we have

(3.11)

$$\begin{aligned}
\left\|D_t T_0^{(\rho)}(t)f\right\|X &= \|D_t \exp(-\rho t)T_0(t)f\|_X \\
&\leqslant \rho \exp(-\rho \operatorname{Re}(t))\|T_0(t)f\|_X + \exp(-\rho \operatorname{Re}(t))\|D_t T_0(t)f\|_X \\
&\leqslant \exp(-\rho \operatorname{Re}(t))\left[\rho + 2|t|^{-1}\right]\|f\|_X \leqslant M|t|^{-1}\exp(-\rho \operatorname{Re}(t)/2).
\end{aligned}$$

Therefore $T_0^{(\rho)}(t)$ is holomorphic in Δ_ϕ by (3.9), (3.11) and strong continuity.

THEOREM 3.1. *$A + B$ generates a holomorphic semigroup $T(t)$ on X_0.*

PROOF. For arbitrarily small $\rho > 0$, $A^{(\rho)} = A - \rho I$ generates the holomorphic semigroup $T_0^{(\rho)}(t)$. Since $B^{(\rho)} = B + \rho I$ is a bounded perturbation of $A^{(\rho)}$, $A^{(\rho)} + B^{(\rho)} = A + B$ generates a holomorphic semigroup on X_0 by [**6**, Theorem 1.3.2]. ∎

Next we shall develop formulas for the representation of the resolvent $R(s, A + B)$. As in §2, we first construct a Green function for a second order system of ordinary differential equations. Then we relate the Green function to the resolvent for the operator equation (3.2).

Let $F: [0, \infty) \to R^2$ in this section. Consider the following system of two ordinary differential equations:

$$D_{xx}U = EU + F(x), \qquad x \in (0, \infty), \tag{3.12}$$

$$U(0) = (0,0)^{\mathrm{T}}, \tag{3.13}$$

where E is the matrix defined in §2. Letting $L \to \infty$ in the defining formula for $G^L(x, y, s)$, we obtain $G(x, y, s) = G_1(s, y, s) + G_3(x, y, s)$, where

$$G_i(x, y, s) = -\left[\lambda_i(\lambda_1^2 - \lambda_3^2)^2\right]^{-1} \sinh(\lambda_i x) \cdot e^{-\lambda_i y} V_i(\lambda_1, \lambda_3) \quad \text{for } x < y, \tag{3.14}$$

where $V_i(\lambda_1, \lambda_3)$ is the matrix below:

$$V_i = \begin{bmatrix} b - (\lambda_j^2 - a)^2 & \omega_i(\lambda_1^2 - \lambda_3^2) \\ \left[b - (\lambda_j^2 - a)^2\right](\lambda_i^2 - a) & \omega_i(\lambda_i^2 - a)(\lambda_1^2 - \lambda_3^2) \end{bmatrix}$$

where $i = 1, 3$ and $j = 3, 1$, respectively, while $\omega_3 = +1$ and $\omega_1 = -1$. When $y < x$, we define $G_i(x, y)$ to be $G_i(y, x)$ in (3.14).

Let $S(x) = \int_0^\infty G(x, y, s) F(y)\, dy$. Substitution into (3.12)–(3.13), followed by computations, shows that $S(x)$ is a solution of (3.12)–(3.13).

Next we relate the kernel to the resolvent. Apply the Laplace transform to HLFN in the time variable t, producing the transform variable s. The new equations are:

$$\begin{cases} s\bar{U}(x, s) - U_0(x) = (DD_{xx} + B)\bar{U}, & (x, s) \in (0, \infty)^2, \\ U(0, s) = (0, 0)^{\mathrm{T}}, & s \in (0, \infty). \end{cases} \tag{3.15}$$

We arrange the system as

$$\begin{cases} D_{xx}\bar{U} = E\bar{U} - D^{-1}U_0(x), & (x, s) \in (0, \infty)^2, \\ \bar{U}(0, s) = (0, 0)^{\mathrm{T}}, & s \in (0, \infty). \end{cases} \tag{3.16}$$

The work above shows that

$$\bar{U}(x, s) = \int_0^\infty G(x, y, s)\left(-D^{-1}U_0(y)\right) dy.$$

THEOREM 3.2. *The resolvent $R(s, A + B)$ is a bounded operator on X_0 for* $\mathrm{Re}(s) > 0$. *Further, the resolvent is represented by the kernel $-G(x, y, s)D^{-1}$.*

PROOF. The solution of the HLFN system or, equivalently, (3.2), is given by $U(x, t) = [T(t)U_0](x)$ or $U(t) = T(t)U_0$, respectively. Applying the Laplace transform, we obtain

$$\bar{U}(s) = L[T(t)U_0](s) = R(s, A + B)U_0 = (sI - (A + B))^{-1}U_0$$

from [4, Corollary II.1.3]. As remarked above, direct calculation shows

$$\bar{U}(x, s) = \int_0^\infty -G(x, y, s)D^{-1}U_0(y)\,dy.$$

Hence $-G(x, y, s)D^{-1}$ represents $R(s, A + B)$. Let g_i be defined by

$$G_i(x, y) = (\lambda_1^2 - \lambda_3^2)^{-2} g_i(x, y) V_i(\lambda_1, \lambda_3).$$

