TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 281, Number 1, January 1984

AN ALGEBRAIC DECOMPOSITION OF THE RECURSIVELY ENUMERABLE DEGREES AND THE COINCIDENCE OF SEVERAL DEGREE CLASSES WITH THE PROMPTLY SIMPLE DEGREES[1]

BY

KLAUS AMBOS-SPIES, CARL G. JOCKUSCH, JR., RICHARD A. SHORE AND ROBERT I. SOARE

ABSTRACT. We specify a definable decomposition of the upper semilattice of recursively enumerable (r.e.) degrees $\mathbf{R}$ as the disjoint union of an ideal $\mathbf{M}$ and a strong filter $\mathbf{NC}$. The ideal $\mathbf{M}$ consists of $\mathbf{0}$ together with all degrees which are parts of r.e. minimal pairs, and thus the degrees in $\mathbf{NC}$ are called noncappable degrees. Furthermore, $\mathbf{NC}$ coincides with five other apparently unrelated subclasses of $\mathbf{R}$: $\mathbf{ENC}$, the effectively noncappable degrees; $\mathbf{PS}$, the degrees of promptly simple sets; $\mathbf{LC}$, the r.e. degrees cuppable to $\mathbf{0}'$ by a low r.e. degree; $\mathbf{SP\overline{H}}$, the degrees of non-*hh*-simple r.e. sets with the splitting property; and $\mathbf{G}$, the degrees in the orbit of an r.e. generic set under automorphisms of the lattice of r.e. sets.

0. Introduction. Let $(\mathbf{R}, \leqslant, \cup, \cap)$ denote the upper semilattice of recursively enumerable (r.e.) degrees with partial ordering induced by Turing reducibility and $\cup$ and $\cap$ the join and meet operations when the latter is defined. (Unless otherwise specified all sets and degrees will be assumed to be r.e.)

Sacks [**1966**, p. 170] asked whether there exists a *minimal pair* namely incomparable r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$. Shoenfield [**1965**] formulated a general conjecture about $\mathbf{R}$ which implies among other things that minimal pairs do not exist. Lachlan [**1966**] and independently Yates [**1966**] refuted Shoenfield's conjecture by constructing a minimal pair. Both minimal pairs and the method for constructing them have played an important role in the study of r.e. degrees. An r.e. degree $\mathbf{a}$ is *cappable* (caps) if there is an r.e. degree $\mathbf{b} > \mathbf{0}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ (i.e. if $\mathbf{a}$ is $\mathbf{0}$ or is part of a minimal pair), and $\mathbf{a}$ is *noncappable* otherwise. Furthermore, $\mathbf{a}$ is *effectively noncappable* if the witness to its noncapping can be found effectively (as defined more precisely in §1). Yates [**1966**] also showed that there exist r.e. degrees $\mathbf{a} < \mathbf{0}'$ which are noncappable, indeed effectively noncappable. Let $\mathbf{M}$, $\mathbf{NC}$ and $\mathbf{ENC}$ denote the classes of cappable, noncappable and effectively noncappable r.e. degrees, respectively.

We prove that $\mathbf{M}$ is an ideal in $\mathbf{R}$ (closed downward and under join) while its complement $\mathbf{NC}$ is a strong filter (closed upwards and for all $\mathbf{a}, \mathbf{b} \in \mathbf{NC}$ there exists

Received by the editors June 14, 1982.

1980 *Mathematics Subject Classification*. Primary 03D25; Secondary 03D30.

[1]The last three authors were partially supported by NSF Grants MCS-8003251, MCS-8003016, MCS-7905782, and MCS-8102443 respectively.

$\mathbf{c} \in \mathbf{NC}$, $\mathbf{c} \leqslant \mathbf{a}$ and $\mathbf{c} \leqslant \mathbf{b}$). This gives the first algebraic decomposition of $\mathbf{R}$ into the disjoint union of a (definable) ideal and a (definable) filter, and like the density theorem of Sacks [**1964**] it emphasizes the regularity of the structure of $\mathbf{R}$ rather than its pathology. In the process we also show that $\mathbf{NC} = \mathbf{ENC}$ which we found surprising because in recursion theory a notion rarely coincides with its effective counterpart. (For example, an r.e. set A is nonrecursive iff A is noncomplemented in $\mathcal{E}$, the lattice of r.e. sets under inclusion, but A is effectively noncomplemented iff A is creative and hence of degree $\mathbf{0}'$, the maximum r.e. degree.)

In a dual fashion we say that an r.e. degree $\mathbf{a}$ is *cuppable* (cups) if there is an r.e. degree $\mathbf{b} < \mathbf{0}'$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$, and $\mathbf{a}$ is *low cuppable* if there is such a $\mathbf{b}$ which is low (i.e. $\mathbf{b}' = \mathbf{0}'$). Let $\mathbf{LC}$ denote the class of low cuppable degrees. The cuppable and noncuppable degrees have also been extensively studied. For example Cooper, Yates and Harrington have constructed various r.e. degrees which are not cuppable (see D. Miller [**1980**]). Furthermore, Harrington has shown (see Fejer and Soare [**1980**]) that every r.e. degree $\mathbf{a}$ either cups or caps and some degrees do both. We sharpen Harrington's results by proving that $\mathbf{NC} = \mathbf{LC}$, namely that every r.e. degree $\mathbf{a}$ either caps or low cups, but no degree can do both, thereby eliminating the overlap in Harrington's second result.

These equivalences were discovered by studying the degrees of promptly simple sets, a computational complexity analogue of Post's simple set introduced by Maass [**1982**] for studying orbits of r.e. sets under automorphisms of $\mathcal{E}$, the lattice of r.e. sets. The notion of a promptly simple set is a dynamic one which takes into account how fast elements appear in A relative to their appearance in other r.e. sets under some standard simultaneous enumeration of all r.e. sets. Maass discovered that this is the correct notion needed to satisfy the Extension Theorem of Soare [**1974**] for generating automorphisms of $\mathcal{E}$, and using this and Soare [**1982a**] Maass showed that any two promptly simple sets A, B which are low (indeed such that $\bar{A}$ and $\bar{B}$ are merely semilow, namely $\{x\colon W_x \cap \bar{A} \neq \varnothing\} \leqslant_T \varnothing'$) are automorphic. Furthermore, Maass introduced the notion of an r.e. generic set, based on effectivizing the construction of a Cohen generic set, and he showed that any r.e. generic set is both promptly simple and low. Hence, the "typical" or generic set constructed by a finite injury priority construction tends to be both promptly simple and low. Let $\mathbf{PS}$ denote the class of degrees of promptly simple sets and $\mathbf{G}$ the degrees of sets automorphic to some promptly simple low set (i.e. automorphic to Maass's r.e. generic set). We prove that $\mathbf{NC} = \mathbf{PS} = \mathbf{G}$. The latter equality uses results by Maass and the authors.

Maass, Shore and Stob [**1981**] showed that prompt simplicity, while not itself $\mathcal{E}$-definable, implies a certain splitting property which is $\mathcal{E}$-definable, and they introduced the class $\mathbf{SP\overline{H}}$ of degrees of r.e. sets with this splitting property but which are not *hh*-simple. They showed that $\mathbf{SP\overline{H}}$ nontrivially splits all the classes $\mathbf{H}_n$, $\mathbf{L}_n$, $n \in \omega$, in the usual high-low hierarchy within $\mathbf{R}$ defined in terms of the jump operator. We prove that $\mathbf{NC} = \mathbf{PS} = \mathbf{SP\overline{H}}$. As a corollary we have an $\mathcal{E}$-definable class of r.e. sets whose degrees $\mathbf{SP\overline{H}}$ are also definable in $\mathbf{R}$ (as $\mathbf{NC}$). Also we have an $\mathbf{R}$-definable class ($\mathbf{NC}$) which nontrivially splits all the classes $\mathbf{H}_n$, $\mathbf{L}_n$, $n \in \omega$, in the high-low hierarchy.

In §1 we prove the equivalences **ENC** = **NC** = **PS** = **LC** = **SP$\overline{\mathbf{H}}$** = **G**. Most of these are easy to prove. The most difficult and the main theorem of §1 is the proof that **NC** $\subseteq$ **PS**. This uses a gap-cogap argument like that introduced by Lachlan and used to prove the Harrington cup or cap theorem. In §2 we see that **NC** is a strong filter by observing that **ENC** trivially is a strong filter. We prove that **M** is an ideal by using a variation of Lachlan's nondiamond theorem [**1966**, Theorem 5] to show that $\overline{\mathbf{PS}}$ is an ideal. For other properties of promptly simple sets and degrees see Maass [**1982**] or Maass, Shore and Stob [**1981**], and for their relation to other computational complexity properties see Soare [**1982b**].

We assume familiarity with the basic definitions and results in recursive function theory as found in Rogers [**1967**] and Soare [**1978**], and we use mostly the notation of the latter with the following additions. Fix a $1:1$ recursive function F from ω onto $\{\langle x, e\rangle : x \in W_e\}$. Let $W_{e,s} = \{x: (\exists t \leqslant s)[F(t) = \langle x, e\rangle]\}$. Hence, at each stage s, $F(s) = \langle x, e\rangle$ causes exactly *one* element x to be enumerated in *one* r.e. set W_e. Let $x \in W_{e,\text{at } s}$ denote that $x \in W_{e,s} - W_{e,s-1}$ i.e. $F(s) = \langle x, e\rangle$. More generally if $\{V_{e,s}: e, s \in \omega\}$ is any (recursive) enumeration of a sequence $\{V_e: e \in \omega\}$ of r.e. sets, $V_e = \bigcup_s V_{e,s}$, then $x \in V_{e,\text{at } s}$ denotes $x \in V_{e,s} - V_{e,s-1}$. We identify a set $A \subseteq \omega$ with its characteristic function and let $A \restriction n$ denote $A \cap \{0, 1, \ldots, n-1\}$. If $\{e\}_s^A(x) = y$ we define the use function $u(A; e, x, s)$ to be $1 +$ the greatest element z used in this computation, and $u(A; e, x, s) = 0$ otherwise. We assume that the definitions are arranged so that if $\{e\}_s^A(x) = y$ then $e, x, y, u < s$ where $u = u(A; e, x, s)$. If we build an r.e. set V or a partial recursive function ψ by a recursive construction, we let V_s (respectively ψ_s) denote those elements enumerated in V (graph ψ) by the end of stage s of the construction. We let $\{e\}^A(x)\downarrow = y$ denote that $\{e\}^A(x)$ converges and yields output y. A strong r.e. (s.r.e.) array $\{F_n: n \in \omega\}$ of finite sets is one for which there exists a recursive function f such that $F_n = D_{f(n)}$ where $y = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$ is the canonical index of the set $D_y = \{x_1 < x_2 < \cdots < x_n\}$.

