THE NULL SPACE AND THE RANGE OF A
CONVOLUTION OPERATOR
IN A FADING MEMORY SPACE

BY

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ABSTRACT. We study the convolution equation

$$(*) \qquad \mu * x'(t) + \nu * x(t) = f(t) \qquad (-\infty < t < \infty),$$

as well as a perturbed version of $(*)$, namely

$$(**) \qquad \mu * x'(t) + \nu * x(t) = F(x)(t) \qquad (-\infty < t < \infty).$$

Here x is a $\mathbf{R}^n$-valued function on $(-\infty, \infty)$, $x'(t) = dx(t)/dt$, and μ and ν are matrix-valued measures. If μ and ν are supported on $[0, \infty)$, with μ atomic at zero, then $(*)$ can be regarded as a linear, autonomous, neutral functional differential equation with infinite delay. However, most of the time we do not consider the ordinary Cauchy problem for the neutral equation, i.e. we do not suppose that μ and ν are supported on $[0, \infty)$, prescribe an initial condition of the type $x(t) = \xi(t)$ $(t \leqslant 0)$, and require $(*)$ and $(**)$ to hold only for $t \geqslant 0$. Instead we permit $(*)$ and $(**)$ to be of "Fredholm" type, i.e. μ and ν need not vanish on $(-\infty, 0)$, we restrict the growth rate of x and f at plus and minus infinity, and we look at the problem of the existence and uniqueness of solutions of $(*)$ and $(**)$ on the whole real line, satisfying conditions like $|x(t)| \leqslant C\eta(t)$ $(-\infty < t < \infty)$, where C is a constant, depending on x, and η is a predefined function. Some authors use the word "admissible" when discussing problems of this type. In the case when the homogeneous version of $(*)$ has nonzero solutions, we decompose the solutions into components with different exponential growth rates, and give a priori bounds on the growth rates of the solutions. As an application of the basic theory, we look at the Cauchy problem for a neutral functional differential equation, and prove the existence of stable and unstable manifolds.

Introduction. We study the convolution equation

$$(1.1) \qquad Lx(t) = f(t) \qquad (t \in \mathbf{R}),$$

as well as a perturbed version of (1.1), namely

$$(1.2) \qquad Lx(t) = F(x)(t) \qquad (t \in \mathbf{R}),$$

where

$$(1.3) \qquad Lx(t) = \mu * x'(t) + \nu * x(t) \qquad (t \in \mathbf{R}).$$

Here x is a $\mathbf{R}^n$-valued function on $\mathbf{R} = (-\infty, \infty)$, $x'(t) = dx(t)/dt$, and μ and ν are measures, which are finite with respect to a suitable weight. If μ and ν are supported on $\mathbf{R}^+ = [0, \infty)$, and μ is atomic at zero, then (1.1) is a linear, autonomous neutral

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functional differential equation with infinite delay, of the type considered in [8]. In particular, if μ and ν are supported on a finite interval $[0, r]$, then (1.1) is a neutral functional differential equation with finite delay, of the type considered in [3]. If μ consists of a single point mass at zero, and ν is absolutely continuous, i.e.

$$Lx(t) = x'(t) + \int_{-\infty}^{t} a(t-s)x(s)\,ds \qquad (t \in \mathbf{R})$$

for some integrable function a, then (1.1) is a Volterra integrodifferential equation (which is a special case of the neutral equation). However, most of the time we do not consider the ordinary Cauchy problem for the neutral equation, i.e. we do not suppose that μ and ν are supported on $[0, \infty)$, prescribe an initial condition of the type $x(t) = \xi(t)$ $(t \leqslant 0)$ and require (1.1) and (1.2) to hold only on $\mathbf{R}^+$. Instead we permit (1.1) and (1.2) to be of "Fredholm type", i.e. μ and ν need not vanish on $(-\infty, 0)$, we restrict the growth rate of x and f at plus and minus infinity, and we look at the problem of the existence and uniqueness of solutions of (1.1) and (1.2) on the whole real line with the specified growth rate. Nonuniqueness of the solution of (1.1) means that the operator L is not one-to-one, and nonexistence of a solution of (1.1) means that L is not onto, regarded as an operator from one function space into another. Some authors use the word "admissible" when discussing problems of this type. In the case when the homogeneous version of (1.1) has nonzero solutions, we decompose the solutions into components with different exponential growth rates, and we give a priori bounds on the growth rates of the solutions. As an application of the basic theory, we look at the Cauchy problem for the neutral functional differential equation and prove the existence of stable and unstable manifolds.

Earlier work on the null space (and the range) of a convolution operator of the type (1.3) has more or less been restricted to the case when x and Lx are bounded, continuous functions. Typically, one has asked the question whether (1.1) has a bounded (almost periodic) [periodic] solution for every bounded (almost periodic) [periodic] forcing function f. Some of our results can also be applied to this situation (see §3).

The original motivation for this work comes from the application in §7, i.e. from the stable and unstable manifold problem for the neutral equation. In a neutral equation the operator L is "causal", i.e. the values of Lx in an interval $(-\infty, T]$ depend only on the restriction of x to the same interval. In spite of this fact, the problem of finding a stable manifold for the equation is not a causal one: To determine whether an initial function (defined on $(-\infty, 0]$) belongs to the stable manifold or not, one has to look at the future values of the solution of the initial value problem. In [8], the author discussed this question, and an examination of the technique in [8] reveals that it is really of "Fredholm" type rather than of "Volterra" type. Here, we have phrased the problem from the very beginning in a Fredholm setting. In some sense, this seems to be the most natural setting for the stable manifold problem, and it leads to a theory which is quite interesting in its own right. For example, with the exception of §7, all our results can be applied equally well to advanced functional differential equations. To make this theory more complete, we have included a section on a "critical" case with "large influence function" which is not needed in the stable manifold theory for the neutral equation.

The outline of the paper is the following. In §2 we define the memory spaces which we use, and recall their most important properties. Basically, these spaces are either weighted L^p-spaces, or spaces of continuous functions with restricted growth rate at plus and minus infinity. Let $\mathscr{B}$ be any one of these spaces, and let $\mathscr{B}^m$ be the space of functions in $\mathscr{B}$ whose (distribution) derivatives up to order m belong to $\mathscr{B}$. The conditions put on μ and ν imply that the convolution operators $\mu *$ and $\nu *$ are well defined, and map $\mathscr{B}^m$ continuously into itself. In particular, the operator L in (1.3) maps $\mathscr{B}^{m+1}$ continuously into $\mathscr{B}^m$.

§3 is devoted to a study of the "noncritical" case. Roughly speaking, (1.1) is noncritical with respect to a memory space $\mathscr{B}^m$ if there is a unique solution $x \in \mathscr{B}^{m+1}$ for each f in $\mathscr{B}^m$, and both x and x' depend continuously on f in the norm of $\mathscr{B}^m$. In other words, the operator L is not only continuous from $\mathscr{B}^{m+1}$ into $\mathscr{B}^m$, it also has a continuous inverse. For simplicity we have throughout assumed that L is of the form (1.3), because that makes it possible to obtain reasonably simple conditions on L which may imply that L is bicontinuous from $\mathscr{B}^{m+1}$ onto $\mathscr{B}^m$. However, the form (1.3) is not really crucial in the sense that in many of our theorems it suffices to assume that L is a linear, bicontinuous map from $\mathscr{B}^{m+1}$ onto $\mathscr{B}^m$ (for all needed different function spaces). In particular, L need not really be autonomous, i.e. it need not commute with translations.

The remainder of the paper is devoted to the "critical" case when L is not a bicontinuous map from $\mathscr{B}^{m+1}$ onto $\mathscr{B}^m$. To obtain any results whatsoever we have to assume that one can make L noncritical by adjusting the growth rates of the functions in $\mathscr{B}^m$ at plus and minus infinity. If L is of the form (1.3), then this assumption is not too difficult to satisfy. We first study the homogeneous linear equation, i.e. (1.1) with f replaced by zero, in §4. Lemma 4.1 contains a formula which generates all the appropriate solutions of the homogeneous equation, and Lemma 4.2 gives an a priori estimate on the growth rate of these solutions. Under certain additional conditions they all lie in a finite dimensional subspace of $\mathscr{B}^{m+1}$, as Lemma 4.3 shows. We give an example on a homogeneous equation whose solutions do not lie in a finite dimensional subspace of $\mathscr{B}^{m+1}$, and show that the solutions of this equation can be split into components with different exponential growth rates.

In §§5 and 6 we again study the nonhomogeneous equation. The basic results can be summarized as follows: If the functions in $\mathscr{B}^m$ are permitted to be "large enough" at plus and minus infinity, then (1.1) has a solution in $\mathscr{B}^{m+1}$ for every $f \in \mathscr{B}^m$, but the solution is not unique. If the functions in $\mathscr{B}^m$ are required to be "small enough" at plus infinity, then the solution of (1.1) is unique whenever it exists, but (1.1) does not have a solution for every $f \in \mathscr{B}^m$. In the former case the operator L has a continuous right inverse, and in the latter case, it has a continuous left inverse. We give a simple example illustrating the problems one can run into when the growth rate at infinity of x and f is neither "small" nor "large".

In §7 we show how the results in §§3–5 can be used to sharpen some of the results to [8] on the existence of stable and unstable manifolds for a neutral functional differential equation with infinite delay. The main improvement, as compared in [8], is that here we permit forcing functions with less restricted growth rates. Also, we show that it is possible to study the equations in a "smooth" setting, i.e. one can

require the solution to be *m* times differentiable, provided one works in a manifold with finite codimension.

The existence of continuous left and right inverses of L is of crucial importance when one studies the nonlinear equation (1.2). As soon as one has good enough estimates on (1.1), one can study (1.2) by means of a perturbation technique, and this is the approach we use. All our results for (1.2) are essentially based on the contraction mapping principle, or on the implicit function theorem. As we show with three examples in §3, sometimes the straightforward perturbation method discussed here gives results which are almost as sharp as, or in some cases even sharper than, results obtained with, e.g., Lyapunov type methods.

2. Memory spaces and fundamental solutions. We use the same memory spaces throughout as in [8]. For more details, see [8].

We call ρ a *dominating function* if ρ is strictly positive, continuous and submultiplicative on $\mathbf{R}$, i.e.

$$(2.1) \qquad \rho(s+t) \leqslant \rho(s)\rho(t) \qquad (s, t \in \mathbf{R}),$$

and $\rho(0) = 1$. We call η an *influence function* dominated by ρ if η is continuous, strictly positive, $\eta(0) = 1$, and

$$(2.2) \qquad \eta(s+t) \leqslant \eta(s)\rho(t) \qquad (s, t \in \mathbf{R}).$$

In particular, ρ is an influence function dominated by itself. For each influence function η we define the *adjoint influence function* $\tilde{\eta}$ by

$$(2.3) \qquad \tilde{\eta}(t) = (\eta(-t))^{-1} \qquad (t \in \mathbf{R}).$$

If η is dominated by ρ, then so is $\tilde{\eta}$. Also, the adjoint function of the adjoint function of η is η itself. It follows from (2.1)–(2.3) and $\eta(0) = 1$ that η is bounded from below and from above by

$$(2.4) \qquad \tilde{\rho}(t) \leqslant \eta(t) \leqslant \rho(t) \qquad (t \in \mathbf{R}).$$

We let $M(\eta; \mathbf{R})$ be the set of all real, locally finite measures μ on $\mathbf{R}$ such that

$$\|\mu\| = \int_{\mathbf{R}} \eta(t) \, d|\mu|(t) < \infty.$$

Here $|\mu|$ is the total variation measure of μ. Let $\mathbf{R}^{n \times n}$ be the set of $n \times n$-dimensional real matrices, and let $M(\eta; \mathbf{R}^{n \times n})$ be the matrix-valued analogue of $M(\eta; \mathbf{R})$. We abbreviate both $M(\eta; \mathbf{R})$ and $M(\eta; \mathbf{R}^{n \times n})$ by $M(\eta)$.

In the following definitions of functions spaces, the functions are defined on $\mathbf{R}$, and their values lie in $\mathbf{R}$, $\mathbf{R}^n$ or $\mathbf{R}^{n \times n}$.

A function φ belongs to $BUC(\eta)$, if $\eta(t)\varphi(t)$ is bounded and uniformly continuous. If in addition $\eta(t)\varphi(t) \to 0$ as $t \to \infty$ ($t \to -\infty$), then φ belongs to $BUC_+(\eta)$ ($BUC_-(\eta)$). The intersection of $BUC_+(\eta)$ and $BUC_-(\eta)$ is $BC_0(\eta)$. The function space $L^p(\eta)$ consists of those functions φ which satisfy $\eta\varphi \in L^p$, where L^p stands for the standard L^p-space over $\mathbf{R}$ (whenever we omit η from the notation this means that we take $\eta(t) \equiv 1$). We let $\mathfrak{B}(\eta)$ stand for any one of the function spaces listed above. Finally, $\mathfrak{B}^m(\eta)$ consists of those functions φ which satisfy $\varphi, \varphi', \dots, \varphi^{(m)} \in \mathfrak{B}(\eta)$ (here $\varphi^{(k)}$ is the kth order distribution derivative of φ), and $\mathfrak{B}^0(\eta) = \mathfrak{B}(\eta)$.