Some simple integration shows $\|g_i\|_{1,\infty}$ is uniformly bounded for $\mathrm{Re}(s) > 0$. Using this fact, for $i = 1, 3$, plus (2.20)–(2.22), we see that $\|G(\cdot, \cdot, s)\|_{1,\infty}$ is uniformly bounded for all positive real s, where G is considered as a point function with values in R^4. Then

$$\|\bar{U}(x, s)\|_X \leqslant \|G\|_{1,\infty}\|U_0\|_X(1 + \eta^{-1}) \tag{3.17}$$

shows that the resolvent is bounded. Our construction of G shows that the resolvent sends X_0 into itself. ■

4. Linear convergence as $L \to \infty$. In §4 we shall discuss the convergence of Green's functions representing the various resolvents. The convergence of Green's functions will yield the convergence of the resolvents in the strong operator topology. A theorem of Kato relates the convergence of resolvents to the convergence of semigroups; the convergence of semigroups relates directly to the convergence of solutions. Lastly, we shall discuss decay rates of norms of differences of solutions.

We have some obstacles to this program. First, the Green's functions do not have common domains. We solve this problem by extending the Green's functions for L finite to be zero outside their natural domains. Second, what are the appropriate means of comparison? We shall employ a variation of the $1, \infty$ norm introduced in Definition 2.4.

DEFINITION 4.1. Let $Q(x, y)$ be defined in $(0, L)^2$. Then for $m < L$,

$$\|Q(\cdot, \cdot)\|_{m,1,\infty} = \sup_{x \in (0, m)} \int_0^L |Q(x, y)|\,dy.$$

The $m, 1, \infty$ "norm" will be a main tool for comparing Green's functions. Since we will usually compare G^L and G, we will have to estimate G on the complement of $(0, L)^2$.

THEOREM 4.2. *The following estimate holds*:

$$\|G^L - G\|_{m,1,\infty} \leqslant c_1 L \cdot \exp(-L\alpha^{1/2}) + c_2 \exp(-[L - m]\alpha^{1/2}) + c_3 \exp\left(-[L - m](\gamma/2\eta)^{1/2}\right) + c_4 L \cdot \exp\left(-L(\gamma/2\eta)^{1/2}\right).$$

PROOF. Let $g_i(x, y)$ be as defined in §3; that is,

$$g_i(x, y) = \begin{cases} \lambda_i^{-1}\sinh(\lambda_i x)\cdot\exp(-\lambda_i y) & \text{for } x < y, \\ \lambda_i^{-1}\sinh(\lambda_i y)\cdot\exp(-\lambda_i x) & \text{for } y < x. \end{cases}$$

Let x be fixed in some compact interval $[0, m]$ where $m < L$. Then

(4.1)
$$\|g_i(x, \cdot)\|_{L^1(L,\infty)} \leqslant |\lambda_i|^{-1}\int_L^\infty \exp(-y\operatorname{Re}(\lambda_i))\,dy\cdot\cosh[x\operatorname{Re}(\lambda_i)]$$
$$\leqslant (|\lambda_i|\operatorname{Re}(\lambda_i))^{-1}\exp(-[L-m]\operatorname{Re}(\lambda_i)) \to 0 \quad \text{as } L\to\infty.$$

Note the second inequality is uniform for x in the interval $[0, m]$. Using the hyperbolic identities, we see that

$$\cosh(\lambda_i[y-L])[\cosh(\lambda_i L)]^{-1} - \exp(-\lambda_i y) = \sinh(\lambda_i y)[1-\tanh(\lambda_i L)]$$
$$= (1-\exp(-2\lambda_i))\exp(-\lambda_i L)[1+\exp(-2\lambda_i L)]^{-1}.$$

Since $\operatorname{Re}(\lambda_i) > 0$ for $\operatorname{Re}(s) > 0$, we have $1+\exp(-2y\operatorname{Re}(\lambda_i)) \leqslant 2$ and

$$|1+\exp(-2L\lambda_i)| \geqslant 1-\exp(-2L\operatorname{Re}(\lambda_i)) \to 1 \quad \text{as } L\to\infty,$$

we get the estimate

$$(*) \qquad |(\cosh(\lambda_i L))^{-1}\cosh[\lambda_i(y-L)] - \exp(-\lambda_i y)| \leqslant 4\exp(-L\operatorname{Re}(\lambda_i))$$

for $y \geqslant 0$. Comparing (2.35), (3.14), and (4.1), we see that $(*)$ gives us the estimates

