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THE FREE BOUNDARY OF A SEMILINEAR ELLIPTIC EQUATION[1]

BY

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ABSTRACT. The Dirichlet problem $\Delta u = \lambda f(u)$ in a domain Ω, $u = 1$ on $\partial\Omega$ is considered with $f(t) = 0$ if $t \leqslant 0$, $f(t) > 0$ if $t > 0$, $f(t) \sim t^p$ if $t \downarrow 0$, $0 < p < 1$; $f(t)$ is not monotone in general. The set $\{u = 0\}$ and the "free boundary" $\partial\{u = 0\}$ are studied. Sharp asymptotic estimates are established as $\lambda \to \infty$. For suitable f, under the assumption that Ω is a two-dimensional convex domain, it is shown that $\{u = 0\}$ is a convex set. Analogous results are established also in the case where $\partial u/\partial\nu + \mu(u - 1) = 0$ on $\partial\Omega$.

Introduction. In this paper we study the Dirichlet problem

$$(0.1)\qquad \Delta u = \lambda f(u) \quad \text{in } \Omega \qquad (\lambda > 0),$$
$$u = 1 \quad \text{on } \partial\Omega$$

where Ω is a bounded domain in R^n and

$$f(t) = 0 \quad \text{if } t \leqslant 0, \qquad f(t) > 0 \quad \text{if } t > 0,$$
$$f(t) \sim t^p \quad \text{as } t \downarrow 0, \qquad 0 < p < 1;$$

$f(t)$ is not assumed to be monotone. The motivation for this problem comes from reaction-diffusion models in which

$$f(t) \uparrow \quad \text{if } 0 < t < t_0, \qquad f(t) \downarrow \quad \text{if } t_0 < t < 1$$

for some $t_0 \in (0, 1)$; see [**2, 3, 7, 18, 19**].

The solution of (0.1) has, in general, more than one solution. We are interested primarily in those solutions which are either minimal, or maximal, or minimizers of the functional

$$(0.2)\qquad \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + \lambda F(v)\right) dx \qquad \left(F(t) = \int_0^t f(s)\, ds\right)$$

subject to the boundary condition $v = 1$.

We shall study the nature of the "dead core" $N_\lambda = \{u_\lambda = 0\}$ (u_λ is a solution of (0.1)); if $p \geqslant 1$ then the dead core is empty, by the maximum principle. One of the

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main results is that if Ω is a two-dimensional convex domain then N_λ is a convex set provided f satisfies

$$f'(t) + \frac{f(t)}{1-t} > 0 \quad \text{for } 0 < t \leqslant 1, \tag{0.3}$$

i.e. every tangent line to $y = f(t)$ intersects the line $\{t = 1\}$ in $\{y > 0\}$ (this condition is satisfied for some reaction-diffusion models). Convexity of level surfaces $\{u_\lambda = c\}$ was proved by Caffarelli and Spruck [**4**] in case $f(t)$ is monotone increasing and negative for $0 \leqslant t < 1$, and $f(1) = 0$; our results and methods are unrelated to theirs; see Remark 5.1.

In §1 we show that for any two minimizers u_1 and u_2, either $u_1 \geqslant u_2$ or $u_2 \geqslant u_1$ and one of the sets N_{u_1}, N_{u_2} is contained in the interior of the other. We also establish comparison theorems for minimal solutions, maximal solutions, and for minimizers corresponding to different f's and different Ω's.

In §2 we establish uniqueness of solutions of (0.1) under the assumption (0.3).

In §3 we show that, for λ large and any solution u_λ of (0.1), ∂N_λ is a smooth surface parallel to $\partial\Omega$ at distance $\gamma/\sqrt{\lambda} + O(1/\lambda)$, γ constant.

In §4 we specialize to domains Ω with $\partial\Omega$ having nonnegative mean curvature and establish a convexity type property of the boundary ∂N_λ. For a smooth portion of ∂N_λ it asserts that the mean curvature is nonnegative.

In §5 we establish, for two-dimensional convex domains, the convexity of the components of N_λ; under the assumption (0.3) N_λ is shown to have at most one component.

Next, in §6 we shall consider the limit case $p = 0$, i.e., $f(t) = \chi_{[t>0]}$ and establish convexity of the coincidence set for some variational inequalities.

Finally, in §7 we extend the results of the preceding sections to the case of the Robin problem, replacing the condition $u = 1$ on $\partial\Omega$ by

$$\frac{\partial u}{\partial \nu} + \mu(u-1) = 0 \quad \text{on } \partial\Omega \qquad (\mu > 0).$$

It is actually this case which primarily arises in applications [**2, 7**]; the Dirichlet problem is viewed as a limit case when $\mu \to \infty$.