1. Equivalences of certain properties of r.e. degrees. Post defined a coinfinite r.e. set A to be *simple* if $W_e \cap A \neq \varnothing$ for every infinite r.e. set W_e. For A to be promptly simple, some element x entering W_e at stage s must enter A "promptly", namely by the end of stage $p(s)$ in the enumeration of A.

DEFINITION 1.1. A coinfinite r.e. set A is *promptly simple* if there is a recursive function p and a recursive enumeration $\{A_s: s \in \omega\}$ of A such that for every e

$$W_e \text{ infinite} \Rightarrow (\exists s)(\exists x)\left[x \in W_{e,\text{at } s} \cap A_{p(s)}\right]. \tag{1.1}$$

(Note that we may assume that p is nondecreasing by replacing $p(s)$ if necessary by $\max\{p(t): t \leqslant s\}$.)

The definition of prompt simplicity is independent of the particular enumeration in the following sense.

PROPOSITION 1.2. *If A is promptly simple and $\{\hat{A}_s: s \in \omega\}$ and $\{V_{e,s}: e, s \in \omega\}$ are s.r.e. arrays of finite sets such that $A = \bigcup_s \hat{A}_s$, $W_e = \bigcup_s V_{e,s}$, $\hat{A}_s \subseteq \hat{A}_{s+1}$, $V_{e,s} \subseteq V_{e,s+1}$, and* $\max(\{e: V_{e,s} \neq \varnothing\})$ *is recursively bounded, then there is a recursive function q such that for all e,*

$$W_e \text{ infinite} \Rightarrow (\exists s)(\exists x)\left[x \in V_{e,\text{at } s} \cap \hat{A}_{q(s)}\right].$$

PROOF. Let A be promptly simple via p with respect to $\{A_s: s \in \omega\}$. Given s, compute for each $x \in V_{e,s} - V_{e,s-1}$ the least t such that $x \in W_{e,t}$ and let $q(s)$ be the least number u such that $\hat{A}_u \supseteq A_{p(t)}$ for all such t. ■

Most simple sets in the literature, such as Post's simple set, are automatically promptly simple, and often with p the identity function. The following characterization which does not mention enumerations shows that prompt simplicity is recursively invariant (i.e. invariant under recursive permutations of ω). This characterization is similar to the analogous recursion theoretic characterization of nonspeedable sets (see Soare [**1977**, Theorem 2.4]) which are related to promptly simple sets by Maass's theorem [**1982**, Theorem 17] that any two sets with both properties are automorphic. The following theorem is proved in Maass [**1982**, Lemma 11] and Maass, Shore and Stob [**1981**, Theorem 1.3].

THEOREM 1.3 (MAASS). *The following are equivalent for an r.e. set A:*
(i) *A is promptly simple;*
(ii) *A is coinfinite and there is a recursive function f such that, for all $e \in \omega$,*

$$W_{f(e)} \subseteq W_e, \tag{1.2}$$

$$W_{f(e)} \cap \bar{A} = W_e \cap \bar{A} \tag{1.3}$$

and

$$W_e \text{ infinite} \Rightarrow W_e - W_{f(e)} \neq \varnothing. \tag{1.4}$$

(iii) *The same as* (ii) *but with* (1.4) *replaced by*

$$W_e \text{ infinite} \Rightarrow W_e - W_{f(e)} \text{ infinite}. \tag{1.5}$$

PROPOSITION 1.4 (MAASS, SHORE AND STOB [**1981**]). *If $A \subseteq B$ are r.e. sets, B is coinfinite and A is promptly simple then B is promptly simple.*

PROOF. Choose enumerations $\{A_s\}_{s\in\omega}$, $\{B_s\}_{s\in\omega}$ such that $A_s \subseteq B_s$. If p satisfies (1.1) for A_s, then a fortiori p satisfies (1.1) for B_s. ■

Indeed it is easy to show (Maass, Shore and Stob [**1981**, Theorem 1.4]) that the promptly simple sets are also closed under intersection, and hence together with the cofinite sets from a filter in $\mathcal{E}$. By a more difficult argument, Maass, Shore and Stob [**1981**, Corollary 1.6] also show that the class **PS** of promptly simple degrees forms a strong filter in **R**, namely: (i) **PS** is upward closed; and (ii) whenever $\mathbf{a}, \mathbf{b} \in$ **PS** there exists $\mathbf{c} \in$ **PS**, $\mathbf{c} \leqslant \mathbf{a}, \mathbf{b}$. We shall obtain (i) as Corollary 1.7 to Theorem 1.6 and (ii) as an immediate corollary of **ENC** = **PS** (Corollary 1.14) since **ENC** is easily seen to satisfy both (i) and (ii) (by Lemma 1.12).

The following lemma will be essential in several theorems below.

LEMMA 1.5 (SLOWDOWN LEMMA). *Let $\{U_{e,s}: e, s \in \omega\}$ be an s.r.e. array of finite sets such that $U_{e,s} \subseteq U_{e,s+1}$ and $U_e = \bigcup_s U_{e,s}$. Then there is a recursive function g such that for all e, $W_{g(e)} = U_e$, and $W_{g(e),s} \cap U_{e,\text{at } s} = \varnothing$ (namely any element enumerated in U_e appears strictly later in $W_{g(e)}$).*

PROOF. By the recursion theorem define

$$W_{g(e)} = \{x: (\exists s)[x \in U_{e,s} - W_{g(e),s}]\}. \ ■$$

Note that by the recursion theorem we may use $W_{g(e)}$ in a construction during which we define the array $\{U_{e,s}\}$.

THEOREM 1.6 (PROMPTLY SIMPLE DEGREE THEOREM). *Let A be an r.e. set and $\{A_s: s \in \omega\}$ a recursive enumeration of A. Then A has promptly simple degree iff there is a recursive function p such that for all s, $p(s) \geqslant s$, and for all e,*

$$(1.6) \qquad W_e \text{ infinite} \Rightarrow (\exists x)(\exists s)\left[x \in W_{e,at\,s} \,\&\, A_s \upharpoonright x \neq A_{p(s)} \upharpoonright x\right],$$

namely A "promptly permits" on some element $x \in W_e$.

PROOF. ($\Rightarrow$) Let $B = \{e\}^A$ where B is promptly simple via $q(s)$ satisfying (1.1), and let $\{B_s\}_{s \in \omega}$ be a recursive enumeration of B. We define p satisfying (1.6) and simultaneously construct an s.r.e. array $\{U_{e,s}: e, s \in \omega\}$ to which we apply Lemma 1.5. Set $p(0) = 0$.

Stage $s > 0$. (Define $p(s)$.) Choose the unique e and x such that $x \in W_{e,\text{at } s}$. If W_e does not yet satisfy (1.6) and there exists $y \in \bar{B}_s \cap \bar{U}_{e,s-1}$ such that $\{e\}_s^{A_s}(y)\downarrow = 0$ and $u(A_s; e, y, s) < x$, then enumerate the least such y in $U_{e,s}$. Find the least t such that $y \in W_{g(e),t}$, where $W_{g(e)}$ is obtained from $\{U_{e,s}: e, s \in \omega\}$ by Lemma 1.5. Let $p(s)$ be the least $v \geqslant q(t)$ such that $B_v(y) = \{e\}_v^{A_v}(y)$. (This ends the construction.)

Now if W_e is infinite but fails to satisfy (1.6), then U_e is infinite (because $\bar{B}$ is infinite). Hence, by the prompt simplicity of B, there exists $y \in W_{g(e),\text{at } t} \cap B_{q(t)}$. But $y \in U_{e,\text{ at } s}$ for some $s < t$ such that $\{e\}_s^{A_s}(y)\downarrow = B_s(y) = 0$. Now $y \in B_{q(t)} - B_s$ implies $A_s \upharpoonright u \neq A_{p(s)} \upharpoonright u$, where $u = u(A_s; e, y, s)$. But y entered U_e only for the sake of some $x \in W_{e,\text{at } s}$, $x > u$, so (1.6) is satisfied for W_e.

($\Leftarrow$) Given $p(s)$ satisfying (1.6) we use the usual permitting and coding methods to construct $B \equiv_T A$ such that B is promptly simple via the identity function. We must meet, for all e, the requirement

$$P_e: W_e \text{ infinite} \Rightarrow (\exists x)(\exists s)[x \in W_{e,\text{at } s} \cap B_s].$$

Define $B_0 = \varnothing$.

Stage $s > 0$. Let $\bar{B}_{s-1} = \{b_0^{s-1} < b_1^{s-1} < \cdots\}$.

Step 1 (for prompt simplicity). Choose the unique x and e such that $x \in W_{e,\text{at } s}$. If $x > b_e^{s-1}$ and P_e is not yet satisfied, compute $A_{p(s)}$ and if $A_s \upharpoonright x \neq A_{p(s)} \upharpoonright x$ enumerate x in B.

Step 2 (to code A into B). For each $x \in A_s - A_{s-1}$, enumerate b_x^{s-1} in B.

This completes the enumeration of B. Now $B \leqslant_T A$ since if $x \in B_s - B_{s-1}$ then $A \upharpoonright x \neq A_{s-1} \upharpoonright x$. But B is promptly simple since if W_e is infinite then the conclusion of P_e is satisfied by the construction, (1.6), and (ii) $\Rightarrow$ (iii) of Theorem 1.8 below. Also $A \leqslant_T B$ since if $b_x = \lim_s b_x^s$ and $B_s \upharpoonright (b_x + 1) = B \upharpoonright (b_x + 1)$ then $x \in A$ iff $x \in A_s$. ■

COROLLARY 1.7 (MAASS, SHORE AND STOB). *If* $\mathbf{b} \in \mathbf{PS}$ *and* $\mathbf{b} \leqslant \mathbf{a} \in \mathbf{R}$ *then* $\mathbf{a} \in \mathbf{PS}$.

PROOF. Suppose B_0 and A are r.e. sets, B_0 is promptly simple, and $B_0 \leqslant_T A$. Then by the proof of ($\Rightarrow$) of Theorem 1.6 there exists p satisfying (1.6) for A. Hence by ($\Leftarrow$) of Theorem 1.6 there is a promptly simple set $B \equiv_T A$. ■

A less direct proof of Corollary 1.7 using a theorem of Lachlan [**1968**, Theorem 1] was given by Maass, Shore and Stob [**1981**, Corollary 1.6]). Still a third proof can be given by combining the proof of ($\Rightarrow$) of Theorem 1.6 and the proof of Theorem 1.10.