It was shown in [8] that if $\mu \in M(\rho)$, and $\varphi \in \mathcal{B}^m(\eta)$, then *the convolution $\mu * \varphi$,* defined formally by

$$\mu * \varphi(t) = \int_{-\infty}^{\infty} d\mu(s)\varphi(t-s),$$

makes sense, and that $\mu * \varphi \in \mathcal{B}^m(\eta)$. Moreover, if $a \in L^1(\rho)$, $a' \in M(\rho)$, and $\varphi \in \mathcal{B}^m(\eta)$, then $a * \varphi \in \mathcal{B}^{m+1}(\eta)$. Below we shall also need a few more results of the same type. For easy reference we have collected all the various combinations of dominating functions and influence functions in convolutions that we shall need into the following list:

$$\mu \in M(\rho), \quad \varphi \in \mathcal{B}^m(\eta) \Rightarrow \mu * \varphi \in \mathcal{B}^m(\eta),$$

$$\mu \in M(\eta), \quad \varphi \in \mathcal{B}^m(\rho) \Rightarrow \mu * \varphi \in \mathcal{B}^m(\eta),$$

$$a \in L^1(\rho), \quad a' \in M(\rho), \quad \varphi \in \mathcal{B}^m(\eta) \Rightarrow a * \varphi \in \mathcal{B}^{m+1}(\eta),$$

$$a \in L^1(\eta), \quad a' \in M(\eta), \quad \varphi \in \mathcal{B}^m(\rho) \Rightarrow a * \varphi \in \mathcal{B}^{m+1}(\eta),$$

$$\mu \in M(\eta), \quad \varphi \in \mathcal{B}^m(\tilde{\eta}) \Rightarrow \mu * \varphi \in \mathcal{B}^m(\tilde{\rho}),$$

$$a \in L^1(\eta), \quad a' \in M(\eta), \quad \varphi \in \mathcal{B}^m(\tilde{\eta}) \Rightarrow a * \varphi \in \mathcal{B}^{m+1}(\tilde{\rho}).$$

These implications are all proved in the same way as in [8].

To the dominating function ρ we adjoin two characteristic numbers

$$\alpha = -\sup_{t<0} \frac{\log \rho(t)}{t}, \qquad \omega = -\inf_{t>0} \frac{\log \rho(t)}{t}$$

(in [8] these two numbers were called ρ^* and ρ_*). They describe the growth rate of ρ at infinity in the sense that $\liminf_{t\to\infty} \rho(t)e^{\omega t} > 0$, but $\rho(t) = O(e^{(-\omega+\varepsilon)t})$ ($t \to \infty$) for every $\varepsilon > 0$, and similarly at $-\infty$. It is always true that $-\infty < \omega \leqslant \alpha < \infty$.

Throughout we assume that the measures μ and ν in (1.1), (1.2) belong to $M(\rho, \mathbf{R}^{n \times n})$. This implies that their bilateral Laplace transforms $\hat{\mu}(z)$ and $\hat{\nu}(z)$ converge absolutely for $\omega \leqslant \operatorname{Re} z \leqslant \alpha$. Formally, if one takes Laplace transforms in (1.1), then one gets

(2.5)
$$D(z)\hat{x}(z) = \hat{f}(z),$$

where

(2.6)
$$D(z) = z\hat{\mu}(z) + \hat{\nu}(z)$$

is the *characteristic function* of L (i.e. the Laplace transform of L). If

(2.7)
$$\det D(z) \neq 0 \qquad (\omega \leqslant \operatorname{Re} z \leqslant \alpha).$$

then it may be possible to find a function $r \in L^1(\rho)$, satisfying

(2.8)
$$\hat{r}(z) = [D(z)]^{-1} \qquad (\omega \leqslant \operatorname{Re} z \leqslant \alpha),$$

and the solution of (2.5) ought to be given by $x = r * f$.

The preceding formal argument was made precise in [8] for the operator L corresponding to the neutral equation, and same reasoning can be applied to the more general operators considered here. In addition to (2.7) we need a technical condition on $\hat{\mu}$, namely "invertibility at infinity".

We call $\hat{\mu}$ *invertible at infinity* (with respect to ρ), if $[\det \hat{\mu}]^{-1}$ is pseudo-locally analytic at infinity in the sense of Definition 8.1 in [4]. If $\hat{\mu}$ is invertible at infinity with respect to ρ, then necessarily

$$(2.9) \qquad \liminf_{\substack{|z| \to \infty \\ \omega \leqslant \operatorname{Re} z \leqslant \alpha}} |\det \hat{\mu}(z)| > 0.$$

Conversely, if μ has no singular part, then (2.9) implies invertibility at infinity of $\hat{\mu}$. There do exist measures μ satisfying (2.9) for which $\hat{\mu}$ is not invertible at infinity. These are measures in which the singular part is large, compared to the discrete part, so they are rather pathological. One should really think of the invertibility at infinity as a technical condition approximately equivalent to (2.9).

If $\hat{\mu}$ is invertible at infinity (with respect to ρ), and if the determinent condition (2.7) is satisfied, then we claim that there exists a function $r \in L^1(\rho)$ satisfying (2.8). This function also satisfies $r' \in M(\rho)$, where r' is the distribution derivative of r, and

$$(2.10) \qquad r' * \mu + r * \nu = \mu * r' + \nu * r = \delta$$

(see [8] for a definition of the convolution of two measures). Here δ is the unit point mass at zero. We call r the *fundamental solution* of (1.1) (corresponding to ρ). The proof of the existence of a fundamental solution r is essentially the same as the proof of Theorem 5.1 in [8] (actually, it is somewhat simpler). Instead of using the measure q in formula (5.17) in [8], one chooses some $\lambda > \omega$, defines $\hat{e}(z) = (z + \lambda)^{-1}$, and writes $\hat{r}$ in the form

$$(2.11) \qquad \hat{r}(z) = \hat{e}(z)[\hat{\mu}(z) + \hat{e}(z)(\hat{\nu}(z) - \lambda\hat{\mu}(z))]^{-1}$$

$$= \hat{e}(z) \frac{\operatorname{adj}[\hat{\mu}(z) + \hat{e}(z)(\hat{\nu}(z) - \lambda\hat{\mu}(z))]}{\det[\hat{\mu}(z) + \hat{e}(z)(\hat{\nu}(z) - \lambda\hat{\mu}(z))]}.$$

That the denominator in (2.11) can be inverted follows directly from [4, Proposition 8.2]. The rest of the argument is the same as in [8].

It is not always true that $\hat{\mu}$ is invertible at infinity with respect to the original dominating function ρ, but it may be invertible at infinity with respect to some other dominating function ρ_1 satisfying $\rho_1(t) \leqslant \rho(t)$ ($t \in \mathbf{R}$). This observation becomes important in §§4–7.

3. The noncritical case. We call (1.1) noncritical (with respect to ρ), if $\hat{\mu}$ is invertible at infinity with respect to ρ (cf. §2), and the determinant condition (2.7) is satisfied.

LEMMA 3.1. *Let (1.1) be noncritical, let η be an influence function dominated by ρ, and let $m \geqslant 0$. Then L, defined in (1.3), maps $\mathfrak{B}^{m+1}(\eta)$ continuously and one-to-one onto $\mathfrak{B}^m(\eta)$. Its inverse is the resolvent mapping which takes $f \in \mathfrak{B}^m(\eta)$ into $x = r * f$, where r is the solution of (2.10).*

PROOF. Clearly L is continuous, so to prove the lemma it suffices to show that for every $f \in \mathfrak{B}^m(\eta)$ the function $x = r * f$ is the unique solution of (1.1) in $\mathfrak{B}^{m+1}(\eta)$.

Use (2.10), the fact that convolution is associative, and Lemma 3.6 in [8] to get

$$L(r * f) = \mu * (r' * f) + \nu * (r * f) = (\mu * r' + \nu * r) * f = f.$$

Thus, $r * f$ is a solution of (1.1) in $\mathfrak{B}^{m+1}(\eta)$. Similarily, if $x \in \mathfrak{B}^{m+1}(\eta)$, and $Lx = 0$, then (also use Lemmas 3.2 and 3.5 in [**8**]).

$$
\begin{aligned}
0 = r * Lx &= r * (\mu * x') + r * (\nu * x) \\
&= r * (\mu * x)' + r * (\nu * x) \\
&= r' * (\mu * x) + (r * \nu) * x \\
&= (r' * \mu + r * \nu) * x = x,
\end{aligned}
$$

so the solution of (1.1) is unique. $\square$

Lemma 3.1 implies e.g. the $\mathfrak{B}(\mathbf{R})$- and AP-cases of [**3**, Lemma 1.2, p. 206].

Lemma 3.1 is also true for $m = -1$, i.e. L is also continuous and one-to-one from $\mathfrak{B}(\eta)$ onto $\mathfrak{B}^{-1}(\eta)$. This is essentially the conclusion of the following lemma. However, we avoid a direct reference to $\mathfrak{B}^{-1}(\eta)$ by using the same formulation as in [**8**], i.e. we allow two perturbation functions, and replace (1.1) by

$$(3.1) \qquad (\mu * x(t) - g(t))' + \nu * x(t) = f(t) \qquad (t \in \mathbf{R}).$$

LEMMA 3.2. *Let* (1.1) *be noncritical. Then for every* $f \in \mathfrak{B}(\eta)$ *and* $g \in \mathfrak{B}(\eta)$ *there is a unique solution* $x \in \mathfrak{B}(\eta)$ *of* (3.1). *This solution is given by* $x = r * f + r' * g$, *where* r *is the solution of* (2.10).

The converse of Lemma 3.2 is trivial: For every $x \in \mathfrak{B}(\eta)$ we can find functions $f \in \mathfrak{B}(\eta)$ and $g \in \mathfrak{B}(\eta)$ (e.g. $g = \mu * x$ and $f = \nu * x$) such that (3.1) holds.

PROOF. That $x = r * f + r' * g$ is a solution follows from the computation (again see (2.10) and the basic lemmas in [**8**])

$$
\begin{aligned}
(\mu * x - g)' &= [\mu * (r * f + r' * g) - g]' = (\mu * r') * f - [\nu * r * g]' \\
&= f - \nu * (r * f + r' * g) = f - \nu * x.
\end{aligned}
$$

The uniqueness computation is the same as in Lemma 3.1, with $\mu * x' + \nu * x = 0$ replaced by $(\mu * x)' + \nu * x = 0$. $\square$

Of course, the proof works equally well if we replace $\mathfrak{B}(\eta)$ by $\mathfrak{B}^{m+1}(\eta)$ for some $m \geq 0$, but that case is already contained in Lemma 3.1 (replace g by zero and f by $f + g'$).

Before we go on to the critical case, let us mention some elementary perturbation type results for (1.2), and for the nonlinear version

$$(3.2) \qquad [\mu * x(t) - G(x)(t)]' + \nu * x(t) = F(x)(t) \qquad (t \in \mathbf{R})$$

of (3.1). By Lemma 3.1, if we take $x \in \mathfrak{B}^{m+1}(\eta)$ for some $m \geq 0$, and let F in (1.2) map $\mathfrak{B}^{m+1}(\eta)$ into $\mathfrak{B}^m(\eta)$, then (1.2) is equivalent to

$$(3.3) \qquad x = r * F(x).$$

Similarly, if $x \in \mathfrak{B}(\eta)$, and F and G map $\mathfrak{B}(\eta)$ into itself, then (3.2) is equivalent to

$$(3.4) \qquad x = r * F(x) + r' * G(x).$$

Applying the global contraction mapping principle to (3.3) and (3.4) we get the following two theorems.

THEOREM 3.3. *Let* (1.1) *be noncritical,* $m \geq 0$, *and let* F *be a Lipschitz continuous mapping from* $\mathfrak{B}^{m+1}(\eta)$ *into* $\mathfrak{B}^m(\eta)$. *If the Lipschitz constant of* F *is small enough, then* (1.2) *has a unique solution* x *in* $\mathfrak{B}^{m+1}(\eta)$.

THEOREM 3.4. *Let* (1.1) *be noncritical, and let F and G be Lipschitz continuous mappings from* $\mathscr{B}(\eta)$ *into itself. If the Lipschitz constants of F and G are small enough, then* (3.2) *has a unique solution* $x \in \mathscr{B}(\eta)$.

Here "small enough" means that the right-hand sides of (3.3) and (3.4) should be (strict) contractions.

We can also apply the (local) implicit function theorem to (3.3) and (3.4). Let Ω be a neighborhood of zero in $\mathscr{B}^k(\eta)$, and let $C^1(\Omega; \mathscr{B}^p(\eta))$ be the space of continuously differentiable functions from Ω into $\mathscr{B}^p(\eta)$, with norm

$$\|F\| = \sup_{x \in \Omega} \|F(x)\| + \sup_{x \in \Omega} \|DF(x)\|.$$

Here the first norm on the right-hand side is the norm in $\mathscr{B}^p(\eta)$, and the second norm is the norm in the space of bounded linear operators from $\mathscr{B}^k(\eta)$ into $\mathscr{B}^p(\eta)$.