$$\|G_i^L - G_i\|_{m,1,\infty} = \sup_{x\in(0,m)}\left[\int_0^L |G^L(x,y) - G_i(x,y)|\,dy + \int_L^\infty |G_i(x,y)|\,dy\right]$$
$$\leqslant C\cdot(L\cdot\exp(-L\operatorname{Re}(\lambda_i)) + \exp(-[L-m]\operatorname{Re}(\lambda_i))),$$

for $i = 1, 3$. Hence the estimate

$$(4.2) \quad \|G^L - G\|_{m,1,\infty} \leqslant c_1 L\cdot\exp(-L\operatorname{Re}(\lambda_3)) + c_2\exp([L-m]\operatorname{Re}(\lambda_3))$$
$$+ c_3 L\cdot\exp(-L\operatorname{Re}(\lambda_1)) + c_4\exp(-[L-m]\operatorname{Re}(\lambda_1)).$$

Recalling the lower bounds (2.16)–(2.17), we get an upper bound on the right-hand side of (4.2) that is equal to the right-hand side of the inequality in the statement of the theorem. ■

Recall the operators F_n converge to an operator F on a Banach space X in the strong operator topology provided the quantity $\|F_n x - Fx\|_X \to 0$ as $n\to\infty$ for each fixed $x \in X$.

THEOREM 4.3. *$R(s, A^L + B) \to R(s, A + B)$ as $L \to \infty$ in the strong operator topology on the space $C^0(0, m; R^2)$ for fixed $m < L$ and* $\operatorname{Re}(s) > 0$.

PROOF. Recall our extension of G^L given at the beginning of the section. Let $f \in C^0(0, \infty; R^2)$. Then

$$(4.3) \quad \|R(s, A^L + B)f - R(s, A + B)f\|_{C^0(0,m;R^2)}$$
$$\leqslant C\cdot\|G^L - G\|_{m,1,\infty}\cdot\|f\|_X \to 0 \quad \text{as } L\to\infty, \text{ for } \operatorname{Re}(s) > 0,$$

by Theorem 4.2. Hence $R(s, A^L + B) \to R(s, A + B)$ strongly in $C^0(0, m; R^2)$ for all s with $\mathrm{Re}(s) > 0$. ∎

THEOREM 4.4. *The semigroups $T^L(t) \to T(t)$ strongly as $L \to \infty$ in the space $C^0(0, m; R^2)$ and uniformly in $t \in [0, \tau]$ for fixed $m, \tau < \infty$.*

PROOF. By [**8**, Theorem 9.2.16] and Theorem 4.3, the results follows. ∎

Speaking more concretely,

$$T^L(t)f \to T(t)f \text{ in the space } C^0(0, m; R^2) \quad \text{as } L \to \infty, \tag{4.4}$$

uniformly in $[0, \tau]$, where m is any fixed finite positive real number, as is τ; f is in $C^0(0, \infty; R^2)$.

Next we get estimates on the decay rates of $\|U^L - U\|$, where U^L is a solution of HLFN for L finite. We shall use $(*)$ above and the estimates of [**2**, Appendix]. Inequalities (2.16)–(2.22) and Equalities (2.23)–(2.26) show us that the coefficients of the matrices

$$M_i = \left[\lambda_i(\lambda_1^2 - \lambda_3^2)^2\right]^{-1} V_i(\lambda_1, \lambda_3)$$

are bounded in each entry away from infinity for $\eta > 0$ fixed. Let M be the maximum of the matrix norms of M_1 and M_3. Then for f in $C^0(0, \infty; R^2)$,

$$\begin{aligned} U^L(x,t) - U(x,t) &= (T^L(t)f)(x) - (T(t)f)(x) \\ &= L^{-1}\left(R(s, A^L + B)f - R(s, A + B)f\right) \\ &= (2\pi i)^{-1}\int_\Gamma \left[R(s, A^L + B) - R(s, A + B)\right] f(x) e^{st}\, ds. \end{aligned}$$

From this we see that the following inequality holds:

$$\begin{aligned} \|U^L - U\| &\leqslant (2\pi)^{-1}\|f\|\int_\Gamma |R(s, A^L + B) - R(s, A + B)|\, e^{t\,\mathrm{Re}(s)} |ds| \\ &\leqslant (2\pi)^{-1}\|f\| e^{tK}\int_{K-i\infty}^{K+i\infty} |R(s, A^L + B) - R(s, A + B)| \cdot |ds| \\ &\leqslant C \cdot \int_{-\infty}^{+\infty} |R(s, A^L + B) - R(s, A + B)|\, dw, \end{aligned}$$

where $w = \mathrm{Im}(s)$, $K = \mathrm{Re}(s)$ and $C = (2\pi)^{-1}\|f\|e^{tK}$. The norm is taken in the space $C^0(0, m; R^2)$. From $(*)$ and (4.1) we get the estimate