Although not used as often as Theorem 1.6, the following theorem gives some additional characterizations of an r.e. set A having promptly simple degree.

THEOREM 1.8. *Let A be an r.e. set and $\{A_s: s \in \omega\}$ a recursive enumeration of A. Then the following are equivalent*:

(i) *A has promptly simple degree.*

(ii) *There is a recursive function p satisfying* (1.6).

(iii) *The same as* (ii) *but with* "$(\exists^\infty x)$" *in place of* "$(\exists x)$" *in* (1.6), *where $\exists^\infty x$ denotes "there exist infinitely many x".*

(iv) *Whenever $\{U_{e,s}: e, s \in \omega\}$ is an s.r.e. array of finite sets such that $W_e = \bigcup_s U_{e,s}$ and $U_{e,s} \subseteq U_{e,s+1}$ there is a recursive function $p(s)$ satisfying* (1.6) *with* "$U_{e,s}$" *in place of* "$W_{e,s}$".

(v) *The same as* (ii) *but with* "$W_e = \omega$" *in place of* "W_e infinite".

PROOF. (i) $\Leftrightarrow$ (ii) was established in Theorem 1.6, and (iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Given $q(s)$ satisfying (ii), we define $p(s)$ satisfying (iii). Using the recursion theorem define

$$W_{g(e,n)} = \{x: x > n \,\&\, (\exists s)[x \in W_{e,s} - W_{g(e,n),s}]\}.$$

For each s, find x and e such that $x \in W_{e,\text{at } s}$. For each $n \leqslant s$, if $x > n$ find the least t such that $x \in W_{g(e,n),t}$. Define $p(s)$ to be the maximum of $q(t)$ over all such t.

Fix n. Now if W_e is infinite then $W_{g(e,n)}$ is infinite so

$$(\exists x)(\exists t)[x \in W_{g(e,n),\text{at } t} \,\&\, A_t \upharpoonright x \neq A_{q(t)} \upharpoonright x].$$

But then $x \in W_{e,\text{at } s}$ for some $s < t$ and $x > n$. However, $q(t) \leqslant p(s)$ so

$$x \in W_{e,\text{at } s} \quad \text{and} \quad A_s \upharpoonright x \neq A_{p(s)} \upharpoonright x.$$

Since n was arbitrary, there are infinitely many such x, so p satisfies (iii).

(iv) $\Rightarrow$ (ii). Define $U_{e,s} = W_{e,s}$.

(iii) $\Rightarrow$ (iv). Let $q(s)$ satisfy (iii). We define $p(s)$ to satisfy (iv). Apply Lemma 1.5 to $\{U_{e,s}: e, s \in \omega\}$ to obtain $g(e)$. Given s, for all x and $e \leqslant s$ such that $x \in U_{e,s}$ find the least t such that $x \in W_{g(e),t}$ and let $p(s)$ be the maximum of $q(t)$ over such t.

(ii) $\Rightarrow$ (v). Immediate.

(v) $\Rightarrow$ (ii). Let $q(s)$ satisfy (v). We define $p(s)$ satisfying (ii), by first defining an s.r.e. array $\{U_{e,s}\}$ and applying Lemma 1.5. If $x \in W_{e,\text{ at } s}$ and $x \notin U_{e,s-1}$ then enumerate in $U_{e,s}$ all $y \leqslant x$, $y \notin U_{e,s-1}$. If there is no such x let $U_{e,s} = U_{e,s-1}$. Apply Lemma 1.5 to $\{U_{e,s}\}$ to obtain g. Given s, find $x \in W_{e,\text{ at } s}$ and then t such that $x \in W_{g(e),\text{at } t}$ (since $W_{g(e)} \supseteq W_e$) and define $p(s) = q(t)$.

Now if W_e is infinite then $W_{g(e)} = \omega$, so

$$(\exists y)(\exists t)[y \in W_{g(e),\text{at } t} \,\&\, A_t \upharpoonright y \neq A_{q(t)} \upharpoonright y].$$

Choose s such that $y \in U_{e,\text{ at } s}$. Then $s < t$ by hypothesis on g. Choose the unique $x \in W_{e,\text{at } s}$. Now $y \leqslant x$ and $s < t \leqslant q(t) \leqslant p(s)$, so $A_s \upharpoonright x \neq A_{p(s)} \upharpoonright x$. ■

DEFINITION 1.9. Let A be an infinite r.e. set and f a $1:1$ recursive function with range A. The *deficiency set* of A with respect to f is the r.e. set

$$B = \{s: (\exists t > s)[f(t) < f(s)]\}.$$

It is well known that $A \equiv_T B$ and that if A is nonrecursive then B is hypersimple (see Rogers [**1967**, p. 140]). It is often the case that if there is some r.e. set $C \in \deg(A)$ with a certain property (such as being simple, hypersimple, or atomless) then the deficiency set B itself has this property, thus providing a very convenient example of such a C. For example, Shoenfield [**1976**] showed that if $\deg(A)$ contains an atomless set (i.e. if $\deg(A)$ is not low_2) then B is atomless. A coinfinite r.e. set A is *atomless* if A is not contained in any maximal set. Next we extend this principle to prompt simplicity by showing that if A has promptly simple degree then B must be promptly simple. (It then follows from a result of Maass [**1982**] that if A_1 and A_2 are any two low r.e. sets of promptly simple degree then their deficiency sets B_1 and B_2 are automorphic.)

THEOREM 1.10. *Let A be an r.e. set of promptly simple degree. Then the deficiency set B of A is a promptly simple set.*

PROOF. Let B be the deficiency set of A with respect to a $1:1$ recursive function f having range A. Let $A_s = \{f(0), f(1), \ldots, f(s)\}$ and

$$B_s = \{t: t \leqslant s \,\&\, (\exists v)[t < v \leqslant s \,\&\, f(v) < f(t)]\}.$$

Let A be of promptly simple degree via $q(s)$ satisfying (1.6). We shall define a prompt simplicity function $p(s)$ for $\{B_s\}_{s\in\omega}$ satisfying (1.1) with B in place of A. As usual let $g(e)$ be the result of applying Lemma 1.5 to the s.r.e. array $\{U_{e,s}: e, s \in \omega\}$ which will be defined during the construction.

Set $p(0) = 0$. Given $s > 0$, find $x \in W_{e,\text{at } s}$. If $x \in B_s$, set $p(s) = s$. Otherwise, enumerate $f(x)$ in $U_{e,s}$, find the least t such that $f(x) \in W_{g(e),t}$, and define $p(s) = \max\{x, q(t)\}$. If W_e is infinite but fails to promptly intersect B then $W_{g(e)}$ is infinite so

$$(\exists x)\big[f(x) \in W_{g(e),\text{at } t} \,\&\, A_t \restriction f(x) \neq A_{q(t)} \restriction f(x)\big].$$

But if $x \in W_{e,\text{at } s}$ then $s < t \leqslant q(t) \leqslant p(s)$, so $A_s \restriction f(x) \neq A_{p(s)} \restriction f(x)$, which implies that $x \in B_{p(s)}$. ■

Promptly simple degrees may now be connected to noncappable and effectively noncappable degrees after a few definitions and elementary properties.

DEFINITION 1.11. (i) An r.e. degree $\mathbf{a}$ is *cappable* if there is an r.e. degree $\mathbf{b} > \mathbf{0}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$, and $\mathbf{a}$ is *noncappable* otherwise.

(ii) An r.e. degree $\mathbf{a}$ is *effectively noncappable* (e.n.c.) if there is an r.e. set $A \in \mathbf{a}$ and a recursive function f such that for all e,

(a) $W_{f(e)} \leqslant_T A$ uniformly in e,

(b) $W_{f(e)} \leqslant_T W_e$ uniformly in e, and

(c) W_e nonrecursive $\Rightarrow$ $W_{f(e)}$ nonrecursive.

(Namely, $W_{f(e)}$ is an effective witness to the fact that the degrees of A and W_e do not cap to $\mathbf{0}$.)

Note that in (b) we may replace A if necessary by $\bigoplus\{W_{f(e)}: e \in \omega\} =_{\text{dfn}} \{\langle x, e\rangle: x \in W_{f(e)}\}$ so we may assume $W_{f(e)} = A^{[e]}$ where

$$A^{[e]} =_{\text{dfn}} \{\langle x, y\rangle: \langle x, y\rangle \in A \;\&\; y = e\}.$$

(To insure that $A \leqslant_T \bigoplus\{W_{f(e)}\}$ we require that $0 \in W_{f(e)}$ iff $e \in A$.) Recall that **NC** and **ENC** denote the degrees of noncappable and effectively noncappable r.e. sets, respectively.

LEMMA 1.12. **ENC** *is a strong filter in* **R**, *namely*
(i) $\mathbf{a} \leqslant \mathbf{b} \in \mathbf{R}$ *and* $\mathbf{a} \in \mathbf{ENC} \Rightarrow \mathbf{b} \in \mathbf{ENC}$, *and*
(ii) $\mathbf{a}, \mathbf{b} \in \mathbf{ENC} \Rightarrow (\exists \mathbf{c} \in \mathbf{ENC})[\mathbf{c} \leqslant \mathbf{a}, \mathbf{b}]$.

PROOF. (i) is immediate from the definition. For (ii) choose r.e. sets $A \in \mathbf{a}$ and $B \in \mathbf{b}$ which are e.n.c. via f and g, respectively. Then $C = \bigoplus\{W_{g(f(e))}: e \in \omega\}$ is e.n.c. via $h = gf$. ■

The next theorem is the most difficult in this paper and yields the main equivalences **ENC** = **NC** = **PS** as well as the fact that these all form strong filters.

THEOREM 1.13. $\mathbf{NC} \subseteq \mathbf{PS}$.

COROLLARY 1.14. **ENC** = **NC** = **PS**.

PROOF. Clearly $\mathbf{ENC} \subseteq \mathbf{NC}$. Maass, Shore and Stob [**1981**, Theorem 1.11] have shown that $\mathbf{PS} \subseteq \mathbf{NC}$, although their proof clearly demonstrates that $\mathbf{PS} \subseteq \mathbf{ENC}$. Equalities thus follow from Theorem 1.13.

COROLLARY 1.15. **NC** *and* **PS** *each form strong filters in* **R**.

PROOF. By Corollary 1.14 and Lemma 1.12. (A different proof for the case of **PS** was given by Maass, Shore and Stob [**1981**, Corollary 1.6].) ■

PROOF OF THEOREM 1.13. Fix an r.e. set B and a recursive enumeration $\{B_s\}_{s\in\omega}$ of B. We shall construct an r.e. set A by a recursive enumeration $\{A_s\}_{s\in\omega}$ such that either A is nonrecursive and $\deg(A) \cap \deg(B) = \mathbf{0}$, or else B has promptly simple degree.