1. Comparison theorems. Let $f(t)$ be a function satisfying

$$f(t) = \begin{cases} t^p f_0(t) & \text{if } 0 \leqslant t < \infty, \text{ for some } 0 < p < 1, \\ 0 & \text{if } -\infty < t < 0, \end{cases} \tag{1.1}$$
$$m \leqslant f_0(t) \leqslant M, \quad 0 < m \leqslant M < \infty, \qquad |f_0''(t)| \leqslant K$$

where m, M, K are positive constants; $f(t)$ is not assumed to be monotone increasing. We are interested in solutions of the Dirichlet problem

$$\begin{aligned} \Delta u &= f(u) \quad \text{in } \Omega, \\ u &= 1 \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

where Ω is a bounded domain in R^n with $C^{2+\alpha_0}$ boundary $\partial\Omega$. One can easily show that there exists a solution u of (1.2) and it belongs to $C^{2+\alpha}(\overline{\Omega})$, where $\alpha = \min(p, \alpha_0)$.

Since $\Delta u \geqslant 0$ in Ω and $u = 1$ on $\partial\Omega$, the maximum principle gives

$$u < 1 \quad \text{in } \Omega. \tag{1.3}$$

We also have

$$u \geqslant 0 \quad \text{in } \Omega. \tag{1.4}$$

Indeed, otherwise u takes negative minimum in the nonempty open set $G = \{u < 0\} \cap \Omega$. Since, however, $\Delta u = f(u) = 0$ in G, the maximum principle gives $u \equiv$ const in G, which is impossible.

Set $F(t) = \int_0^t f(s)\,ds$ and consider the functional

$$J(v) = \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + F(v)\right) dx \tag{1.5}$$

over the class of admissible functions

$$K = \left\{v \in H^{1,2}(\Omega), v = 1 \text{ on } \partial\Omega\right\}. \tag{1.6}$$

Problem (J). Find $u \in K$ such that $J(u) = \min_{v \in K} J(v)$. If u is a solution of this problem then u is clearly also a solution of (1.2).

DEFINITION 1.1. A solution of Problem (J) is called a *minimizer* solution of (1.2).

We shall need some standard comparison lemmas.

LEMMA 1.1. *Let* $w \in C^0(\bar{\Omega})$,

$$\Delta w + b_i w_{x_i} + cw = h \quad \text{in } \Omega,$$
$$w = \phi \quad \text{on } \partial\Omega$$

where b_i *are bounded functions, and suppose that* $\Omega \subset \{\bar{x} < x_1 < \bar{x} + a\}$ *and* $\sup_\Omega c(x) < (e^{\gamma a} - 1)^{-1}$ *where* $\gamma > 1$, $\gamma^2 > (\sup_\Omega |b_1|)\gamma + 1$. *Then*

$$w \geqslant \frac{\min_{\partial\Omega} \phi - (\sup_\Omega h)(e^{\gamma a} - 1)}{1 - (\sup_\Omega c)(e^{\gamma a} - 1)}. \tag{1.7}$$

$$\max_{\bar{\Omega}} |w| \leqslant \frac{\max_{\partial\Omega}|\phi| + (\sup_\Omega |h|)(e^{\gamma a} - 1)}{1 - (\sup_\Omega c)(e^{\gamma a} - 1)}. \tag{1.8}$$

For the proof see [**6**, p. 330]. Notice that $c(x)$ is not required to be continuous or bounded from below in Ω.

COROLLARY 1.2. *Suppose*

$$\Delta u - h_1(u)|\nabla u|^2 \leqslant h_2(u) \quad \text{in } \Omega,$$
$$\Delta v - h_1(v)|\nabla v|^2 \geqslant h_2(v) \quad \text{in } \Omega,$$
$$u \geqslant v \quad \text{on } \partial\Omega$$

where $h_i(t)$ *are monotone increasing in* t. *Then* $u \geqslant v$ *in* Ω.

Indeed, the function $w = u - v$ satisfies

$$\Delta w - h_1(u)\nabla(u + v) \cdot \nabla w - \left(c_1|\nabla v|^2 + c_2\right)w \leqslant 0$$

where

$$c_i(x) = \begin{cases} \dfrac{h_i(u(x)) - h_i(v(x))}{u(x) - v(x)} & \text{if } u(x) \neq v(x), \\ 0 & \text{if } u(x) = v(x). \end{cases}$$

Since $c_i \geqslant 0$, we can apply (1.7).

