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NONSTABLE REFLEXIVE SHEAVES ON $\mathbf{P}^3$

BY

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ABSTRACT. The spectrum is defined for nonstable rank two reflexive sheaves on $\mathbf{P}^3$ and is used to establish vanishing theorems for intermediate cohomology in terms of the Chern classes and the order of nonstability. These results are shown to be best possible and the extremal cases are classified. Some applications to Cohen-Macaulay generically local complete intersection curves in $\mathbf{P}^3$ are given.

0. Introduction. We say that a coherent sheaf F on a scheme X is *reflexive* if the natural map $F \to F^{**}$ from F to its double dual is an isomorphism. Here we use the usual definition $F^* := \mathcal{H}om(F, \mathcal{O}_X)$. The concept of reflexive sheaves can be viewed as a natural generalization of the concept of vector bundles.

Let X be a normal projective variety and H a very ample divisor on X. We say that a coherent sheaf F on X is *stable* (in the sense of Mumford and Takemoto [**11**]) if for any coherent subsheaf F' of F with $0 < \operatorname{rank} F' < \operatorname{rank} F$ we have

$$\deg c_1(F')/\operatorname{rank}(F') < \deg c_1(F)/\operatorname{rank}(F),$$

where $c_1(F)$ is the first Chern class of F and deg denotes the degree with respect to H.

The systematic study of stable reflexive sheaves was begun by Hartshorne in [**5**] with emphasis on the case of rank two stable reflexive sheaves on $\mathbf{P}^3$. In the following we use similar methods to study the remainder of the rank two reflexive sheaves on $\mathbf{P}^3$.

We will call a reflexive sheaf *nonstable* if it fails to be stable in the above sense. Schwarzenberger classified nonstable vector bundles on $\mathbf{P}^2$ in [**10**]. There he called them "almost indecomposable". In higher dimensions the picture is unclear. For example, except for direct sums of line bundles no nonstable vector bundles are known on $\mathbf{P}^n$ for $n \geqslant 4$. Grauert and Schneider claimed in [**4**] that such bundles do not exist, but their proof remains incomplete.

In our study we prove vanishing theorems for the intermediate cohomology of nonstable reflexive sheaves on $\mathbf{P}^3$ in terms of the Chern classes and an invariant r measuring the order of nonstability. For example, if F is a rank two reflexive sheaf with $c_1 = 0$, then r is the largest integer for which $H^0(F(-r)) \neq 0$, and

$$H^1(\mathbf{P}^3, F(m)) = 0 \quad \text{for } m \leqslant -\tfrac{1}{2}c_2 - \tfrac{3}{2} - \tfrac{1}{2}r^2 - r, \tag{1}$$

and

$$H^2(\mathbf{P}^3, F(m)) = 0 \quad \text{for } m \geqslant c_2 + r^2 + r - 2. \tag{2}$$

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Further, we can bound c_3 in terms of c_2 and r. If $c_1 = 0$ we have

$$c_3 \leq (c_2 + r^2)(c_2 + (r+1)^2).$$

Using the reduction step of Hartshorne, we classify all sheaves whose cohomology is best possible for these vanishing theorems.

After the preliminary material in §§1 and 2, we define the spectrum of a nonstable rank two reflexive sheaf in §3 and use it to prove the vanishing theorems. The concept of spectrum was introduced by Barth and Elencwajg in [**1**] and was later generalized by Hartshorne in [**5**]. In §4 we show the vanishing theorems and bounds on c_3 of §3 are best possible by giving examples of extremal cases, and in §5 we use the reduction step of Hartshorne to classify extremal cases. In the final section we give some applications to Cohen-Macaulay generically local complete intersection curves in $\mathbf{P}^3$. In particular, we determine all possible arithmetic genera p_a of such curves of any given degree d, and we classify those curves with large p_a. Further, we decide when the union of two plane curves of degrees a and b, meeting at b points, can be smoothly deformed.

I would like to thank my thesis advisor, Robin Hartshorne, for suggesting this study and for providing help and inspiration along the way.

1. Preliminaries. Throughout the paper we work over an algebraically closed ground field k of arbitrary characteristic. Let F be a coherent sheaf on a scheme X of finite type over k. We say a coherent sheaf F on X is *reflexive* if the natural map $F \to F^{**}$ from F to its double dual is an isomorphism. See [**5**, §§1–4] for generalities on reflexive sheaves.

We denote n-dimensional projective space over k by $\mathbf{P}^n$. Suppose F is a rank two reflexive sheaf on $\mathbf{P}^n$. We call F *nonstable* if it fails to be stable in the sense of Mumford and Takemoto [**11**], i.e., if there exists an invertible subsheaf L of F such that $c_1(L) \geq \frac{1}{2}c_1(F)$, where we consider the Chern classes of F as integers. By tensoring F with the hyperplane bundle $\mathcal{O}_{\mathbf{P}^n}(1)$, we may assume $c_i(F)$ is 0 or -1. Then F is nonstable if and only if $H^0(\mathbf{P}^n, F(-s)) \neq 0$ for some nonnegative integer s.

As a measure of the nonstability of a sheaf F with $c_1 = 0$ or -1, we use the following definition. F is *nonstable of order* r if r is the integer maximal for the property $H^0(\mathbf{P}^n, F^*(-r)) \neq 0$. Since $F^* \cong F(-c_1)$ [**5**, Proposition 1.10], F nonstable implies $r + c_1 \geq 0$.

Let F be a rank two reflexive sheaf on $\mathbf{P}^3$, nonstable of order r. Let σ be a nonzero global section of $F^*(-r)$. We may consider the induced map $\sigma\colon \mathcal{O} \to F^*(-r)$; the dual map σ^* is the quotient map of an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(2r + c_1) \to F(r) \to I_{Y,\mathbf{P}^3} \to 0,$$

where Y is a subscheme of $\mathbf{P}^3$ of codimension two, by the maximality of r. It is clear from this sequence that unless $F \cong \mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}$, $H^0(F(-r - c_1))$ is one-dimensional, so to F is associated a unique one-dimensional subscheme Y of $\mathbf{P}^3$, Cohen-Macaulay and generically a local complete intersection. When there is no possibility of confusion we will refer to Y as the *curve associated to* F. Note that Y may be the empty curve, in which case the sequence splits and F is a direct sum of line bundles. It is clear that any rank two direct sum of line bundles is nonstable.

We can use the exact sequence above to determine the degree and arithmetic genus of Y in terms of the invariant r and the Chern classes c_1, c_2, c_3 of F. Again, we assume $c_1 = 0$ or -1.

$$\deg Y = d = c_2 + c_1 r + r^2, \qquad p_a(Y) = \tfrac{1}{2}(c_3 + 2 - d(c_1 + 2r + 4)).$$

Therefore, if we fix r, c_1, c_2, c_3 of the sheaf F, the Hilbert polynomial of Y is determined.

The question of finding necessary and sufficient conditions on the set of integers c_1, c_2, c_3, r for the existence of a nonstable reflexive sheaf on $\mathbf{P}^3$ seems complicated; but there are some obvious necessary restrictions. For example:

(1) $c_3 \equiv c_1 c_2 \pmod 2$;

(2) $c_3 \geqslant 0$;

(3) $c_2 + r^2 \geqslant 0$ for $c_1 = 0$; $c_2 + r^2 - r \geqslant 0$ for $c_1 = -1$.

(1) follows from the Riemann-Roch formula for rank two reflexive sheaves on $\mathbf{P}^3$,

$$\chi(F) = \binom{c_1 + 3}{3} - 2c_2 + \tfrac{1}{2}(c_3 - c_1 c_2) + 1.$$

(2) follows from [**5**, Proposition 2.6], the idea being that c_3 represents the number of points, counted with multiplicity, at which F fails to be locally free. More precisely, $c_3(F) = h^0(\mathcal{E}xt^1(F, \omega_{\mathbf{P}^3}))$. (3) is the statement that the degree of Y is nonnegative.

There are further restrictions on the invariants c_1, c_2, c_3, r, as can be seen from the following elementary example.

EXAMPLE 1.0.1. If F is a rank two reflexive sheaf on $\mathbf{P}^3$, nonstable of order r, with $c_1 = 0$ and $c_2 + r^2 = 1$, then it is clear that the corresponding subscheme Y is a line. The identity $0 = p_a(Y) = \frac{1}{2}(-2r - 4 + c_3 + 2)$ implies $c_3 = 2r + 2$, showing that in this situation not all nonnegative even c_3 may occur. We see that sheaves with $c_1 = 0$, $c_2 + r^2 = 1$ are in one-to-one correspondence with pairs (Y, ξ), where Y is a line in $\mathbf{P}^3$ and

$$\xi \in H^0(\mathcal{O}_Y(2r + 2)) \cong \operatorname{Ext}^1(I_Y(-r), \mathcal{O}_{\mathbf{P}^3}(r))$$

determines the extension corresponding to F. The choice of line in $\mathbf{P}^3$ has four parameters and the choice of ξ has $2r + 2$; so the dimension of the family is $2r + 6$. Similarly, the family of reflexive sheaves with $c_1 = -1$, $c_2 + r^2 - r = 1$ (and necessarily $c_3 = 2r + 1$) has dimension $2r + 5$.

In Proposition 3.8 we will provide a bound for c_3 in terms of c_1, c_2, and r. It turns out there are also "gaps" below this bound; see, for example, Remark 6.1.1.

To complete this section we prove results on the restriction of nonstable sheaves to hyperplanes.

PROPOSITION 1.1. *Let F be a rank two reflexive sheaf on $\mathbf{P}^n$, $n \geqslant 2$, nonstable of order r. Then for a general hyperplane H in $\mathbf{P}^n$, $F|_H$ is reflexive and nonstable of order r.*

PROOF. $F|_H$ is reflexive by [**2**, Proposition 1.1]. We have an exact sequence $0 \to \mathcal{O}(r + c_1) \to F \to I_Y(-r) \to 0$, where Y is codimension-two in $\mathbf{P}^n$. Tensoring with the structure sheaf of a hyperplane H we have

$$\operatorname{Tor}_1(I_Y(-r), \mathcal{O}_H) \to \mathcal{O}_H(r + c_1) \to F|_H \to I_Y \otimes \mathcal{O}_H(-r) \to 0.$$

If we choose H so that it misses the associated points of Y, then $\operatorname{Tor}_1(I_Y(-r), \mathcal{O}_H) = 0$ and $I_Y \otimes \mathcal{O}_H(-r) \cong I_{Y \cap H, H}(-r)$. Since $Y \cap H$ is of codimension two in H, the order of nonstability of $F|_H$ must be exactly r.

