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DECOMPOSITIONS OF THE MAXIMAL IDEAL SPACE OF L^∞

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ABSTRACT. In this paper we show the existence of one point maximal antisymmetric sets for $H^\infty + C$.

Let X be a compact Hausdorff space and A be a closed unital subalgebra of $C(X)$. There are several decompositions of X relative to A. We shall be primarily concerned with the Bishop and Shilov decompositions for the case where $X = M(L^\infty)$ and $A = H^\infty + C$. We first present a summary of our results. Definitions and further details are presented in §1.

In [**14**] D. Sarason shows that the Bishop decomposition of $M(L^\infty)$ into maximal antisymmetric sets for $H^\infty + C$ is a proper refinement of the Shilov decomposition. In [**13**] Sarason refined Bishop's theorem to support sets for representing measures of multiplicative linear functionals in $M(H^\infty + C)$. In [**13** and **14**] Sarason asked for the precise relation between support sets and sets of antisymmetry for $H^\infty + C$. Is every maximal antisymmetric set for $H^\infty + C$ the support set of the representing measure of some multiplicative linear functional on $H^\infty + C$? This question is still open. In fact it was unknown whether any maximal antisymmetric set equals the support set of a multiplicative linear functional on $H^\infty + C$. We shall show the existence of a maximal antisymmetric set consisting of a single point. It is easy to see that a maximal antisymmetric set consisting of a single point must be a support set. We shall give many examples of one point maximal antisymmetric sets and shall show that many of these are contained in QC level sets consisting of more than one point, extending a result of Sarason's that will appear in [**14**].

These results were part of the author's thesis. I would like to express my gratitude to Sheldon Axler for his help.

1. Preliminaries. The space of essentially bounded, measurable, complex valued functions on the unit circle $\partial\mathbf{D}$ with normalized Lebesgue measure will be denoted by L^∞. The space L^∞ is a Banach algebra when it is given pointwise multiplication and the essential supremum norm. Let $f \in L^\infty$. We define f in the unit disc by

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P_r(\theta - t)\,dt$$

where

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

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The space of continuous, complex valued functions on $\partial\mathbf{D}$ will be denoted by C. By H^∞ is meant the space of bounded analytic functions on the unit disc $\mathbf{D}$. We will frequently identify a function in H^∞ with its boundary function on the circle. When we do this, we may view H^∞ as a (uniformly closed) subalgebra of L^∞. The algebra $H^\infty + C = \{f + g : f \in H^\infty, g \in C\}$ is a closed subalgebra of L^∞ [**12**]. Finally, the largest C^*-subalgebra of $H^\infty + C$ will be denoted by QC.

The maximal ideal space $M(B)$ of a commutative Banach algebra B with a unit 1 is the set of multiplicative linear functionals of B. We give $M(B)$ the weak-* topology. With this topology, $M(B)$ is a compact Hausdorff space. For $f \in B$, the Gelfand transform of f is the complex valued function $\hat{f} \in C(M(B))$ defined by $\hat{f}(\varphi) = \varphi(f)$ for all $\varphi \in M(B)$. In the cases we are interested in here, the Gelfand transform is an isometry and we write f for $\hat{f}$, since the meaning will be clear from the context.

In the case of $M(L^\infty)$, L^∞ is isometrically isomorphic (via the Gelfand transform) to $C(M(L^\infty))$. In [**7** and **9**] it is shown that $M(L^\infty)$ is an extremally disconnected, compact Hausdorff space. For these and other relevant facts about the topology of $M(L^\infty)$ the reader is referred to [**7** and **9**].

We will also use facts about $M(H^\infty)$. Further information is available in [**7**–**9**]. As usual, we regard $\mathbf{D}$ as an open subset of $M(H^\infty)$ and write

$$M(H^\infty) = \mathbf{D} \cup \{\varphi \in M(H^\infty) : |\varphi(z)| = 1\}.$$

The Corona Theorem [**4**] states that $\mathbf{D}$ is dense in $M(H^\infty)$.

For each $\varphi \in M(H^\infty)$, there is a unique positive Borel measure μ_φ on $M(L^\infty)$ such that

$$\varphi(f) = \int_{M(L^\infty)} f \, d\mu_\varphi \quad \text{for all } f \in H^\infty.$$

If $\varphi \in M(H^\infty + C)$ the closed support of μ_φ is denoted $\operatorname{supp} \mu_\varphi$, or simply $\operatorname{supp} \varphi$.

Let B denote a closed subalgebra of L^∞ containing the constant functions which separates the points of $M(L^\infty)$. A closed subset $S \subseteq M(L^\infty)$ is called a peak set for B if there is a function $f \in B$ such that f equals one on S and $|f|$ is less than one off S. The function f will be called a peaking function for S. A closed subset S of $M(L^\infty)$ is called a weak peak set for B if it is the intersection of peak sets. If S is a weak peak set for B, the restriction algebra $B|S$ is a Banach algebra [**7**, p. 57].

Let B denote a closed subalgebra of L^∞ containing the function z. For $\lambda \in \partial\mathbf{D}$ we let $M_\lambda(B) = \{\varphi \in M(B) : \varphi(z) = \lambda\}$. We call $M_\lambda(B)$ the B-fiber over λ. We note that

$$M(H^\infty + C) = \bigcup_{\lambda \in \partial\mathbf{D}} \{\varphi \in M(H^\infty) : \varphi(z) = \lambda\} = M(H^\infty) \sim \mathbf{D}.$$

The L^∞-fiber over λ is a weak peak set for H^∞, hence for $H^\infty + C$. A function f in C is constant on each fiber and its value on the fiber over λ is simply $f(\lambda)$. Therefore $H^\infty + C|M_\lambda(L^\infty) = H^\infty|M_\lambda(L^\infty)$.