(4.5)

$$\begin{aligned} \|R(s, A^L + B) - R(s, A + B)\| &\leqslant M'\|G^L - G\|_{m,1,\infty} \\ &\leqslant M'M\big[L \cdot \exp(-L\,\mathrm{Re}(\lambda_1)) + \exp(-[L - m]\,\mathrm{Re}(\lambda_1)) \\ &\quad + L \cdot \exp(-L\,\mathrm{Re}(\lambda_3)) + \exp(-[L - m]\,\mathrm{Re}(\lambda_3))\big] \\ &\leqslant 2M'ML\big[\exp(-[L - m]\,\mathrm{Re}(\lambda_1)) + \exp(-[L - m]\,\mathrm{Re}(\lambda_3))\big] \end{aligned}$$

where M' is the matrix norm of D^{-1}. Hence

(4.6)

$$\|U^L - U\| \leqslant 2M'ML\left(\int_{-R}^{R} + \int_{-\infty}^{-R} + \int_{R}^{\infty}\right)[\exp(-[L-m]\operatorname{Re}(\lambda_1)) + \exp(-[L-m]\operatorname{Re}(\lambda_3))]\,dw$$
$$\leqslant 4RM'ML\left[\exp\left(-\tfrac{1}{2}[L-m](\gamma/\eta)^{1/2}\right) + \exp\left(-[L-m]\tfrac{1}{2}\gamma^{1/2}\right)\right]$$
$$+4M'ML\left[\int_R^\infty \exp\left(\tfrac{1}{2}[(1+\eta)w/\eta]^{1/2}(-L+m)\right)dw + \int_R^\infty \exp\left((-L+m)\left(\tfrac{1}{2}w^{1/2}\right)\right)dw\right]$$

where the second inequality follows from (2.16)–(2.19). Here R is such that (2.19) holds.

Let r be a fixed negative real number. Then

$$\int_R^\infty \exp(rw^{1/2})\,dw = \int_{R^{1/2}}^\infty 2u\cdot\exp(ru)\,du = -2r^{-1}(R^{1/2}-1)\exp(rR^{1/2}). \tag{4.7}$$

THEOREM 4.5. *Let $\xi \in (0,1)$. Then there exists $C = C(\xi)$ and $\theta = \theta(\xi)$, both positive constants for fixed ξ, such that*

$$\|U^L - U\|_{C^0(0,\,\xi L;\,R^2)} \leqslant C\cdot\exp(-\theta L) \tag{4.8}$$

holds for all large L.

PROOF. We first consider norms in the space $C^0(0, m; R^2)$ for fixed m. We apply (4.7) twice to obtain

$$I_1 = \int_R^\infty \exp\left(\tfrac{1}{2}(-L+m)\left[\frac{1+\eta}{\eta}\right]^{1/2} w^{1/2}\right)dw$$
$$= 4(L-m)^{-1}\left[\frac{\eta}{1+\eta}\right]^{1/2}(R^{1/2}-1)\exp\left(\tfrac{1}{2}R^{1/2}(-L+m)\left[\frac{1+\eta}{\eta}\right]^{1/2}\right),$$
$$I_2 = \int_R^\infty \exp\left(\tfrac{1}{2}(-L+m)w^{1/2}\right)dw = 4(L-m)^{-1}(R^{1/2}-1)\exp\left(\tfrac{1}{2}R^{1/2}(-L+m)\right).$$

Then we may continue (4.6) as

(4.9)

$$\|U^L - U\| \leqslant 4M'ML[I_1 + I_2] + 4RM'ML\left[\exp\left(\tfrac{1}{2}[m-L](\gamma/\eta)^{1/2}\right) + \exp\left(\tfrac{1}{2}(m-L)\gamma^{1/2}\right)\right].$$

Note that all four terms on the right-hand side of (4.9) decay exponentially as $L \to \infty$.

To finish the proof, we note that $m < L$ is arbitrary. If $\xi \in (0,1)$, then $m = \xi L < L$. Replace $m - L$ in (4.9) by $(\xi - 1)L < 0$. Let $\tilde{\theta} = (\xi - 1)/2$. Since $L\cdot\exp(z\tilde{\theta}L)$ is bounded as $L \to \infty$, for z a positive constant, we may bound $L\cdot\exp(z\tilde{\theta}L)$ by a fixed number C_z. Applying this reasoning in (4.9) for the various constants z, we see that each term decays exponentially as L becomes infinite. Let θ equal $-y\tilde{\theta}$, where y is the smallest constant z dealt with in the exponents, and C is the sum of the constants C_z. These choices of C and θ provide the required bound. ■

REMARK. As $\xi \to 1^-$, $(\xi - 1)L \to 0^-$ for any given L. Hence the convergence is poorer near $x = L$.