LEMMA 1.3. *If*

$$\Delta v + c(x)v \leqslant 0 \quad \text{in } \Omega,$$
$$v \geqslant 0 \quad \text{in } \Omega, \qquad v \not\equiv 0,$$

then $v > 0$ *in* Ω.

Indeed, writing $c = c^+ - c^-$ and noting that $c^+ v \geqslant 0$, we get $\Delta v - c^-(x)v \leqslant 0$ in Ω, so that $v > 0$ by the strong maximum principle.

We now return to Problem (J).

THEOREM 1.4. *If* u, v *are two minimizers and* $u \not\equiv v$, *then either* $u > v$ *in* $\{v > 0\}$ *or* $v > u$ *in* $\{u > 0\}$.

PROOF. Notice that $u \wedge v = \min\{u, v\}$ and $u \vee v = \max\{u, v\}$ belong to K and therefore $J(u \wedge v) \geqslant J(u) = J(v)$, $J(u \vee v) \geqslant J(u)$. Since, as easily seen, $J(u \wedge v) + J(u \vee v) = J(u) + J(v)$, we conclude that $J(u \wedge v) = J(u)$. Consequently, the function $w = u \wedge v$ satisfies $\Delta w = f(w)$ in Ω.

Next, $u - w \geqslant 0$ in Ω and $\Delta(u - w) = f(u) - f(w) = c(u - w)$, where $c = c(x)$ is a bounded function in any compact subset of $\{w > 0\}$. By Lemma 1.3 we then have

$$\text{either } u - w > 0 \text{ in } \{w > 0\}, \text{ i.e., } v < u \text{ in } \{v > 0\},$$
$$\text{or } u - w \equiv 0 \text{ in } \{w > 0\}, \text{ i.e., } u \leqslant v \text{ in } \{u > 0\}.$$

In the second case we repeat the preceding argument with v, u (instead of u, w) and deduce that $v > u$ in $\{u > 0\}$.

THEOREM 1.5. *If* w_1, w_2 *are two solutions of* (0.1) *corresponding to* λ_1 *and* λ_2 *respectively, and if* $\lambda_1 \leqslant \lambda_2$, $w_2 \leqslant w_1$ *then either* $w_2 \equiv w_1$ *and* $\lambda_2 = \lambda_1$ *or* $w_2 < w_1$ *on* $\{w_2 > 0\}$.

PROOF. Suppose there exists a point $x_0 \in \Omega$ such that $0 < w_1(x_0) = w_2(x_0)$. If $\lambda_1 < \lambda_2$, then

$$\Delta(w_2 - w_1)(x_0) = (\lambda_2 - \lambda_1)f(w_2(x_0)) > 0$$

which contradicts the fact that $w_2 - w_1$ attains its maximum at x_0.

If $\lambda_1 = \lambda_2$ then $\Delta(w_2 - w_1) = \lambda_2 c(x)(w_2 - w_1)$, where

$$c(x) = \begin{cases} \dfrac{f(w_2) - f(w_1)}{w_2 - w_1} & \text{if } w_2 - w_1 \neq 0, \\ 0 & \text{if } w_2 - w_1 = 0. \end{cases}$$

By Lemma 1.3, $w_2 - w_1 \equiv 0$ in a neighborhood of x_0. Hence the set $\{x;\, w_2 = w_1\}$ is open and, consequently, $w_2 \equiv w_1$.

DEFINITION 1.2. The set $N_u = \{u = 0\} \cap \Omega$ is called the *dead core* of u; this terminology is taken from the reaction-diffusion problem which is modelled by (1.2).

Notice that (1.1) implies that

$$f(t) \text{ is strictly increasing in some interval } 0 < t < \varepsilon_0. \tag{1.9}$$

THEOREM 1.6. *If u, v are two solutions of* (0.1) *with $u \leqslant v$, $u \not\equiv v$, and if N_v is nonempty, then $N_v \subset \operatorname{int} N_u$, i.e.,*

$$\operatorname{dist}(N_v, R^n \setminus N_u) > 0. \tag{1.10}$$

PROOF. Set $G_t = \{u < t\}$ for some small $t > 0$. By Theorem 1.5, $v \geqslant u + \delta$ on ∂G_t, for some $\delta > 0$. Suppose (1.10) is not true. Then for any $\varepsilon > 0$ there is a unit vector e such that the function $v_\varepsilon(x) = v(x + \varepsilon e)$ satisfies

$$v_\varepsilon = 0 < u \quad \text{at some point } x_\varepsilon \in G_t. \tag{1.11}$$

Now, if ε is small enough (depending on δ) then

$$v_\varepsilon \geqslant u + \delta/2 > u \quad \text{on } \partial G_t. \tag{1.12}$$

Also,

$$\Delta u = f(u), \qquad \Delta v_\varepsilon = f(v_\varepsilon) \quad \text{in } G_t, \tag{1.13}$$

and $f(s)$ is monotone increasing in s in the range of $u(x)$ and of $v_\varepsilon(x)$, $x \in G_t$, provided t is sufficiently small at the outset. Thus Corollary 1.2 (with $h_1 \equiv 0$) can be applied to (1.12), (1.13), and we conclude that $v_\varepsilon \geqslant u$ in G_t, which contradicts (1.11).