COROLLARY 1.2. *Let F be a rank two reflexive sheaf on $\mathbf{P}^n$, nonstable of order r, and L a general line. Then the restriction of F to every linear space containing L is nonstable of order r.*

PROOF. By Proposition 1.1 the restriction of F to a general line L is nonstable of order r. Each linear subspace of $\mathbf{P}^n$ containing L is caught in a chain $L = H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{n-1} \subseteq H_n = \mathbf{P}^n$ of linear subspaces H_i of dimension i. The orders of nonstability of the $F|_{H_i}$ are nonincreasing as a function of i, since for any linear space H and hyperplane K of H, $h^0(E_K) = 0 \Rightarrow h^0(E) = 0$ for any sheaf E on H. Since $F|_H$ and $F|_{H_n}$ are nonstable of order r, so are $F|_{H_i}$ for all i.

2. Cohomology of nonstable bundles on $\mathbf{P}^2$. In this section we study the intermediate cohomology of nonstable vector bundles on $\mathbf{P}^2$. Given a submodule of the graded module $H^1(E(*))$, we prove results describing the "rate of growth" of the dimensions of the submodule. We will use this information in §3, where we will restrict a sheaf on $\mathbf{P}^3$ to a general plane and investigate the restriction of the intermediate cohomology to $\mathbf{P}^2$. At the end of §2 we prove a cohomology vanishing theorem for nonstable bundles on $\mathbf{P}^2$.

PROPOSITION 2.1. *Let E be a rank two vector bundle on $\mathbf{P}^2$, $c_1 = 0$ or -1, nonstable of order r. Suppose further that E is not a direct sum of line bundles. Denote* dim H^i *by h^i. Then*:

$$h^0(E(m)) = \begin{cases} 0 & \text{for } m < -r - c_1, \\ \dbinom{m + r + c_1 + 2}{2} & \text{for } -r - c_1 \leqslant m \leqslant r, \end{cases}$$

$$h^2(E(m)) = \begin{cases} 0 & \text{for } m > r - 3, \\ \dbinom{r - m - 1}{2} & \text{for } -r - c_1 - 3 \leqslant m \leqslant r - 3, \end{cases}$$

$$h^1(E(m)) \begin{cases} \leqslant r^2 + rc_1 + c_2 - 1 & \text{for } m \leqslant -r - c_1 - 3, \\ = r^2 + rc_1 + c_2 - 1 & \text{for } m = -r - c_1 - 3, \\ = r^2 + rc_1 + c_2 & \text{for } -r - c_1 - 2 \leqslant m \leqslant r - 1 \\ = r^2 + rc_1 + c_2 - 1 & \text{for } m = r, \\ \leqslant r^2 + rc_1 + c_2 - 1 & \text{for } m \geqslant r. \end{cases}$$

PROOF. This proposition follows from the exact sequence

$$0 \to \mathcal{O}(r + c_1) \to E \to I_Y(-r) \to 0$$

and the Riemann-Roch formula on $\mathbf{P}^2$, which says, for $c_1 = 0$ or -1,

$$\chi(E(m)) = (m+1)(m+2+c_1) - c_2. \quad \square$$

Later in this section we will refine our information about $h^1(E(m))$. See Remark 2.3.1.

The following "bilinear map lemma" was used by Hartshorne in [5].

LEMMA 2.2. *Let ϕ: $V_1 \times V_2 \to W$ be a map of nonzero finite-dimensional vector spaces which is bilinear and nondegenerate, i.e, for each $v_1 \neq 0$ in V_1 and each $v_2 \neq 0$ in V_2, $\phi(v_1, v_2) \neq 0$. Then*

$$\dim W \geqslant \dim V_1 + \dim V_2 - 1.$$

LEMMA 2.3. *Let E be a rank two vector bundle on $\mathbf{P}^2$, $c_1 = 0$ or -1, nonstable of order r. Let N be a graded submodule of the graded module $\oplus_{l \in \mathbf{Z}} H^1(E(l))$. Denote* $\dim N_m$ *by* n_m. *Then*

(a) $n_m \leqslant n_{m+1}$ *for* $m \leqslant r - 2$;

(b) $n_m < n_{m+1}$ *for* $m \leqslant -r - c_1 - 3$, *if* $n_m \neq 0$;

(c) *if* $n_m + 1 = n_{m+1}$ *for* $m \leqslant -r - c_1 - 4$, *then there exists a linear form* x *annihilating* N_k *for* $k \leqslant m$.

PROOF. The lemma is vacuous if E is a direct sum of line bundles. In the remaining case there is an exact sequence

$$0 \to \mathcal{O}(r + c_1) \to E \to I_Y(-r) \to 0,$$

where Y is a zero-dimensional nonempty subscheme of $\mathbf{P}^2$.

The restriction of this sequence to any line L missing Y is

$$0 \to \mathcal{O}_L(r + c_1) \to E_L \to \mathcal{O}_L(-r) \to 0.$$

This sequence splits since, by definition, $r + c_1 \geqslant 0$. So $E_L \cong \mathcal{O}_L(r + c_1) \oplus \mathcal{O}_L(-r)$ and we calculate

$$h^0(E_L(m)) = \begin{cases} 0 & \text{for } m \leqslant -r - c_1 - 1, \\ m + r + c_1 + 1 & \text{for } -r - c_1 - 1 \leqslant m \leqslant r - 1. \end{cases}$$

Now consider the exact sequence of cohomology

$$0 \to H^0(E(m)) \to H^0(E(m+1)) \to H^0(E_L(m+1))$$
$$\xrightarrow{\delta_m} H^1(E(m)) \to H^1(E(m+1)).$$

We will show δ_m is the zero map for $m \leqslant r - 2$ and a general line L, proving (a). If $m \leqslant -r - c_1 - 2$, then $H^0(E_L(m+1)) = 0$, so δ_m is zero. If $-r - c_1 - 1 \leqslant m \leqslant r - 2$, the first three terms of this exact sequence have dimensions

$$\binom{m + r + c_1 + 2}{2}, \quad \binom{m + r + c_1 + 3}{2}, \quad m + r + c_1 + 2,$$

respectively, so δ_m is again zero.

To prove (b), consider the bilinear map

$$H^0(\mathcal{O}_{\mathbf{P}^2}(1)) \otimes N_m \to N_{m+1}.$$

The left-hand side is nonzero so Lemma 2.2 applies to show that either $n_{m+1} \geqslant n_m + 2$ or there exists an $x \in H^0(\mathcal{O}_{\mathbf{P}^2}(1))$ such that $N_m \xrightarrow{x} N_{m+1}$ is not injective. In the latter case, denoting by L the line defined by x, there is a diagram

$$\begin{tikzcd} H^0(E(m+1)) \arrow[r] & H^0(E_L(m+1)) \arrow[r, "\delta_m"] & H^1(E(m)) \arrow[r, "x"] & H^1(E(m+1)) & \\ & N'_{m+1} \arrow[u, phantom, "\cup|"] \arrow[r] & N_m \arrow[u, phantom, "\cup|"] \arrow[r] & N''_{m+1} \arrow[u, phantom, "\cup|"] \arrow[r] & 0 \end{tikzcd}$$

where N' is the preimage of N by δ, and N'' the image of N by x. By assumption, $N'_{m+1} \neq 0$.

Assume $m \leqslant -r - c_1 - 3$. Then $m + 2 \leqslant -r - c_1 - 1$, so that $H^0(E(m+1))$ and $H^0(E(m+2))$ are zero, and δ_m and δ_{m+1} are injective. This implies $n_m = n'_{m+1} + n''_{m+1}$ and $n_{m+1} = n'_{m+2} + n''_{m+2}$. Since $m + 1 \leqslant r - 2$ we can use (a) to find $n''_{m+1} \leqslant n''_{m+2}$. N' is a torsion-free graded $H^0(\mathcal{O}_L(1))$-module and $n'_{m+1} \neq 0$, so $n'_{m+1} < n'_{m+2}$. It follows that $n_m < n_{m+1}$.

Finally, if $N''_{m+1} \neq 0$, then $n''_{m+1} < n''_{m+2}$ for $m \leqslant -r - c_1 - 4$ by (b). This would imply $n_{m+1} \geqslant n_m + 2$. So $n_{m+1} = n_m + 1$ implies N''_{m+1} is zero, or, in other words, that x annihilates N_m. This proves (c).

Remark 2.3.1. Setting $N = \bigoplus_{m \in \mathbf{Z}} H^1(E(m))$ and using Serre duality, we see, in particular, that

$$h^1(E(m)) < h^1(E(m+1)) \quad \text{for } m \leqslant -r - c_1 - 3,$$
$$h^1(E(m)) > h^1(E(m+1)) \quad \text{for } m \geqslant r - 1,$$

improving Proposition 2.1. Thus, we have the following vanishing result:

Corollary 2.4. *Let E be a rank two vector bundle on $\mathbf{P}^2$, $c_1 = 0$ or -1, nonstable of order r. Then $H^1(E(m)) = 0$ for*

$$m \leqslant \begin{cases} -c_2 - r^2 - r - 2, & c_1 = 0, \\ -c_2 - r^2 - 1, & c_1 = -1. \end{cases}$$

3. The spectrum. The notion of spectrum was developed for stable reflexive sheaves by Barth and Elencwajg in [**1**] and Hartshorne in [**5**]. We make the analogous construction for the nonstable case, and find that to each nonstable rank two reflexive sheaf we can associate a set of $d = c_2 + c_1 r + r^2$ integers, the spectrum, which gives information about the intermediate cohomology of the sheaf. This set of integers has certain connectedness properties (3.4), and in case the sheaf is locally free, symmetry properties (3.5), which put restrictions on the cohomology of the sheaf. In particular, we prove cohomology vanishing theorems and restrictions on the values of c_3 (3.7, 3.8).