THEOREM 3.5. *Let* (1.1) *be noncritical,* $m \geqslant 0$, *and let* Ω *be a neighborhood of zero in* $\mathscr{B}^{m+1}(\eta)$. *Then there are neighborhoods V and W of zero in* $C^1(\Omega; \mathscr{B}^m(\eta))$ *and* Ω, *respectively, such that for every* $F \in V$ *there is a unique solution* $x(F)$ *of* (1.2) *in W. Furthermore,* $x(F)$ *is a continuously differentiable function of F,* $x(0) = 0$, *and the derivative* $Dx(F)$ *applied to a function* $\tilde{F}$ *is the solution* $\tilde{x}$ *of the variational equation*

$$(3.5) \qquad\qquad L\tilde{x} = DF(x(F))\tilde{x} + \tilde{F}(x(F)).$$

THEOREM 3.6. *Let* (1.1) *be noncritical, and let* Ω *be a neighborhood of zero in* $\mathscr{B}(\eta)$. *Then there are neighborhoods V and W of zero in* $C^1(\Omega; \mathscr{B}(\eta))$ *and* Ω, *respectively, such that for every* $F \in V$ *and* $G \in V$ *there is a unique solution* $x(F, G)$ *of* (3.2) *in W. Furthermore,* $x(F, G)$ *is a continuously differentiable function of F and G,* $x(0, 0) = 0$, *and the derivative* $Dx(F, G)$ *applied to the functions* $(\tilde{F}, \tilde{G})$ *is the solution* $\tilde{x}$ *of the variational equation*

$$\left[\mu * \tilde{x} - DG(x(F, G))\tilde{x} - \tilde{G}(x(F, G))\right]' = \nu * \tilde{x} + DF(x(F, G))\tilde{x} + \tilde{F}(x(F, G)).$$

The proof of Theorems 3.5 and 3.6 are the same as e.g. the proof of Theorem 2.1 in [3, p. 211]. (The term $DF(x(F))\tilde{x}$ in (3.5) has dropped out in [3]. Of course, it is zero when $DF = 0$.)

In Theorems 3.5 and 3.6 one can replace W by $\mathscr{B}^{m+1}(\eta)$ if either $\Omega = \mathscr{B}^{m+1}(\eta)$, or if we (without loss of generality) take V to be connected, and require x to be a continuous function of F, and to satisfy $x(0) = 0$ (see [2, Theorem (10.2.1), p. 270]). Similar remarks apply to all our following applications of the implicit function theorem, too.

Theorems 3.5 and 3.6 generalize [3, Theorem 2.1, p. 211] in several respects. Here the equation can be of Fredholm type (i.e. anticipatory), the delay may be infinite, also influence functions other than $\eta = 1$ are permitted, the local smoothness of x is less restricted, and the continuity assumption on the perturbation term is much weaker. In particular, here $F(x)$ can be a (small) differential operator, which is clearly not possible in [3].

Let us end this section with three examples on how the perturbation results discussed above can be applied. The equations to which we apply them all happen to

be Volterra integrodifferential equations, but clearly the same technique works equally well for more general equations. In the first two equations we study the existence of bounded, almost periodic and periodic solutions, and in the last example we discuss the rate of convergence to zero of a given solution.

In [1], Burton applies a Liapunov technique to the linear, nonconvolution Volterra integrodifferential equation

$$(3.6) \qquad x'(t) = Ax(t) + \int_0^t G(t, s)x(s)\, ds + F(t) \qquad (t \in \mathbf{R}^+).$$

Roughly, A is negative definite, and "dominates" the integral term, so one can think of (3.6) as a perturbation of the ordinary differential equation

$$x'(t) = Ax(t) \qquad (t \in \mathbf{R}).$$

This equation is of the form (1.1), and one can apply the perturbation technique developed above. It does not give exactly the same results as in [1], but they are fairly close to each other. For simplicity, let us only discuss Burton's Examples 3 and 6, in which the equation is scalar and of the form

$$(3.7) \qquad x'(t) = -x(t) + \int_{-\infty}^t C(t, s)x(s)\, ds + f(t) \qquad (t \in \mathbf{R}).$$

(Extend x in Burtons Example 3 to all of $\mathbf{R}$ in such a way that (3.7) holds on all of $\mathbf{R}$.) Take $\eta = \sigma = 1$, and $\mathfrak{B}(\eta) = BUC$. The fundamental solution r of the unperturbed equation

$$(3.8) \qquad\qquad x'(t) = -x(t) \qquad (t \in \mathbf{R}),$$

is

$$(3.9) \qquad\qquad r(t) = \begin{cases} 0 & (t \in \mathbf{R}^-), \\ e^{-t} & (t \in \mathbf{R}^+), \end{cases}$$

and convolving (3.7) with r we get

$$(3.10) \qquad\qquad x(t) = r * f(t) + r * F(x)(t),$$

where $F(x)(t) = \int_{-\infty}^t C(t, s)x(s)\, ds$ $(t \in \mathbf{R})$. We follow Burton, and suppose that $|C(t, s)| \leqslant \delta e^{-q(t-s)}$ $(-\infty < s \leqslant t < \infty)$. Then

$$|F(x)(t)| \leqslant \left(\int_{-\infty}^t |C(t, s)|\, ds \right) \|x\| \leqslant (\delta/q)\|x\|,$$

so also $|r * F(x)(t)| \leqslant (\delta/q)\|x\|$. If

$$(3.11) \qquad\qquad\qquad \delta < q,$$

then the right-hand side of (3.10) is a contraction in BUC, so (3.10) has a unique solution in BUC. This is roughly the same as Burton's conclusion in his Example 3. If f in (3.10) is periodic or almost periodic, and F maps the set of all T-periodic functions P_T or the set of all almost periodic functions AP into itself, then the solution of (3.10) will automatically be periodic or almost periodic (P_T and AP are closed in BUC). In particular, there is no need to strengthen (3.11) in the periodic case, and Burtons condition $\delta < [(e-1)/e]q$ in his Example 6 can be weakened to (3.11).

Seifert [6] gives conditions which imply that the equation

$$(3.12) \quad x'(t) = 1 - e^{x(t)} - p(t)e^{x(t)} - q(t)\int_0^\infty e^{x(t-s)}\, d\gamma(s) \qquad (t \in \mathbf{R})$$

has an almost periodic solution. Take $\eta = \sigma \equiv 1$, and suppose that $p \in BUC$, $q \in BUC$, $\gamma \in M$. Again we linearize, and consider (3.12) as a perturbation of (3.8). Define $g(x) = 1 - e^x$, $C = \int_0^\infty d\gamma(s)$,

$$F(x)(t) = [g(x(t)) + x(t)] + p(t)g(x(t))$$
$$+ q(t)\int_0^\infty g(x(t-s))\, d\gamma(s),$$

and convolve (3.12) with r defined in (3.9) to get

$$(3.13) \qquad\qquad x = -r * (p + Cq) + r * F(x).$$

The norm of the operator $r *$ from BUC into itself is one, and for every $\delta > 0$, F is Lipschitz continuous in the ball $B_\delta = \{x \in BUC \,|\, \|x\| \le \delta\}$, with Lipschitz constant at most

$$C_F = e^\delta - 1 + e^\delta(\|p\| + \|\gamma\|\,\|q\|).$$

Thus, if $C_F < 1$, then the right-hand side of (3.13) is a contraction. If in addition

$$(3.14) \qquad\qquad \|p + Cq\| \le \delta(1 - C_F),$$

then the right-hand side of (3.13) maps B_δ into itself. By the contraction mapping principle, if (3.14) holds, then (3.12) has a unique solution in B_δ. This solution is actually unique in the somewhat bigger ball where $C_F < 1$. Again, if p and q are almost periodic or periodic, then so is x. This solution is stable within the class of bounded solutions of (3.12) in the sense that if we perturb the right-hand side of (3.12) with a small function in BUC, then the new solution remains close to the old solution (because it is a differentiable function of F). It is also asymptotically stable in the sense that if the added perturbation tends to zero as $t \to \infty$, then the new solution tends to the old solution as $t \to \infty$ (apply Theorem 3.5 with BUC replaced by BUC_+). It follows from the discussion in §7 that this solution is stable even with respect to "smooth" perturbations of the initial function (because the unstable subspace of (3.8) is zero dimensional).

If we estimate $\|p + Cq\|$ in (3.14) by $\|p\| + \|\gamma\|\|q\|$, then (3.14) becomes

$$(3.15) \qquad\qquad \|p\| + \|\gamma\|\,\|q\| \le \frac{\delta(2 - e^\delta)}{1 + \delta e^\delta},$$

a condition similar to Seifert's condition (3.7), which roughly requires

$$(3.16) \qquad\qquad \|p\| + \|\gamma\|\,\|q\|e^\delta \le (1 - \|p\|)e^{-2\delta}.$$

These two conditions overlap each other in the sense that (3.15) is sharper when $\|p\|$ is small compared to $\|\gamma\|\|q\|$, and (3.16) is sharper when $\|\gamma\|\|q\|$ is small compared to $\|p\|$.

In our last example we show how Theorem 3.3 can be used to get a sharpened estimate on the rate of convergence to zero of the solutions of a Volterra integrodifferential equation. This example was suggested to us by Daniel Shea, in a personal

discussion (Helsinki, August 1978). The equation is

$$(3.17) \qquad x'(t) + \int_0^t a(t-s)g(x(s))\,ds = f(t) \qquad (t \in \mathbf{R}^+).$$

Suppose that g is continuously differentiable, $g(0) = 0$, and that we (in one way or another) know that $x(t) \to 0$ as $t \to \infty$. At infinity, equation (3.17) begins to look like

$$(3.18) \qquad x'(t) + k\int_{-\infty}^t a(t-s)x(s)\,ds = 0,$$

where $k = g'(0)$. Let ρ be a dominating function on $\mathbf{R}$ with $\rho(t) = 1$ ($t \in \mathbf{R}^-$), and suppose that a satisfies

$$(3.19) \qquad a \in L^1(\rho), \quad z + k\hat{a}(z) \neq 0 \qquad (\omega \leqslant \mathrm{Re}\, z \leqslant 0)$$

(define $a(t) = 0$ for $t < 0$). Then (3.18) is noncritical (with respect to ρ). Choose e.g. $\eta = \rho$, and suppose that f is the restriction to $\mathbf{R}^+$ of a function in $\mathscr{B}(\rho)$, i.e.

$$(3.20) \qquad f \in \mathscr{B}(\mathbf{R}^+ ; \rho),$$

where $\mathscr{B}(\rho)$ is one of the spaces $BUC(\rho)$, $BC_0(\rho)$ or $L^p(\rho)$, $1 \leqslant p \leqslant \infty$. We claim that this implies

$$(3.21) \qquad x \in \mathscr{B}^1(\mathbf{R}^+ ; \rho).$$

In other words, the a priori knowledge that $x(t) \to 0$ as $t \to \infty$ together with the listed assumptions on g, a and f imply that x must tend to zero at infinity with the a priori determined minimal rate which is built into (3.21).

To prove (3.21) we first extend x to all of $\mathbf{R}$ so that $x(t) = 0$ for $t \leqslant -1$, and x belongs locally to $\mathscr{B}^1$. Next define h by

$$(3.22) \qquad x'(t) + \int_{-\infty}^t a(t-s)g(x(s))\,ds = h(t) \qquad (t \in \mathbf{R}).$$

Then $h \in \mathscr{B}(\rho)$ (for $t \in \mathbf{R}^+$ the difference between h and f is the convolution of a with a bounded function with compact support, and $h(t) = 0$ for $t \leqslant -1$). Let r be the fundamental solution of (3.18), and write (3.22) in the form

$$x = r * h - r * a * [g \circ x - kx].$$

This looks promising, but the right-hand side need not yet be a contraction from $\mathscr{B}^1(\rho)$ into itself (it need not even map $\mathscr{B}^1(\rho)$ into itself). Therefore, choose any function g_1 such that

$$g_1(x) = g(x) - kx \quad (|x| \leqslant \varepsilon), \qquad |g'(x)| \leqslant \varepsilon \quad (x \in \mathbf{R}),$$

where ε is so small that the mapping which takes x into $r * a * (g_1 \circ x)$ is a contraction from $\mathscr{B}^1(\rho)$ into itself. Define h_1 by

$$x'(t) + \int_{-\infty}^t a(t-s)\big[kx(s) + g_1(x(s))\big]\,ds = h_1(t) \qquad (t \in \mathbf{R}).$$

Then $h_1 \in \mathscr{B}(\rho)$, and we are finally in a positive where we can apply Theorem 3.3 to get (3.21).