To attempt to meet the first alternative we use the usual minimal pair method as presented in Soare [**1980**, Theorem 4.2]. Namely we must construct A to satisfy for every $e \in \omega$ the requirements P_e: W_e infinite $\Rightarrow W_e \cap A \neq \varnothing$ and N_e: $\{e\}^A = \{e\}^B = f_e \Rightarrow f_e$ is recursive. From $\{A_t: t \leqslant s\}$ and $\{B_t: t \leqslant s\}$ define the recursive functions

$$l(e, s) = \max\{x: (\forall y < x)[\{e\}_s^{A_s}(y)\downarrow = \{e\}_s^{B_s}(y)]\}$$

and

$$m(e, s) = \max\{l(e, t): t < s\}.$$

Call s an *e-expansion* stage if $l(e, s) > m(e, s)$. (Thus, if $\{e\}^A = \{e\}^B$ and $\{e\}^A$ is total then there are infinitely many e-expansion stages.)

Simultaneously with the construction of A we define for each e a recursive function p_e such that either requirement N_e is met or else p_e witnesses that B has promptly simple degree, namely (using Theorem 1.6) that p_e satisfies for all $i \in \omega$ the requirement

$$R_{e,i}\colon W_i \text{ infinite} \Rightarrow (\exists x)(\exists s)\Big[x \in W_{i,\text{at } s} \,\&\, B_s \upharpoonright x \neq B_{p_e(s)} \upharpoonright x\Big].$$

Before giving the full construction we sketch the basic module for a single requirement. Fix e. We must satisfy the requirement

$$R_e\colon N_e \quad \text{or} \quad (\forall i) R_{e,i}.$$

For each i we define a partial recursive function $\psi_{e,i}$ such that if $\{e\}^A = \{e\}^B$ is total and i is minimal such that p_e fails to satisfy $R_{e,i}$ then $\psi_{e,i} = \{e\}^B$ so that N_e is met. To accomplish this we define a recursive "restraint" function $r(e, i, s)$ which is the restraint imposed by $R_{e,i}$, and which tends to restrain elements from entering A. We define $r(e, s) = \max\{r(e, i, s)\colon i \leqslant s\}$, which is the restraint imposed by R_e, and it prevails against the positive requirements of lower priority; namely $P_j, j \geqslant e$.

At stage $s + 1$ we open an $R_{e,i}$-gap by choosing the least i (if one exists) such that $R_{e,i}$ is not yet satisfied and such that there exist $x \in W_{i,\text{ at } s}$ and $y \in \operatorname{dom}\{e\}_s^{B_s} - \operatorname{dom}(\psi_{e,i,s})$ with $y < l(e, s)$ and $u_y < x$ where

$$u_y = \tilde{u}(B_s; e, y, s) =_{\text{dfn}} \max\{u(B_s; e, y', s)\colon y' \leqslant y\}.$$

We define $\psi_{e,i}(z) = \{e\}_s^{B_s}(z)$ for all $z \leqslant y$, $z \notin \operatorname{dom}(\psi_{e,i,s})$ and $r(e, j, s + 1) = 0$ for all $j \geqslant i$.

This gap is later *closed* at stage $t + 1$ where $t > s$ is the next e-expansion stage after s. At stage $t + 1$ we define $p_e(s) = t$ and set $r(e, i, t) = t$ as A-restraint (since by convention $u(A_s; e, z, t) < t$). Notice that if $B_s \upharpoonright u_y \neq B_t \upharpoonright u_y$ then $B_s \upharpoonright x \neq B_t \upharpoonright x$, so p_e satisfies requirement $R_{e,i}$ via x. Thus, if $R_{e,i}$ is never satisfied then any value $\psi_{e,i}(y) = w$ once defined is protected by the B-side, $\{e\}_v^{B_v}(y) = w$ at all later stages v in this gap, i.e. $s + 1 \leqslant v \leqslant t$.

Hence, if $\{e\}^A = \{e\}^B$ and B is not of p.s. degree, choose the least i such that W_i is infinite, but $R_{e,i}$ is never satisfied. Then $\psi_{e,i} = \{e\}^B$ because once $\psi_{e,i}(y) = w$ is defined at some stage, the B-side holds the computation at all later stages which lie in some $R_{e,i}$-gap and the A-side holds the computation (because of the A-restraint $r(e, i, s)$) during all the corresponding cogaps (intervals between gaps). Furthermore, $\liminf_s r(e, s) < \infty$ since at each stage s when an $R_{e,i}$-gap is opened,

$$r(e, s) = \max\{r(e, j, s)\colon j < i\}$$

(since all restraint imposed by $R_{e,j}$, $j \geqslant i$, is dropped when $R_{e,i}$ opens a gap). Of course if there is no i such that infinitely many $R_{e,i}$-gaps are opened then we may have $\lim_s r(e, i, s) > 0$ for all i so that $\liminf_s r(e, s) = \infty$, but in this case B is of promptly simple degree and we need not meet the requirements $R_{e'}$, $e' \geqslant e$. This ends the basic module for a single requirement R_e.

The strategy σ_e just given for meeting a single requirement R_e, say R_0, produces an A-restraint function $r(0, s)$ such that $\liminf_s r(0, s) < \infty$. As in the minimal pair construction (Soare [**1980**, Theorem 4.2] or Fejer and Soare [**1980**]) we modify the strategy σ_e for R_e, when $e > 0$, so that the various restraint functions $r(i, s)$, $i \leqslant e$, drop back simultaneously, namely $\liminf_s \tilde{r}(e, s) < \infty$ where $\tilde{r}(e, s) = \max\{r(j, s): j \leqslant e\}$. To do this R_e must guess the value of $k = \liminf_s \tilde{r}(e-1, s)$ (the maximum restraint imposed at stage s by any $R_{e'}$, $e' < e$) and must simultaneously play infinitely many strategies σ_e^k, $k \in \omega$, one for each possible value of k. Each strategy σ_e^k is played like σ_0 but with $S^k = \{s: \tilde{r}(e-1, s) = k\}$ in place of ω as the set of stages on which it is active. Strategy σ_e^k still succeeds if any restraint it imposes is maintained during the intermediate stages $s \notin S^k$ namely those stages when σ_e^k is dormant. Thus, at stage s if $k = \tilde{r}(e-1, s)$, we play σ_e^k, maintain the restraints previously imposed by the (dormant) σ_e^i, $i < k$, and discard restraints imposed by σ_e^j, $j > k$. Therefore, if $k = \liminf_s \tilde{r}(e-1, s)$, then: (1) the strategy σ_e^k succeeds in meeting R_e; (2) the strategies σ_e^i, $i < k$, impose finitely much restraint over the course of the whole construction; and (3) the strategies σ_e^i, $i > k$, drop all restraint at each stage $s \in S^k$. Thus, the entire restraint $\tilde{r}(e, s)$ imposed by all the R_j, $j \leqslant e$, *together* has $\liminf_s \tilde{r}(e, s) < \infty$.

In addition, as in Fejer and Soare [**1980**, §3] we arrange that σ_e^k is allowed to open an $R_{e,i}^k$-gap (and drop its A-restraint) only at a stage $s \in S^k$. However, σ_e^k is allowed to *close* that gap (thereby reimposing A-restraint and defining $p_e^k(s)$) at *any* stage t (providing t is an e-expansion stage). Thus, we have a sufficiently small amount of restraint so that $\liminf_s \tilde{r}(e, s) < \infty$, and yet we close the gaps often enough so that if $\{e\}^A = \{e\}^B$ then every $R_{e,i}^k$-gap is closed (at the next e-expansion stage) so that p_e^k is total. In the following proof we use the notation $r(e, s)$ in place of $\tilde{r}(e, s)$ to denote the maximum restraint imposed by all R_j for $j \leqslant e$.

Construction of A, p_e^k and $\psi_{e,i}^k$.

Stage $s = 0$. Do nothing.

Stage $s + 1$. For each $e \leqslant s$ perform in increasing order of e the following steps.

Step 1. Let $k = r(e-1, s+1)$. (We define $r(-1, t) = 0$ for all t.) For each $j > k$ cancel any gap or restraint previously imposed by $R_{e,i}^j$, for any i.

Step 2 (closing gaps). If s is not an e-expansion stage go to Step 3. Otherwise, if there is an $R_{e,i}^j$-gap which was opened at some stage $v < s$ and has not been closed or cancelled, then declare the gap to be *closed*, define $p_e^j(t) = s$ for all $t \leqslant v$ not in domain p_e^j, and let $R_{e,i}^j$ assign s as A-restraint (since $s > u(A_s; e, x, s)$ for all $x \leqslant l(e, s)$).

Step 3 (opening gaps). Let $s' = \max\{t < s: r(e-1, t) = k\}$ if such t exists and $= 0$ otherwise. Choose the least $i \leqslant s$ such that $R_{e,i}^k$ is not yet satisfied and

$$(\exists x)(\exists y)\big[x \in W_{i,s} - W_{i,s'} \,\&\, y \in \operatorname{dom}\big(\{e\}_s^{B_s}\big) - \operatorname{dom}\big(\psi_{e,i,s}^k\big) \\ \&\, l(e, s) > y \,\&\, \tilde{u}(B_s; e, y, s) < x\big],$$

where

$$\tilde{u}(B_s; e, y, s) = \max\{u(B_s; e, z, s): z \leqslant y\}.$$

Choose the least such x and y and open an $R_{e,i}^k$-gap by defining $\psi_{e,i}^k(y) = \{e\}_s^{B_s}(y)$, and cancelling for all $j \geqslant i$ any A-restraint associated with $R_{e,j}^k$. If i fails to exist do nothing. (Note that some $R_{e,i}^k$-gap may have been closed at step 2 and a new one opened at step 3 in which case any A-restraint put on by $R_{e,i}^k$ at step 2 is cancelled at step 3.) Let $r(e, s+1)$ be the maximum of the A-restraint still imposed by $R^{k'}_{e',i'}$ for some $e' \leqslant e$, $k' \leqslant s$, $i' \leqslant s$.

Step 4 (Making A simple). If $W_{e,s} \cap A_s = \varnothing$ and

$$(\exists y)[y \in W_{e,s} \,\&\, y > 2e \,\&\, y > r(e, s+1)],$$

choose the least such y and enumerate y in A.

This completes the construction.

Assume that B is not of p.s. degree. Hence, for all e, there exists i such that W_i is infinite but $R_{e,i}$ is not satisfied. We must show that for all e requirement N_e is satisfied and $\liminf_s r(e, s) < \infty$, since then it is automatic by step 4 of the construction that A is simple.