5. Nonlinear theory; results for $0 < L \leqslant \infty$. In §§5 and 6 we shall use the abstract operator-theoretic approach started in earlier sections. The method of contracting rectangles shall be employed as well. In §5 we will view the FN from the (1.6)–(1.7) approach as well as the (1.3)–(1.4) approach. Using previous work, we may now formulate (1.6)–(1.7) in operator language as

$$\begin{cases} D_t W^L = (A^L + B)W^L + F(t, W^L), & t \in (0, \infty), \\ W^L(0) = W_0^L, \quad \text{for } L < \infty, \end{cases} \tag{5.1}$$

$$\begin{cases} D_t W = (A + B)W + F(t, W), & t \in (0, \infty), \\ W(0) = W_0 \quad \text{for } L = \infty. \end{cases} \tag{5.2}$$

We shall follow Henry [**6**] in showing that (5.1) and (5.2) are equivalent to certain integral equations. Note that F means the nonlinearity mentioned in §1.

DEFINITION 5.1. Let X be a Banach space. Suppose U is open in $R \times X$. The function $f\colon U \to X$ is locally Hölder continuous in t and locally Lipschitz in x in U provided: if $(t_1, x_1) \in U$, there exists $L > 0$, $0 < \theta < 1$ and an open set V such that $(t, x) \in V$ and $(s, y) \in V$ implies

$$\|f(t, x) - f(s, y)\| \leqslant L\left(|t - s|^\theta + \|x - y\|\right).$$

LEMMA 5.2. *$F(t, W)$ is locally Hölder continuous in t and locally Lipschitz in W when X is taken to be X_0.*

PROOF. Fix $(t_0, \Gamma) \in (0, \infty) \times X_0$. Let V be a bounded neighborhood of this point. Let (t, W_1) and (s, W_3) be in B. Then

$$\begin{aligned} \|F(t, W_1) - F(s, W_3)\|_X \leqslant &|(BG - G')(t) - (BG - G')(s)|_\infty \\ &+ \|f(w_1 + g(t)) - f(w_3 + g(s))\|. \end{aligned} \tag{5.3}$$

Note

$$\begin{aligned} f(u) - f(v) &= u^2(1 + \alpha - u) - v^2(1 + \alpha - v) \\ &= \left[(1 + \alpha)(u + v) - (u^2 + uv + v^2)\right](u - v) = q(u, v)[u - v], \end{aligned}$$

where q is the indicated polynomial in u and v, so q is bounded if u and v are; f is the polynomial defined in §1. Then

$$\begin{aligned} &f(w_1 + g(t)) - f(w_3 + g(s)) \\ &\qquad = q(w_1 + g(t), w_3 + g(s))\left[(w_1 - w_3) + (g(t) - g(s))\right]. \end{aligned}$$

Since $(g, h)^{\mathrm{T}} \in Y$, the functions g and h are globally bounded. Hence (t, W_1) and (s, W_3) in V imply that the sup norm of the quantity $q(w_1 + g(t), w_3 + g(s))$ is bounded by a constant C depending on (t_0, Γ). Since $G \in Y$, G and G' are locally

Lipschitz. Therefore the first term on the right-hand side of (5.3) has some Lipschitz constant $M = M(t_0, \Gamma)$. Hence we may continue (5.3) as

$$\|F(t, W_1) - F(s, W_3)\|_X \leqslant (CM + M)|t - s| + C\|W_1 - W_3\|_X. \qquad \blacksquare$$

DEFINITION 5.3. Let U be a neighborhood in $(0, \infty) \times X_0$ where $F(t, W)$ is defined. A solution of (5.12) or of (5.2) on $(0, \tau)$ is a continuous function $W: [0, \tau) \to X_0$ such that $W(0) = W_0$, $(t, W(t)) \in U$ for $t \in (0, \tau)$, $W(t) \in D(A^L + B)$ or $D(A + B)$ for $t \in (0, \tau)$, $D_t W$ exists, the differential equation is satisfied and $\int_0^\rho \|F(t, W(t))\|\, dt < \infty$ for some $\rho > 0$.

REMARK. Below we shall prove that solutions of FN are globally bounded. Hence $F(t, W)$ is bounded; the integral condition follows. The other conditions in the definition will also be taken care of subsequently. For now we assume these conditions, so as to get to the integral equation setting.

THEOREM 5.4. (5.1) *is equivalent to the integral equation*

$$W^L(t) = T^L(t) W_0^L + \int_0^t T^L(t - s) F(s, W^L(s))\, ds. \tag{5.4}$$

The differential equation (5.2) *is equivalent to*

$$W(t) = T(t) W_0 + \int_0^t T(t - s) F(s, W(s))\, ds. \tag{5.5}$$

PROOF. Since the operators $A^L + B$ and $A + B$ generate homomorphic semigroups $T^L(t)$ and $T(t)$, respectively, the result follows from [**6**, Lemma 3.3.2] with $\alpha = 0$. ■

We now address existence and uniqueness.