THEOREM 3.1. *Let F be a rank two reflexive sheaf on $\mathbf{P}^3$, $c_1 = 0$ or -1, nonstable of order r. Let $d = c_2 + c_1 r + r^2$ be the degree of the unique associated curve Y. Then there exists a unique set of integers $k_1, \dots, k_d$, called the* spectrum *of F, such that if we set $\mathcal{K} := \bigoplus_{i=1}^{d} \mathcal{O}_{\mathbf{P}^1}(k_i)$, we have*

(a) $$h^1(\mathbf{P}^3, F(m)) = h^0(\mathbf{P}^1, \mathcal{K}(m+1)) \quad \text{for } m \leq r-1,$$

(b) $$h^2(\mathbf{P}^3, F(m)) = h^1(\mathbf{P}^1, \mathcal{K}(m+1)) \quad \text{for } m \geq -r - c_1 - 3.$$

PROOF. Let H be a general plane in $\mathbf{P}^3$. Since H misses the zero-dimensional support of $\mathcal{E}xt^1(F, \omega_{\mathbf{P}^3})$, F_H is locally free. By Proposition 1.1, F_H is nonstable of order r. Let x be the linear form vanishing on H and consider the sequence

$$0 \to H^0(F(m-1)) \xrightarrow{x} H^0(F(m)) \to H^0(F_H(m)) \xrightarrow{\delta_{m-1}} H^1(F(m-1)).$$

We will show that δ_{m-1} is zero for $m \leq r$.

For $m \leq -r - c_1 - 1$, this is obvious, since $H^0(F_H(m)) = 0$. For $-r - c_1 \leq m \leq r$, the first three terms of this exact sequence have dimensions

$$\binom{m+r+c_1+2}{3}, \quad \binom{m+r+c_1+3}{3}, \quad \binom{m+r+c_1+2}{2},$$

respectively, so δ_{m-1} is again zero.

Define the graded submodule N of $\bigoplus_{m\in\mathbf{Z}} H^1(F_H(m))$ to be in degree m the image of the natural map $H^1(F(m)) \to H^1(F_H(m))$. We have an exact sequence

(1) $$0 \to H^1(F(m-1)) \to H^1(F(m)) \to N_m \to 0$$

for $m \leq r$.

Secondly, consider the exact sequence

$$H^2(F(m+1)) \xrightarrow{\varepsilon_m} H^2(F_H(m+1)) \to H^3(F(m)) \to H^3(F(m+1)) \to 0.$$

This is the dual of the sequence

$$0 \to H^0(F(-m-5-c_1)) \to H^0(F(-m-4-c_1))$$
$$\to H^0(F_H(-m-4-c_1)) \xrightarrow{\varepsilon_m^*} H^1(F(-m-5-c_1)).$$

Counting dimensions of global sections as above we see that ε_m is zero for $-m-4-c_1 \leq r$, i.e., $m \geq -r - c_1 - 4$.

Let S be the quotient module of $\bigoplus_{m\in\mathbf{Z}} H^1(F_H(m))$ defined in degree m to be the kernel of the map $H^2(F(m-1)) \xrightarrow{x} H^2(F(m))$. We have an exact sequence

(2) $$0 \to S_{m+1} \to H^2(F(m)) \to H^2(F(m+1)) \to 0$$

for $m \geq -r - c_1 - 4$ and, as graded modules,

$$0 \to N \to \bigoplus_{m\in\mathbf{Z}} H^1(F_H(m)) \to S \to 0.$$

We are now ready to define the k_i. Let

$$\#\{k_i = m\} = \begin{cases} n_{-m-1} - n_{-m-2} & \text{for } m \geq -r, \\ s_{-m-2} - s_{-m-1} & \text{for } m \leq r + c_1. \end{cases}$$

To show this is well defined we need to show

$$n_{-m-1} - n_{-m-2} = s_{-m-2} - s_{-m-1}$$

for $-r \leqslant m \leqslant r + c_1$. In this range,

$$-r - 2 \leqslant -r - 2 - c_1 \leqslant -m - 2 < -m - 1 \leqslant r - 1,$$

and, by Proposition 2.1, $n_t + s_t = h^1(F_H(t))$ is constant for $-r - 2 \leqslant t \leqslant r - 1$. Also, these numbers are nonnegative because, by Lemma 2.3, $n_{-m-1} - n_{-m-2} \geqslant 0$ for $-m - 2 \leqslant r - 2$, i.e., $m \geqslant -r$, and $s_{-m-2} - s_{-m-1} = s^*_{m-1-c_1} - s^*_{m-2-c_1} \geqslant 0$ for $m - 2 - c_1 \leqslant r - 2$, i.e., $m \leqslant r + c_1$, where we treat S^* as a submodule of $\bigoplus_{m \in \mathbf{Z}} H^1(F_H(m))$.

We count the k_i as follows: $\#\{k_i \geqslant 0\} = n_{-1}$, $\#\{k_i < 0\} = s_{-1}$, and $n_{-1} + s_{-1} = h^1(F_H(-1)) = c_2 + rc_1 + r^2 = d$.

It remains to prove (a) and (b). To check (a), we have for $m \leqslant r - 1$:

$$\begin{aligned} h^0(\mathbf{P}^1, \mathcal{K}(m+1)) &= \sum_{k_i \geqslant -m-1} (k_i + m + 2) = \sum_{t \geqslant -m-1} (n_{-t-1} - n_{-t-2})(t + m + 2) \\ &= \sum_{t \geqslant -m-1} n_{-t-1}(t + m + 2) - \sum_{t \geqslant -m} n_{-t-1}(t + m + 1) \\ &= \sum_{t \leqslant m} n_t = h^1(F(m)), \end{aligned}$$

the last equality coming from exact sequence (1). To check (b), we have for $m \geqslant -r - c_1 - 3$:

$$\begin{aligned} h^1(\mathbf{P}^1, \mathcal{K}(m+1)) &= -\sum_{k_i \leqslant -m-3} (k_i + m + 2) \\ &= -\sum_{t \leqslant -m-3} (s_{-t-2} - s_{-t-1})(t + m + 2) \\ &= -\sum_{t \leqslant -m-3} s_{-t-2}(t + m + 2) + \sum_{t \leqslant -m-4} s_{-t-2}(t + m + 3) \\ &= \sum_{t \geqslant m+1} s_t = h^2(F(m)), \end{aligned}$$

the last equality coming from exact sequence (2). This completes the proof of the theorem.

PROPOSITION 3.2. *There is a $k_i = -r - 1$ in the spectrum unless F is a direct sum of line bundles.*

PROOF. $\#\{k_i = -r - 1\} = s_{r-1} - s_r$. We must show $s_{r-1} > s_r$, i.e., $s^*_{-r-2-c_1} > s^*_{-r-3-c_1}$. This follows from Lemma 2.3(b), as long as $s^*_{-r-2-c_1} = s_{r-1}$ is nonzero.

Assume $s_{r-1} = 0$. By Lemma 2.3(a) we must have $s_m = 0$ for $m \geqslant r - 1$. Now exact sequence (2) in the proof of the above theorem shows that $H^2(F(r-2)) = 0$. From the exact sequence

$$0 \to \mathcal{O}(r + c_1) \to F \to I_Y(-r) \to 0,$$

we get the exact sequence of cohomology

$$0 = H^2(F(r-2)) \to H^2(I_Y(-2)) \to H^3(\mathcal{O}_{\mathbf{P}^3}(2r - 2 + c_1)) = 0,$$

so $H^1(\mathcal{O}_Y(-2)) = H^2(I_Y(-2)) = 0$. It remains to prove the following:

LEMMA 3.3. *Let Y be a one-dimensional projective scheme, L a very ample invertible sheaf on Y. Then $H^1(Y, L^{\otimes -2}) \neq 0$.*

PROOF. Embed Y in $\mathbf{P}^n$ by L. We must show $H^1(\mathcal{O}_Y(-2)) \neq 0$. We may assume Y is connected. If Y is also reduced, then $p_a(Y) \geqslant 0$. $\chi(\mathcal{O}_Y(-2)) = -2d + 1 - p_a < 0$, where d is the degree of Y in $\mathbf{P}^n$, so $H^1(\mathcal{O}_Y(-2)) \neq 0$.

In the general case, setting $Z = Y_{\text{red}}$, we have

$$0 \to I_{Z,Y} \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0.$$

Since $H^1(\mathcal{O}_Z(-2)) \neq 0$ and $H^2(I_{Z,Y}(-2)) = 0$, we have $H^1(\mathcal{O}_Y(-2)) \neq 0$.

PROPOSITION 3.4. (1) *If $k > r + 1 + c_1$ is in the spectrum, then so are $r + 1 + c_1$, $r + 2 + c_1, \ldots, k$.*

(2) *If $k < -r - 1$ is in the spectrum, then so are $k, k + 1, \ldots, -r - 1$.*

PROOF. (1) Lemma 2.3(b) says that $n_m - n_{m-1} > 0$ for $m \leqslant -r - c_1 - 2$, as long as $n_{m-1} \neq 0$. If $k > r + 1 + c_1$ is in the spectrum, then $n_{-k-1} \neq 0$, so $n_{-k-1} < n_{-k} < \cdots < n_{-r-c_1-2}$, showing that

$$\#\{k_i = m\} = n_{-m-1} - n_{-m-2} > 0 \qquad \text{for } r + c_1 + 1 \leqslant m \leqslant k - 1.$$

(2) If $k < -r - 1$ is in the spectrum, then $s^*_{k-1-c_1} = s_{-k-2} \neq 0$. Again by Lemma 2.3(b), $s^*_{k-1-c_1} < s^*_{k-c_1} < \cdots < s^*_{-r-2-c_1}$, i.e., $s_{-k-2} < s_{-k-3} < \cdots < s_{r-1}$, so

$$\#\{k_i = m\} = s_{-m-2} - s_{-m-1} > 0 \quad \text{for } k + 1 \leqslant m \leqslant -r - 1.$$

PROPOSITION 3.5. *Assume F is locally free. Then $\{-k_i\} = \{k_i\}$ if $c_1 = 0$ and $\{-k_i\} = \{k_i + 1\}$ if $c_1 = -1$.*

PROOF. Serre duality shows that

$$\begin{aligned} s_m &= h^2(F(m-1)) - h^2(F(m)) = h^1(F(-m-3-c_1)) - h^1(F(-m-4-c_1)) \\ &= n_{-m-3-c_1} \end{aligned}$$

for $m \geqslant 0$. Thus we have

$$\#\{k_i = m\} = s_{-m-2} - s_{-m-1} = n_{m-1-c_1} - n_{m-2-c_1} = \#\{k_i = -m + c_1\}.$$

PROPOSITION 3.6.

$$c_3 = \begin{cases} -2\sum k_i & \text{for } c_1 = 0, \\ -2\sum k_i - c_2 - r^2 + r & \text{for } c_1 = -1. \end{cases}$$

PROOF. Essentially the same as the proof of Proposition 7.3 in [5], except that in the $c_1 = -1$ case, $m_{-1} = c_2 + r^2 - r$.