Fix e and assume by induction that for all $e' < e$, $N_{e'}$ is met and that

$$\liminf_s r(e', s) < \infty.$$

Let $k = \liminf_s r(e-1, s)$. Let $S^k = \{s\colon r(e-1, s) = k\}$. Assume that $\{e\}^A = \{e\}^B$, a total function. Choose the least i such that W_i is infinite but $R^k_{e,i}$ is not satisfied. Since there are infinitely many $R^k_{e,i}$-gaps, $\psi^k_{e,i}$ and p^k_e are total and recursive. Choose s_0 such that for all $s \geqslant s_0$: (1) no $R^{k'}_{e,i'}$-gap is opened or closed at stage s if $k' < k$ or $i' < i$; and (2) $P_{e'}$ does not contribute an element to A at stage s if $e' \leqslant e$.

Now suppose that $\psi^k_{e,i}(y) = z$ is defined at some stage $s+1 > s_0$. We claim by induction on $v > s$ that either

(1) $\{e\}^{B_v}_v(y) = z$, or

(2) $\{e\}^{A_v}_v(y) = z$,

and hence that $f_e(y) = z$. (Thus, $f_e(y) = \psi^k_{e,i}(y)$ for almost all y so N_e is satisfied.)

To prove the claim note that at stage $s+1$ we open an $R^k_{e,i}$-gap via y and some $x \in W_{i,s} - W_{i,s'}$. Choose v, $s' < v \leqslant s$, such that $x \in W_{i,\text{ at } v}$. Necessarily $p^k_{e,s}(t)$ is undefined for all t, $s' < t$ (since $p^k_e(t)$ is defined only when a gap begun at a stage $\geqslant t$ is closed, $R^k_{e,i'}$-gaps are only begun at stages $\in S^k$, and s' is the most recent such stage $< s$). By choice of s_0, no $R^k_{e,i}$-gap is ever cancelled after s_0, so the above gap must be closed at stage $t+1$, where t is the next e-expansion stage $> s$. Now $p^k_e(v) = t$, and $v \leqslant s < t$, so $B_s \upharpoonright x = B_t \upharpoonright x$ because $R_{e,i}$ is never satisfied. But $\tilde{u}(B_s; e, y, s) < x$, so (1) holds for all v, $s+1 \leqslant v \leqslant t$, namely those stages v in the $R^k_{e,i}$-gap.

Now at stage $t+1$, this $R^k_{e,i}$-gap for y is closed, and $R^k_{e,i}$ sets $t \geqslant \tilde{u}(A_t; e, y, t)$ as A-restraint. But by choice of s_0, no such A-restraint is ever injured after s_0. Hence, this A-restraint remains in force until the next stage $s_1 + 1 \geqslant t+1$ at which the next $R^k_{e,i}$-gap is opened via $y_1 = y+1$. But since $\tilde{u}(X; e, y_1, v) \geqslant \tilde{u}(X; e, y, v)$ for all v, the above argument shows that (1) holds for all stages v in the $R^k_{e,i}$-gap opened by y_1. Thus,

(1) holds for all v in the $R^k_{e,i}$-gaps and

(2) holds for all v in the $R^k_{e,i}$-cogaps.

Finally, let $r(e)$ be the maximum of k and the restraint imposed by $R^{k'}_{e,i'}$ for $k' < k$ or $i' < i$. Now $r(e, s) = r(e)$ at every stage s when an $R^k_{e,i}$-gap is opened. Hence, $r(e) = \liminf_s r(e, s) < \infty$. ∎

COROLLARY 1.16. *There is a recursive function f such that if* $\deg(W_e)$ *is part of a minimal pair then* $\deg(W_e)$ *and* $\deg(W_{f(e)})$ *form a minimal pair.*

PROOF. Apply Theorem 1.13 with $B = W_e$ to obtain uniformly $W_{f(e)} = A$. ∎

Harrington proved the cup or cap theorem (see Fejer and Soare [**1980**, Corollary 2.4]) which asserts that every r.e. degree caps (to $\mathbf{0}$) or cups (to $\mathbf{0}'$) and also proved that some degrees do both. We now prove that **NC** = **LC**, namely that every r.e. degree either caps or low cups but none does both, thereby eliminating the overlap in Harrington's Theorem.

THEOREM 1.17. **NC** ⊆ **LC**, *namely if an r.e. degree* $\mathbf{a}$ *is noncappable then* $\mathbf{a}$ *low cups to* $\mathbf{0}'$ *in that* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$ *for some low r.e. degree* $\mathbf{b}$.

COROLLARY 1.18. **NC** = **LC**.

PROOF. Ambos-Spies [**1980**, Theorem 4.1] proved an extension of Lachlan's nondiamond theorem [**1966**] by showing that there are no r.e. degrees $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{c}_0$, $\mathbf{c}_1$ such that $\mathbf{b}_0 \cup \mathbf{b}_1$ is low, $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$, $\mathbf{b}_i < \mathbf{c}_i$ and $\mathbf{a}_i \cap \mathbf{c}_i = \mathbf{b}_i$ for $i = 0, 1$. It follows (taking $\mathbf{b}_0 = \mathbf{0}$, $\mathbf{b}_1 = \mathbf{a}_1$ low, and $\mathbf{c}_1 = \mathbf{0}'$) that no cappable r.e. degree $\mathbf{a}_0$ can be cupped to $\mathbf{0}'$ by any low r.e. degree $\mathbf{a}_1$. Therefore, **LC** ⊆ **NC**. ∎

PROOF OF THEOREM 1.17. By Theorem 1.13 it suffices to show that **PS** ⊆ **LC**. Choose B contained in the odd numbers, $2\omega + 1$, and p according to Theorem 1.8(iii) (i.e. p satisfies (1.6) for infinitely many x). We wish to build a low r.e. set $A \subseteq 2\omega$ such that $K \leqslant_T A \oplus B$, where $K = \{e : e \in W_e\}$. Note that $A \oplus B \equiv_T A \cup B$. Choose an index j for K and let $K_s = W_{j,s}$. We have a list of "coding markers" $\{\Gamma_n\}_{n \in \omega}$, and we let Γ_n^s denote the position of Γ_n at the end of stage s. We arrange that for all n and s, Γ_n^s is even and

(1) $n \in K - K_s \Rightarrow (A_s \cup B_s) \restriction (\Gamma_n^s + 1) \neq (A \cup B) \restriction (\Gamma_n^s + 1)$,

(2) $\Gamma_n^s \leqslant \Gamma_n^{s+1}$ and $\Gamma_n^s \leqslant \Gamma_{n+1}^s$,

(3) $\Gamma_n^s < \Gamma_n^{s+1} \Rightarrow (A_s \cup B_s) \restriction (\Gamma_n^s + 1) \neq (A \cup B) \restriction (\Gamma_n^s + 1)$, and

(4) $(\forall n)[f(n) =_{\mathrm{dfn}} \lim_s \Gamma_n^s < \infty]$.

These conditions clearly guarantee that $K \leqslant_T A \cup B$ since $f \leqslant_T A \cup B$ by (3) and (4), and for each n if s is such that $(A \cup B) \restriction (f(n) + 1) = (A_s \cup B_s) \restriction (f(n) + 1)$ then $n \in K$ iff $n \in K_s$ by (1).

To make A low we meet for all e the lowness requirement

$$N_e \colon (\exists^\infty s)\left[\{e\}_s^{A_s}(e) \text{ converges}\right] \Rightarrow \{e\}^A(e) \text{ converges}.$$

We accomplish this by attempting to clear from the A-use $A \restriction u(A_s; e, e, s)$ of the computation $\{e\}_s^{A_s}(e)$ all markers Γ_n, $n \geqslant e$, by using the prompt simplicity of B to force $B \restriction \Gamma_n^s$ to change. During the construction we define r.e. sets U_e, $e \in \omega$. Let g be the corresponding recursive function obtained by Lemma 1.5.

Construction of A.

Stage $s = 0$. Set $\Gamma_n^0 = 2n$ for all $n \in \omega$.

Stage $s + 1$.

Step 1. Find the least e such that $\{e\}_s^{A_s}(e)$ converges and $\Gamma_e^s \leqslant u(A_s; e, e, s)$. (If no such e exists go to step 2.) Enumerate Γ_e^s in U_e. Find the least stage t such that $\Gamma_e^s \in W_{g(e),t}$. (By Lemma 1.5, $s < t$.)

Case 1 (Free clear). $B_s \upharpoonright \Gamma_e^s \neq B_{p(t)} \upharpoonright \Gamma_e^s$. Move all markers Γ_i, $i \geqslant e$, maintaining their order to new even positions in $\overline{A}_s$ and greater than $u(A_s; e, e, s)$.

Case 2 (Capricious destruction). $B_s \upharpoonright \Gamma_e^s = B_{p(t)} \upharpoonright \Gamma_e^s$. Enumerate Γ_e^s in A (thereby capriciously destroying the computation $\{e\}_s^{A_s}(e)$), and move all markers Γ_i, $i \geqslant e$, maintaining their order, to new even positions in $\overline{A}_s$ greater than their old positions.

Step 2. If $n \in K_{s+1} - K_s$ enumerate the current position of Γ_n into A and move all markers Γ_m, $m \geqslant n$, to new even positions not yet in A.

This ends the construction.

LEMMA 1. $(\forall n)[f(n) = \lim_s \Gamma_n^s < \infty]$.

PROOF. If not find the least n such that Γ_n moves infinitely often, and choose s_0 such that $K_s \upharpoonright (n+1) = K_{s_0} \upharpoonright (n+1)$, and $\Gamma_m^{s_0} = \lim_s \Gamma_m^s$ for all $m < n$ and $s > s_0$. Now since Γ_n moves infinitely often after s_0, $W_{g(n)}$ is infinite but case 1 never applies after stage s_0 (else the computation would remain cleared forever). At each stage $s+1 > s_0$ when the construction applies to Γ_n^s, $x = \Gamma_n^s$ is enumerated in $U_{n,s+1}$ and hence in $W_{g(n),t}$ for some $t > s$ but $B_t \upharpoonright x = B_{p(t)} \upharpoonright x$, so $W_{g(n)}$ violates p being a prompt simplicity function for B satisfying Theorem 1.8(iii).