The sets in the Shilov decomposition are called QC level sets. Thus for $\psi \in M(L^\infty)$ (or $\psi \in M(QC)$) we let

$$E_\psi = \{\varphi \in M(L^\infty) : \varphi(q) = \psi(q) \text{ for all } q \in QC\}.$$

We call E_ψ the QC level set corresponding to ψ. Each QC level set is a weak peak set for $H^\infty + C$ and is contained in some L^∞ fiber. In this context a theorem of Shilov [**15**] specializes to give:

THEOREM 1.1. *Let $f \in L^\infty$. If $f \mid E_\psi \in H^\infty \mid E_\psi$ for each QC level set E_ψ, then $f \in H^\infty + C$.*

The sets in Bishop's decomposition are called antisymmetric sets. A set $S \subseteq M(L^\infty)$ is called an antisymmetric set for $H^\infty + C$ if whenever $f \in H^\infty + C$ and $f \mid S$ is real valued, then $f \mid S$ is constant. A maximal antisymmetric set for $H^\infty + C$ is a weak peak set for $H^\infty + C$. It is easy to see that each antisymmetric set is contained in some QC level set. A special case of Bishop's theorem [**2**] says the following:

THEOREM 1.2. *Let $\{S_\alpha\}$ denote the maximal antisymmetric sets for $H^\infty + C$. If $f \in L^\infty$ is such that $f \mid S_\alpha \in H^\infty \mid S_\alpha$ for each maximal antisymmetric set S_α, then $f \in H^\infty + C$.*

Sarason [**14**] has given an example of a QC level set that is not an antisymmetric set for $H^\infty + C$. Thus Bishop's decomposition for $M(L^\infty)$ is strictly finer than Shilov's decomposition for $M(L^\infty)$.

The third theorem along these lines is due to Sarason [**13**].

THEOREM 1.3. *Let $f \in L^\infty$. If $f \mid \operatorname{supp} \varphi \in H^\infty \mid \operatorname{supp} \varphi$ for each $\varphi \in M(H^\infty + C)$, then $f \in H^\infty + C$.*

It is not difficult to show that for $\varphi \in M(H^\infty + C)$ the support of φ is an antisymmetric set for $H^\infty + C$. Therefore Sarason's theorem is a refinement of Bishop's theorem. This paper is concerned with the relation of Theorem 1.3 to Theorem 1.2.

2. The main result.

THEOREM 2.1. *Let $\{\lambda_n\}$ be a sequence of distinct points of $\partial \mathbf{D} \sim \{1\}$ with $\lambda_n \to 1$. Let $\psi_n \in M_{\lambda_n}(L^\infty)$ and $\psi \in \overline{\{\psi_n\}}^{M(L^\infty)} \cap M_1(L^\infty)$. Then $\{\psi\}$ is a maximal antisymmetric set for $H^\infty + C$.*

An unpublished result of K. Hoffman shows that any point of $M(L^\infty)$ in the closure of a sequence of points from distinct L^∞ fibers is a maximal support set. Our proof is independent of this fact, although Hoffman's result follows easily from Theorem 2.1.

In order to prove Theorem 2.1, we need the result given below.

THEOREM 2.2. *Let $\{\lambda_n\}$ be a sequence of distinct points of $\partial \mathbf{D} \sim \{1\}$ such that $\lambda_n \to 1$. Let $\{I_n\}$ be a sequence of intervals of $\partial \mathbf{D}$ with $\bar{I}_n \cap \overline{\bigcup_{m \neq n} I_m} = \varnothing$ and $\lambda_n \in I_n$. Then there exists $q \in QC$ satisfying:*

(1) *q is continuous except at $\lambda = 1$;*

(2) *$|\arg q(\lambda_n) - \pi| < \frac{1}{4}$ for all n;*

(3) *$|\arg q(\lambda)| < \frac{1}{4}$ for $\lambda \in \partial \mathbf{D} \sim \overline{\bigcup I_n}$.*

Before proceeding with the proof we prove a lemma which will be used in the proof of Theorem 2.2.

Let $C^1_{\mathbf{R}}$ denote the space of real valued, continuously differentiable functions on $\partial\mathbf{D}$. Each function u in $C^1_{\mathbf{R}}$ has a unique extension to a harmonic function on $\mathbf{D}$ (which we continue to denote by u) whose boundary values are the given function. The harmonic conjugate $\tilde{u}$ of u is harmonic on $\mathbf{D}$ and extends continuously to $\overline{\mathbf{D}}$.

LEMMA 2.3. *Let I be an open interval contained in $\partial\mathbf{D}$ and let $w \in I$. Then given $\varepsilon > 0$ and $\lambda_0 \in \mathbf{R}^+$, there exists $u \in C^1_{\mathbf{R}}$ with $\|u\|_\infty < \varepsilon$, $|\tilde{u}(z)| < \varepsilon$ for $z \in \partial\mathbf{D} \sim I$ and $\tilde{u}(w) = \lambda_0$.*

PROOF. By choosing $\delta > 0$ sufficiently small and rotating, we may assume $w = 1$ and $I = \{e^{i\theta}: -2\delta < \theta < 2\delta\}$. It is enough to show that there exists $v \in C^1_{\mathbf{R}}$ with $\|v\| < 1$, $|\tilde{v}(z)| < 1$ for $z \in \partial\mathbf{D} \sim I$ and $\tilde{v}(1) = \lambda_0/\varepsilon$, for then $u = \lambda_0 v/\tilde{v}(1)$ satisfies $\|u\|_\infty < \varepsilon$, $|\tilde{u}(z)| < \varepsilon$ for $z \in \partial\mathbf{D} \sim I$ and $\tilde{u}(1) = \lambda_0$.

It is not hard to show that

$$\lim_{x \to \infty} \frac{1 - (1/k)^{1/x}}{1/x} = \ln k \quad \text{for } k > 0.$$

We use this fact below. To find v, let $\varepsilon > 0$ and $\lambda_0 \in \mathbf{R}^+$ be given. Choose k so that $\ln k > 2\pi\lambda_0/(\varepsilon \tan \frac{\delta}{2})$. Choose an odd integer m satisfying (i) $m[1 - (1/k)^{1/m}] > 2\pi\lambda_0/(\varepsilon \tan \frac{\delta}{2})$, (ii) $1/m < \delta$ and (iii) $\cos(1/2m) > 1/2$.