The same equation and essentially the same problem was studied in [7] with a different technique. There it was not assumed that g is continuously differentiable; only that g satisfies

$$\liminf_{\xi \to 0} g(\xi)/\xi > 0, \qquad \limsup_{\xi \to 0} g(\xi)/\xi < \infty.$$

At the expense of assuming differentiability of g, the result presented here is sharper and much more general than in [7].

The same type of problem has also been studied in [5].

4. The general setup for the critical case. What can then be said if (1.1) is critical rather than noncritical? At least for the moment, in the general case, virtually nothing is known. The answers given below can be regarded as abstractions of some ideas presented in [8]. They all assume that although (1.1) is critical with respect to ρ, it is noncritical with respect to some other dominating functions ρ_1 and ρ_2, satisfying $\rho_i(t) \leqslant \rho(t)$, $i = 1, 2$. For simplicity we restrict ourselves to the smooth case, i.e. the case when $x \in \mathfrak{B}^{m+1}(\eta)$ for some $m \geqslant 0$.

To be more specific, in the sequel we assume throughout that there exist two dominating functions ρ_+ and ρ_-, and a number $T \geqslant 0$ such that

$$\rho_+(t) = \rho(t) \quad (t \leqslant -T), \qquad \rho_+(t) \leqslant \rho(t) \quad (t \geqslant T),$$

$$\rho_-(t) \leqslant \rho(t) \quad (t \leqslant -T), \qquad \rho_-(t) = \rho(t) \quad (t \geqslant T),$$

and that $\rho(t) = \max\{\rho_+(t), \rho_-(t)\}$ ($t \in \mathbf{R}$). Then $M(\rho) = M(\rho_+) \cap M(\rho_-)$ (in [8], ρ_- and ρ_+ were chosen to be $\rho_-(t) = e^{-\omega t}$ and $\rho_+(t) = e^{-\alpha t}$). We also assume throughout that (1.1) is noncritical with respect to ρ_+ and ρ_-. If we let α_+, ω_+, α_- and ω_- denote the characteristic growth rates of ρ_+ and ρ_- at $\pm\infty$ (see §2), then $\omega \leqslant \omega_+ \leqslant \alpha_+ = \alpha$ and $\omega = \omega_- \leqslant \alpha_- \leqslant \alpha$. If $\alpha_- \geqslant \omega_+$, then (1.1) will automatically be noncritical with respect to ρ itself, and we are back in the situation in §3. Therefore, we also take

(4.1) $$\omega = \omega_- \leqslant \alpha_- < \omega_+ \leqslant \alpha_+ = \alpha.$$

In particular, $\omega < \alpha$. Again, sufficient conditions for the existence of the dominating functions ρ_+ and ρ_- are given in [4, 8].

This time we have two fundamental solutions of (1.1), r_+ and r_-, corresponding to ρ_+ and ρ_-. In other words, $r_+ \in L^1(\rho_+)$, $r'_+ \in M(\rho_+)$, $r_- \in L^1(\rho_-)$, $r'_- \in M(\rho_-)$ and r_+ and r_- both satisfy (2.10). If they are identical, then we are back in the noncritical case, so we can assume that their difference $q(t) = r_+(t) - r_-(t)$ ($t \in \mathbf{R}$) does not vanish identically. Define

(4.2) $$\tau(t) = \min\{\rho_+(t), \rho_-(t)\} \quad (t \in \mathbf{R}).$$

Then it is easy to show that τ is an influence function dominated by ρ. Clearly $q \in L^1(\tau)$, $q' \in M(\tau)$, and, as both r_+ and r_- satisfy (2.10), q is a solution of $\mu * q' + \nu * q = q' * \mu + q * \nu = 0$. We call q the fundamental zero solution of (1.1), because of the fact that it generates all the solutions of the homogeneous version

(4.3) $$Lx = 0$$

of (1.1):

LEMMA 4.1. *If $f \in \mathcal{B}^m(\tilde{\tau})$ for some $m \geq 0$, then $x = q * f \in \mathcal{B}^{m+1}(\tilde{\rho})$, and x is a solution of (4.3). Conversely, every solution $x \in \mathcal{B}^{m+1}(\tilde{\rho})$ of (4.3) is of this form.*

Note that, because of (2.4), $\mathcal{B}^{m+1}(\eta) \subset \mathcal{B}^{m+1}(\tilde{\rho})$ for every possible influence function that we consider. This means that all "possible" solutions of (4.3) are of the form given in Lemma 4.1.

PROOF. The proof of the first claim is easy. Take $f \in \mathcal{B}^m(\tilde{\tau})$. As we observed in §2, $q * f \in \mathcal{B}^{m+1}(\tilde{\rho})$. Moreover,

$$\mu * (q * f)' + \nu * (q * f) = (\mu * q' + \nu * q) * f = 0,$$

so $q * f$ is a solution of (4.3).

Conversely, let $x \in \mathcal{B}^{m+1}(\tilde{\rho})$ be a solution of (4.3). Pick any real-valued function $\varphi \in C^\infty$ such that $\varphi(t) = 0$ $(t \leq -1)$, $\varphi(t) = 1$ $(t \geq 1)$, and define

$$(4.4) \qquad x_+(t) = \varphi(t)x(t), \quad x_-(t) = (1 - \varphi(t))x(t) \qquad (t \in \mathbf{R}).$$

Then, $x_+ \in \mathcal{B}^{m+1}(\tilde{\rho}_+)$ and $x_- \in \mathcal{B}^{m+1}(\tilde{\rho}_-)$. Define

$$(4.5) \qquad\qquad\qquad\qquad f = Lx_+ .$$

Then $f \in \mathcal{B}^m(\tilde{\rho}_+)$. On the other hand, by (4.3) and (4.4),

$$(4.6) \qquad\qquad\qquad\qquad f = -Lx_- ,$$

and this implies that $f \in \mathcal{B}^m(\tilde{\rho}_-)$. In other words, $f \in \mathcal{B}^m(\tilde{\rho}_+) \cap \mathcal{B}^m(\tilde{\rho}_-) = \mathcal{B}^m(\tilde{\tau})$. We can solve (4.5) for x_+ by using the fundamental solution r_+, and (4.6) for x_- by using the fundamental solution r_-. This gives

$$x = x_+ + x_- = r_+ * f - r_- * f = q * f,$$

and the proof of Lemma 4.1 is complete. $\square$

If ρ_+ and ρ_- satisfy the regularity condition

$$(4.7) \qquad\qquad \begin{aligned} &e^{\omega_+ t}\rho_+(t) \text{ is nonincreasing for } t \in \mathbf{R}^-, \\ &e^{\alpha_- t}\rho_-(t) \text{ is nondecreasing for } t \in \mathbf{R}^+, \end{aligned}$$

then we can get a stronger result than Lemma 4.1.

LEMMA 4.2. *If (4.7) is true and $x \in \mathcal{B}^{m+1}(\tilde{\rho})$ is a solution of (4.3), then $x = q * f$ for some function $f \in \mathcal{B}^m(\rho)$. In particular, $x \in \mathcal{B}^{m+1}(\tau)$, where τ is defined as in (4.2).*

PROOF. Because of (4.7), the functions

$$\eta_+(t) = \begin{cases} \rho_+(t) & (t \in \mathbf{R}^-), \\ e^{-\omega_+ t} & (t \in \mathbf{R}^+), \end{cases} \qquad \eta_-(t) = \begin{cases} e^{-\alpha_- t} & (t \in \mathbf{R}^-), \\ \rho_-(t) & (t \in \mathbf{R}^+), \end{cases}$$

are influence functions dominated by ρ_+ and ρ_-, respectively. Clearly we can replace $\tilde{\rho}_+$ and $\tilde{\rho}_-$ in the proof of Lemma 3.1 by $\tilde{\eta}_+$ and $\tilde{\eta}_-$ to get $f \in \mathcal{B}^m(\tilde{\eta}_+) \cap \mathcal{B}^m(\tilde{\eta}_-) = \mathcal{B}^m(\eta_{\omega_+}) \cap \mathcal{B}^m(\eta_{\alpha_-})$, where we have defined $\eta_\lambda(t) = e^{-\lambda t}$ $(t \in \mathbf{R})$, for $\lambda = \omega_+$ and $\lambda = \alpha_-$ (recall that $\alpha_- < \omega_+$). Moreover, $r_+ \in M(\eta_{\omega_+})$ and $r_- \in M(\eta_{\alpha_-})$, so we must have $x_+ \in \mathcal{B}^{m+1}(\eta_{\omega_+})$, $x_- \in \mathcal{B}^{m+1}(\eta_{\alpha_-})$. But $x_+(t) = 0$ for $t \leq -1$, and $x_-(t) = 0$ for $t \geq 1$, so that in turn implies that $x_+ \in \mathcal{B}^{m+1}(\eta_+)$, $x_- \in \mathcal{B}^{m+1}(\eta_-)$. Repeating

the same argument as before one more time we finally get $f \in \mathscr{B}^m(\eta_+) \cap \mathscr{B}^m(\eta_-) = \mathscr{B}^m(\rho)$, and the proof of Lemma 4.2 is complete. $\square$

If $\hat{\mu}$ is invertible at infinity when respect to ρ and

$$(4.8) \qquad \det D(z) \neq 0 \quad \text{for Re } z = \omega, \text{Re } z = \alpha,$$

then the set of solutions of (4.3) in $\mathscr{B}^{m+1}(\tilde{\rho})$ is finite dimensional:

LEMMA 4.3. *Let $\hat{\mu}$ be invertible at infinity with respect to ρ, and let* (4.8) *hold. Then q is an exponential polynomial of the form*

$$q(t) = \sum_{j=1}^{k} p_j(t) e^{z_j t}.$$

Here z_j $(1 \leq j \leq k)$ are the zeros of $\det D(z)$ in the strip $\omega < \text{Re } z < \alpha$, and p_j are certain polynomials in t of degree at most one less than the order of the zero z_j, with coefficients in $\mathbf{R}^{n \times n}$. In particular, the set of solutions of (4.3) *in $\mathscr{B}^{m+1}(\tilde{\rho})$ is finite dimensional and independent of $\mathscr{B}$ and m.*

The proof of Lemma 4.3 is essentially the same as the proof of [**8**, Lemma 6.4].

Sometimes there may be several "noncritical bands" in the strip $\omega \leq \text{Re } z \leq \alpha$. More specifically, say e.g. that we still have another dominating function $\rho_0(t)$, with characteristic constants α_0 and ω_0, that (1.1) is noncritical with respect to ρ_0, and that $\alpha_- < \omega_0 \leq \alpha_0 < \omega_+$. Then we get one more fundamental solution r_0, and we can split the fundamental zero solution q into two zero solutions, i.e. $q = q_+ + q_-$, where

$$q_+(t) = r_+(t) - r_0(t) \quad (t \in \mathbf{R}), \qquad q_-(t) = r_0(t) - r_-(t) \quad (t \in \mathbf{R}).$$

Clearly, this together with Lemma 4.1 implies that the solutions x of (4.3) in $\mathscr{B}^{m+1}(\tilde{\rho})$ can be split in two parts

$$x = x_+ + x_- = q_+ * f + q_- * f,$$

where $x_+ \in \mathscr{B}^{m+1}(\eta_+)$ and $x_- \in \mathscr{B}^{m+1}(\eta_-)$, with

$$\eta_+(t) = \min\{\tilde{\rho}_0(t), \tilde{\rho}_+(t)\} \qquad (t \in \mathbf{R}),$$

and

$$\eta_-(t) = \min\{\tilde{\rho}_-(t), \tilde{\rho}_0(t)\} \qquad (t \in \mathbf{R}).$$

This splitting is unique, because if $x \in \mathscr{B}^{m+1}(\eta_+) \cap \mathscr{B}^{m+1}(\eta_-)$, then $x \in \mathscr{B}^{m+1}(\tilde{\rho}_0)$, and by (4.3) and Lemma 3.1, $x = 0$. This means that the set of solutions of (4.3) in $\mathscr{B}^{m+1}(\tilde{\rho})$ is the direct sum of those solutions of (4.3) which belong to $\mathscr{B}^{m+1}(\eta_+)$, and those which belong to $\mathscr{B}^{m+1}(\eta_-)$.

Let us illustrate how one can decompose solutions into components with different growth rates by applying the theory to the scalar, neutral functional differential equation

$$(4.9) \qquad x'(t) - 3x'(t-1) + 2x'(t-2)$$
$$= -\{x(t) - 3x(t-1) + 2x(t-2)\} \qquad (t \in \mathbf{R}).$$

This example is quite easy to analyze, but it is general enough to show what kind of solutions one can expect to get.