LEMMA 2. $(\forall e)[N_e$ *is satisfied*] *and hence A is low.*

PROOF. Choose a stage s such that $\{e\}_s^{A_s}(e)$ converges and for all $i \leqslant e$, $\Gamma_i^s = \lim_t \Gamma_i^t$. Now $\Gamma_e^s > u = u(A_s;\ e, e, s)$ else some Γ_i, $i \leqslant e$, moves at stage $s+1$ (whether case 1 or 2 applies) contrary to the choice of s. Hence $A_s \upharpoonright u = A \upharpoonright u$ so $\{e\}^A(e)$ converges. ∎

Another class of r.e. degrees was introduced by Maass, Shore and Stob [**1981**] in applying prompt simplicity to study the lattice $\mathcal{E}$ of r.e. sets. Although prompt simplicity is neither definable in $\mathcal{E}$ nor invariant under automorphisms of $\mathcal{E}$ it implies a certain splitting property which is definable in $\mathcal{E}$.

DEFINITION 1.19. An r.e. set A has the *splitting property* if for every r.e. set B there are r.e. sets B_0 and B_1 such that

(i) $B_0 \cup B_1 = B$,

(ii) $B_0 \cap B_1 = \varnothing$,

(iii) $B_0 \subseteq A$,

(iv) if W is r.e. but $W - B$ is not r.e., then $W - B_0$ and $W - B_1$ are not r.e.

Maass, Shore and Stob proved [**1981**, Theorem 2.2] that if A is promptly simple then A has the splitting property. Let $\mathbf{SP\overline{H}}$ denote the class of r.e. sets which have the splitting property but are not hyperhypersimple. They proved [**1981**, Corollary 3.6] that $\mathbf{SP\overline{H}}$ nontrivially splits each class $\mathbf{H}_n$, $\mathbf{L}_n$, $n \geqslant 1$, of the usual high-low hierarchy, thereby giving the first example of a high degree which omits some nontrivial automorphism type. This class of degrees (which is lattice definable in $\mathcal{E}$) is now shown to coincide with **NC** which is, of course, definable in **R**.

THEOREM 1.20. $\mathbf{NC} = \mathbf{SP\overline{H}}$.

PROOF. Maass, Shore and Stob prove [**1981**, Theorem 3.1] that $\mathbf{SP\overline{H}} \subseteq \mathbf{NC}$. For the reverse inclusion let A be promptly simple and B be the deficiency set of A. Thus,

$B \equiv_T A$, B is promptly simple by Theorem 1.10 (and thus has the splitting property) but B is not hyperhypersimple because $\bar{B}$ is retraceable and therefore not hyperhyperimmune (see Rogers [**1967**, pp. 158, 251]). ∎

COROLLARY 1.21. *There is a nontrivial $\mathcal{E}$-definable class of r.e. sets whose degrees* $\mathbf{SP\bar{H}}$ *are also* **R**-*definable* (*as* **NC**). ∎

PROPOSITION 1.22. *There is an* **R**-*definable class of r.e. degrees* **NC** *which nontrivially splits all the classes* $\mathbf{H}_{n+1} - \mathbf{H}_n$, $\mathbf{L}_{n+1} - \mathbf{L}_n$, $n \geqslant 0$, *of the high-low hierarchy.*

PROOF. Choose $\mathbf{a} \in \mathbf{NC} \cap \mathbf{L}_1$. Now for any $\mathbf{b} \geqslant \mathbf{0}'$, $\mathbf{b}$ r.e. in $\mathbf{0}'$, there exists r.e. $\mathbf{c} \geqslant \mathbf{a}$ (and hence $\mathbf{c} \in \mathbf{NC}$) such that $\mathbf{c}' = \mathbf{b}$ by the Robinson jump interpolation theorem (see Robinson [**1971**] or Soare [**1976**, p. 528]). The result for **NC** now follows from the fact that $\mathbf{H}_{n+1} - \mathbf{H}_n \neq \varnothing$, and $\mathbf{L}_{n+1} - \mathbf{L}_n \neq \varnothing$ for all n.

For $\mathbf{M} = \mathbf{R} - \mathbf{NC}$, choose $\mathbf{a} \in \mathbf{H}_1 \cap \mathbf{M}$. Choose $\mathbf{b} > \mathbf{0}'$ which relative to $\mathbf{a}$ is r.e., high ($\mathbf{b}' = \mathbf{a}''$), and incomplete ($\mathbf{b} < \mathbf{a}'$). By the jump interpolation theorem $\mathbf{b} = \mathbf{c}'$ for some r.e. $\mathbf{c} \leqslant \mathbf{a}$. Clearly $\mathbf{c} \in \mathbf{H}_2 - \mathbf{H}_1$ and $\mathbf{c} \in \mathbf{M}$. By iterating this procedure we can find for each n a degree $\mathbf{d} \in (\mathbf{H}_{n+1} - \mathbf{H}_n) \cap \mathbf{M}$. To produce $\mathbf{c} \in (\mathbf{L}_2 - \mathbf{L}_1) \cap \mathbf{M}$ we choose b low relative to $\mathbf{a}$ ($\mathbf{b}' = \mathbf{a}'$) instead of high and apply the same method. ∎

Finally, let **G** denote the degrees of r.e. sets which are in the orbit of Maass's r.e. generic set G [**1981**], i.e. which are the image under an automorphism of $\mathcal{E}$ of some promptly simple set A whose complement is semilow, i.e. $\{n\colon W_n \cap \bar{A} \neq \varnothing\} \leqslant_T \varnothing'$. Maass and the authors proved that this sixth and final class coincides with **PS**.

LEMMA 1.23. *If B is promptly simple then there exists a promptly simple set $A \equiv_T B$ such that $\bar{A}$ is semilow.*

COROLLARY 1.24. $\mathbf{G} = \mathbf{PS}$.

PROOF. First $\mathbf{G} \subseteq \mathbf{SP\bar{H}}$ because promptly simple sets have the splitting property and the latter is invariant under Aut($\mathcal{E}$), and if $\bar{A}$ is semilow then $\mathcal{E}^*(\bar{A}) \cong \mathcal{E}$ by Soare [**1982 a**] so A is not hyperhypersimple. To see that $\mathbf{PS} \subset \mathbf{G}$, apply Lemma 1.23 to an arbitrary promptly simple set B to obtain A promptly simple with $\bar{A}$ semilow and $A \equiv_T B$. Now by Maass [**1982**] A is automorphic to any other such set and in particular to G. ∎

PROOF OF LEMMA 1.23. Fix a recursive enumeration $\{B_s\}_{s\in\omega}$ of B and a recursive function q satisfying (1.6). We construct $\{A_s\}_{s\in\omega}$ and p to meet the requirements

$$P_e\colon W_e \text{ infinite} \Rightarrow (\exists x)(\exists s)\left[x \in W_{e,\text{at } s} \cap A_{p(s)}\right].$$

$$N_e\colon \lim c_e^s \text{ exists, where}$$

$$c_e^s = \begin{cases} \mu x\left[x \in W_{e,s} \cap \bar{A}_s\right], & \text{if such } x \text{ exists,} \\ -1, & \text{otherwise.} \end{cases}$$

Stage $s = 0$. Enumerate 0 in A_0.

Stage $s + 1$. Let $\bar{A}_s = \{a_0^s < a_1^s < a_2^s < \cdots\}$.

Step 1. Choose the least e such that P_e is not yet satisfied and there exists $x \in W_{e,\text{at } s}$, $x \neq a_i^s, c_i^s$, all $i \leqslant e$. Enumerate $x \in U_{e,s}$ and choose $t > s$ such that $x \in W_{g(e),\text{ at } t}$, where $W_{g(e)}$ is obtained from $\{U_{e,s}\colon s \in \omega\}$ by Lemma 1.5. Define

$p(s) = q(t)$. If $B_t \upharpoonright x \neq B_{q(t)} \upharpoonright x$ enumerate x in A. Otherwise (or if no such e exists), simply set $p(s) = 0$.

Step 2. Let $k = \mu y[y \in B_{s+1} - B_s]$. Choose the least a_i^s, $k \leqslant i \leqslant 2k$, which is not equal to c_j^s, for any $j < k$, and enumerate a_i^s in A.

Now $B \leqslant_T A$ since if $k \in B_{s+1} - B_s$ then $A_{s+1} \upharpoonright a_{2k}^s + 1 \neq A_s \upharpoonright a_{2k}^s + 1$, and $A \leqslant_T B$ by permitting as usual. Now $\lim_s c_i^s$ and $\lim_s a_i^s$ exist because a_i^s or c_i^s can be enumerated in A_{s+1} only for the sake of P_k or $k \in B_{s+1} - B_s$ for some $k \leqslant i$, and each contributes at most one element. Let $c_e = \lim_s c_e^s$. Now $\lambda e[c_e]$ is a function recursive in $\varnothing'$ so $\bar{A}$ is semilow because $W_e \cap \bar{A} \neq \varnothing$ iff $c_e > -1$. Finally A is promptly simple because if W_e is infinite but fails to satisfy P_e then $W_{g(e)}$ violates (1.6) for B via $q(s)$. ■

The above theorems raise the question of whether there are any other well-known classes of r.e. degrees equivalent to this ubiquitous class **PS**. It follows by Corollary 1.18 that any degree $\mathbf{a} \in \mathbf{NC}$ cups (to $\mathbf{0}'$). It is natural to ask whether such an $\mathbf{a}$ cups to every $\mathbf{b} > \mathbf{a}$ (i.e. for every $\mathbf{b} > \mathbf{a}$ does there exist $\mathbf{c} < \mathbf{b}$ such that $\mathbf{a} \cup \mathbf{c} = \mathbf{b}$?). Ambos-Spies [to appear] has shown this to be false for some degrees $\mathbf{a} \in \mathbf{NC}$.

Variations of these properties may be investigated for other reducibilities. We say that A is *weak truth table reducible* to B ($A \leqslant_{\mathrm{wtt}} B$) if there is an $e \in \omega$ and a recursive function f such that $A = \{e\}^B$ and $y < f(x)$ for all numbers y used in the computation $\{e\}^B(x)$. The Harrington cup and cap theorem asserts that there exist r.e. degrees which both cup and cap. The next theorem asserts that this is false for "wtt-cup" in place of "cup". Namely, if a set A can be nontrivially wtt-cupped (to $\mathbf{0}'$) then A cannot be nontrivially capped (to $\mathbf{0}$) or even wtt-capped (to $\mathbf{0}$). This theorem, which was discovered before Harrington's Theorem, made the latter appear even more surprising, and the method has been used by Fejer [**1980**].