Let

$$v(z) = \begin{cases} 0 & \text{if } z \in \partial\mathbf{D} \sim \dfrac{I}{2}, \\ \left(-\dfrac{1}{2} \tan \dfrac{\delta}{2}\right)(mt)^{1/m} & \text{if } z = e^{it} \in \left\{e^{is}: \dfrac{1}{km} < s < \dfrac{1}{m}\right\}, \end{cases}$$

and extend v so that $v \in C^1_{\mathbf{R}}$, $v(1) = 0$, $v(e^{is}) \leqslant 0$ for $0 \leqslant s \leqslant \pi$, $v(e^{-is}) = -v(e^{is})$, and $\|v\|_\infty < \tan \frac{\delta}{2} \leqslant 1$.

Writing $v(\theta)$ for $v(e^{i\theta})$ we have [9, p. 79]

$$\begin{aligned}
\tilde{v}(0) &= \int_{-\pi}^{\pi} \frac{v(-t) - v(t)}{2 \tan \frac{t}{2}} \frac{dt}{2\pi} = 2\int_0^{\pi} \frac{v(-t)}{\tan \frac{t}{2}} \frac{dt}{2\pi} \\
&\geqslant 2\int_{1/km}^{1/m} \frac{v(-t)}{\tan \frac{t}{2}} \frac{dt}{2\pi} \\
&= 2\int_{1/km}^{1/m} \frac{(\frac{1}{2} \tan \frac{\delta}{2})(mt)^{1/m}}{\sin \frac{t}{2}} \cos \frac{t}{2} \frac{dt}{2\pi} \\
&\geqslant \frac{1}{2}\int_{1/km}^{1/m} \frac{\tan \frac{\delta}{2} m^{1/m} t^{1/m}}{\frac{t}{2}} \frac{dt}{2\pi} \\
&= \left(\tan \frac{\delta}{2}\right) m^{1/m} \int_{1/km}^{1/m} t^{1/m-1} \frac{dt}{2\pi} \\
&= \frac{(\tan \frac{\delta}{2}) m^{1/m+1}}{2\pi} \cdot \left[\left(\frac{1}{m}\right)^{1/m} - \left(\frac{1}{k}\right)^{1/m}\left(\frac{1}{m}\right)^{1/m}\right] \\
&= \frac{(\tan \frac{\delta}{2})}{2\pi} m\left[1 - \left(\frac{1}{k}\right)^{1/m}\right] > \frac{\lambda_0}{\varepsilon}.
\end{aligned}$$

Hence $\tilde{v}(0) > \lambda_0/\varepsilon$.

Suppose $z = e^{i\theta} \notin I$. Since the (closed) support of v is contained in $I/2$ we have

$$\begin{aligned}
|\tilde{v}(\theta)| &\leqslant \int_{-\pi}^{\pi}\left|\frac{v(\theta+t)-v(\theta-t)}{2\tan\frac{t}{2}}\right|\frac{dt}{2\pi} \\
&= \int_{|t|\leqslant\delta}\left|\frac{v(\theta+t)-v(\theta-t)}{2\tan\frac{t}{2}}\right|\frac{dt}{2\pi} \\
&\quad + \int_{\delta<|t|\leqslant\pi}\left|\frac{v(\theta+t)-v(\theta-t)}{2\tan\frac{t}{2}}\right|\frac{dt}{2\pi} \\
&= \int_{\delta\leqslant|t|\leqslant\pi}\left|\frac{v(\theta+t)-v(\theta-t)}{2\tan\frac{t}{2}}\right|\frac{dt}{2\pi} \leqslant \frac{\|v\|_\infty}{\tan\frac{\delta}{2}} \leqslant 1.
\end{aligned}$$

Therefore $|\tilde{v}(z)| \leqslant 1$ if $z \in \partial\mathbf{D} \sim I$, as desired.

PROOF OF THEOREM 2.2. Given intervals I_n with $I_n \cap \overline{\bigcup_{m\neq n} I_m} = \varnothing$, $\lambda_n \in I_n$ and $\lambda_n \to 1$, choose functions $u_n \in C^1_{\mathbf{R}}$ with $\|u_n\|_\infty < 1/2^{n+3}$, $|\tilde{u}_n(z)| < 1/2^{n+3}$ for $z \in \partial\mathbf{D} \sim I_n$ and $\tilde{u}_n(\lambda_n) = (2n+1)\pi$. Let $u = \sum_{n=1}^\infty u_n$. Then $u \in C_{\mathbf{R}}$ and since the map $T\colon L^2 \to L^2$ defined by $T(f) = \tilde{f}$ is continuous, $\tilde{u} = \sum_{n=1}^\infty \tilde{u}_n$ in L^2 norm. Since each $u_n \in C^1_{\mathbf{R}}$, $\tilde{u}_n \in C_{\mathbf{R}}$. It is easy to see that $\{\sum_{n=1}^m \tilde{u}_n\}_m$ converge uniformly to $\tilde{u}$ on compact subsets of $\partial\mathbf{D} \sim \{1\}$. Hence $\tilde{u}$ is continuous except possibly at $\lambda = 1$.

Let $q = e^{i\tilde{u}}$. Then

$$q = e^{u+i\tilde{u}}e^{-u} \in H^\infty + C \quad \text{and} \quad \bar{q} = e^{-u-i\tilde{u}}e^{u} \in H^\infty + C.$$

Therefore $q \in QC$.

For any n we have

$$\begin{aligned}
|\arg q(\lambda_n) - \pi| &= |\arg e^{i\tilde{u}(\lambda_n)} - \pi| \\
&= \left|\arg\exp\left\{i\sum_m \tilde{u}_m(\lambda_n)\right\} - \pi\right| \\
&= \left|\arg\exp\left\{i\left[(2n+1)\pi + \sum_{m\neq n}\tilde{u}_m(\lambda_n)\right]\right\} - \pi\right| \\
&= \left|\arg\left(-\exp\left\{i\sum_{m\neq n}\tilde{u}_m(\lambda_n)\right\}\right) - \pi\right| < \frac{1}{4},
\end{aligned}$$

and if $\lambda \in \partial\mathbf{D} \sim \overline{\bigcup I_n}$, then

$$|\arg q(\lambda)| = \left|\arg\exp\left\{i\sum_m \tilde{u}_m(\lambda)\right\}\right| < \frac{1}{4}.$$

Before we present the proof of Theorem 2.1 we prove a proposition that will be used frequently.