The characteristic function $D(z)$ of (4.9) is

$$D(z) = (1 + z)(1 - e^{-z})(1 - 2e^{-z}) \qquad (z \in \mathbf{C}).$$

We have three critical lines of the type Re $z = \lambda$, namely at $\lambda_+ = \log 2$ (simple zeros at $z = \log 2 + n2\pi i$, $n = 0, \pm 1, \pm 2, \ldots$), at $\lambda_0 = 0$ (simple zeros at $z = n2\pi i$, $n = 0, \pm 1, \pm 2, \ldots$), and at $\lambda_- = -1$ (simple zero at $z = -1$). Define $\rho_\lambda(t) = e^{-\lambda t}$ ($t \in \mathbf{R}$), and let r_λ be the fundamental solution of (4.9) corresponding to ρ_λ for $\lambda \notin \{\lambda_+, \lambda_0, \lambda_-\}$. Then r_λ is independent of λ in each noncritical interval. More specfically, define

$$e_+(t) = 0 \quad (t \in \mathbf{R}^-), \qquad e_+(t) = e^{-t} \quad (t \in \mathbf{R}^+),$$

$$e_-(t) = e^{-t} \quad (t \in \mathbf{R}^-), \qquad e_-(t) = 0 \quad (t \in \mathbf{R}^+),$$

and

$$r_+(t) = \sum_{n=0}^{\infty} (2^{n+1} - 1)e_+(t - n) \qquad (t \in \mathbf{R}),$$

$$r_{0+}(t) = -\sum_{n=-\infty}^{-1} 2^{n+1}e_+(t - n) - \sum_{n=0}^{\infty} e_+(t - n) \qquad (t \in \mathbf{R}),$$

$$r_{0-}(t) = \sum_{n=-\infty}^{-1} (1 - 2^{n+1})e_+(t - n) \qquad (t \in \mathbf{R}),$$

$$r_-(t) = \sum_{n=-\infty}^{-1} (1 - 2^{n+1})e_-(t - n) \qquad (t \in \mathbf{R}).$$

Then

$$r_\lambda = r_+ \quad (\lambda > \lambda_+), \qquad r_\lambda = r_{0+} \quad (\lambda_0 < \lambda < \lambda_+),$$

$$r_\lambda = r_{0-} \quad (\lambda_- < \lambda < \lambda_0), \qquad r_\lambda = r_- \quad (\lambda < \lambda_-).$$

We get three fundamental zero solutions $q_+ = r_+ - r_{0+}$, $q_0 = r_{0+} - r_{0-}$, and $q_- = r_{0-} - r_-$, which are given by

$$q_+(t) = \sum_{n=-\infty}^{\infty} 2^{n+1}e_+(t - n) \qquad (t \in \mathbf{R}),$$

$$q_0(t) = -\sum_{n=-\infty}^{\infty} e_+(t - n) \qquad (t \in \mathbf{R}),$$

$$q_-(t) = ((e - 1)(2e - 1))^{-1}e^{-t} \qquad (t \in \mathbf{R}).$$

Let Y_+^{m+1}, Y_0^{m+1} and Y_-^{m+1} be the subspace of solutions of (4.9) which are locally in $\mathcal{B}^{m+1}$, and which are generated by q_+, q_0 and q_-. Clearly, Y_-^{m+1} consists of scalar multiples of e^{-t}, and is one dimensional. The function q_0 is periodic with period one, so every $x \in Y_0^{m+1}$ must also be periodic with period one. A direct substitution into (4.9) shows that functions which are periodic with period one satisfy (4.9). This means that Y_0^{m+1} is exactly the set of functions x which are locally in $\mathcal{B}^{m+1}$, and which are periodic with period one. A similar argument shows that Y_+^{m+1} is the set of functions x which are locally in $\mathcal{B}^{m+1}$, and which are of the form $x(t) = 2^t y(t)$, where y is periodic with period one.

In this particular example we get very detailed information on the global behavior of the solutions. For example, the solutions in Y_-^{m+1} are characterized by the fact that they tend to zero at infinity, so every sufficiently smooth solution of (4.9) tending to zero at infinity must be a scalar multiple of e^{-t}. Likewise, every bounded solution must be periodic with period one, and every solution tending to zero at minus infinity must be of the form $2^t y(t)$, where y is periodic with period one. A general solution is a sum of solutions of these particular types.

In the preceding example, the fact that we obtained periodic solutions was irrelevant (it was due to the fact that $D(z)$ had simple zeros at $z = n2\pi i, n = 0, \pm 1, 2, \ldots$). However, observe that the two infinite dimensional subspaces Y_+^{m+1} and Y_0^{m+1} are not smooth as in the finite dimensional case. Also, they are not invariant if we replace $\mathscr{B}$ or m in our basic space $\mathscr{B}^{m+1}$ by another $\mathscr{B}$ or m, as in the finite dimensional case. In this particular example it would be tempting to call Y_-^{m+1} the stable subspace, Y_0^{m+1} the central subspace, and Y_+^{m+1} the unstable subspace of (4.9) (see also §7).

5. The critical case with small influence function. In the noncritical case it did not really matter which influence function η was used, as long as it was dominated by ρ. Here the situation is quite different. We first describe the case when the influence function is "small".

Let η be of the form

$$(5.1) \qquad \eta(t) = \min\{\eta_+(t), \eta_-(t)\} \qquad (t \in \mathbf{R}),$$

where η_+ is an influence function dominated by ρ_+, and η_- an influence function dominated by ρ_-. Then $\mathscr{B}(\eta) = \mathscr{B}(\eta_+) + \mathscr{B}(\eta_-)$ in the sense that $\mathscr{B}(\eta_+) \subset \mathscr{B}(\eta)$, $\mathscr{B}(\eta_-) \subset \mathscr{B}(\eta)$, and every $\varphi \in \mathscr{B}(\eta)$ can be written (in a nonunique way) as $\varphi = \varphi_+ + \varphi_-$, where $\varphi_+ \in \mathscr{B}(\eta_+)$ and $\varphi_- \in \mathscr{B}(\eta_-)$. For instance, we could take $\eta_+ = \tilde{\rho}_+, \eta_- = \tilde{\rho}_-$ and $\eta = \tilde{\rho}$. This is the smallest possible choice of η, here, as well as in §3 (see (2.4)). On the other hand, we could also take $\eta_+ = \rho_+, \eta_- = \rho_-$, and $\eta = \tau$, defined in (4.2). This is the largest possible choice of η here (again, see (2.4)). In particular, η has to be small in the sense that

$$\eta(t) = O(e^{(-\alpha_- - \varepsilon)t}) \quad (t \to -\infty), \qquad \eta(t) = O(e^{(-\omega_+ + \varepsilon)t}) \quad (t \to \infty)$$

for every $\varepsilon > 0$. It follows from the strict inequality $\alpha_- < \omega_+$ in (4.1) together with (2.4) that

$$\eta_-(t) = o(\eta_+(t)) \quad (t \to -\infty), \qquad \eta_+(t) = o(\eta_-(t)) \quad (t \to \infty),$$

so there exists a $T \geq 0$ such that

$$(5.2) \qquad \eta(t) = \eta_-(t) \quad (t \leq -T), \qquad \eta(t) = \eta_+(t) \quad (t \geq T).$$

LEMMA 5.1. *Let η be of the form* (5.1), *and let $m \geq 0$. Then L maps $\mathscr{B}^{m+1}(\eta)$ continuously onto $\mathscr{B}^m(\eta)$. The null space of L consists of all functions x of the form $x = q * f$, where q is the fundamental zero solution, and $f \in \mathscr{B}^m(\eta_+) \cap \mathscr{B}^m(\eta_-)$ is arbitrary.*

PROOF. Most of the statement of Lemma 5.1 follows from the discussion in §4. In this case x_+ and x_- in (4.4) satisfy $x_+ \in \mathscr{B}^{m+1}(\eta_+)$ and $x_- \in \mathscr{B}^{m+1}(\eta_-)$, so (4.5), (4.6) imply that $f \in \mathscr{B}^m(\eta_+) \cap \mathscr{B}^m(\eta_-)$.

To show that L is onto, let $f \in \mathcal{B}^m(\eta)$ be arbitrary, and split f into $f = f_+ + f_-$, where $f_+ \in \mathcal{B}^m(\eta_+)$ and $f_- \in \mathcal{B}^m(\eta_-)$. Let x_+ be the (unique) solution of $Lx_+ = f_+$ in $\mathcal{B}^{m+1}(\eta_+)$, and similarly, let x_- be the solution of $Lx_- = f_-$ in $\mathcal{B}^{m+1}(\eta_-)$ (cf. Lemma 3.1). Then $x = x_+ + x_- \in \mathcal{B}^{m+1}(\eta)$, and x is a solution of (1.1). $\square$

The same method as was used in the proof of Lemma 5.1 can be used to construct projections of $\mathcal{B}^{m+1}(\eta)$ onto the null space of L, and onto a complementary subspace. Fix an arbitrary cut-off function φ, i.e. any real-valued function $\varphi \in C^\infty$ satisfying $\varphi(t) = 0$ ($t \leqslant -1$), $\varphi(t) = 1$ ($t \geqslant 1$), and for every $f \in \mathcal{B}^m(\eta)$, define the operator Rf by

$$(5.3) \qquad Rf = r_+ * (\varphi f) + r_- * \left[(1 - \varphi)f\right].$$

Here $\varphi f(t) = \varphi(t)f(t)$, and $(1 - \varphi)f(t) = (1 - \varphi(t))f(t)$ ($t \in \mathbf{R}$). Indeed, R is well defined, it is linear, and it is continuous from $\mathcal{B}^m(\eta)$ into $\mathcal{B}^{m+1}(\eta)$, because by (5.2), $\varphi f \in \mathcal{B}^m(\eta_+)$ and $(1 - \varphi)f \in \mathcal{B}^m(\eta_-)$. By (2.10) and (5.3),

$$LRf = (Lr_+) * (\varphi f) + (Lr_-) * (1 - \varphi)f = \varphi f + (1 - \varphi)f = f,$$

so R is a right inverse of L. Define

$$(5.4) \qquad P_Z x = RLx, \qquad P_Y x = x - P_Z x.$$

Applying L to $P_Z x$ we get $LP_Z x = LRLx = Lx$, so

$$(5.5) \qquad LP_Z x = Lx, \qquad LP_Y x = 0.$$

In particular, $P_Z^2 x = RLP_Z x = RLx = P_Z x$, so $P_Z^2 = P_Z$, which implies $P_Y^2 = P_Y$. In other words, P_Y and P_Z are projection operators.

Let us restate the preceding argument in the following form.

Lemma 5.2. *The operators P_Y and P_Z defined in* (5.4) *are projection operators, which split $\mathcal{B}^{m+1}(\eta)$ into $Y \oplus Z$, where Y is the range of P_Y, and Z is the range of P_Z. They satisfy* (5.5). *In particular, L restricted to Y is identically zero, and L maps Z one-to-one onto $\mathcal{B}^m(\eta)$. The inverse of L restricted to Z is the operator R defined in* (5.3).

Maybe one should point out the fact that Z is not determined uniquely by L itself in the same way as Y is. If we replace the cut-off function φ by another cut-off function, then we get a new Z. The projections P_Y and P_Z do not commute with translations.

Thanks to the fact that we can split $\mathcal{B}^{m+1}(\eta)$ into $Y \oplus Z$, we can perturb (1.1) and still get the same type of splitting. Split x in (1.2) into $x = y + z$, where $y \in Y$ and $z \in Z$. Then (1.2) becomes $Lz = F(y, z)$, which by Lemma 5.2 is equivalent to

$$(5.6) \qquad z = RF(y, z).$$

For example, if F maps $\mathcal{B}^{m+1}(\eta)$ into $\mathcal{B}^m(\eta)$, and F is Lipschitz continuous with respect to z, with a sufficiently small Lipschitz constant, then the right-hand side of (5.6) is a contraction with respect to z, and for every $y \in Y$ we get a unique solution of (1.2). If we instead apply the implicit function theorem to (5.6), then we get the following theorem.

THEOREM 5.3. *Split* $\mathfrak{B}^{m+1}(\eta)$ *into* $Y \oplus Z$ *as in Lemma* 5.2, *and let* Ω *be a neighborhood of zero in* $\mathfrak{B}^{m+1}(\eta)$. *Then there are neighborhoods* $U, V,$ *and* W *of zero in* Y, $C^1(\Omega, \mathfrak{B}^m(\eta))$ *and* Z, *respectively, such that for every* $y \in U$, $F \in V$ *there is a unique solution* $x(y, F) = y + z(y, F)$ *of* (1.2) *with* $z(y, F) \in W$. *Furthermore,* $x(y, F)$ *is continuously differentiable in* (y, F), $x(y, 0) = y$, *and* $Dx(y, F)$ *applied to* $(\tilde{y}, \tilde{F})$ *is of the form* $\tilde{x} = \tilde{y} + \tilde{z}$, *where* $\tilde{z} \in Z$ *is the solution of the variational equation*

$$L\tilde{z} = D_z F(y, z(y, F))\tilde{z} + D_y F(y, z(y, F))\tilde{y} + \tilde{F}(y, z(y, F)).$$

The proof of Theorem 5.3 is the same as the proof of Theorem 2.3 in [**3**, p. 211]. Note that if $F(0) = 0$, $D_y F(0) = 0$, then the manifold $\{x(y, F) \mid y \in U\}$ is tangent to Y at zero.