THEOREM 3.7. *Suppose F is a rank two reflexive sheaf on $\mathbf{P}^3$, nonstable of order r.*

(1) *If $c_1 = 0$, then $H^1(F(m)) = 0$ for $m \leqslant -\frac{1}{2}c_2 - \frac{3}{2} - \frac{1}{2}r^2 - r$.*

(2) *If $c_1 = -1$, then $H^1(F(m)) = 0$ for $m \leqslant -\frac{1}{2}c_2 - \frac{1}{2} - \frac{1}{2}r^2 - \frac{1}{2}r$.*

PROOF. Set $k = \max\{k_i\}$. Then $H^1(F(m)) = 0$ for $m + k + 1 < 0$ by Theorem 3.1.

(1) $c_1 = 0$. By Proposition 3.2 there exists a $k_i = -r - 1$. By Proposition 3.4 the spectrum is connected below this point, and the length of the spectrum is $d = c_2 + r^2$. Using Proposition 3.6 we see that

$$k \leqslant \begin{cases} \frac{1}{2}(c_2 + r^2) + r & \text{if } c_2 + r^2 \text{ is even,} \\ \frac{1}{2}(c_2 + r^2 - 1) + r & \text{if } c_2 + r^2 \text{ is odd.} \end{cases}$$

Thus $H^1(F(m)) = 0$ for

$$m < -k - 1 \leqslant \begin{cases} -\frac{1}{2}(c_2 + r^2) - r - 1 & \text{if } c_2 + r^2 \text{ is even,} \\ -\frac{1}{2}(c_2 + r^2) - r - \frac{1}{2} & \text{if } c_2 + r^2 \text{ is odd,} \end{cases}$$

or, in either case, for $m \leqslant -\frac{1}{2}(c_2 + r^2) - r - \frac{3}{2}$.

(2) $c_1 = -1$. By the same reasoning, $k \leqslant \frac{1}{2}(c_2 + r^2 - r) + r - 1$. So

$$H^1(F(m)) = 0 \quad \text{for } m < -k - 1 \leqslant -\tfrac{1}{2}(c_2 + r^2 + r - 1),$$

i.e., for integers $m \leqslant -\frac{1}{2}c_2 - \frac{1}{2}r^2 - \frac{1}{2}r - \frac{1}{2}$.

THEOREM 3.8. *Suppose F is a rank two reflexive sheaf on $\mathbf{P}^3$, nonstable of order r.*

(1) *If $c_1 = 0$, then $H^2(F(m)) = 0$ for $m \geqslant c_2 + r^2 + r - 2$ and*

$$c_3 \leqslant (c_2 + r^2)(c_2 + (r + 1)^2).$$

(2) *If $c_1 = -1$, then $H^2(F(m)) = 0$ for $m \geqslant c_2 + r^2 - 2$ and*

$$c_3 \leqslant (c_2 + (r - 1)r)(c_2 + r(r + 1)).$$

PROOF. Both problems are solved by finding the "most negative" possible spectrum.

(1) $c_1 = 0$. By the connectedness property (3.4) of the spectrum,

$$-r - 1, -r - 2, \ldots, -r^2 - r - c_2$$

is the most negative spectrum. Set $k = \min\{k_i\} = -r - r^2 - c_2$. Then

$$H^2(F(m)) = 0 \quad \text{for } m \geqslant -k - 2 = r + r^2 + c_2 - 2,$$

and

$$c_3 = -2\sum k_i = 2(r + 1 + r + 2 + \cdots + r + r^2 + c_2) = (c_2 + (r + 1)^2)(c_2 + r^2).$$

(2) $c_1 = -1$. The most negative spectrum is $-r - 1, -r - 2, \ldots, -c_2 - r^2$. Thus $k = \min\{k_i\} = -c_2 - r^2$, and

$$H^2(F(m)) = 0 \quad \text{for } m \geqslant -k - 2 = c_2 + r^2 - 2.$$

Finally,

$$c_3 = -2\sum k_i - c_2 - r^2 + r = (c_2 + (r - 1)r)(c_2 + r(r + 1)).$$

4. Some examples. In this section we construct examples to show that the results of Theorems 3.7 and 3.8 are best possible. In Examples 4.1.3 and 4.1.5 the following theorem of Ferrand [3] will be useful:

THEOREM 4.1. *Let X be a local complete intersection curve in $\mathbf{P}^3$. Let M be an invertible sheaf on X, u: $I_X \to M \to 0$ a surjective map, and Y the scheme defined by $I_Y = \ker u$. Let m be an integer, and assume $M \cong \omega_X(m)$ and the map*

$$\bar{u}: H^1(\mathbf{P}^3, I_X(-m)) \to H^1(X, \omega_X)$$

induced by u is the zero map. Then $\omega_Y \cong \mathcal{O}_Y(-m)$.

EXAMPLE 4.1.1. For each $r \geq 0$, $d = c_2 + r^2 > 0$, a reflexive sheaf F, nonstable order r, with $c_1 = 0$, $c_3 = (c_2 + r^2)(c_2 + (r+1)^2)$ and $H^2(F(c_2 + r^2 + r - 3)) \neq 0$.

Let Y be a plane curve of degree d, $0 \neq \xi \in H^0(\mathcal{O}_Y(2r + d + 1))$. Then ξ vanishes at $d(2r + d + 1)$ points, counted with multiplicity. Since

$$H^0(\mathcal{O}_Y(2r + d + 1)) \cong H^0(\omega_Y(2r + 4)) \cong \operatorname{Ext}^1(I_Y, \mathcal{O}_{\mathbf{P}^3}(2r)),$$

ξ determines an extension

$$0 \to \mathcal{O}(r) \to F \to I_Y(-r) \to 0$$

where F is locally free except at $c_3 = d(2r + d + 1) = (c_2 + r^2)(c_2 + (r+1)^2)$ points.

The spectrum of F is $-r - d, -r - d + 1, \ldots, -r - 1$, so clearly

$$h^2(F(c_2 + r^2 + r - 3)) = h^1(\mathcal{K}(d + r - 2)) \neq 0.$$

EXAMPLE 4.1.2. For each $r \geq 1$, $d = c_2 + r^2 - r > 0$, a reflexive sheaf F, nonstable of order r, with $c_1 = -1$, $c_3 = (c_2 + r(r-1))(c_2 + r(r+1))$ and

$$H^2(F(c_2 + r^2 - 3)) \neq 0.$$

Let Y be a plane curve of degree d, $0 \neq \xi \in H^0(\mathcal{O}_Y(2r + d + 2)) \cong \operatorname{Ext}^1(I_Y, \mathcal{O}_{\mathbf{P}^3}(2r - 1))$. ξ determines an extension

$$0 \to \mathcal{O}(r - 1) \to F \to I_Y(-r) \to 0$$

where F is locally free except at $c_3 = d(2r + d) = (c_2 + r(r-1))(c_2 + r(r+1))$ points.

The spectrum of F is $-r - d, -r - d + 1, \ldots, -r - 1$, so $h^2(F(c_2 + r^2 - 3)) = h^1(\mathcal{K}(d + r - 2)) \neq 0$.

EXAMPLE 4.1.3. For each $r \geq 0$, $d = c_2 + r^2$ even positive integer, a vector bundle F nonstable of order r with $c_1 = 0$ and $H^1(F(-\frac{1}{2}c_2 - \frac{1}{2}r^2 - r - 1)) \neq 0$.

We can construct such bundles in two different ways:

Method 1. Set $m = \frac{1}{2}(c_2 + r^2) + r + 1$. Let Z_1 be a degree $2m - 1$ plane curve and Z_2 a complete intersection of two surfaces of degrees $m + r + 1$, $m - r - 1$, respectively. Then $\omega_{Z_1} \cong \mathcal{O}_{Z_1}(2m - 4)$ and $\omega_{Z_2} \cong \mathcal{O}_{Z_2}(2m - 4)$. Set $Z = Z_1 \amalg dZ_2$. Then $\omega_Z \cong \mathcal{O}_Z(2m - 4)$ and $\deg Z = 2m - 1 + m^2 - (r+1)^2$. Let ξ be a nowhere-vanishing global section of $\omega_Z(-2m + 4) \cong \mathcal{E}xt^1(I_Z, \mathcal{O}_{\mathbf{P}^3}(-2m))$. Then ξ determines a vector bundle F' as an extension

$$0 \to \mathcal{O} \to F' \to I_Z(2m) \to 0$$

where $c_1(F') = 2m$, $c_2(F') = \deg Z = m^2 + 2m - 1 - (r+1)^2$. Setting $F = F'(-m)$ we have

$$0 \to \mathcal{O}(-m) \to F \to I_Z(m) \to 0$$

where $c_1(F) = 0$, $c_2(F) = 2m - 1 - (r+1)^2$. It is clear that the least degree of a surface containing Z is $m - r$, so F is nonstable and $h^0(F(-r)) = 1$. Further, $d(F) = c_2 + r^2 = 2(m - r - 1)$. Z has two connected components so $h^1(F(-m)) = h^1(I_Z) = 1$. Thus $H^1(F(-\frac{1}{2}(c_2 + r^2) - r - 1)) \neq 0$. The spectrum of F is

$$-r - \tfrac{1}{2}\left(c_2 + r^2\right), -r - \tfrac{1}{2}\left(c_2 + r^2\right) + 1, \dots, -r - 1, r + 1, r + 2, \dots, r + \tfrac{1}{2}\left(c_2 + r^2\right).$$

Method 2. Set $e = \frac{1}{2}(c_2 + r^2)$ and let X be a plane curve of degree e. Choose a map

$$I_{X,\mathbf{P}^3} \underset{\mathcal{O}_{\mathbf{P}^3}}{\otimes} \mathcal{O}_X \cong \mathcal{O}_X(-e) \oplus \mathcal{O}_X(-1) \to \mathcal{O}_X(2r + e + 1)$$

which is surjective. Composing with the surjective map $I_{X,\mathbf{P}^3} \to I_X \otimes_{\mathcal{O}_{\mathbf{P}^3}} \mathcal{O}_X$, we get a surjective map whose kernel defines a multiplicity-two structure Y on X:

$$0 \to I_Y \to I_X \to \mathcal{O}_X(2r + e + 1) \to 0.$$

Using Ferrand's theorem (4.1) with $M = \mathcal{O}_X(2r + e + 1) \cong \omega_X(2r + 4)$, we find that $\omega_Y \cong \mathcal{O}_Y(-2r - 4)$.