PROPOSITION 2.4. *Let $t \in M_1(QC)$ and $\{\lambda_n\}$ be a sequence of distinct points of $\partial\mathbf{D} \sim \{1\}$ such that t is in the $M(QC)$ closure of a sequence of points $\{t_n\}$, where $t_n \in M_{\lambda_n}(QC)$ and $\lambda_n \to 1$. Then $E_t \subseteq \overline{\bigcup_n M_{\lambda_n}(L^\infty)}$.*

PROOF. Suppose $\varphi \in M(L^\infty) \sim \overline{\bigcup_n M_{\lambda_n}(L^\infty)}$. If $\varphi \in M(L^\infty) \sim M_1(L^\infty)$, then $\varphi \in M(L^\infty) \sim E_t$. Therefore we may assume $\varphi \in M_1(L^\infty)$. Since $M(L^\infty)$ has a basis of clopen sets (sets that are both closed and open), we can find a clopen set $F \subseteq M(L^\infty)$ with $\varphi \in F \subseteq M(L^\infty) \sim \overline{\bigcup_n M_{\lambda_n}(L^\infty)}$. For each n, $M_{\lambda_n}(L^\infty) \subseteq M(L^\infty) \sim F$ and therefore

$$\bigcup_{m=1}^{\infty} \left\{\varphi' \in M(L^\infty): |\varphi'(z) - \lambda_n| > \frac{1}{m}\right\} \supseteq F.$$

Since F is compact, there exists N such that

$$\bigcap_{m=1}^{N} \left\{\varphi' \in M(L^\infty): |\varphi'(z) - \lambda_n| \leqslant \frac{1}{m}\right\}$$

is contained in $M(L^\infty) \sim F$. Thus there exists an interval I_n with $\lambda_n \in I_n$ satisfying $\underline{M_\lambda(L^\infty) \subseteq M(L^\infty) \sim F}$ for all $\lambda \in \overline{I_n}$. By choosing I_n sufficiently small we may assume $\overline{I_n} \cap \overline{\bigcup_{m \neq n} I_m} = \varnothing$. Note that

$$(*) \qquad \bigcup_n \{M_\lambda(L^\infty): \lambda \in \overline{I_n}\} \subseteq M(L^\infty) \sim F.$$

Thus there is a QC function q satisfying conditions (1)–(3) of Theorem 2.2.

For any n and any $\psi \in M_{\lambda_n}(L^\infty)$ we have, by (1) and (2) of Theorem 2.2, that $|\arg q(\psi) - \pi| \leqslant \frac{1}{4}$. Passing to $M(QC)$ we have $|\arg q(t) - \pi| \leqslant \frac{1}{4}$. Therefore for any $\Psi' \in E_t$ we have $|\arg q(\Psi') - \pi| \leqslant \frac{1}{4}$.

To see that $\varphi \in M(L^\infty) \sim E_t$, we shall show that $|\arg q(\varphi)| \leqslant \frac{1}{4}$. Choose $\varepsilon > 0$ and let $F_\varepsilon = \{\eta \in M(L^\infty): |\arg \varphi(q) - \arg \eta(q)| < \varepsilon\}$. Then $F_\varepsilon \cap F$ is an open set in $M(L^\infty)$ containing φ. Since $M_1(L^\infty)$ has no interior in $M(L^\infty)$ there exists $\lambda_0 \neq 1$ such that $M_{\lambda_0}(L^\infty) \cap F \cap F_\varepsilon \neq \varnothing$.

Choose $\lambda_0 \in \partial\mathbf{D}$ satisfying $\lambda_0 \neq 1$ and $M_{\lambda_0}(L^\infty) \cap F \cap F_\varepsilon \neq \varnothing$. By $(*)$, $\lambda_0 \in \partial\mathbf{D} \sim \overline{\bigcup I_n}$. Hence $|\arg q(\lambda_0)| \leqslant \frac{1}{4}$. Let $\psi_{\varepsilon,0} \in M_{\lambda_0}(L^\infty) \cap F \cap F_\varepsilon$. Then $|\arg \psi_{\varepsilon,0}(q)| \leqslant \frac{1}{4}$. Therefore $|\arg \varphi(q)| \leqslant \frac{1}{4} + \varepsilon$. Since ε was arbitrary, $|\arg \varphi(q)| \leqslant \frac{1}{4}$. Therefore $\varphi \in M(L^\infty) \sim E_t$, so

$$M(L^\infty) \sim \left(\overline{\bigcup M_\lambda(L^\infty)}\right) \subseteq M(L^\infty) \sim E_t,$$

which implies the result.

PROOF OF THEOREM 2.1. Choose $\varphi \in M(L^\infty)$ with $\varphi \neq \psi$ such that φ and ψ are in the same QC level set. If no such φ exists, then $E_\psi = \{\psi\}$ and hence the maximal antisymmetric set containing ψ, S_ψ, satisfies $S_\psi = \{\psi\}$ and we are done. We assume then that such a φ exists. Since $\varphi \neq \psi$, there exists a clopen set F with $\varphi \in F$ and $\psi \in M(L^\infty) \sim F$. Thus passing to a subsequence of $\{\psi_n\}$ if necessary, we may assume $\{\psi_n\} \subseteq M(L^\infty) \sim F$. By a theorem of Axler [**1**], for each n we can find $f_n \in H^\infty + C$ with $\|f_n\|_\infty = 1$ such that $|\psi_n(f_n)| = 1$ and $\eta(f_n) = 0$ for all $\eta \in F$. Using an idea of Sarason, we let G_n denote the open ellipse with major axis $[-1, 1]$ and minor axis $[-i/n, i/n]$. Let T_n denote a conformal mapping of the open unit disc $\mathbf{D}$ onto G such that $T_n(0) = 0$, and by [**11**, p. 309] we may assume $T_n \in C$. Choose $z_n \in \mathbf{D}$ with $|z_n| > n/(n+1)$, $T_n(z_n)$ real and $T_n(z_n) > n/(n+1)$. By multiplying f_n by a constant of modulus one, we may assume $|z_n|\psi_n(f_n) = z_n$.