THEOREM 1.24. *If A and B are r.e. sets such that $K \leqslant_{\mathrm{wtt}} A \oplus B$ but $K \not\leqslant_{\mathrm{wtt}} B$, then* $\deg(A)$ *is noncappable.*

PROOF. Suppose that C is a nonrecursive r.e. set. We shall build a simple set E such that $E \leqslant_T A$ and $E \leqslant_T C$, so that $\deg(A)$ and $\deg(C)$ do not form a minimal pair. Fix A and B as in the theorem so that $A \subseteq \{2x\colon x \in \omega\}$ and $B \subseteq \{2x+1\colon x \in \omega\}$ so that $A \cup B \equiv_{\mathrm{wtt}} A \oplus B$. Let $\{A_s\}_{s\in\omega}$, $\{B_s\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$ be recursive enumerations of A, B and C. It suffices for each e to meet the positive requirement

$$P_e\colon W_e \text{ infinite} \Rightarrow W_e \cap E \neq \varnothing.$$

To satisfy P_e we define a certain r.e. set S_e in order to "force" numbers into A or B. The sets $\{S_e\}_{e\in\omega}$ will be uniformly r.e. and hence uniformly reducible to $A \cup B$, say by wtt-reductions $\{\Psi_e\}_{e\in\omega}$. Let $\psi_e(x)$ be the use function for Ψ_e as a wtt-reduction. By the recursion theorem Ψ_e and ψ_e may be used in the construction.

We say that P_e *requires attention at stage* $s+1$ via n if $W_{e,s} \cap E_s = \varnothing$, and for some $x > 2e$,

(i) $x \in W_{e,s}$,

(ii) $C_{s+1} \upharpoonright x \neq C_s \upharpoonright x$,

(iii) $\psi_e(n) < x$,

(iv) $n \in K_s - S_{e,s}$, and

(v) $\Psi_{e,s}(A_s \cup B_s; n) = 0$.

Stage $s = 0$. Let $E_0 = S_{e,0} = \varnothing$ for all e.

Stage $s + 1$. Enumerate into S_e all n (if any) via which P_e requires attention. For each such n, find the least $t > s$ such that

$$(A_s \cup B_s)\restriction \psi_e(n) \neq (A_t \cup B_t)\restriction \psi_e(n).$$

If the change is on the A-side (namely $A_s\restriction \psi_e(n) \neq A_t\restriction \psi_e(n)$) and P_e is not yet satisfied at s, then put the least $x > 2e$ satisfying (i)–(iii) into E. This completes the construction of E.

Now $E \leqslant_T A$ by permitting since for all x if $A\restriction x = A_s\restriction x$ then $E\restriction x = E_s\restriction x$ and similarly $E \leqslant_T C$. (Note that $E \leqslant_{\text{wtt}} A$ and $E \leqslant_{\text{wtt}} C$, so A is not wtt-cappable.) Clearly $\bar{E}$ is infinite since each P_e contributes at most one number x to E and $x > 2e$. Finally, since $K \nleqslant_{\text{wtt}} B$ there exist n and s such that $n \in K_s - K_{s-1}$ and $B\restriction \psi_e(n) = B_s\restriction \psi_e(n)$. If W_e is infinite and P_e is not satisfied by stage s then P_e will be attacked via n (using the nonrecursiveness of C to achieve (ii)). The attack succeeds since some $y < \psi_e(n)$ must later enter $A \cup B$ but it cannot enter B. ∎

The same proof shows that if $K \nleqslant_T B$, and $K \leqslant_T A \oplus B$ say $K = \Psi(A \oplus B)$, and the use function ψ is B-recursive (for example if K is coded into $A \oplus B$ using coding markers which move only when B permits) then $\deg(A)$ is noncappable.

It can be shown that not every noncappable degree has a wtt-cuppable representative. This follows because wtt-cuppable degrees cannot be contiguous, but there are contiguous noncappable degrees. (A r.e. Turing degree is *contiguous* if it contains only one r.e. wtt-degree. Ladner and Sasso [**1975**] have shown the existence of nonzero contiguous r.e. degrees, and their construction may easily be combined with Yates construction [**1966**] of a noncappable r.e. degree.)

2. Ideals and filters in R. We showed in Lemma 1.12 that **ENC**, and thus by Corollary 1.14 also each of **PS** and **NC**, form a strong filter in **R**. Next we show that their complement **M** forms an ideal and we explore some density type results involving **M** and **NC**. The first result in the direction of showing **M** an ideal is Lachlan's nondiamond theorem (Lachlan [**1966**, Theorem 5]). Next Jockusch used Lachlan's method to show that $\mathbf{0}'$ cannot be expressed as a sup of degrees in **M**, so that the ideal generated by **M** is proper.

THEOREM 2.1. *The cappable r.e. degrees* **M** *form an ideal in* **R** (*namely are closed downward and under join*).

COROLLARY 2.2. *The r.e. degrees* **R** *can be decomposed into the disjoint union of a definable strong filter* **NC** *and a definable ideal* **M**. ∎

PROOF OF THEOREM 2.1. The proof is similar to that of Lachlan's nondiamond theorem Lachlan [**1966**] (see the proof in Soare [**1980**, §6]).

Clearly **M** is closed downwards. Thus by Corollary 1.14 it suffices to show that $\overline{\mathbf{PS}}$ is closed under join, namely that if $\mathbf{a}, \mathbf{b} \in \mathbf{R}$ and $\mathbf{a} \cup \mathbf{b} \in \mathbf{PS}$ then either $\mathbf{a} \in \mathbf{PS}$ or $\mathbf{b} \in \mathbf{PS}$. Choose r.e. sets A, B, C such that $C = A \cup B$ is of promptly simple degree via $\{C_s\}_{s\in\omega}$ and $q(s)$ meeting (1.6), $A \subseteq 2\omega$, and $B \subseteq 2\omega + 1$.

Let $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ be recursive enumerations of A and B such that $C_s = A_s \cup B_s$. We shall define a recursive function $p(s)$ and partial recursive functions $\hat{p}^i(t)$, all $i \in \omega$, such that either A is of promptly simple degree via p satisfying (1.6) or else any witness W_i to the failure of p guarantees that $\hat{p}^i$ is total and that B is of promptly simple degree via $\hat{p}^i$ satisfying (1.6). Applying Theorem 1.6 we attempt to satisfy, for all i and j, the requirement

$$P_{i,j}: (\exists x)(\exists s)\left[x \in W_{i,\mathrm{at}\,s} \,\&\, A_s\restriction x \neq A_{p(s)}\restriction x\right]$$

or

$$(\exists y)(\exists t)\left[x \in W_{j,\mathrm{at}\,t} \,\&\, B_t\restriction y \neq B_{\hat{p}^i(t)}\restriction y\right].$$

During the construction we define r.e. sets $U_{i,j}$ and assume that $g(i, j)$ is the corresponding function satisfying Lemma 1.5. The sets $U_{i,j}$ are used to "force" numbers to enter C (and hence A or B) promptly.

Construction of p and $\hat{p}^i$.

Stage $s = 0$. Set $p(0) = 0$.

Stage $s + 1$. Find the unique x and i such that $x \in W_{i,\mathrm{at}\,s}$. For each $j \leqslant s$ find the least $t < s$ and $y < s$ such that

$$(2.1) \qquad y \in W_{j,\mathrm{at}\,t} \,\&\, t \notin \mathrm{dom}\, \hat{p}^i_s \,\&\, z_{i,j} < y,$$

where $z_{i,j} = (\mu z)[z \notin U_{i,j,s}]$. If t and y exist, enumerate $z_{i,j}$ in $U_{i,j,s+1}$, and let $v_{i,j}$ be the least v such that $z_{i,j} \in W_{g(i,j),v}$, and otherwise let $v_{i,j} = s + 1$. (Necessarily $s < v_{i,j}$ by Lemma 1.5.) Define $p(s) = \max\{q(v_{i,j}): j \leqslant s\}$. Define $\hat{p}^i(t) = p(s)$ for all $t \leqslant s$, $t \notin \mathrm{dom}(\hat{p}^i_s)$.

This ends the construction.

We claim that if A is not of promptly simple degree then B is. Choose i such that W_i is infinite but for all x

$$(2.2) \qquad x \in W_{i,\mathrm{at}\,s} \Rightarrow A_s\restriction x = A_{p(s)}\restriction x.$$

Since W_i is infinite it follows that $\hat{p}^i$ is total. If W_j is infinite then $U_{i,j}$ is infinite, so $W_{g(i,j)}$ is infinite. Hence, there exist x, s and y such that $x \in W_{i,\mathrm{at}\,s}$, y satisfies (2.1), and $C_s\restriction z_{i,j} \neq C_{q(v)}\restriction z_{i,j}$ where $v = v_{i,j}$. But $p(s) \geqslant q(v)$ and $A_s\restriction x = A_{p(s)}\restriction x$ so $B_t\restriction y \neq B_{\hat{p}^i(t)}\restriction y$ since $z_{i,j} \leqslant y$ and $t < s \leqslant v < q(v) \leqslant p(s) = \hat{p}^i(t)$. Hence, B is of promptly simple degree via $\hat{p}^i$. ■

Note that Theorem 2.1 can also be proved from $\mathbf{NC} = \mathbf{LC}$ (Corollary 1.18) and Corollary 4.1 of Ambos-Spies **[1980]** or Corollary 1 of Ambos-Spies **[1983]**.

COROLLARY 2.3. *If* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbf{M}$ *then there exists an r.e. degree* $\mathbf{b} > \mathbf{0}$ *such that* $\mathbf{a}_i \cap \mathbf{b} = \mathbf{0}$ *for all* $i \leqslant n$, *and furthermore an index of a representative of* $\mathbf{b}$ *may be found effectively from indices of representatives of* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

PROOF. This is proved by Theorem 2.1 and Corollary 1.16. ■

Theorem 2.1 shows that the join of two cappable degrees is cappable. Also any $\mathbf{a} \in \mathbf{M}$ is equal to $\mathbf{b} \cup \mathbf{c}$ for strictly smaller degrees $\mathbf{b}, \mathbf{c} \in \mathbf{M}$ by the Sacks splitting theorem and the downward closure of $\mathbf{M}$. However, we cannot necessarily choose $\mathbf{b}$ and $\mathbf{c}$ to form a minimal pair because Lachlan **[1979]** has constructed an r.e. degree $\mathbf{a}$

which bounds no minimal pair. It is easy to see that any r.e. degree $\mathbf{d} > \mathbf{0}$ either bounds a minimal pair or is part of one, so Lachlan's degree $\mathbf{a}$ is in $\mathbf{M}$. L. Welch has shown [**1981**] that there is no r.e. degree $\mathbf{a} < \mathbf{0}'$ such that $\mathbf{a} \geqslant \mathbf{b}$ for every $\mathbf{b} \in \mathbf{M}$. Thus, $\mathbf{M}$ is not contained in any proper principal ideal of $\mathbf{R}$. Dually, a straightforward cone-avoidance argument shows that $\mathbf{NC}$ is not contained in any proper principal filter of $\mathbf{R}$. Also note that $\mathbf{M}$ is not a maximal ideal since by Corollary 1.18 the ideal generated by $\mathbf{M} \cup \{\mathbf{a}\}$ is proper for every low r.e. degree $\mathbf{a}$. (There are low r.e. degrees $\mathbf{a} \notin \mathbf{M}$ since there are low r.e. degrees with join $\mathbf{0}'$ so low r.e. degrees can be low cuppable and hence noncappable.) However, $\mathbf{M}$ is a prime ideal (i.e. $\mathbf{a} \cap \mathbf{b} \in \mathbf{M}$ implies $\mathbf{a} \in \mathbf{M}$ or $\mathbf{b} \in \mathbf{M}$) because its complement $\mathbf{NC}$ is a filter.