Choosing a nowhere-vanishing global section $\xi \in H^0(\omega_Y(2r + 4)) \cong \operatorname{Ext}^1(I_Y, \mathcal{O}_{\mathbf{P}^3}(2r))$, we construct a vector bundle $0 \to \mathcal{O}(r) \to F \to I_Y(-r) \to 0$. Now

$$H^1\left(F\left(-\tfrac{1}{2}\left(c_2 + r^2\right) - r - 1\right)\right) \cong H^1(I_Y(-e - 2r - 1))$$

is nonzero because of the exact sequence

$$\begin{aligned} 0 = H^0(I_X(-2r - e - 1)) &\to H^0(\mathcal{O}_X) \to H^1(I_Y(-2r - e - 1)) \\ &\to H^1(I_X(-2r - e - 1)) = 0. \end{aligned}$$

The spectrum of F is again $-r - e, -r - e + 1, \dots, -r - 1, r + 1, r + 2, \dots, r + e$.

EXAMPLE 4.1.4. For each $r \geqslant 0$, $d = c_2 + r^2$ odd positive integer, a reflexive sheaf nonstable of order r, $c_1 = 0$, and $H^1(F(-\frac{1}{2}(c_2 + r^2) - r - \frac{1}{2})) \neq 0$.

Set $m = \frac{1}{2}(c_2 + r^2 + 1) + r$. Let Z_1 be a degree $2m$ plane curve and Z_2 a complete intersection of two surfaces of degrees $m + r + 1$ and $m - r - 1$, respectively. Then $\omega_{Z_1} \cong \mathcal{O}_{Z_1}(2m - 3)$ and $\omega_{Z_2} \cong \mathcal{O}_{Z_2}(2m - 4)$. Set $Z = Z_1 \amalg Z_2$. Then $\deg Z = 2m + m^2 - (r+1)^2$. Choose

$$\xi \in H^0\left(\mathcal{O}_{Z_1}(1) \oplus \mathcal{O}_{Z_2}\right) \cong H^0(\omega_Z(-2m + 4)) \cong \operatorname{Ext}^1\left(I_{Z_1}, \mathcal{O}(-2m)\right)$$

vanishing at $2m$ points. Then ξ determines an extension $0 \to \mathcal{O} \to F' \to I_Z(2m) \to 0$ where F' is reflexive, $c_1(F') = 2m$, $c_2(F') = \deg Z = 2m + m^2 - (r+1)^2$, and $c_3(F') = 2m$. Setting $F := F'(-m)$ we have $0 \to \mathcal{O}(-m) \to F \to I_Z(m) \to 0$ where $c_1(F) = 0$, $c_2(F) = 2m - (r+1)^2$, $c_3(F) = 2m$. The least degree of a surface containing Z is $m - r$, so F is nonstable of order r. Further, $d(F) = c_2 + r^2 = 2m - 2r - 1$. Z has two connected components so $h^1(F(-m)) = h^1(I_Z) = 1$. Since $m = \frac{1}{2}(c_2 + r^2 + 1) + r$, we have $H^1(F(-\frac{1}{2}(c_2 + r^2 + 1) - r)) \neq 0$. The spectrum of F is

$$-\tfrac{1}{2}\left(c_2 + r^2 + 1\right) - r, -\tfrac{1}{2}\left(c_2 + r^2 - 1\right) - r, \dots,$$
$$-r - 1, r + 1, r + 2, \dots, \tfrac{1}{2}\left(c_2 + r^2 - 1\right) + r.$$

EXAMPLE 4.1.5. For each $r \geqslant 1$, $d = c_2 + r^2 - r$ even positive integer, a vector bundle F, $c_1 = -1$, nonstable of order r, and $H^1(F(-\frac{1}{2}(c_2 + r^2 + r))) \neq 0$.

As before, we can construct these bundles in two different ways:

Method 1. Set $m = \frac{1}{2}(c_2 + r^2 - r) + r$. Let Z_1 be a degree $2m$ plane curve, and Z_2 a complete intersection of two surfaces of degrees $m + r + 1$ and $m - r$, respectively. Set $Z := Z_1 \coprod Z_2$. Then $\omega_Z \cong \mathcal{O}_Z(2m - 3)$. A nowhere-vanishing section of $\omega_Z(-2m - 3)$ determines a vector bundle F' as an extension

$$0 \to \mathcal{O} \to F' \to I_Z(2m + 1) \to 0.$$

Setting $F := F'(-m - 1)$ we get $0 \to \mathcal{O}(-m - 1) \to F \to I_Z(m) \to 0$ where $c_1(F) = -1$, $c_2(F) = 2m - r(r + 1)$. The least degree of a surface containing Z is $m - r + 1$, so F is nonstable of order r. Z has two connected components so $h^1(F(-m)) = h^1(I_Z) = 1$. Thus we have $H^1(F(-\frac{1}{2}(c_2 + r^2 + r))) \neq 0$. The spectrum of F is

$$-r - \tfrac{1}{2}\left(c_2 + r^2 - r\right), -r - \tfrac{1}{2}\left(c_2 + r^2 - r\right) + 1, \dots,$$
$$-r - 1, r, r + 1, \dots, r + \tfrac{1}{2}\left(c_2 + r^2 - r\right) - 1.$$

Method 2. Set $e = \frac{1}{2}(c_2 + r^2 - r)$ and let X be a plane curve of degree e. Choose a surjective map $I_{X,\mathbf{P}^3} \to \mathcal{O}_X(2r + e)$ as before (4.1.3). The kernel defines a multiplicity-two structure Y on X:

$$0 \to I_Y \to I_X \to \mathcal{O}_X(2r + e) \to 0.$$

Again by 4.1, $\omega_Y \cong \mathcal{O}_Y(-2r - 3)$. Choosing a nowhere-vanishing section $\xi \in H^0(\omega_Y(2r + 3)) \cong \operatorname{Ext}^1(I_Y, \mathcal{O}(2r - 1))$, we construct a vector bundle

$$0 \to \mathcal{O}(r - 1) \to F \to I_Y(-r) \to 0.$$

Now $H^1(F(-\frac{1}{2}(c_2 + r^2 + r))) \cong H^1(I_Y(-2r - e)) \neq 0$ because of the exact sequence of cohomology

$$0 = H^0\left(I_X(-2r - 3)\right) \to H^0\left(\mathcal{O}_X\right) \to H^1\left(I_Y(-2r - e)\right) \to H^1\left(I_X(-2r - 3)\right) = 0.$$

The spectrum of F is again

$$-r - \tfrac{1}{2}\left(c_2 + r^2 - r\right), -r - \tfrac{1}{2}\left(c_2 + r^2 - r\right) + 1, \dots,$$
$$-r - 1, r, r + 1, \dots, r + \tfrac{1}{2}\left(c_2 + r^2 - r\right) - 1.$$

EXAMPLE 4.1.6. For each $r \geqslant 1$, $d = c_2 + r^2 - r$ odd positive integer, a reflexive sheaf F with $c_1 = -1$, nonstable of order r, and $H^1(F(-\frac{1}{2}(c_2 + r^2 + r - 1))) \neq 0$.

Set $m = \frac{1}{2}(c_2 + r^2 - r - 1) + r$. Let Z_1 be a degree $2m + 1$ plane curve and Z_2 a complete intersection of two surfaces of degrees $m + r + 1$ and $m - r$, respectively.

Then $\omega_{Z_1} \cong \mathcal{O}_{Z_1}(2m-2)$ and $\omega_{Z_2} \cong \mathcal{O}_{Z_2}(2m-3)$. Set $Z = Z_1 \coprod Z_2$. Then $\deg Z = 2m+1+m^2+m-(r+1)r$. Choose

$$\xi \in H^0\big(\mathcal{O}_{Z_1}(1) \oplus \mathcal{O}_{Z_2}\big) \cong H^0\big(\omega_Z(-2m+3)\big) \cong \operatorname{Ext}^1\big(I_Z, \mathcal{O}_{\mathbf{P}^3}(-2m-1)\big),$$

vanishing at $2m+1$ points. Then ξ determines an extension

$$0 \to \mathcal{O} \to F' \to I_Z(2m+1) \to 0$$

where F' is reflexive, $c_1(F') = 2m+1$, $c_2(F') = \deg Z = m^2+3m+1-r(r+1)$, and $c_3(F') = 2m+1$. Setting $F := F'(-m-1)$ we have $0 \to \mathcal{O}(-m-1) \to F \to I_Z(m) \to 0$ where $c_1(F) = -1$, $c_2(F) = 2m+1-r(r+1)$, $c_3(F) = 2m+1$. The least degree of a surface containing Z is $m-r+1$, so F is nonstable of order r. Further, $h^1(F(-m)) = h^1(I_Z) = 1$, so $H^1(F(-\frac{1}{2}(c_2+r^2+r-\frac{1}{2}))) \neq 0$.

5. Classification of extremal cases. In this section we classify reflexive sheaves having the maximum c_3 for given c_1, c_2, r, and vector bundles which are extremal for the H^1-vanishing theorem (3.7). Our main technique is the reduction step 5.1, introduced by Hartshorne in [5]. In case the restriction of a rank two nonstable reflexive sheaf F to a plane H is nonstable of order strictly greater than r, we may make a reduction step, which provides a new rank two nonstable reflexive sheaf F' with a strictly smaller value of $d = c_2 + c_1 r + r^2$. In 5.2 we describe the relation between the unique curves associated to F and F'. This allows us to determine the curves associated to the extremal sheaves of the last section.