Since $H^\infty + C|M_{\lambda_n}(L^\infty) = H^\infty|M_{\lambda_n}(L^\infty)$, there exists an H^∞ function whose restriction to $M_{\lambda_n}(L^\infty)$ is $f_n|M_{\lambda_n}(L^\infty)$. Multiplying that function by a suitable peaking function for $M_{\lambda_n}(L^\infty)$, we obtain a function $g_n \in H^\infty$ such that $\|g_n\|_\infty < 1/|z_n|$ and $g_n|M_{\lambda_n}(L^\infty) = f_n|M_{\lambda_n}(L^\infty)$. Thus $T \circ (|z_n|g_n) \in H^\infty$. Let $\eta \in M(L^\infty)$. We claim that

$$\eta(T_n \circ |z_n|g_n) = T_n(|z_n|\eta(g_n)).$$

To see this, note that T_n is a uniform limit of polynomials $p_{m,n}$. If $f \in H^\infty$ with $\|f\|_\infty < 1$, then

$$\eta(T_n \circ f) = \eta\Big(\lim_m p_{m,n}(f)\Big) = \lim_m p_{m,n}(\eta(f)) = T_n(\eta(f)).$$

Therefore for each n we have

$$\begin{aligned}\psi_n\big(T_n \circ |z_n|g_n\big) &= T_n\big(|z_n|\psi_n(g_n)\big)\\ &= T_n\big(|z_n|\psi_n(f_n)\big) = T_n(z_n) > n/(n+1).\end{aligned}$$

If $\tau \in F \cap M_{\lambda_n}(L^\infty)$ for some n, then

$$\tau\big(T_n \circ |z_n|g_n\big) = T_n\big(|z_n|\tau(f_n)\big) = T_n(0) = 0.$$

For each λ_n choose intervals I_n centered at λ_n with $I_n \cap \overline{\bigcup_{m\neq n} I_m} = \varnothing$, where the Lebesgue measure of I_n, $|I_n|$, satisfies $|I_n| < 1/2^{n+4}$ and $1 \in \partial\mathbf{D} \sim (\bigcup_n I_n)$. Let $\mathcal{O}(I_n) = \{z \in \overline{\mathbf{D}}: |z - \lambda_n| < |I_n|/2\}$ and let h_n be a peaking function for $M_{\lambda_n}(L^\infty)$. By raising h_n to a sufficiently large power, we may assume $\|h_n|\overline{\mathbf{D}} \sim \mathcal{O}(I_n)\|_\infty < 1/2^{n+4}$.

Let K_n be a linear fractional transformation such that $\|K_n\|_\infty = 1$, $K_n(1) = 0$ and $K_n(\lambda_n) = 1$. Let $l_n = h_n(T_n \circ |z_n|g_n)K_n$. Then

$$l_n\big|M_{\lambda_n}(L^\infty) = \big(T_n \circ |z_n|g_n\big)\big|M_{\lambda_n}(L^\infty) \quad \text{for all } n \tag{1}$$

and

$$\big\|l_n|\overline{\mathbf{D}} \sim \mathcal{O}(I_n)\big\|_\infty < 1/2^{n+4}. \tag{2}$$

Let $L_m = \sum_{n=1}^m l_n$ and $L = \sum_{n=1}^\infty l_n$. It is easy to see that L_m converges to L uniformly on compact subsets of $\overline{\mathbf{D}} - \{1\}$. Furthermore, $\|L_m\| \leqslant 2$ and thus $L \in H^\infty(\mathbf{D})$.

To see that $L|E_\psi$ is real valued, let $\varepsilon > 0$ be given. Choose N such that $\sum_{n=N}^\infty 1/2^n < \varepsilon/3$. Let I be an open interval of $\partial\mathbf{D}$ containing 1 such that $\max_{1\leqslant j\leqslant N}\|K_j|I\|_\infty < \varepsilon/3N$. Then $\|l_j|I\|_\infty < \varepsilon/3N$, $j = 1,2,\ldots,N$. Choose ψ_0 in the QC level set corresponding to ψ, E_ψ. Let

$$V = \Big\{\eta \in M(L^\infty): |\eta(L) - \psi_0(L)| < \frac{\varepsilon}{3}\Big\} \cap \bigcup_{\lambda\in I} M_\lambda(L^\infty).$$

Then V is an open set about ψ_0. By Proposition 2.4 there exists an integer m satisfying $m > \max(N, 3/\varepsilon)$ and such that $\bar{V} \cap M_{\lambda_m}(L^\infty) \neq \varnothing$. Let $\varphi_0 \in V \cap M_{\lambda_m}(L^\infty)$. Since $\Sigma_n l_n$ converges uniformly on $\bar{I}_m$, we have

$$|\operatorname{Im}\varphi_0(L)| = \left|\operatorname{Im}\sum_n \varphi_0(l_n)\right| = \left|\sum_{n=1}^{N} \operatorname{Im}\varphi_0(l_n) + \operatorname{Im}\varphi_0(l_m) + \sum_{\substack{n=N+1\\ n\neq m}}^{\infty} \operatorname{Im}\varphi_0(l_n)\right|$$

$$\leqslant \sum_{n=1}^{N} |\operatorname{Im}\varphi_0(l_n)| + \left|\operatorname{Im}\varphi_0\left(T_m \circ \left(|z_m| g_m\right)\right)\right| + \sum_{\substack{n=N+1\\ n\neq m}}^{\infty} |\operatorname{Im}\varphi_0(l_n)|$$

$$\leqslant \sum_{n=1}^{N} \frac{\varepsilon}{3N} + \left|\operatorname{Im}\left(T_m \circ \left(\varphi_0\left(|z_m| g_m\right)\right)\right)\right| + \sum_{\substack{n=N+1\\ n\neq m}} \frac{1}{2^{n+4}}$$

$$< \varepsilon/3 + 1/m + \varepsilon/3 < \varepsilon.$$

Therefore $|\operatorname{Im}\psi_0(L)| < 4\varepsilon/3$. Since ε was arbitrary, $\psi_0(L)$ must be real valued.