6. The critical case with large influence function. In the preceding section the influence function was "small", and here we look at the situation where the influence function is "large".

Again, let η_+ be an influence function dominated by ρ_+, and let η_- be an influence function dominated by ρ_-, but this time suppose that η is of the form

$$(6.1) \qquad\qquad \eta(t) = \max\{\eta_+(t), \eta_-(t)\} \qquad (t \in \mathbf{R}).$$

Then η is again an influence function dominated by ρ, and $\mathfrak{B}(\eta) = \mathfrak{B}(\eta_+) \cap \mathfrak{B}(\eta_-)$. If we take $\eta_+ = \rho_+$ and $\eta_- = \rho_-$, then $\eta = \rho$, which is that largest possible choice of η here, as well as in §3. On the other hand, if we take $\eta_+(t) = \tilde{\rho}_+(t)$ and $\eta_-(t) = \tilde{\rho}_-(t)$, then $\eta(t) = \tilde{\tau}(t)$, where τ is the function in (4.2). This is the smallest possible choice of η here. Clearly, if η is of the type (5.1), then $\tilde{\eta}$ is of the type (6.1), and vice versa.

LEMMA 6.1. *Let* η *be of the form* (6.1), *and let* $m \geq 0$. *Then the operator* Lx *is continuous and one-to-one from* $\mathfrak{B}^{m+1}(\eta)$ *into* $\mathfrak{B}^m(\eta)$. *Its range consists of those functions* $f \in \mathfrak{B}^m(\eta)$ *which satisfy* $q * f \equiv 0$, *where* q *is the fundamental zero solution. If* f *belongs to range of* L, *then the unique solution* x *of* (1.2) *is given by*

$$(6.2) \qquad\qquad x = r_- * f = r_+ * f.$$

PROOF. That L is one-to-one, and that a solution $x \in \mathfrak{B}^{m+1}(\eta)$ must satisfy (6.2) follows from Lemma 3.1 and the fact that (1.1) is noncritical with respect to both ρ_+ and ρ_-. This implies that we must have $q * f \equiv 0$ whenever f is in the range of L. On the other hand, if f satisfies $q * f \equiv 0$, then (6.2) defines a solution $x \in \mathfrak{B}^{m+1}(\eta)$ of (1.1). $\square$

Let G denote the range of L in $\mathfrak{B}^m(\eta)$ (this has nothing to do with the function G in (3.2)). We can again construct projections P_G and P_H of $\mathfrak{B}^m(\eta)$ onto G and onto a complementary subspace H. Let φ be a cut-off function of the same type as was used in §5, and, for every $f \in \mathfrak{B}^m(\eta)$, define

$$Qf = (1 - \varphi)(r_+ * f) + \varphi(r_- * f).$$

By Lemma 6.1, $QLx = x$, so Q is a left inverse of L. Define

$$(6.3) \qquad\qquad P_G f = LQf, \qquad P_H f = f - P_G f.$$

Then $P_G Lx = Lx$ and $P_H Lx = 0$, where P_H is a projection onto H, and P_G is a projection onto a complementary subspace G.

This time we write the perturbed equation (1.2) in a slightly different way. If one wants to study the range of the operator $Lx - F(x)$ instead of its null space, it is natural to replace (1.2) by

$$(6.4) \qquad Lx - F(x) = f,$$

and look at the functions f which can be solutions of this equation.

THEOREM 6.2. *Split $\mathcal{B}^m(\eta)$ into $G \oplus H$, as above, and let Ω be a neighborhood of zero in $\mathcal{B}^{m+1}(\eta)$. Then there are neighborhoods U, V and W of zero in G, $C^1(\Omega; \mathcal{B}^m(\eta))$ and Ω, respectively, such that for $g \in U$, $F \in V$ there is a unique solution $x(g, F)$ of the equation*

$$(6.5) \qquad P_G[Lx(g, F) - F(x(g, F))] = g$$

in W. Furthermore, $x(g, F)$ is continuously differentiable in (g, F), $x(g, 0) = r_- * g = r_+ * g$, and $Dx(g, F)$ applied to $(\tilde{g}, \tilde{F})$ is the solution $\tilde{x}$ of the variational equation

$$(6.6) \qquad L\tilde{x} - P_G DF(x(g, F))\tilde{x} = P_G F(x(g, \tilde{F})) + \tilde{g}.$$

Define $h(g, F) = -P_H F(x(g, F))$ and $f(g, F) = g + h(g, F)$. Then $x(g, F)$ and $f(g, F)$ satisfy (6.4), $f(g, F)$ is continuously differentiable in (g, F), $h(g, 0) = 0$, and $DF(g, F)$ applied to $(\tilde{g}, \tilde{F})$ is given by

$$f = \tilde{g} - P_H[\tilde{F}(x(g, F)) + DF(x(g, F))\tilde{x}],$$

where $\tilde{x}$ is the solution of (6.6).

Again, Theorem 6.2 follows immediately, if one applies the implict function theorem to (6.5), rewritten e.g. in the form

$$x(g, F) = r_- * g + QF(x(g, F)).$$

If $F(0) = 0$, $DF(0) = 0$, then the manifold $\{f(g, F)|g \in U\}$ is tangent to G at zero.

Above we have treated only two critical cases, namely when η is small enough to be of the form (5.1), and when η is large enough to be of the form (6.1). The intermediate case when neither (5.1) nor (6.1) holds is more difficult, with one exception, namely the periodic case. If (1.1) is noncritical with respect to the dominating function $\rho(t) \equiv 1$, then, as we saw in §3, the periodic case is just as easy (or just as difficult) as the case when the solutions are almost periodic, or just bounded and uniformly continuous. On the other hand, if we still take $\rho(t) \equiv 1$, and $\hat{\mu}$ is invertible at infinity with respect to ρ, but $\det D(z) = 0$ for some z with $\text{Re } z = 0$, then the cases of almost periodic and bounded, uniformly continuous solutions cause problems, whereas the periodic case is still relatively easy. One can, e.g., rescale t so that the solutions are 2π-periodic, and then interpret (1.1) and (1.2) as equations on the unit circle (in (1.1) this essentially means that one replaces $d\mu(t)$ by $\sum_{k=0}^{\infty} d\mu(t + 2\pi k)$, restricted to $0 \leqslant t < 2\pi$, and similarly for ν). Then it is quite easy to show that the range of L is closed, and that the dimension of the null space of L is finite and equals the codimension of the range of L. However, we shall not discuss the periodic case here in any great detail. Instead we give an example of "bad" behavior when η is neither sufficiently small, nor sufficiently large, and $\mathcal{B} = BC_0$.

Take the simplest possible equation, namely the scalar equation

$$(6.7) \qquad\qquad x'(t) = f(t) \qquad (t \in \mathbf{R}).$$

Here $Lx(t) = x'(t)$, and the characteristic function of L is $D(z) = z$ ($z \in \mathbf{C}$), with a zero at $z = 0$. This means that (6.7) is critical with respect to every dominating function ρ for which $\omega \leqslant 0 \leqslant \alpha$. If we choose

$$\eta(t) = (1 + |t|)^{\kappa} \qquad (t \in \mathbf{R}),$$

where $\kappa \in \mathbf{R}$, then η is neither small enough at infinity to be of the type (5.1), nor large enough to be of the type (6.1). Take $\mathfrak{B} = BC_0$, and $m \geqslant 0$ arbitrary. Then L is continuous from $\mathfrak{B}^{m+1}(\eta)$ into $\mathfrak{B}^m(\eta)$. It is one-to-one if $\kappa \geqslant 0$, and its null space is one dimensional if $\kappa < 0$ (the set of all constant functions). In both cases we can decompose $\mathfrak{B}^{m+1}(\eta)$ into the null space of L, and a complementary subspace, and so far this looks just like the other cases which we have treated earlier.

However, we get problems if we try to decompose $\mathfrak{B}^m(\eta)$ into the range $R(L)$ of L and a complementary subspace, because we claim that $R(L)$ is not closed. That this is true one can see as follows. First of all, it is easy to show that L is not onto. Choose any $f \in \mathfrak{B}^m(\eta)$ such that

$$(6.8) \qquad\qquad f(t) = (1 + t)^{-(\kappa + \varepsilon)} \qquad (t \geqslant 1),$$

where $0 < \varepsilon < 1$, and $\kappa + \varepsilon \neq 1$. Then a solution x of (6.7) must be of the form

$$x(t) = \frac{(1 + t)^{1 - (\kappa + \varepsilon)}}{1 - (\kappa + \varepsilon)} + C \qquad (t \geqslant 1),$$

where C is a constant. However, this implies $\eta(t)x(t) \to \infty$ as $t \to \infty$, so $x \notin \mathfrak{B}^{m+1}(\eta)$, and $f \notin R(L)$.

On the other hand, if $\kappa \leqslant 1$, then we claim that $R(L)$ is dense in $\mathfrak{B}^m(\eta)$. Let $f \in \mathfrak{B}^m(\eta)$ be arbitrary, and let $\varepsilon > 0$. Then we can find $g \in \mathfrak{B}^m(\eta)$ with compact support such that $\|f - g\| \leqslant \varepsilon$. As $\kappa \leqslant 1$, we can find a function $h \in \mathfrak{B}^m(\eta)$ with compact support and with $\|h\| \leqslant \varepsilon$ such that $\int_{-\infty}^{\infty} h(t)\, dt = \int_{-\infty}^{\infty} g(t)\, dt$. Then

$$\int_{-\infty}^{\infty} (g(t) - h(t))\, dt = 0,$$

so $g - h \in R(L)$ (take x to be $\int_{-\infty}^{t}(g(s) - h(s))\, ds$ and note that x has compact support). Clearly, $\|f - (g - h)\| \leqslant 2\varepsilon$, and this shows that $R(L)$ is dense in $\mathfrak{B}^m(\eta)$.

If $\kappa > 1$, then $R(L)$ is not dense in $\mathfrak{B}^m(\eta)$, because every $f \in R(L)$ must satisfy $\int_{-\infty}^{\infty} f(t)\, dt = 0$. Let Y be the subspace

$$Y = \left\{ f \in \mathfrak{B}^m(\eta) \,\middle|\, \int_{-\infty}^{\infty} f(t)\, dt = 0 \right\}.$$

Then Y has codimension one. Every $f \in Y$ with compact support belongs to $R(L)$, and these functions are dense in Y, so $R(L)$ is dense in Y. However, $R(L) \neq Y$, because we can make the function f in (6.8) odd, so that it belongs to Y.

In other words, $R(L)$ is never closed. It is dense if $\kappa \leqslant 1$, and its closure has codimension one if $\kappa > 1$. As $R(L)$ is not closed, the "inverse" of L from the range

of L into a complementary subspace of the null space of L is not continuous, and one cannot use the same straightforward perturbation technique as we have done above.

7. An initial value problem. Below we shall apply the same type of arguments to decompose the solutions of the neutral functional differential equation

$$(7.1) \qquad\qquad Lx(t) = f(t) \qquad (t \in \mathbf{R}^+ = [0, \infty)),$$

with initial condition

$$(7.2) \qquad\qquad x(t) = \xi(t) \qquad (t \in \mathbf{R}^- = (-\infty, 0]),$$

and with L defined by (1.3), into stable and unstable components. This time μ and ν belong to $M(\mathbf{R}^+ ; \rho_-; \mathbf{R}^{n \times n})$, i.e. they belong to $M(\rho_-; \mathbf{R}^{n \times n})$ for some dominating function ρ_-, and they are supported on $\mathbf{R}^+$. We suppose that the corresponding whole line equation (1.1) is noncritical with respect to ρ_-, and let η_- be an influence function dominated by ρ_-. We take ξ and f in $\mathfrak{B}^{m+1}(\mathbf{R}^-; \eta_-; \mathbf{R}^n)$ and $\mathfrak{B}^m(\mathbf{R}^+ ; \eta_-; \mathbf{R}^n)$, respectively, i.e. they are restrictions to $\mathbf{R}^-$ and $\mathbf{R}^+$ of functions in $\mathfrak{B}^{m+1}(\eta_-; \mathbf{R}^n)$ and $\mathfrak{B}^n(\eta_-; \mathbf{R}^n)$. For simplicity we discuss only the "smooth" case when x is locally in $\mathfrak{B}^{m+1}$ for some $m \geq 0$ (equation (7.1) with x locally in $\mathfrak{B}$ was discussed in [**8**]).