DEFINITION [5]. Let F be a rank two reflexive sheaf on $\mathbf{P}^3$ with $c_1 = 0$ or -1. A plane H in $\mathbf{P}^3$ is a *nonstable plane for F* if there exists a $t > 0$ such that $H^0(F_H^*(-t)) \neq 0$. This condition is equivalent to $H^2(F_H(t-3)) \neq 0$. The *order of nonstability* of H is the maximal such integer t.

PROPOSITION 5.1 (REDUCTION STEP, SEE [5, PROPOSITION 9.1]). *Let F be a rank two reflexive sheaf on $\mathbf{P}^3$, with $c_1 = 0$ or -1, nonstable of order r, and let H be a nonstable plane for F of order $t > r$. Then there is an exact sequence*

$$0 \to F'(c_1) \to F \to I_{Z,H}(-t) \to 0$$

where Z is a zero-dimensional subscheme of H. Let $s = \operatorname{length} \mathcal{O}_Z$. Then F' is a rank two reflexive sheaf with $c_1' = 0$ or -1, nonstable of order $r+1+c_1$. The Chern classes of F' are

$$c_1' = -c_1 - 1, \quad c_2' = c_2 - t, \quad c_3' = c_3 - c_2 - c_1 t - t^2 + 2s,$$

where c_1, c_2, c_3 are the Chern classes of F. There is also a dual exact sequence

$$0 \to F^* \to F'^*(-c_1) \to I_{W,H}(t+1) \to 0,$$

where W is a zero-dimensional subscheme of H of length $c_2 + c_1 t + t^2 - s$. Furthermore, if F is a vector bundle, then $s = c_2 + c_1 t + t^2$ and W is empty.

REMARK 5.1.1. The degree of the unique associated curve Y drops by exactly $t - r$, i.e., if we set $d = c_2 + c_1 r + r^2$, the degree of Y, and $d' = c_2' + c_1' r' + r'^2$, the degree of the curve Y' associated to F', we have $d' = d - t + r$.

The following proposition further describes the relation between Y and Y'.

PROPOSITION 5.2. *Suppose F is a rank two reflexive sheaf on $\mathbf{P}^3$, nonstable of order r, with first Chern class $c_1 = 0$ or -1. Let H be a nonstable plane of order $t > r$. Let F' be the reflexive sheaf arising from the reduction step* 5.1, *and let Y and Y' be the unique schemes associated to F and F', respectively. Then Y' is a closed subscheme of Y, there is a plane curve C of degree $t - r$ in H such that Y is the set-theoretic union of Y' and C, and there are exact sequences*

$$\text{(a)} \qquad 0 \to I_{W,C}(r+t+1+c_1) \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0,$$

$$\text{(b)} \qquad 0 \to I_{Y'}(-1) \to I_Y \to I_{Z,H}(r-t) \to 0,$$

where W and Z are the zero-dimensional subschemes of 5.1.

PROOF. We will give a proof for the case $c_1 = 0$. The $c_1 = -1$ case is similar.

Exact sequence (b) follows directly from the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}(r) & = & \mathcal{O}(r) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & F' & \to & F & \to & I_{Z,H}(-t) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & I_{Y'}(-r-1) & & I_Y(-r) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Next, dualize the above diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}(r) & \to & F & \to & I_Y(-r) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}(r+1) & \to & F'(1) & \to & I_{Y'}(-r) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_H(r+1) & \xrightarrow{f} & I_{W,H}(t+1) & & I_{Y',Y}(-r) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Thus f is a nonzero form of degree $t - r$ on H, which defines a plane curve C. It follows that $I_{W,C}(t+1) \cong I_{Y',Y}(-r)$, which establishes (a).

REMARK 5.2.1. If F is a vector bundle, then $W = \varnothing$, so $I_{Y',Y} \cong \mathcal{O}_C(t+r+1+c_1)$.

REMARK 5.2.2. If F has a nonstable plane of order $r + 1 = t$, then C is a line L and $I_{Y',Y}$ is a torsion-free $\mathcal{O}_L$-module. The length of W is $c_2 + c_1 t + t^2 - s$, therefore $I_{Y',Y} \cong \mathcal{O}_L(s+1-d)$.

PROPOSITION 5.3. *Let F be a rank two reflexive sheaf on $\mathbf{P}^3$ with Chern classes $c_1 = 0$, c_2, c_3, nonstable of order r, with $c_3 = (c_2 + r^2)(c_2 + (r+1)^2)$. Then the curve Y associated to F is a plane curve of degree $d = c_2 + r^2$.*

PROOF. We may assume $d = c_2 + r^2 \geqslant 1$. The spectrum of F must be $-r-1$, $-r-2, \ldots, -r-d$, as seen in the proof of 3.8. We calculate $h^2(F(d+r-3)) = 1$ and $h^2(F(d+r-4)) = 3$. Therefore there exists a linear form $x \in H^0(\mathcal{O}(1))$ such that the multiplication map $H^2(F(d+r-4)) \xrightarrow{x} H^2(F(d+r-3))$ is zero. From the long exact sequence of cohomology we have $H^2(F_H(d+r-3)) \neq 0$, where H is the plane defined by x. Thus H is a nonstable plane of order $t = r + d > r$, and we may apply 5.2.

Making a reduction step with H, we produce a reflexive sheaf F' with first Chern class $c_1' = -1$ and $d' = d - t + r = 0$. Thus $F' \cong \mathcal{O}(r) \oplus \mathcal{O}(-r-1)$ and $Y' = \varnothing$.

Proposition 5.2(a) gives an exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_C(t + r_, + 1) \to \mathcal{O}_W \to 0,$$

where C is a plane curve of degree d. This shows that $\mathcal{O}_Y$ is supported on H, and therefore the natural map $\mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_Y$ factors through $\mathcal{O}_H$. The depth of $\mathcal{O}_Y$ is one at every point, so the homological dimension of $\mathcal{O}_Y$ as an $\mathcal{O}_{\mathbf{P}^2}$-module is one. This, together with the surjective map $\mathcal{O}_H \to \mathcal{O}_Y$, proves that $\mathcal{O}_Y$ is the structure sheaf of the plane curve C.

REMARK 5.3.1. We now see that all sheaves with Chern classes $c_1 = 0$, c_2, nonstable of order r, and maximal third Chern class c_3 can be constructed by an extension $\xi \in \operatorname{Ext}^1(I_Y(-r), \mathcal{O}(r)) \cong H^0(\mathcal{O}_Y(d + 2r + 1))$, where Y is a plane curve of degree $d = c_2 + r^2$, as in 4.1.1. The analogous statement for $c_1 = -1$ is also true and follows from the next proposition.

PROPOSITION 5.4. *Let F be a rank two reflexive sheaf on $\mathbf{P}^3$, with Chern classes $c_1 = -1$, c_2, c_3, nonstable of order r, with $c_3 = (c_2 + r(r-1))(c_2 + r(r+1))$. Then Y is a plane curve of degree $d = c_2 + r^2 - r$.*

PROPOSITION 5.5. *Let F be a rank two reflexive sheaf on $\mathbf{P}^3$, $c_1 = 0$, nonstable of order r, with $d = c_2 + r^2$ even and positive. Suppose $H^1(F(-\frac{1}{2}d - r - 1)) \neq 0$. Then F is locally free, the spectrum of F is*

$$-r - \tfrac{1}{2}d, -r - \tfrac{1}{2}d + 1, \ldots, -r-1, r+1, r+2, \ldots, r + \tfrac{1}{2}d,$$

and the curve Y associated to F is a multiplicity-two structure on a plane curve C of degree $\frac{1}{2}d$ of the form

$$0 \to I_Y \to I_C \to \mathcal{O}_C\left(2r + \tfrac{1}{2}d + 1\right) \to 0.$$

PROOF. $H^1(F(-\frac{1}{2}d - r - 1)) \neq 0$ implies there is a $k_i \geqslant r + \frac{1}{2}d$ in the spectrum by 3.1. By the property of connectedness (3.4) and the fact that the sum of the elements of the spectrum is negative (3.6), the spectrum must be as claimed and $c_3(F) = 0$.

The spectrum of F implies $h^2(F(r + \frac{1}{2}d - 3)) = 1$ and $h^2(F(r + \frac{1}{2}d - 4)) = 3$. There exists a plane H such that $h^2(F_H(r + \frac{1}{2}d - 3)) \neq 0$, i.e., H is nonstable of order $t = r + \frac{1}{2}d$. Making a reduction step we have

$$0 \to F' \to F \to I_{Z,H}(-t) \to 0$$

where $c_1' = -1$, $d' = \frac{1}{2}d$. By Remark 5.2.1 we see $W = \varnothing$, implying that $s = c_2 + t^2$ and $c_3' = c_2 + t^2 = \frac{1}{2}d(\frac{1}{2}d + 2r + 2)$, which is the maximum possible c_3'. Thus by

5.4, Y' is a plane curve of degree $\frac{1}{2}d$, and 5.2 gives the exact sequence

$$0 \to I_Y \to I_{Y'} \to \mathcal{O}_C\left(2r + \tfrac{1}{2}d + 1\right) \to 0,$$

where Y' and C are plane curves of degree $\frac{1}{2}d$. It remains to show $Y' = C$, which follows from

LEMMA 5.6. *If Y and Y' are curves in $\mathbf{P}^3$ and C is a plane curve in $\mathbf{P}^3$ satisfying $0 \to \mathcal{O}_C(n) \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0$ for $n > 0$, then C is contained in Y'.*

PROOF. Let Z be an irreducible component of C which is not contained in Y'. Tensoring with $\mathcal{O}_Z$ we have an exact sequence $\mathcal{O}_Z(n) \xrightarrow{f} \mathcal{O}_Z \to T \to 0$ where T is a sheaf of finite length. Since Z is a plane curve we have $H^0(\mathcal{O}_Z(-n)) = 0$, so $f = 0$, a contradiction.

In the case $c_1 = -1$, similar methods prove the following.