Recall that we chose φ to be a point in $M(L^\infty)$ with $\varphi \neq \psi$ such that φ and ψ are in the same QC level set and F was a clopen subset of $M(L^\infty)$ with $\varphi \in F$ and $\psi \in \overline{\{\psi_n\}}^{M(L^\infty)} \subseteq M(L^\infty) \sim F$. Since $\{\eta \in M(L^\infty): |\eta(L) - \psi(L)| < 1/8\}$ is an open subset containing ψ, there exists n with $n \geqslant 7$ and $\psi_n \in M_{\lambda_n}(L^\infty)$ such that $|\psi_n(L) - \psi(L)| < 1/8$. Thus

$$|\psi_n(L)| = \left|\psi_n(l_n) + \sum_{m\neq n} \psi_n(l_m)\right| \geqslant \frac{n}{n+1} - \frac{1}{4} \geqslant \frac{7}{8} - \frac{1}{4} = \frac{5}{8}.$$

Therefore $|\psi(L)| \geqslant 1/2$.

To determine $\varphi(L)$ note that $U = \{\eta \in M(L^\infty): |\eta(L) - \varphi(L)| < 1/8\} \cap F$ is an open set in $M(L^\infty)$ containing φ. By Proposition 2.4 there exists m such that $M_{\lambda_m}(L^\infty) \cap U \neq \varnothing$. Let $\varphi_m \in M_{\lambda_m}(L^\infty) \cap U$. Then we have

$$|\varphi_m(L)| = \left|\varphi_m(l_m) + \sum_{n\neq m} \varphi_m(l_n)\right| \leqslant |\varphi_m(l_m)| + \sum_{n\neq m} |\varphi_m(l_n)| \leqslant 0 + \frac{1}{4}.$$

Therefore $|\varphi(L)| \leqslant 3/8$ and $\psi(L) \neq \varphi(L)$.

The maximal antisymmetric set S_ψ containing ψ is contained in E_ψ, so $L|S$ is real valued. Thus $L|S$ is constant. Therefore $\varphi \notin S$. Since φ was an arbitrary point of E_ψ distinct from ψ, $S = \{\psi\}$ and the proof is complete.

We will show that many of the points that are in the $M(L^\infty)$ closure of a sequence of points from distinct L^∞ fibers are contained in QC level sets consisting of more than one point. We will also show that not every QC level set contains such a point.

We call a sequence $\{z_j\}_{j=1}^\infty$ of distinct points in $\mathbf{D}$ an interpolating sequence if whenever $\{w_j\}_{j=1}^\infty$ is a bounded sequence of complex numbers, there exists a function $f \in H^\infty$ with $f(z_j) = w_j$ for all j. It is well known **[4]** that a sequence $\{z_j\}_{j=1}^\infty$ is an interpolating sequence if and only if there exists a constant $\delta > 0$ such that

$$\prod_{j\neq k} \left|\frac{z_k - z_j}{1 - \bar{z}_j z_k}\right| \geqslant \delta > 0 \quad \text{for } k = 1, 2, 3, \ldots.$$

A Blaschke product with zeroes $\{z_j\}_{j=1}^\infty \subseteq \mathbf{D}$ is a function $b \in H^\infty(\mathbf{D})$ of the form

$$b(z) = \lambda z^k \prod_{|z_j| \neq 0} \frac{-\bar{z}_j}{|z_j|}\left(\frac{z - z_j}{1 - \bar{z}_j z}\right) \quad \text{for } z \in \mathbf{D},$$

where $|\lambda| = 1$ and $\Sigma_n(1 - |z_n|) < \infty$. If the zeroes of b form an interpolating sequence, b is called an interpolating Blaschke product. We will use the theorem stated below. The proof is given in [**9**, p. 205].

THEOREM 2.5. *Let $\{z_j\}_{j=1}^\infty$ be an interpolating sequence and let b be the Blaschke product with zeroes $\{z_j\}_{j=1}^\infty$. Then $\overline{\{z_j\}_{j=1}^\infty}^{M(H^\infty)}$ is homeomorphic to the Stone-Čech compactification of $\{z_j\}_{j=1}^\infty$ and every zero of b in $M(H^\infty) \sim \mathbf{D}$ is in the $M(H^\infty)$ closure of $\{z_j\}_{j=1}^\infty$.*

The following lemma was proven by K. Clancey and J. A. Gosselin [**5**] and by R. G. Douglas [**6**].

LEMMA 2.6. *Let u be an inner function. If $t \in M(QC)$ with $u|E_t$ invertible in $H^\infty|E_t$, then $u|E_t$ is constant. In fact $\{t \in M(QC)\colon u|E_t$ is constant$\} = \{t \in M(QC)\colon u|E_t$ is invertible in $H^\infty|E_t\}$ is an open set in $M(QC)$.*

Using Lemma 2.6 together with the following result of D. E. Marshall [**10**, p. 15], we prove a similar result about characteristic functions. In what follows, $H^\infty[f]$ denotes the closed subalgebra of L^∞ generated by H^∞ and f.

THEOREM 2.7. *Let χ_E be a nonconstant characteristic function in L^∞. Then there is an inner function u such that $H^\infty[\chi_E] = H^\infty[\bar{u}]$.*

THEOREM 2.8. *Let χ_E be a nonconstant characteristic function in L^∞. If $\chi_E|E_{t_0} \in H^\infty|E_{t_0}$ for some QC level set E_{t_0}, then $\chi_E|E_{t_0}$ is constant.*

Sarason has given a proof of Theorem 2.8. Since his proof is unpublished we include a proof below. Our proof is different from Sarason's; his did not use Theorem 2.7.