Even if $\xi \in \mathfrak{B}^{m+1}(\mathbf{R}^-; \eta_-)$ and $f \in \mathfrak{B}^m(\mathbf{R}^+ ; \eta_-)$, it is not automatically true that x belongs locally to $\mathfrak{B}^{m+1}$. If we extend f to all of $\mathbf{R}$ by

$$(7.3) \qquad\qquad f(t) = L\xi(t) \qquad (t < 0),$$

then f belongs to $\mathfrak{B}^m(\eta_-)$, except for a possible discontinuity at zero. We cannot allow such a discontinuity if we want x to be locally in $\mathfrak{B}^{m+1}$, and therefore we choose $\xi \in \mathfrak{B}^{m+1}(\mathbf{R}^-; \eta_-)$ in such a way that

$$(7.4) \qquad\qquad f \in \mathfrak{B}^m(\eta_-),$$

with $f(t)$ defined by (7.3) for $t < 0$. This is no restriction if $m = 0$ and $\mathfrak{B} = L^p$, but otherwise it means that ξ has to satisfy the following equations. Define $K = \{0, \ldots, m\}$ if $\mathfrak{B}$ is one of the subspaces of BUC, and $K = \{0, \ldots, m-1\}$ if $\mathfrak{B} = L^p$ (take $K = \varnothing$ when $m = 0$ and $\mathfrak{B} = L^p$). Then (7.4) is equivalent to

$$(7.5) \qquad\qquad L\xi^{(k)}(0) = f^{(k)}(0) \qquad (k \in K),$$

If

$$(7.6) \qquad\qquad \mu \text{ is atomic at zero,}$$

i.e. μ has an invertible point mass at zero, then by taking d large enough, we can assure that (1.1) is noncritical with respect to the dominating function e^{-dt} ($t \in \mathbf{R}$) (see [**8**]). Define

$$\rho_+(t) = \eta_+(t) = e^{-dt} \qquad (t \in \mathbf{R}),$$

and

$$(7.7) \qquad\qquad \rho(t) = \max\{\rho_+(t), \rho_-(t)\} \qquad (t \in \mathbf{R}),$$

$$\eta(t) = \min\{\eta_+(t), \eta_-(t)\} \qquad (t \in \mathbf{R}).$$

Then ρ is a dominating function of the form considered in §4, and η is an influence function of the type considered in §5. Recall that there exist two fundamental

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solutions r_- and r_+ of the whole line version of (7.1), one corresponding to ρ_-, and the other corresponding to ρ_+. This time we do not exclude the possibility that $r_- = r_+$. (If $r_- = r_+$, then throughout one can replace η by η_- below.)

The general assumptions listed above, including (7.5), (7.6), imply that (7.1), (7.2) has a unique solution $x \in \mathcal{B}^{m+1}(\eta)$, with η defined as in (7.7). This was proved for $m = 0$ in [8, §§4 and 7], and for $m > 0$ one can differentiate (7.1) m times, and apply the case $m = 0$ to get the desired smoothness. One gets this solution by first extending ξ to a function y in $\mathcal{B}^{m+1}(\eta)$ in an arbitrary way, then defining g to be the function $g(t) = f(t) - Ly(t)$ $(t \in \mathbf{R}^+)$, $g(t) = 0$ $(t \in \mathbf{R}^-)$, and finally defining $x = y + r_+ * g$.

At this point we can without loss of generality take $f \equiv 0$ in (7.1), and replace (7.1) by

$$(7.8) \qquad\qquad Lx(t) = 0 \qquad (t \in \mathbf{R}^+).$$

If f is nonzero, then we extend f to a function in $\mathcal{B}^m(\eta_-)$ in an arbitrary way, and replace x by $\tilde{x} = x - r_- * f$. The new function $\tilde{x}$ is a solution of (7.8) instead of (7.1), and the initial function $\tilde{\xi}$ of $\tilde{x}$ satisfies

$$(7.9) \qquad\qquad L\xi^{(k)}(0) = 0 \qquad (k \in K)$$

instead of (7.5). For simplicity we drop the tilde, and call the solution of (7.8) x instead of $\tilde{x}$, and its initial function ξ instead of $\tilde{\xi}$.

(The preceding transformation shifts the solution manifold and the stable and unstable manifolds defined below by the same amount, so that they all pass through zero, and become subspaces rather than translates of subspaces. The locations of the solution manifold and the stable manifold depend only on the values of $f(t)$ for $t \in \mathbf{R}^+$, but the location of the unstable manifold depends also on the values of $f(t)$ for $t < 0$. In particular, the location of the unstable manifold is not known, unless one knows that values of $f(t)$ for $t < 0$. See the discussion of the nonlinear equation (7.10) below.)

Before we continue the discussion of (7.8), let us introduce one more notation. We shall quite frequently need the subspaces of functions in $\mathcal{B}^{m+1}(\eta)$ and $\mathcal{B}^m(\eta)$ which vanish on either $\mathbf{R}^-$ and $\mathbf{R}^+$, so let us denote for every index k and every influence function η,

$$\mathcal{B}_-^k(\eta) = \left\{ f \in \mathcal{B}^k(\eta) \,|\, f(t) = 0 \text{ for } t \in \mathbf{R}^+ \right\},$$

$$\mathcal{B}_+^k(\eta) = \left\{ f \in \mathcal{B}^k(\eta) \,|\, f(t) = 0 \text{ for } t \in \mathbf{R}^- \right\}.$$

We let $\mathcal{B}_-^k(\mathbf{R}^-; \eta)$ be the restriction of $\mathcal{B}_-^k(\eta)$ to $\mathbf{R}^-$.

Let S be the set of all possible solutions x of (7.8) in $\mathcal{B}^{m+1}(\eta)$, and let S_- be the restriction of S to $\mathbf{R}^-$. We call S_- the subspace of smooth initial functions and S the subspace of smooth solutions of (7.8).

Clearly S_- consists of those initial functions $\xi \in \mathcal{B}^{m+1}(\mathbf{R}^-; \eta_-)$ which satisfy (7.9). This implies that S_- has codimension nm in the continuous case and $n(m-1)$ in the L^p-case, which one can see as follows. The subspace $\mathcal{B}_-^m(\eta_-)$ has codimension nm or $n(m-1)$ in $\mathcal{B}^m(\mathbf{R}^-; \eta_-)$. We can choose a complementary subspace of dimension nm

or $n(m-1)$ with basis f_{ij}. Without loss of generality, let each f_{ij} vanish on $(-\infty, -1]$. Let H_- be the subspace of $\mathfrak{B}^{m+1}(\mathbf{R}^-; \eta_-)$ which is spanned by $r_+ * f_{ij}$. Then each function in H_- vanishes on $(-\infty, -1]$, and as $L(r_+ * f_{ij}) = f_{ij}$, this collection of functions is linearly independent, so also H_- has dimension nm or $n(m-1)$. Moreover, it is obvious that H_- is a complementary subspace to S_-. We call the projections corresponding to the splitting of $\mathfrak{B}^{m+1}(\mathbf{R}^-; \eta_-)$ into $S_- \oplus H_-$ for P_{S_-} and P_{H_-}.

It is also true that S is a closed subspace of $\mathfrak{B}^{m+1}(\eta)$, because $\mathfrak{B}_-^m(\eta)$ is closed in $\mathfrak{B}^m(\eta)$, and L is continuous. As a matter of fact, it even has a complementary subspace in $\mathfrak{B}^{m+1}(\eta)$. To see that this is so one argues as follows. The subspace of functions $x \in \mathfrak{B}^{m+1}(\eta)$ which satisfy $L^{(k)}x(0) = 0$ ($k \in K$) has codimension nm or $n(m-1)$, and by extending the functions in our subspace H_- of $\mathfrak{B}^{m+1}(\mathbf{R}^-; \eta_-)$ to all of $\mathbf{R}$ we get a complementary subspace H in $\mathfrak{B}^{m+1}(\eta)$. Without loss of generality, do the extension in such a way that all the functions in H vanish on $[1, \infty)$. In the subspace of functions x satisfying $L^{(k)}x(0) = 0$ ($k \in K$) we can define a projection P_S onto S by setting $y = P_S x$ to be the solution of (7.8) which satisfies $y(t) = x(t)$ ($t \in \mathbf{R}^-$). Define $P_T x = x - P_S x$. Then P_T is a projection, and the range T of P_T is $\mathfrak{B}_+^{m+1}(\eta)$. The projections P_S, P_T and P_H split $\mathfrak{B}^{m+1}(\eta)$ into $S \oplus \mathfrak{B}_+^{m+1}(\eta) \oplus H$. Observe that L maps $\mathfrak{B}_+^{m+1}(\eta)$ one-to-one and continuously onto $\mathfrak{B}_+^m(\eta)$, and that the inverse of L in this case is the convolution operator $r_+ *$.

It is possible to further decompose the solution subspace S into a stable subspace Y and an unstable subspace Z. Observe that $x \in S$ implies $Lx \in \mathfrak{B}^m(\eta_-)$. For every $x \in S$, define

$$P_Y x = r_- * Lx, \qquad P_Z x = x - P_Y x.$$

Then $P_Y^2 x = P_Y x$, so P_Y and P_Z are projections onto subspaces Y and Z of S. As we mentioned above, we call Y the stable and Z the unstable subspace of smooth solutions of (7.8). Clearly $Y \in \mathfrak{B}^{m+1}(\eta_-)$. Also, $z \in Z$ iff $z \in \mathfrak{B}^{m+1}(\eta)$, and $Lz = 0$, so Z is the null space of L in $\mathfrak{B}^{m+1}(\eta)$. The stable subspace Y is always infinite dimensional, but the dimension of the unstable subspace Z may or may not be finite. If $r_- = r_+$, then the dimension of Z is zero.

Let Y_- and Z_- be the restrictions of Y and Z to $\mathbf{R}^-$. We call Y_- the stable and Z_- the unstable subspace of smooth initial functions of (7.8).

We can summarize the preceding argument into the following theorem.

THEOREM 7.1. *Decompose $\mathfrak{B}^{m+1}(\eta)$ into $S \oplus \mathfrak{B}_+^{m+1}(\eta) \oplus H$ as above, with $S = Y \oplus Z$, where S is the set of smooth solutions of (7.1). Then the operator L vanishes on Z, it maps Y one-to-one and continuously onto $\mathfrak{B}_-^m(\eta_-)$, it maps $\mathfrak{B}_+^{m+1}(\eta)$ one-to-one continuously onto $\mathfrak{B}_+^m(\eta)$, and it maps H one-to-one and continuously onto LH. Its inverse from $\mathfrak{B}_-^m(\eta_-)$ into Y is the convolution operator $r_- *$, its inverse from $\mathfrak{B}_+^m(\eta)$ into $\mathfrak{B}_+^{m+1}(\eta)$ is the convolution operator $r_+ *$, and both $r_- *$ and $r_+ *$ are inverses of L from LH into H. Moreover, L vanishes on Z_-, it maps Y_- one-to-one and continuously onto $\mathfrak{B}_-^m(\mathbf{R}^-; \eta_-)$, and it maps H_- one-to-one and continuously onto the restriction $(LH)_-$ of LH to $\mathbf{R}^-$. The inverse of L from $(LH)_-$ to H_- is the convolution operator $r_+ *$.*

At this point maybe we should warn the reader that the name "unstable subspace" for Z in some cases is misleading. Sometimes it would be more appropriate to call Z a central or a central-unstable subspace.

By now we are ready to allow a perturbation in (7.1). If we let f in (7.1) depend on x, then we get the perturbed equation

$$(7.10) \qquad Lx(t) = F(x)(t) \qquad (t \in \mathbf{R}^+).$$

This is not the standard way of writing a nonlinear neutral functional differential equation. Usually one thinks of F as a mapping from the space of initial functions into $\mathbf{R}^n$, and writes the perturbation term in the form $F(t, x_t)$, where $x_t(s) = x(s + t)$ ($s \in \mathbf{R}^-$; $t \in \mathbf{R}^+$). There are two reasons for choosing the formulation (7.10) rather than the standard one. We do not run into the standard difficulties which one encounters when the initial functions are (discontinuous) L^p-functions, and the basic theory for (7.10) is quite elementary.

There is one choice which we have to make sooner or later. It is quite natural to require F to map some set of functions locally in $\mathfrak{B}^{m+1}$ into functions locally in $\mathfrak{B}^m$, but the choice of the growth rate at infinity for the functions in the domain of F is less obvious. It depends on what type results one wants. For the stable manifold theory, the correct growth rate is the one determined by η_-.

Before we can prove the existence of a stable manifold we need a decomposition of $\mathfrak{B}^{m+1}(\eta_-)$ similar to the decomposition of $\mathfrak{B}^{m+1}(\eta)$ into $S \oplus \mathfrak{B}_+^{m+1}(\eta) \oplus H$, developed earlier in this section. In the same way as we can split $\mathfrak{B}^m(\eta)$ into complementary subspaces $\mathfrak{B}_-^m(\eta) \oplus LH \oplus \mathfrak{B}_+^m(\eta)$, we can also split $\mathfrak{B}^m(\eta_-)$ into $\mathfrak{B}_-^m \oplus LH \oplus \mathfrak{B}_+^m(\eta_-)$. Mapping each of these subspaces into $\mathfrak{B}^{m+1}(\eta_-)$ with $r_- *$, we get a decomposition of $\mathfrak{B}^{m+1}(\eta_-)$ into $Y \oplus H \oplus L^{-1}\mathfrak{B}_+^m(\eta_-)$, where $L^{-1}\mathfrak{B}_+^m(\eta_-)$ is the image of $\mathfrak{B}_+^m(\eta_-)$ under $r_- *$. The important conclusion of this reasoning is that the projection P_Y which we used to map $\mathfrak{B}^{m+1}(\eta)$ onto Y also can be regarded as a projection mapping $\mathfrak{B}^{m+1}(\eta_-)$ onto Y.