PROPOSITION 5.7. *Let F be a rank two reflexive sheaf on $\mathbf{P}^3$, $c_1 = -1$, nonstable of order r, with $d = c_2 + r^2 - r$ even and positive. Suppose $H^1(F(-\frac{1}{2}d - r)) \neq 0$. Then F is locally free, the spectrum of F is*

$$-r - \tfrac{1}{2}d, -r - \tfrac{1}{2}d + 1, \ldots, -r - 1, r, r + 1, \ldots, r + \tfrac{1}{2}d - 1,$$

and the curve Y associated to F is a multiplicity-two structure on a plane curve C of degree $\frac{1}{2}d$, of the form

$$0 \to I_Y \to I_C \to \mathcal{O}_C\left(2r + \tfrac{1}{2}d\right) \to 0.$$

6. Applications to curves. If we continue the study of the spectrum of nonstable sheaves, we can prove some results concerning Cohen-Macaulay generically local complete intersection curves in $\mathbf{P}^3$. This is the class of curves that seems natural to study in liaison theory; see the articles by Peskine and Szpiro [**8**] and Rao [**9**].

Given such a curve Y, we twist ω_Y sufficiently high so there exists a global section vanishing at a zero-dimensional set. Using this section $\xi \in H^0(\omega_Y(2r + 4)) \cong \operatorname{Ext}^1(I_Y(-r), \mathcal{O}(r))$, we construct a reflexive sheaf, nonstable of order r, such that Y is associated to F. By studying F, we get information about Y.

The following lemma is the analogue of a result of Hartshorne [**6**, 5.1] which treated the semistable case.

LEMMA 6.1. *Suppose F is a rank two reflexive sheaf on $\mathbf{P}^3$, nonstable of order r, $c_1 = 0$ or -1. Suppose k_0 occurs exactly once in the spectrum, and $k_0 \leqslant -r - 2$. Then for all k_i in the spectrum such that $k_i < k_0$, k_i occurs exactly once.*

PROOF. Let H be a general plane, and define $M = \bigoplus_{m \in \mathbf{Z}} H^1(F_H(m))$ and the quotient module S as in Theorem 3.1. By definition of the k_i, $\#\{k_i = m\} = s_{-m-2} - s_{-m-1}$ for $m \leqslant r + c_1$. Set $k = \min\{k_i\}$.

If $k = k_0$ there is nothing to prove. If $k < k_0$, then $k_0 - 1$ occurs in the spectrum by the connectedness property 3.4, and we have $s_{-k_0-1} > s_{-k_0}$. By hypothesis, $s_{-k_0-2} - s_{-k_0-1} = \#\{k_i = k_0\} = 1$.

The dual of S is a submodule of M^*, and we have $S^*_{k_0-2-c_1} \neq 0$ and $s^*_{k_0-1-c_1} - s^*_{k_0-2-c_1} = 1$. We are in a position to apply Lemma 2.3(c), since $k_0 - 2 - c_1 \leqslant -r - c_1 - 4$, i.e., $k_0 \leqslant -r - 2$. This lemma provides a nonzero linear form x annihilating s^*_m for $m \leqslant k_0 - 2 - c_1$. In particular x annihilates $s^*_{k-1-c_1} \neq 0$, which says precisely that $L = \{x = 0\}$ is a jumping line of order k.

We now perform a reduction step [3, 5.2] for F_H using the jumping line L. We have

$$0 \to E' \to F_H \to \mathcal{O}_L(-k) \to 0,$$

where $c_1(E') = -1$ or -2, or, dually,

$$0 \to F_H \to E'(1) \to \mathcal{O}_L(k + c_1 + 1) \to 0.$$

The associated cohomology sequence is

$$H^0(E'(m+1)) \to H^0(\mathcal{O}_L(m + k + c_1 + 1)) \to H^1(F_H(m)).$$

By construction $1 \in H^0(\mathcal{O}_L)$ maps into $S^* \subseteq \bigoplus_{m \in \mathbf{Z}} H^1(F_H(m))$, so if we set $T_L := \bigoplus_{m \in \mathbf{Z}} H^0(\mathcal{O}_L(m))$ we can define N' by

$$T_L \xrightarrow{\delta} S^* \to N' \to 0.$$

The map δ is injective in degrees $m \leqslant -r - 2$ since $H^0(E'(\leqslant -r - 1)) = 0$.

By hypothesis, $s^*_{k_0-1-c_1} - s^*_{k_0-2-c_1} = 1$, and the dimension of T_L increases by 1 in every degree. So $n'_{k_0-1-c_1} = n'_{k_0-2-c_1}$, and Lemma 2.3(b) implies $n'_{\leqslant k_0-2-c_1} = 0$, since we have $k_0 - 2 - c_1 \leqslant -r - c_1 - 3$. Thus $T_L \xrightarrow{\delta} S^*$ is an isomorphism in degrees $m \leqslant k_0 - 2 - c_1$, showing that $\#\{k_i = m\} = 1$ for $m \leqslant k_0$.

REMARK 6.1.1. If F is a rank two reflexive sheaf with $c_1 = 0$, nonstable of order r, then the maximal c_3 is $(c_2 + r^2)(c_2 + (r+1)^2) = d(d + 2r + 1)$ by Theorem 3.8. For a given d, consider the spectrum $-r - d + 1, -r - d + 2, \ldots, -r - 2, -r - 2, -r - 1$. A sheaf with this spectrum has $c_3 = 2rd + d^2 - d + 4$. It is clear from 6.1 and the connectedness property 3.4 that the only allowable spectrum with a higher c_3 is $-r - d, -r - d + 1, \ldots, -r - 1$, with the maximal $c_3 = d(d + 2r + 1)$. We conclude there is a gap in the possible values of c_3, in that $d^2 - d + 4 + 2rd < c_3 < d^2 + d + 2rd$ is impossible. This causes a restriction on the arithmetic genus of curves:

COROLLARY 6.2. *There exists a Cohen-Macaulay generically local complete intersection curve Y of degree $d \geqslant 2$ and arithmetic genus p_a in $\mathbf{P}^3$ if and only if $p_a(Y) = \frac{1}{2}(d-1)(d-2)$ or $p_a(Y) \leqslant \frac{1}{2}(d-2)(d-3)$.*

PROOF. Suppose Y is a curve with $p_a > \frac{1}{2}(d-2)(d-3)$. Choose an even integer n such that $n \geqslant 4$ and $\omega_Y(n)$ is generated by global sections. Then there exists $\xi \in H^0(\omega_Y(n)) \cong \operatorname{Ext}^1(I_Y, \mathcal{O}(n-4))$ giving rise to a rank two reflexive sheaf F on $\mathbf{P}^3$, $c_1 = 0$, nonstable of order $r = \frac{1}{2}(n-4)$, such that the unique section of $F(-r)$ vanishes at Y.

We calculate $c_3 = 2p_a - 2 + d(2r + 4) > d(d-1) + 4 + 2rd$. By Remark 6.1.1, this is impossible unless $c_3 = d(d+1) + 2rd$ and Y is a plane curve.

It remains to give examples of curves of degree $d \geqslant 2$ and all arithmetic genera less than or equal to $\frac{1}{2}(d-2)(d-3)$. Let C be a plane curve of degree $d - 1$ which contains a line L as an irreducible component. The conormal bundle is $I_C \otimes \mathcal{O}_C \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-d+1)$. For $m \geqslant -1$ we can choose a surjective map to $\mathcal{O}_L(m)$ and compose with the natural map $I_C \to I_C \otimes \mathcal{O}_C$ to get a surjective map $I_C \to \mathcal{O}_L(m)$. If we define Y by $0 \to I_Y \to I_C \to \mathcal{O}_L(m) \to 0$ for $m \geqslant -1$, we get a curve Y of arithmetic genus $p_a(Y) = \frac{1}{2}(d-2)(d-3) - m - 1$.

PROPOSITION 6.3. *Suppose F is a rank two reflexive sheaf on $\mathbf{P}^3$, nonstable of order r, $c_1 = 0$, with spectrum*

$$-r-a, -r-a+1, \ldots, -r-b-2; -r-b-1, -r-b-1, -r-b, -r-b, \ldots, -r-2, -r-2, -r-1,$$

where a and b are integers satisfying $b \geqslant 1$ and $a \geqslant b+3$. Let Y be the scheme associated to F. Then there exist plane curves C and Y' of degrees a and b, respectively, such that

$$0 \to I_Y \to I_C \to \mathcal{O}_{Y'}(-1) \to 0.$$

Thus Y is either:

(1) *the union of two plane curves, of degrees a and b, meeting at b points*;

(2) *a plane curve of degree a with a local complete intersection multiplicity-two structure on a degree b subcurve*; *or*

(3) *the union of two plane curves, of degrees a and b, meeting along a common subline.*

PROOF. Since $a \geqslant b+3$ there is a nonstable plane H of order $t = r + a$. We make a reduction step

$$0 \to F' \to F \to I_{Z,H}(-t) \to 0$$

where the invariants of F' are

$$c_1' = -1, \quad d' = b, \quad r' = r+1,$$
$$c_3' = c_3 - c_2 - t^2 + 2s = b(b+2r+2) + 2s.$$

By Theorem 3.8, the maximum c_3' for F' is $b(b+2r+2)$. Thus the nonnegative integer s is zero, $Z = \varnothing$, and Y' is a plane curve of degree b. By Proposition 5.2 we have an exact sequence

$$0 \to I_{Y'}(-1) \to I_Y \to \mathcal{O}_H(-a) \to 0.$$

Since the codimension of Y' in $\mathbf{P}^3$ is two, the inclusion $I_{Y'}(-1) \to I_Y \to \mathcal{O}_{\mathbf{P}^3}$ factors through $\mathcal{O}_{\mathbf{P}^3}(-1)$ and we have a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & I_{Y'}(-1) & \to & I_Y & \to & \mathcal{O}_H(-a) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
0 & \to & \mathcal{O}_{\mathbf{P}^3}(-1) & \to & \mathcal{O}_{\mathbf{P}^3} & \to & \mathcal{O}_{H'} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{Y'}(-1) & & \mathcal{O}_Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It is clear that $f \neq 0$. Thus $H = H'$, f defines a plane curve C, and we have an exact sequence

$$0 \to I_Y \to I_C \to \mathcal{O}_{Y'}(-1) \to 0.$$

To finish, there is the following fact: suppose C is a curve in the plane H and Y' is a plane curve in $\mathbf{P}^3$, and there exists a surjective map $I_C \to \mathcal{O}_{Y'}(-1)$; then C contains $Y' \cap H$.