Since $\mathbf{M}$ forms an ideal in $\mathbf{R}$ it is natural to study the quotient structure $\mathbf{R}/\mathbf{M}$ as first suggested by Jockusch. S. Schwarz [**1982**] has shown that the Friedberg-Muchnik theorem and even the Sacks splitting theorem hold for $\mathbf{R}/\mathbf{M}$ but the existence of minimal pairs fails. It is unknown whether density or, more generally, the Shoenfield conjecture holds in $\mathbf{R}/\mathbf{M}$. S. Schwarz [**1982**] has also classified the index sets $\{e\colon W_e$ is promptly simple$\}$ and $\{e\colon W_e$ is of promptly simple degree$\}$ as each Σ_4-complete. It is unknown whether the previous results can be strengthened to show that $\mathbf{M}(\mathbf{NC})$ forms an effective Σ-ideal (effective Σ-filter) namely whether for any r.e. sequence $\{\mathbf{a}_n\}_{n\in\omega}$ of degrees in $\mathbf{M}(\mathbf{NC})$ there exists a degree $\mathbf{b} \in \mathbf{M}(\mathbf{NC})$ such that $\mathbf{b} > \mathbf{a}_n$ ($\mathbf{b} < \mathbf{a}_n$) for all $n \in \omega$.

We turn now to some density type results. Since $\mathbf{M}$ forms an ideal we have immediately

$$\mathbf{a} \in \mathbf{M} \Rightarrow (\exists \mathbf{b} > \mathbf{a})[\mathbf{b} \in \mathbf{M}] \tag{2.3}$$

and

$$\mathbf{a} \in \mathbf{NC} \Rightarrow (\exists \mathbf{b} < \mathbf{a})[\mathbf{b} \in \mathbf{NC}]. \tag{2.4}$$

For (2.4) apply the Sacks splitting theorem to get $\mathbf{a} = \mathbf{a}_0 \cup \mathbf{a}_1$, $\mathbf{a}_i < \mathbf{a}$, and note that we cannot have both $\mathbf{a}_0, \mathbf{a}_1 \in \mathbf{M}$. The following local version sharpens these results.

THEOREM 2.4. *If* $\mathbf{a} < \mathbf{b}$, $\mathbf{a} \in \mathbf{M}$ *and* $\mathbf{b} \in \mathbf{NC}$ *then*

(i) $(\exists \mathbf{c})[\mathbf{a} < \mathbf{c} < \mathbf{b} \,\&\, \mathbf{c} \in \mathbf{M}]$,

(ii) $(\exists \mathbf{d})[\mathbf{a} < \mathbf{d} < \mathbf{b} \,\&\, \mathbf{d} \in \mathbf{NC}]$,

and indeed (ii) *can be strengthened to*

(iii) $(\exists \mathbf{d}_0, \mathbf{d}_1)[\mathbf{a} < \mathbf{d}_0, \mathbf{d}_1 < \mathbf{b} \,\&\, \mathbf{d}_0, \mathbf{d}_1 \in \mathbf{NC} \,\&\, \mathbf{b} = \mathbf{d}_0 \cup \mathbf{d}_1]$

so that every noncappable degree splits over every lesser cappable degree.

PROOF. (i) Since $\mathbf{a} \in \mathbf{M}$ choose $\mathbf{e} > \mathbf{0}$ such that $\mathbf{a} \cap \mathbf{e} = \mathbf{0}$. Since $\mathbf{b} \in \mathbf{NC}$ we may assume without loss of generality that $\mathbf{e} < \mathbf{b}$. Now take $\mathbf{c} = \mathbf{a} \cup \mathbf{e}$.

(ii) This follows by (iii).

(iii) By Corollary 1.18, $\mathbf{NC} = \mathbf{LC}$, so there exists a low r.e. degree $\mathbf{f}$ such that $\mathbf{b} \cup \mathbf{f} = \mathbf{0}'$. Using the technique of the Robinson splitting theorem (Robinson, [**1971**]) split $\mathbf{b}$ into degrees $\mathbf{b}_0$ and $\mathbf{b}_1$ such that $\mathbf{b}_0 \cup \mathbf{f}$ and $\mathbf{b}_1 \cup \mathbf{f}$ are both low. Let

$\mathbf{d}_0 = \mathbf{a} \cup \mathbf{b}_0$ and $\mathbf{d}_1 = \mathbf{a} \cup \mathbf{b}_1$. Both are low cuppable ($\mathbf{0}' = \mathbf{b} \cup \mathbf{f} = \mathbf{b}_i \cup (\mathbf{b}_{1-i} \cup \mathbf{f}) \leqslant \mathbf{d}_i \cup (\mathbf{b}_{1-i} \cup \mathbf{f})$) and thus noncappable. Hence, it suffices to show that $\mathbf{d}_i < \mathbf{b}$. If not then $\mathbf{d}_i = \mathbf{b}$, and hence

$$\mathbf{a} \cup (\mathbf{b}_i \cup \mathbf{f}) = (\mathbf{a} \cup \mathbf{b}_i) \cup \mathbf{f} = \mathbf{d}_i \cup \mathbf{f} = \mathbf{b} \cup \mathbf{f} = \mathbf{0}',$$

i.e. $\mathbf{a}$ is low cuppable (by $\mathbf{b}_i \cup \mathbf{f}$) and hence noncappable. ■

References

[1980] K. Ambos-Spies, *On the structure of the recursively enumerable degrees*, Ph.D. dissertation, Univ. of Munich.

[1983] ______, *An extension of the non-diamond theorem in classical and α-recursion theory*, J. Symbolic Logic.

[to appear] ______, *Non-cappability in the r.e. weak truth table and Turing degrees*,

[1980] P. Fejer, *The structure of definable subclasses of the recursively enumerable degrees*, Ph.D. dissertation, Univ. of Chicago.

[1980] P. Fejer and R. I. Soare, *The plus-cupping theorem for the recursively enumerable degrees*, Proceedings of Logic Year 1979-80 (M. Lerman, J. H. Schmerl and R. I. Soare, eds.), Lecture Notes in Math., no. 859, Springer-Verlag, Berlin and New York, pp. 49–62.

[1966] A. H. Lachlan, *Lower bounds for pairs of r.e. degrees*, Proc. London Math. Soc. **16**, 537–569.

[1968] ______, *Degrees of recursively enumerable sets which have no maximal superset*, J. Symbolic Logic **33**, 431–443.

[1979] ______, *Bounding minimal pairs*, J. Symbolic Logic **44**, 626–642.

[1975] R. E. Ladner and L. P. Sasso, *The weak truth table degrees of recursively enumerable sets*, Ann. of Math. Logic **4**, 429–448.

[1982] W. Maass, *Recursively enumerable generic sets*, J. Symbolic Logic **47**, 809–823.

[1981] W. Maass, R. A. Shore and M. Stob, *Splitting properties and jump classes*, Israel J. Math. **39**, 210–224.

[1980] D. Miller, *High recursively enumerable degrees and the anti-cupping property*, Proceedings of Logic Year 1979-80, (M. Lerman, J. H. Schmerl and R. I. Soare, eds.), Lecture Notes in Math., no. 859, Springer-Verlag, Berlin and New York, pp. 230–267.

[1971] R. W. Robinson, *Jump restricted interpolation in the r.e. degrees*, Ann. of Math. (2) **93**, 586–596.

[1967] H. Rogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York.

[1964] G. E. Sacks, *The recursively enumerable degrees are dense*, Ann. of Math. (2) **80**, 300–312.

[1966] ______, *Degrees of unsolvability*, (rev. ed.), Ann. of Math. Studies, no. 55, Princeton Univ. Press, Princeton, N. J.

[1982] S. Schwarz, *Index sets of recursively enumerable sets, quotient lattices, and recursive linear orderings*. Ph.D. dissertation, Univ. of Chicago.

[1965] J. R. Shoenfield, *Applications of model theory to degrees of unsolvability*, Sympos. Theory of Models (J. W. Addison, L. Henkin and A. Tarski, eds.), North-Holland, Amsterdam, pp. 359–363.

[1976] ______, *Degrees of classes of r.e. sets*, J. Symbolic Logic **41**, 695–696.

[1974] R. I. Soare, *Automorphisms of the lattice of recursively enumerable sets. Part* I: *Maximal sets*, Ann. of Math. (2) **100**, 80–120.

[1976] ______, *The infinite injury priority method*, J. Symbolic Logic **41**, 513–530.

[1977] ______, *Computational complexity, speedable, and levelable sets*, J. Symbolic Logic **42**, 545–563.

[1978] ______, *Recursively enumerable sets and degrees*, Bull. Amer. Math. Soc. **84**, 1149–1181.

[1980] ______, *Fundamental methods for constructing recursively enumerable degrees. Recursion theory: its generalisations and applications*, (F. R. Drake and S. S. Wainer, eds.), Cambridge Univ. Press, Cambridge, pp. 1–51.

[1982a] ______, *Automorphisms of the lattice of recursively enumerable sets*. II: *Low sets*, Ann. of Math. Logic **22**, 69–107.

[1982b] ______, *Computational complexity of recursively enumerable sets*, Information and Control **52**, 8–18.

[1981] L. Welch, *A hierarchy of families of recursively enumerable degrees and a theorem on bounding minimal pairs*, Ph.D. dissertation, Univ. of Illinois.

[1966] C. E. M. Yates, *A minimal pair of r.e. degrees*, J. Symbolic Logic **31**, 159–168.

Lehrstuhl Informatik II, Universität Dortmund, Dortmund, West Germany

Department of Mathematics, University of Illinois, Urbana, Illinois 61801

Department of Mathematics, Cornell University, Ithaca, New York 14853

Department of Mathematics, University of Chicago, Chicago, Illinois 60637