THEOREM 5.5. (5.4) *and* (5.5) *have unique local solutions for each starting time* $t_0 \geqslant 0$.

PROOF. §§2 and 3 showed that $A^L + B$ and $A + B$ generate holomorphic semigroups on X. Lemma 5.2 showed that $F(t, W)$ is locally Hölder continuous in t and locally Lipschitz in W. Hence the hypotheses of [**6**, Theorem 3.3.3] are satisfied with $\alpha = 0$; the proof is complete. ■

Considering FN as (1.3)–(1.4) and using contracting rectangles, we arrive at

THEOREM 5.6. *The system* (1.3)–(1.4) *has a unique globally defined and bounded solution for* $0 < L \leqslant \infty$.

PROOF. See [**2**, Theorem 7.2.8]. ■

We note that the methods involved in the above proof are those of Rauch and Smoller with modifications for $L < \infty$ and differing boundary conditions.

Next, we consider smoothness.

THEOREM 5.7. *The solution of FN is a* C^∞ *function of time.*

PROOF. Note that [**6**] assumes that its sectorial operator A has positive spectrum; we assume $-A$ has positive spectrum. We note that $A + B$ and $A^L + B$ are sectorial, and

$$F(t, W) = BG(t) - G'(t) + \left(f(w_1 + g(t)), 0\right)^{\mathrm{T}}.$$

Since $G \in Y$ and $f(u) = u^2(1 + \alpha - u)$ is cubic, the hypotheses of [**6**, Theorem 3.4.2] are met. Hence the solution is continuously differentiably dependent on its initial function. Further, the map $(t, W) \to F(t, W)$ is C^∞, so [**6**, Corollary 3.4.3] yields the result. ■

REMARK. If G is in C^k rather than C^∞, then $F(t, W)$ is C^{k-1} rather than C^∞. The reasoning above then indicates that the solutions are C^{k-1} in time for $t > 0$.

REMARK. Let $t_0 = \sup \operatorname{spt}(G)$. Then for $t > t_0$, the equations are independent of, and hence analytic in, the time variable. The corollary then tells us that the solutions are analytic in time for $t > t_0$.

REMARK. From the forms of the resolvents in §§2 and 3, we see that the resolvents are C^∞ in the space variable x. Applying the inverse Laplace transform to the resolvents applied to the initial data, we obtain the desired solutions to the original problems. Standard theorems from Laplace transform theory show that the solutions to our object equations are C^∞ in the spatial variable x.

We conclude the section with some decay results.

THEOREM 5.8. *Let $L = \infty$. If the initial and boundary data (W_0, G) are sufficiently small, then $\|W(t)\| \to 0$ as $t \to \infty$.*

PROOF. Let $t_0 = \sup \operatorname{spt}(G)$. We wish to estimate $\|W(t_0)\|$. Note that

$$F(W + G) = \left(g^2[1 + \alpha - g], 0\right)^{\mathrm{T}} + \left(q(w_1, g)\cdot w_1, 0\right)^{\mathrm{T}}.$$

Define $H(t)$ to be the first vector in the sum above plus $BG(t) - G'(t)$. Let $\tilde{F}(t, W)$ be the second vector in the sum. Let $m = |q(w_1, g)|_\infty$. We estimate using the integral equation

$$W(t) = T(t)W_0 + \int_0^t T(t-s)\left[H(s) + \tilde{F}(s, W(s))\right] ds.$$

The result $\|T(t)\| \leqslant C \cdot \exp(-\delta t)$, for real t, is proved in the same style as (2.15). Applying this result to the integral equation, we have the following:

$$\|W(t)\|\exp(\delta t) \leqslant C\|W_0\| + C\cdot \exp(\delta t)\|H\| + mC \cdot \int_0^t \exp(\delta s)\|W(s)\|\, ds$$

for t in $[0, t_0]$. By Gronwall's inequality,

$$\|W(t)\|\exp(\delta t) \leqslant \left[\|W_0\| + \exp(\delta t)\|H\|\right]\cdot C \cdot \exp(mCt).$$

In particular, this holds for $t = t_0$; hence

$$\|W(t_0)\| \leqslant C\cdot\left[\|W_0\| + \exp(\delta t_0)\|H\|\right]\exp([mC - \delta]t_0). \tag{5.6}$$

Let

$$O_1(\rho) = \left\{W_0 : \|W_0\| < \rho\left(2\left[\max\{C, 1\}^2\right]\right)^{-1}\exp([\delta - mC]t_0)\right\},$$

$$O_3(\rho) = \left\{H : \|H\| < \rho\left(2\left[\max\{C, 1\}^2\right]\right)^{-1}\exp(-mCt_0)\right\},$$

where O_1, O_3 are subsets of C^0. For $G \in O_2$ contained in C^1 with O_2 sufficiently small, we have $H \in O_3$ which is contained in C^0. Hence $(W_0, G) \in O_1 \times O_3$ implies

$$\|W(t_0)\| \leqslant \rho/[2\max(C, 1)] \tag{5.7}$$

which follows from (5.6) and the definitions of O_1, O_2 and O_3.