THEOREM 7.2. *Let Ω be a neighborhood of zero in $\mathfrak{B}^{m+1}(\eta)$. Then there are neighborhoods U, V and W of zero in Y, $C^1(\Omega; \mathfrak{B}^m(\eta_-))$ and Ω, respectively, such that for every $y \in U$ and every $F \in V$ there is a unique solution $x(y, F)$ of (7.10) in W with $P_Y x(y, F) = y$. Furthermore $x(y, F)$ is continuously differentiable in (y, F), and $x(y, 0) = y$.*

Theorem 7.2 generalizes [3, Theorem 2.4, p. 213] in roughly the same sense as Theorems 3.5 and 3.6 generalize [3, Theorem 2.1]. We call the manifold $Y_F = \{x(y; F)|y \in U\}$ the stable manifold of smooth solutions of (7.10). If $F(0) = 0$, $DF(0) = 0$, then the stable manifold Y_F is tangent to Y at zero.

The important thing to observe before one tries to prove Theorem 7.2 is that if x is a solution of (7.10), then $Lx - F(x) \in \mathfrak{B}_-^m(\eta_-)$. Actually, the statement of Theorem 7.2 tells us something about the range of a nonlinear operator, namely the range of the inverse of $Lx - F(x)$, mapping $\mathfrak{B}_-^m(\eta_-)$ into a perturbed version of Y. This means that the proof of Theorem 7.2 is analogous to the proof of Theorem 6.2, although it uses the inverse of $Lx - F(x)$ rather than $Lx - F(x)$ itself. First one

defines $f(y, F)$ to be the solution in $\mathcal{B}^m_-(\eta_-)$ of the equation $y = P_Y R(f)$, where R is the mapping in Theorem 3.5 which takes f in a neighborhood of zero in $\mathcal{B}^m_-(\eta_-)$ into the solution x of the equation $Lx - F(x) = f$. To see that the equation $y = P_Y R(f)$ has a unique solution in $\mathcal{B}^m_-(\eta_-)$, which is a differentiable function of (y, F), it suffices to write it in the form

$$r_- * f = y - P_Y R(f) + r_- * f = y - P_Y[R(f) - r_- * f],$$

observe that this is equivalent to the equation $f = Ly - LP_Y[R(f) - r_- * f]$, note that $R(f) - r_- * f = 0$ when $F = 0$, and use the implicit function theorem. Defining $x(y, F)$ to be $R(f(y, F))$ we get the conclusion of the theorem.

The easiest way to get the existence of an unstable manifold is to apply Theorem 5.3. For this to be possible we have to suppose that not only does F map $\mathcal{B}^{m+1}(\eta_-)$ into $\mathcal{B}^m(\eta_-)$, but also $\mathcal{B}^{m+1}(\eta)$ into $\mathcal{B}^m(\eta)$. In many applications this assumption will not be automatically satisfied. Instead however, it will frequently be true that it is possible to redefine the original operator F for functions x which are "large" at infinity in such a way that the new operator F satisfies our assumption, and that it coincides with the original operator for functions which are "small" at infinity. This idea of redefining the nonlinearity is used quite commonly in proofs of (local) central manifolds for ordinary and abstract differential equations. Of course, if one does so, then all the conclusions which we make are only valid for those solutions of the original equation which are "small" at infinity.

THEOREM 7.3. *Let Ω be a neighborhood of zero in $\mathcal{B}^{m+1}(\eta)$. Then there are neighborhoods U, V and W of zero in Z, $C^1(\Omega; \mathcal{B}^m(\eta))$ and Ω, respectively, such that for every $z \in U$ and every $F \in V$ there is a unique solution $x(z, F)$ in W of the equation $Lx - F(x) = 0$ with $P_Z x(z, F) = z$. Furthermore, $x(z, F)$ is continuously differentiable in (z, F), and $x(z, 0) = z$.*

This is more or less a restatement of Theorem 5.3. We call the manifold $Z_F = \{x(z, F) | z \in U\}$ the unstable manifold of smooth solutions of (7.10). Again, if $F(0) = 0$, $DF(0) = 0$, then Z_F is tangent to Z at zero.

Theorem 7.3 generalizes [3, Theorem 2.3, p. 2.11].

In the same way as we may regard Y_F to be the image of $\mathcal{B}^m_-(\eta_-)$ under the inverse of $Lx - F(x)$, we can regard the set of smooth solutions of (7.10) to be the image of the set of all possible initial functions ξ in (7.2) under the mapping which to every initial function ξ assigns the solution $x \in \mathcal{B}^{m+1}(\eta)$ of (7.10) satisfying (7.2). In other words, before we can study the set of all solutions of (7.10), we have to look at the initial value problem which one gets by combining (7.10) with (7.2).

For the initial value problem to be well posed, we have to assume that F is nonanticipative, i.e.

(7.11) $x(t) = y(t) \, (t \leqslant T)$ implies $F(x)(t) = F(y)(t) \, (t \leqslant T)$ for every $T \in \mathbf{R}$.

For instance, f in (7.1) is of this form if we extend f to $\mathbf{R}$ so that $f \in \mathcal{B}^m(\eta_-)$. Also, the operator L itself is of the same type.

Generally it is not true that (7.10) has a "smooth" solution (i.e. one which is locally in $\mathcal{B}^{m+1}$) for every initial function $\xi \in \mathcal{B}^{m+1}(\mathbf{R}^-; \eta_-)$ and for every nonanticipative F (cf. the discussion of (7.1)). Our first goal is to find the "manifold of smooth initial functions" for which (7.10) has a smooth solution.

If x is a solution of (7.10), then necessarily $Lx^{(k)}(0) - F^{(k)}(x)(0)$ ($k \in K$), where K is the same set as in (7.9) (use right-derivatives at zero, and (7.10)). If F is nonanticipative, then $F^{(k)}(x)(0)$ depends only on the initial function ξ of x (use left-derivatives at the point zero, and (7.11)). Thus, an initial function ξ which gives rise to a smooth solution x must necessarily satisfy

$$(7.12) \qquad\qquad L\xi^{(k)}(0) = F^{(k)}(\xi)(0) \qquad (k \in K).$$

Split $\xi \in \mathcal{B}^{m+1}(\mathbf{R}^-; \eta)$ into $\xi = s + h$, where $s \in S_-$ and $h \in H_-$. Then (7.12) becomes

$$(7.13) \qquad\qquad Lh^{(k)}(0) = F^{(k)}(s + h)(0) \qquad (k \in K).$$

The left-hand side of (7.13) is invertible in H_-, and if the Lipschitz constant of F with respect to h is small enough, then we can apply the contraction mapping principle to get a unique solution $h(s, F)$ for every sufficiently small $s \in S_-$. In other words, we have the following result.

Theorem 7.4. *Let Ω be a neighborhood of zero in $\mathcal{B}^{m+1}(\eta_-)$. Then there are neighborhoods U, V and W of zero in S, $C^1(\Omega; \mathcal{B}^m(\eta_-))$ and H_-, respectively, such that for every $s \in U$ and every nonanticipatory $F \in V$ there is a unique initial function $\xi(s, F) = s + h(s, F)$ satisfying (7.12) with $h(s, F) \in W$. Furthermore, $\xi(s, F)$ is continuously differentiable in (s, F), and $\xi(s, 0) = s$.*

We call the manifold $S_{F_-} = \{\xi(s, F) | s \in U\}$ the manifold of smooth initial functions of (7.10). If $F(0) = 0$, $D_h F(0) = 0$, then S_{F_-} is tangent to S_- at zero.

The next thing to show is that for every $\xi \in S_{F_-}$ there is a unique solution x of (7.10). Here again, we have to work with the influence function η rather than with η_-.

Theorem 7.5. *Let Ω be a neighborhood of zero in $\mathcal{B}^{m+1}(\eta)$. Then there are neighborhoods U, V, and W of zero in S_{F_-}, $C^1(\Omega; \mathcal{B}^m(\eta))$ and Ω, respectively, such that for every nonanticipative $F \in V$ and every $\xi \in U$, there is a unique solution $x(\xi, F)$ of (7.10) in $\mathcal{B}^{m+1}(\eta)$ satisfying (7.2). Furthermore, $x(\xi, F)$ is continuously differentiable in (ξ, F).*

Again, this is just another application of the implicit function theorem. If we split ξ into $s_- + h_-$, where $s_- \in S_-$, and $h_- \in H_-$, then s_- and h_- have extensions $s \in S$ and $h \in H$. Define $y = s + h$, and write the unknown solution x in the form $x = y + z$, where $z \in \mathcal{B}_+^{m+1}(\eta)$. If we let $g(y, z)$ be the function $g(y, z)(t) = 0$ ($t \in \mathbf{R}^-$), $g(y, z)(t) = -Ly(t) + F(y + z)(t)$ ($t \in \mathbf{R}^+$), then z has to satisfy the equation $Lz = g(y, z)$, or equivalently, $z = r_+ * g(y, z)$. By the implicit function theorem, this equation has a unique solution z, and $x = y + z$ is the unique solution of (7.10) with initial condition (7.2).

We call the set of all solutions x in Theorem 7.5 the manifold of smooth solutions of (7.10), and denote it by S_F. Above we have parametrized S_F by using S_{F_-} as the parameter space. One could also parametrize S_F by using S as a parameter space, i.e. one could write each $x \in S_F$ as a differentiable function $x(s, F)$ of $s \in S$ and F. This is true because by Theorem 7.4, there is an invertible, continuously differentiable mapping between S_{F_-} and S_-, and we also know that there is an invertible, continuously differentiable mapping between S_- and S. With this formulation, one has $x(s, 0) = s$.

In the sequel, we shall suppose that F satisfies all the continuity assumptions which are needed in Theorems 7.2–7.5. We define the stable manifold Y_{F_-} and the unstable manifold Z_{F_-} of smooth initial functions to be the restrictions of Y_F and Z_F to $\mathbf{R}^-$. By Theorem 7.5, there is an invertible, continuously differentiable correspondence between Y_{F_-} and Y_F, and between Z_{F_-} and Z_F. This also means that the $\mathbf{R}^-$ versions of Theorems 7.2 and 7.3 are true. The $\mathbf{R}^-$ version of Theorem 7.2 can be proved by mapping a function $y_- \in Y_-$ onto a function $y \in Y$, mapping this function onto a solution $x(y, F) \in Y_F$, and restricting this solution to $\mathbf{R}^-$. As all these mappings are invertible and continuously differentiable, so is the combined mapping. The $\mathbf{R}^-$ version of Theorem 7.3 is proved in a completely similar way.

What can one then say about a general smooth solution of (7.10), i.e. one which belongs neither to the stable nor to the unstable manifold? At minus infinity, one expects the solutions to be close to the "stable" ones, and this is indeed the case.

THEOREM 7.6. *Every $\xi \in S_F$ is of the form $\xi = y + z + h$, where $y \in Y_{F_-}$, $z \in Z_-$, and h vanishes on $(-\infty, -1]$.*

This is quite obvious, because every $\xi \in S_{F_-}$ can be split into three components $\xi = y_1 + z_1 + h_1$, with $y_1 \in Y_-$, $z_1 \in Z_-$, and $h_1 \in H_-$. If we let y be the function in S_{F_-} corresponding to y_1, given in the $\mathbf{R}^-$ version of Theorem 7.2, then y is of the form $y = y_1 + z_2 + h_2$, with $z_2 \in Z_-$ and $h_2 \in H_-$. Thus, the conclusion of Theorem 7.6 holds with $z = z_1 - z_2$ and $h = h_1 - h_2$.

At plus infinity, we expect a general solution to be close to an "unstable" solution. To prove that this is the case, we need one more continuity assumption on F.

THEOREM 7.7. *Let Ω be a neighborhood of zero in $\mathfrak{B}^{m+1}(\eta_-))$. Then there is a neighborhood V of zero in $C^1(\Omega; \mathfrak{B}^m(\eta_-)$, such that if the mapping $y \mapsto F(x + y) - F(x)$ belongs to V for all $x \in S_F$, then every $x \in S_F$ is of the form $x = z + y$, where $z \in Z_F$ and $y \in \mathfrak{B}^{m+1}(\eta_-)$.*

This is true, because if $x \in S_F$, then we can use the implicit function theorem to solve the equation

$$Ly = -[Lx - F(x)] + F(x + y) - F(x)$$

in $\mathfrak{B}^{m+1}(\eta_-)$ (observe that $Lx - F(x) \in \mathfrak{B}^m(\eta_-)$ for every $x \in S_F$). If y is the solution of this equation, then $x + y$ satisfies $L(x + y) - F(x + y) = 0$, so $x + y \in Z_F$. Thus indeed, we have $x = (x + y) - y$, with $x + y \in Z_F$, and $-y \in \mathfrak{B}^{m+1}(\eta_-)$.

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