PROOF. By Theorem 2.7 there exists an inner function u such that $H^\infty[\chi_E] = H^\infty[\bar{u}]$. Therefore $M(H^\infty[\chi_E]) = M(H^\infty[\bar{u}])$. Thus

$$M(H^\infty|E_{t_0}) \subseteq M(H^\infty[\chi_E]) = M(H^\infty[\bar{u}]).$$

Hence $|\varphi(u)| = 1$ for all $\varphi \in M(H^\infty|E_t)$. Therefore $u|E_t$ is invertible in $H^\infty|E_t$. By Lemma 2.6 there exists $\mathcal{O}$ open in $M(QC)$ containing t_0 with $\bar{u}|E_t \in H^\infty|E_t$ for all $t \in \mathcal{O}$. Let $q \in QC$ with $q(t_0) = 1$, $q(s) = 0$ for $s \in M(QC) \sim \mathcal{O}$ and $0 \leq q \leq 1$. Choose $\psi \in M(H^\infty + C)$. If supp $\psi \subseteq E_s$ and $s \in M(QC) \sim \mathcal{O}$, then $q|\text{supp}\,\psi = 0$ and therefore $q\chi_E|\text{supp}\,\psi = 0$. If supp $\psi \subseteq E_t$ and $t \in \mathcal{O}$, then $\bar{u}|\text{supp}\,\psi \in H^\infty|\text{supp}\,\psi$ and, hence, $\chi_E|\text{supp}\,\psi \in H^\infty|\text{supp}\,\psi$. By Theorem 1.3 $q\chi_E \in H^\infty + C$. Since $q\chi_E$ is real valued, $q\chi_E \in QC$. Thus $q\chi_E|E_{t_0}$ is constant. Since $\varphi(q) = 1$ for all $\varphi \in E_{t_0}$ we must have $\chi_E|E_{t_0}$ constant, as desired.

By the Shilov Idempotent Theorem we obtain the corollary below, answering a question of R. G. Douglas in [**6**].

COROLLARY 2.9. *If $t \in M(QC)$, then $M(H^\infty \mid E_t)$ is connected.*

It is a consequence of the following result of T. Wolff [**17**] that any function $f \in L^\infty$ is constant on some QC level set.

THEOREM 2.10. *Let $f \in L^\infty$. There exists an outer function $q \in QC \cap H^\infty$ such that $qf \in QC$.*

We will show that for any $\lambda \in \partial\mathbf{D}$ and any clopen set F contained in $M_\lambda(L^\infty)$ there exists a QC level set contained in F. By [**9**, p. 171] we see that for $f \in L^\infty$ and $\lambda \in \partial\mathbf{D}$, f is constant on some QC level set contained in $M_\lambda(L^\infty)$.

THEOREM 2.11 [**14, 16**]. *Let f and g be functions in L^∞. If for each $\varphi \in M(H^\infty + C)$ either $f \mid \operatorname{supp}\varphi \in H^\infty \mid \operatorname{supp}\varphi$ or $g \mid \operatorname{supp}\varphi \in H^\infty \mid \operatorname{supp}\varphi$, then for each QC level set E_t either $f \mid E_t \in H^\infty \mid E_t$ or $g \mid E_t \in H^\infty \mid E_t$.*

This theorem, together with the following unpublished result of K. Hoffman, provides us with much more information about the points in $M(L^\infty)$ that are in the closure of a sequence of points from distinct L^∞ fibers.

THEOREM 2.12. *Let $\{z_j\}_{j=1}^\infty$ be an interpolating sequence such that*

$$\lim_{n\to\infty} \prod_{m\neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| = 1.$$

If $\varphi \in \overline{\{z_n\}}^{M(H^\infty)}$ and $\varphi \in M(H^\infty + C)$, then $\operatorname{supp}\varphi$ is a maximal support set.

THEOREM 2.13. *Let E be a nonempty clopen subset of $M_1(L^\infty)$. Then E contains a QC level set that is not a maximal antisymmetric set for $H^\infty + C$.*

As we remarked earlier Sarason [**14**] has shown that each L^∞ fiber contains a QC level set that is not a maximal antisymmetric set for $H^\infty + C$. Some of the ideas used to prove Theorem 2.13 are similar to techniques communicated by T. Wolff (private communication).

PROOF. Let F be a clopen subset of $M(L^\infty)$ such that $E = F \cap M_1(L^\infty)$. Then there exists a measurable subset G of $\partial\mathbf{D}$ of positive measure such that $\chi_F = \hat{\chi}_G$. Let $\{\lambda_n\}$ be a sequence of distinct points of $\partial\mathbf{D}$ with $\lambda_n = e^{i\theta_n} \to 1$ and $\lim_{r\to 1}\chi_G(re^{i\theta_n}) = 1$. We claim that there exists an interpolating sequence $\{z_m\}$ with the following properties:

(1)

$$\lim_{n\to\infty} \prod_{m\neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| = 1.$$

(2) $\{z_m\}$ is the disjoint union of interpolating sequences $\{z_{m,n}\}_{n=1}^\infty$ such that $z_{m,n} = r_{m,n}e^{i\theta_n}$ for suitable choices of $r_{m,n}$ and

(3) $\chi_G(z_m) \to 1$ as $m \to \infty$.

We construct such a sequence as follows: Let $z_1 = z_{1,1} = r_{1,1}e^{i\theta_1}$, where $0 < r_{1,1} < 1$. Choose $z_2 = z_{2,1} = r_{2,1}e^{i\theta_2}$ such that $\chi_G(z_2) > \frac{1}{2}$ and $|(z_2 - z_1)/(1 - \bar{z}_2 z_1)| > e^{-1/2}$. Choose $z_3 = z_{1,2} = r_{1,2}e^{i\theta_1}$ such that $\chi_G(z_3) > 1 - \frac{1}{3}$ and $|(z_3 - z_j)/(1 - \bar{z}_j z_3)| > e^{-1/2^{3+j}}$, $j = 1, 2$. We choose $z_4 = z_{2,2} = r_{2,2}e^{i\theta_2}$ satisfying $\chi_G(z_4) > 1 - \frac{1}{4}$ and

$|(z_4 - z_j)/(1 - \bar{z}_j z_4)| \geqslant e^{-1/2^{4+j}}$, $j = 1,2,3$. We continue to choose z_n satisfying (2) such that $\chi_G(z_n) > 1 - 1/n$ and $|(z_n - z_j)/(1 - \bar{z}_j z_n)| > e^{-1/2^{n+j}}$ for $j < n$. It is not hard to see that (1) and (3) also hold.