Suppose $t > t_0$. For these t's, $G(t) = (0,0)^T$, so the differential equation [DE] becomes

$$\begin{cases} D_t W = (A + B)W + F(W), & t > t_0, \\ W(t_0) = \text{the solution arising from the earlier DE at } t = t_0. \end{cases}$$

The associated integral equation for this differential equation is

$$W(t) = T(t - t_0)W(t_0) + \int_{t_0}^{t} T(t - t_0 - s)F(W(s))\, ds.$$

Note that

$$\lim_{\|W\| \to 0} \frac{\|F(W)\|}{\|W\|} = 0;$$

hence for $\varepsilon > 0$, there exists $\rho > 0$ small enough that $\|W\| < \rho$ implies $\|F(W)\| \leqslant \varepsilon\|W\|$. In particular, we choose $\varepsilon = \delta_0/C$ with $0 < \delta_0 < \delta$. Next we choose $\rho > 0$ small enough that $\|W\| < \rho$ implies $\|F(W)\| < \delta_0\|W\|/C$.

Since $W(t)$ is continuous, $\|W(t_0)\| < \rho/[2\max(C, 1)] < \rho$ implies there exists positive h such that $\|W(t)\| < \rho$ for t in $[t_0, t_0 + h]$. For these t,

$$\|W(t)\|\exp(\delta(t - t_0)) \leqslant C\|W(t_0)\| + \int_{t_0}^{t} \delta_0 \exp(\delta(s - t_0))\|W(s)\|\, ds.$$

Gronwall's inequality then yields

$$\|W(t)\| \leqslant C \cdot \exp[(\delta_0 - \delta)(t - t_0)]\|W(t_0)\|.$$

Choosing $(W_0, G) \in O_1 \times O_3$, this inequality further simplifies to

$$\|W(t)\| < (\rho/2)\cdot \exp[(\delta_0 - \delta)(t - t_0)] < \rho/2 \tag{5.8}$$

using (5.7); note this holds for t in $[t_0, t_0 + h]$. Continuity of W and (5.8) then imply that $\|W(t)\| < \rho$ in $[t_0, t_0 + h + h_1]$ for some $h_1 > 0$. This allows us to extend (5.8) to the larger interval, which in turn allows us to extend $\|W(t)\| < \rho$ to an even larger interval $[t_0, t_1]$, where $t_1 > t_0 + h + h_1$. Hence we may repeat the process indefinitely, so (5.8) holds for all $t \geqslant t_0$. Then $\|W(t)\| \to 0$ as $t \to \infty$ whenever (W_0, G) is in the given open set containing the zero solution. ■

THEOREM 5.9. *Let* $0 < L < \infty$. *If the initial and boundary data are sufficiently small, then* $\|W^L(t)\| \to 0$ *at* $t \to \infty$.

PROOF. Use the same argument as in the proof of Theorem 5.8, with W^L replacing W and $T^L(t)$ replacing $T(t)$. ■

6. Nonlinear convergence as $L \to \infty$. As in §4, we extend definitions for the sake of making comparisons. In particular, if W^L is a solution of (5.1), we extend it by setting it equal to zero for $x > L$. In this section we will use $z(L)$ to refer to a quantity that exhibits exponential decay as $L \to \infty$. That is, $z(L)$ decays in the style of the estimates (4.8). Let $Z(t) = W^L(t) - W(t)$.

THEOREM 6.1. *There exist constants* c_1, c_2, c_3 *and* $\delta > 0$ *such that*

$$\|Z(t)\| \leqslant [c_1\|Z_0\| + z(L)c_2 \exp(\delta t)]\exp(c_3 T_1) \tag{6.1}$$

for $t \in [0, T_1]$; *the norm is that of* $C^0(0, m; R^2)$ *or of* $C^0(0, \xi L; R^2)$ *for fixed finite* m *and for* ξ *in* $(0, 1)$.