COROLLARY 6.4. *If Y is a Cohen-Macaulay generically local complete intersection curve in $\mathbf{P}^3$ of degree $d \geqslant 5$ and $p_a(Y) = \frac{1}{2}(d-2)(d-3)$, then Y is either*

(1) *a plane curve plus a line meeting at one point, or*

(2) *a plane curve with a local complete intersection multiplicity-two structure on a subline.*

In either case Y is defined by the exact sequence $0 \to I_Y \to I_C \to \mathcal{O}_L(-1) \to 0$, where C is a plane curve.

PROOF. Choose n sufficiently large as in the proof of Corollary 6.2 and construct a rank two reflexive sheaf F on $\mathbf{P}^3$, $c_1 = 0$, nonstable of order r, such that Y is the unique curve associated to F. Calculate $c_3(F) = 2p_a - 2 + d(2r+4) = d(d-1) + 4 + 2rd$. It follows from Lemma 6.1 that the only spectrum allowing this c_3 is

$$-r-d+1, -r-d+2, \ldots, -r-3, -r-2, -r-2, -r-1.$$

Now we can use Proposition 6.3 with $b = 1$, $a = d-1$ to finish.

PROPOSITION 6.5. *Suppose F is a rank two reflexive sheaf on $\mathbf{P}^3$, $c_1 = 0$, nonstable of order r, with spectrum $-r-d+1$, $-r-d+2, \ldots, -r-2$, $-r-1$, n, where $d = c_2 + r^2$ and $n \geqslant -r-1$ is an integer. Let Y be the unique scheme associated to F. Then Y is either*

(1) *the disjoint union of a line L and a plane curve C, and $n = -r-1$, or*

(2) *a plane curve C with a local complete intersection multiplicity-two structure on a subline L.*

In either case Y is defined by the exact sequence

$$0 \to I_Y \to I_C \to \mathcal{O}_L(r+n+1) \to 0.$$

PROOF. There is a nonstable plane of order $t = r + d + 1$. We make a reduction step

$$0 \to F' \to F \to I_{Z,H}(-t) \to 0$$

where the invariants of F' are

$$c_1' = -1, \quad d' = 1, \quad r' = r+1, \quad c_3' = -2n + 1 + 2s.$$

By Example 1.0.1, $c_3' = 2r' + 1 = 2r + 3$, so $s = r + n + 2$, where $s = \operatorname{length} \mathcal{O}_Z$. Since $n \geqslant -r-1$, Z is not empty.

As in Proposition 6.3 we get a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & I_L(-1) & \to & I_Y & \to & I_{Z,H}(1-d) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \to & \mathcal{O}_{\mathbf{P}^3}(-1) & \to & \mathcal{O}_{\mathbf{P}^3} & \to & \mathcal{O}_H & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \mathcal{O}_L(-1) & & \mathcal{O}_Y & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since $f \neq 0$, f is injective. Let G be the cokernel defined by

$$0 \to \mathcal{O}_L(-1) \to \mathcal{O}_Y \to G \to 0.$$

Let C be the plane curve in H defined by the equation f. Then there is an exact sequence

$$0 \to \mathcal{O}_L(-1) \to I_{C,Y} \to \mathcal{O}_Z \to 0.$$

Since depth $I_{C,Y} = 1$ at each point and the Hilbert polynomial of $I_{C,Y}$ is $P(k) = k + s$, we conclude $I_{C,Y} \cong \mathcal{O}_L(s-1) \cong \mathcal{O}_L(r+n+1)$, so there is an exact sequence

$$0 \to I_Y \to I_C \to \mathcal{O}_L(r+n+1) \to 0.$$

Now Lemma 5.6 shows that either $n = -r-1$ or $L \subseteq C$ and supports a multiplicity-two structure.

COROLLARY 6.6. *If Y is a Cohen-Macaulay generically local complete intersection curve in $\mathbf{P}^3$ of degree d and $p_a(Y) \geqslant \frac{1}{2}(d-3)(d-4)+2$, then Y is one of the following*:

(1) *a plane curve of degree d, $p_a(Y) = \frac{1}{2}(d-1)(d-2)$;*

(2) *the union of a degree $d-1$ plane curve and a line, meeting at one point, $p_a(Y) = \frac{1}{2}(d-2)(d-3)$;*

(3) *the disjoint union of a plane curve and a line, $p_a(Y) = \frac{1}{2}(d-2)(d-3)-1$; or*

(4) *a plane curve C of degree $d-1$ with a l.c.i. multiplicity-two structure on a subline L, of the form $0 \to I_Y \to I_C \to \mathcal{O}_L(n) \to 0$, where $n \geqslant -1$,*

$$p_a(Y) = \tfrac{1}{2}(d-2)(d-3) - n - 1.$$

PROOF. For $d \leqslant 4$ this follows from Corollary 6.2. For $d \geqslant 5$, the condition $p_a(Y) \geqslant \frac{1}{2}(d-3)(d-4)+2$ implies by Lemma 6.1 that the spectrum of a nonstable sheaf to which Y is associated is of the form treated in Proposition 6.5 or in Proposition 6.3 with $b = 1$. $\square$

Suppose Y is the union of two plane curves of degrees a and b, respectively, where $b \leqslant a$, intersecting at b points. We may ask whether there exists a smooth deformation of the curve Y, i.e., is Y the specialization of nonsingular curves in $\mathbf{P}^3$?

It is clear from Castelnuovo's well-known bound on the genus of space curves that Y cannot be smoothed if $a \geqslant b+3$. On the other hand, if $a = b$, then Y is a complete intersection and therefore has smooth deformations. An argument provided by Hartshorne allows us to decide the remaining two cases, $a = b + 1$ and, at least in characteristic 0, $a = b + 2$.

PROPOSITION 6.7 (char $k = 0$). *Let Y be the union of two plane curves of degrees a and b meeting at b points. Then there exists a smooth deformation of Y if and only if $a \leq b + 2$.*

PROOF. It remains to look at the cases $a = b + 1$ and $a = b + 2$. First, assume $a = b + 1$. Let C and Y' be the plane curves of degrees a and $a - 1$, respectively. Then there is an exact sequence

$$0 \to \mathcal{O}_{Y'}(-1) \to \mathcal{O}_Y \to \mathcal{O}_C \to 0$$

and $p_a(Y) = \frac{1}{4}d^2 - d + \frac{3}{4}$. Applying $\mathcal{E}xt^2_{\mathcal{O}_{\mathbf{P}^3}}(\cdot, \omega_{\mathbf{P}^3})$ we have

$$0 \to \mathcal{O}_C(a-3) \to \omega_Y \to \mathcal{O}_{Y'}(a-3) \to 0.$$

Thus $\omega_Y(3-a)$ is generated by global sections, and there exists $\xi \in H^0(\omega_Y(3-a)) \cong \operatorname{Ext}^1(I_Y(a+1), \mathcal{O}_{\mathbf{P}^3})$ vanishing at a zero-dimensional set of length $2p_a - 2 + d(3-a) = a - 1$. For ease of calculation, assume a is odd. Then ξ corresponds to an exact sequence

$$0 \to \mathcal{O} \to F\left(\tfrac{1}{2}(a+1)\right) \to I_Y(a+1) \to 0$$

where the Chern classes of F are $c_1 = 0$, $c_2 = 2a - 1 - \frac{1}{4}(a+1)^2$, $c_3 = a - 1$, and where Y is the zero set of a global section of $F(\frac{1}{2}(a+1))$. Since $H^0(I_Y(1)) = 0$ and $H^0(I_Y(2)) \neq 0$, it is clear that F is nonstable of order $r = \frac{1}{2}(a-3)$. We can now compute $d(F) = c_2 + r^2 = 1$. By 1.0.1 there exists a line L and an exact sequence

$$0 \to \mathcal{O}\left(\tfrac{1}{2}(a-3)\right) \to F \to I_L\left(-\tfrac{1}{2}(a-3)\right) \to 0.$$

Since $I_L(1)$ is generated by global sections, so is $F(n)$ for $n \geq \frac{1}{2}(a-1)$. In particular, $F(\frac{1}{2}(a+1))$ is generated by global sections and one can show as in [**17**, Proposition 1.4] that a general global section of $F(\frac{1}{2}(a+1))$ will vanish along an irreducible nonsingular curve. Thus Y is a specialization of these curves. The same argument works if a is even, with the obvious changes.

In the case $a = b + 2$, let C and Y' be the plane curves of degrees a and $a - 2$, respectively. By the same reasoning as above there is an exact sequence

$$0 \to \mathcal{O}_C(a-3) \to \omega_Y \to \mathcal{O}_{Y'}(a-4) \to 0$$

so that $\omega_Y(4-a)$ is generated by global sections. Choose an element $\xi \in H^0(\omega_Y(4-a))$ so that the zero-dimensional zero set of ξ, of length $2a - 2$, avoids the singular points of Y. Again we assume a is odd. Then ξ gives rise to an exact sequence

$$0 \to \mathcal{O} \to F\left(\tfrac{1}{2}(a+1)\right) \to I_Y(a) \to 0$$

where the Chern classes of F are $c_1 = -1$, $c_2 = 2a - 2 - \frac{1}{4}(a^2 - 1)$, $c_3 = 2a - 2$, and where Y is the zero set of a global section of $F(\frac{1}{2}(a+1))$. We compute $r(F) = \frac{1}{2}(a-3)$ and $d(F) = 2$. Since $c_3 = 4r + 4$, the maximal value for $d = 2$, we use

Proposition 5.4 to give an exact sequence

$$0 \to \mathcal{O}\left(\tfrac{1}{2}(a-5)\right) \to F \to I_Z\left(-\tfrac{1}{2}(a-3)\right) \to 0$$

where Z is a degree two plane curve. Since $I_Z(2)$ is generated by global sections, so is $F(\frac{1}{2}(a+1))$, and therefore in char 0 one can show as in [**17**, 1.4] that a general global section of $F(\frac{1}{2}(a+1))$ vanishes along a nonsingular curve. Again, if a is even, the same method works.

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