Let $\varphi_n \in \overline{\{z_{m,n}\}_{m=1}^{\infty}}^{M(H^\infty + C)}$. Then $\varphi_n \in M_{\lambda_n}(H^\infty)$. Let $\varphi_0 \in \overline{\{\varphi_n\}}^{M(H^\infty + C)} \cap M_1(H^\infty)$. By Theorem 2.12 we have that, for each n, $\operatorname{supp} \varphi_n$ is a maximal support set. Let b be the interpolating Blaschke product with zeroes $\{z_m\}$. Choose $\psi \in M(H^\infty + C)$. If $\bar{b} | \operatorname{supp} \psi \notin H^\infty | \operatorname{supp} \psi$, then $|\psi(b)| < 1$. Since $b | \operatorname{supp} \psi$ is not invertible in $H^\infty | \operatorname{supp} \psi$, there exists $\tau \in M(H^\infty | \operatorname{supp} \psi) = \{\eta \in M(H^\infty)\colon \operatorname{supp} \eta \subseteq \operatorname{supp} \psi\}$ with $\tau(b) = 0$. By Theorem 2.5, $\tau \in \overline{\{z_m\}}$ and hence by Theorem 2.12, $\operatorname{supp} \tau$ is a maximal support set. Therefore $\operatorname{supp} \tau = \operatorname{supp} \psi$. Since $\chi_G(z_m) \to 1$ as $m \to \infty$, we have $\tau(\chi_G) = 1$. Thus $\operatorname{supp} \tau \subseteq G$, so $\operatorname{supp} \psi \subseteq G$. Thus for any $\psi \in M(H^\infty + C)$ either $\bar{b} | \operatorname{supp} \psi \in H^\infty | \operatorname{supp} \psi$ or $\chi_F | \operatorname{supp} \psi \in H^\infty | \operatorname{supp} \psi$. By Theorem 2.11 on each QC level set E_t we have $\bar{b} | E_t \in H^\infty | E_t$ or $\chi_F | E_t \in H^\infty | E_t$. Consider the QC level set E_{t_0} containing $\operatorname{supp} \varphi_0$. Since $\bar{b} | \operatorname{supp} \varphi_0 \notin H^\infty | \operatorname{supp} \varphi_0$, we must have $\chi_F | E_{t_0} \in H^\infty | E_{t_0}$. By Theorem 2.8, $\chi_F | E_{t_0}$ is constant. By (3) above, we must have $E_{t_0} \subseteq F$.

Let $\{\varphi_{n_\alpha}\}$ be a subnet of $\{\varphi_n\}$ such that $\varphi_{n_\alpha} \to \varphi_0$. Choose $\psi_n \in \operatorname{supp} \varphi_n$. Then some subnet of $\{\psi_n\}$ converges. We may assume without loss of generality that $\psi_{n_\alpha} \to \psi_0$. Therefore $\psi_0 | QC = \varphi_0 | QC$. Hence $\psi_0 \in E_{t_0}$. By Theorem 2.1, $\{\psi_0\}$ is a maximal antisymmetric set for $H^\infty + C$. Since $\varphi_0 \in M(H^\infty) \sim M(L^\infty)$, $\operatorname{supp} \varphi_0$ consists of more than one point. Since $\psi_0 \in E_{t_0}$ and $\operatorname{supp} \varphi_0 \subseteq E_{t_0}$, E_{t_0} is not a maximal antisymmetric set for $H^\infty + C$.

One may ask whether every QC level set contains a point that is in the closure of a sequence of points from distinct L^∞ fibers. The answer to this question is no, as we shall see. We first state a lemma due to Sarason that appears in [**14**].

LEMMA 2.14. *Let b be an inner function. If $\varphi \in M(H^\infty + C)$ and $b | \operatorname{supp} \varphi$ is nonconstant, then* $b(\operatorname{supp} \varphi) = \partial \mathbf{D}$.

EXAMPLE 2.15. Let $\{z_n\}_{n=1}^{\infty}$ be an interpolating sequence with $z_n \to 1$ and let $\varphi \in \overline{\{z_n\}}^{M(H^\infty)}$, $\varphi \in M(H^\infty + C)$. Let E_ψ denote the QC level set containing $\operatorname{supp} \varphi$. Then E_ψ does not contain a point which is in the closure of a sequence of points from distinct fibers.

PROOF. Suppose not, that is, suppose $\varphi_0 \in E_\psi$ and $\varphi_0 \in \overline{\{\varphi_n\}}^{M(L^\infty)}$, where $\varphi_n \in M_{\lambda_n}(L^\infty)$ and $\lambda_n \neq \lambda_m$ for $n \neq m$. Let b be the interpolating Blaschke product with zeroes $\{z_n\}_{n=1}^{\infty}$. By assumption, $z_n \to 1$. Thus b is continuous at λ for $\lambda \neq 1$. Hence $b | M_\lambda(L^\infty)$ is constant for $\lambda \neq 1$. Let $U = \{\psi \in M(L^\infty)\colon |\psi(b) - \varphi_0(b)| < \frac{1}{2}\}$. Then U is an open set in $M(L^\infty)$ containing φ_0. Therefore $\varphi_0 \in \overline{\{\varphi_n\colon \varphi_n \in U\}}^{M(L^\infty)}$. By Corollary 2.4,

$$E_\psi = E_{\varphi_0} \subseteq \overline{\bigcup \{M_{\lambda_n}(L^n) L\, \varphi_n \in M_{\lambda_n}(L^\infty) \cap U\}}^{M(L^\infty)}.$$

Hence $E_{\varphi_0} \subseteq \bar{U}$. If $b | E_{\varphi_0}$ were nonconstant, then by Lemma 2.14 we have $b(E_{\varphi_0}) = \partial \mathbf{D}$. Therefore there exists $\eta \in E_{\varphi_0}$ with $\eta(b) = -\varphi_0(b)$. Hence $\eta \notin \bar{U}$. Thus $b | E_{\varphi_0}$ is constant. Therefore $|\varphi(b)| = 1$ and, hence, φ cannot be in the closure of the zeroes of b. This contradiction implies the result.

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