PROOF. We consider the integral equations

$$W(t) = T(t)W_0 + \int_0^t T(t-s)\left[H(s) + \tilde{F}(s, W(s))\right] ds, \tag{6.2}$$

$$W^L(t) = T^L(t)W_0^L + \int_0^t T^L(t-s)\left[H(s) + \tilde{F}(s, W^L(s))\right] ds. \tag{6.3}$$

Subtraction (see above remarks) yields

$$\begin{aligned} Z(t) = (T^L - T)(t)W_0^L &+ \int_0^t (T^L - T)(t-s)H(s)\, ds \\ &+ \int_0^t (T^L - T)(t-s)\tilde{F}(s, W^L(s))\, ds \\ &+ T(t)Z_0 + \int_0^t T(t-s)\left[\tilde{F}(s, W^L(s)) - \tilde{F}(s, W(s))\right] ds. \end{aligned}$$

Let $T_1 < \infty$ be fixed but arbitrary. We estimate the last equation:

$$\begin{aligned} \|Z(t)\| \leqslant z(L) + t \cdot z(L) + M \cdot t \cdot z(L) + C \cdot \exp(-\delta t)\|Z_0\| \\ + \int_0^t M''C \cdot \exp(-\delta[t-s])\|Z(s)\|\, ds \quad \text{as } L \to \infty, \end{aligned}$$

where M'' is the Lipschitz constant for $\tilde{F}(s, W)$ and M is $\max_s \|\tilde{F}\|$, where $\tilde{F}$ is evaluated at $(s, W^L(s))$. The above inequality becomes

$$\begin{aligned} \|Z(t)\|\exp(\delta t) \leqslant [1 + t + \hat{M}t]\exp(\delta t) \cdot z(L) + \|Z_0\| \\ + \int_0^t CM''\exp(\delta s)\|Z(s)\|\, ds \quad \text{as } L \to \infty. \end{aligned}$$

Applying Gronwall's inequality to this last inequality, we obtain

$$\begin{aligned} \|Z(t)\| &\leqslant \left[C\|Z_0\| + z(L) \cdot (1 + t + t\hat{M})\exp(\delta t)\right]\exp[(M''C - \delta)t] \\ &\leqslant \left[C\|Z_0\| + z(L) \cdot \left(1 + T_1 + T_1\hat{M}\right)\exp(\delta T_1)\right]\exp[(M''C - \delta)T_1] \end{aligned}$$

for $t \in [0, T_1)$. Again referring to §4, the $z(L)$ estimates are uniform in $[0, T_1]$ in the spaces $C^0(0, m; R^2)$ or $C^0(0, \xi L; R^2)$. ∎

COROLLARY 6.2. *If* $\|Z_0\| = 0$, *then* $\|Z(t)\| \to 0$ *as* $L \to \infty$, *uniformly in* $t \in [0, T_1]$.

PROOF. If $\|Z_0\| = 0$, then $z(L)$ is a multiplier of the entire right-hand side of (6.4). ∎

REMARK. $W^L(t) \to W(t)$ implies $U^L(t) \to U(t)$ in the spaces $C^0(0, m; R^2)$ and $C^0(0, \xi L; R^2)$, uniformly in $[0, T]$. Hence we have convergence of our original functions.

REMARK. $W^L \to W$ in C^0 implies convergence in the spaces $L^p(0, m; R^2)$ uniformly in $[0, T_1]$ for each $1 < p < \infty$ for any fixed finite T_1 as $L \to \infty$.

REMARK. For large L_1 and L_2, both W^{L_1} and W^{L_2} will be uniformly close to W in C^0 and L^p. Hence $\|W^{L_1} - W^{L_2}\|$ will be small as $L_1, L_2 \to \infty$; that is, $\{W^L(t) : L > 0\}$ is Cauchy in C^0 and L^p for t in $[0, T_1]$.

7. Conclusions. The results of §§1–4 were mainly tools needed to produce the theorems of §§5 and 6. §§5 and 6 convey information about FN instead of HLFN, and hence the results may be given physiological interpretation. Theorem 5.6 agrees with the empirical observations that nerve axon impulses do not "blow up". Theorem 5.7 tells us that stimulation (at the left end of the nerve) that is smooth in time results in smooth impulses. The first remark after Theorem 5.7 tells us that less smooth stimulation gives rise to less smooth (in time) impulses. The following remark means that after a "jagged" stimulation ceases, the impulses become analytic in time, as one would expect, since diffusion processes are present. The smoothness of solutions in the spatial variable (last remark of §5) is also to be expected.

Theorem 5.8 is a threshold result. That is, sufficiently small stimulus results in exponential decay (in time) of any action along the axon.

§6 contains the results of main interest in the paper. That is, results that have not been treated to any great depth in the literature. Theorem 6.1 gives a general estimate on the difference of the solutions for L infinite and a solution for L large but finite. Corollary 6.2 is the main result. Given the same initial data and stimulus on the left-hand side, the impulses on a nerve axon of length L are uniformly close to the impulses on the semi-infinite nerve axon for t in $[0, T_1]$, for any fixed finite time T_1. Here closeness is measure from the left end to the length ξL for $\xi \in (0, 1)$. Bounds on the closeness are given; the bounds exhibit exponential decay in L. The dependence on ξ is described in §4; the closer ξ is to 1, the worse the convergence. That is, the impulses in the finite axon approximate well those of the semi-infinite nerve except near the right endpoint. This is to be expected, since the right endpoint of the finite nerve is a complicated affair, while in the semi-infinite model, no right endpoint is present.

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