We shall denote the functional in (1.5) also by $J_f(v)$ in order to emphasize the dependence on f; the corresponding Problem (J) is denoted by (J_f).

THEOREM 1.7. *Suppose $\tilde{f}$ satisfies the same properties as f in* (1.1) *and let $u, \tilde{u}$ be minimizers of (J_f) and $(J_{\tilde{f}})$ respectively. If $\tilde{f}(t) > f(t)$ for all $0 < t < 1$ then $\tilde{u} < u$ in $\{\tilde{u} > 0\}$ and* $\operatorname{dist}(N_u, R^n \setminus N_{\tilde{u}}) > 0$.

PROOF. Denote $J_f(v)$ by $J(v)$ and $J_{\tilde{f}}(v)$ by $\tilde{J}(v)$. Let $\tilde{F}(t) = \int_0^t \tilde{f}(s)\, ds$. Then we have $F(t) - F(s) < \tilde{F}(t) - \tilde{F}(s)$ if $0 \leqslant s < t < 1$. Consequently,

$$\tilde{F}(u \wedge \tilde{u}) + F(u \vee \tilde{u}) < \tilde{F}(\tilde{u}) + F(u) \tag{1.14}$$

at each point x for which $\tilde{u}(x) > u(x)$. If $\tilde{u}(x) \leqslant u(x)$ then equality holds in (1.14). Thus, provided the set $\{\tilde{u} > u\}$ is nonempty we have

$$\int_\Omega \tilde{F}(u \wedge \tilde{u}) + \int_\Omega F(u \vee \tilde{u}) < \int_\Omega \tilde{F}(\tilde{u}) + \int_\Omega F(u)$$

and, consequently, also

$$\tilde{J}(u \wedge \tilde{u}) + J(u \vee \tilde{u}) < J(\tilde{u}) + J(u). \tag{1.15}$$

Since, however, $u \wedge \tilde{u}$ and $u \vee \tilde{u}$ belong to K, we must have $\tilde{J}(u \wedge \tilde{u}) \geqslant \tilde{J}(\tilde{u})$ and $J(u \vee \tilde{u}) \geqslant J(u)$, contradicting (1.15).

We have thus proved that $u \geqslant \tilde{u}$ in Ω. We can now proceed as in Theorems 1.4 and 1.5 in order to deduce that $u > \tilde{u}$ in $\{\tilde{u} > 0\}$ and that $N_u \subset \operatorname{int} N_{\tilde{u}}$.

DEFINITION 1.3. A solution $\bar{u}_\lambda$ ($\underline{u}_\lambda$) of (0.1) is called *maximal* (*minimal*) if for any solution u_λ of (0.1) there holds $\bar{u}_\lambda \geqslant u_\lambda$ (respectively, $\underline{u}_\lambda \leqslant u_\lambda$).

THEOREM 1.8. *For each* $\lambda > 0$ *there exist maximal and minimal solutions of* (0.1).

PROOF. We shall apply a monotone iteration method as in Keller [10]. Choose $M > 0$ such that the function $h(u) = \lambda f(u) - Mu^p$ satisfies

$$h(u) \leqslant 0 \quad \text{if } 0 \leqslant u \leqslant 1, \qquad h'(u) \leqslant 0 \quad \text{if } 0 < u \leqslant 1. \tag{1.16}$$

Introduce the functional

$$H_q(v) = \int_\Omega \left[\tfrac{1}{2} |\nabla v|^2 + \frac{M}{p+1} |v|^{p+1} + h(q(w))\, v \right] dv$$

for a given function $q \in C^1(\bar{\Omega})$, $0 \leqslant q(x) \leqslant 1$, and consider the problem: Find z satisfying

$$\min_{v \in K} H_q(v) = H_q(z), \qquad z \in K. \tag{1.17}$$

Since the functional is lower semicontinuous, there exists a solution $z = z_q$ of (1.17), and since the function

$$A(p, z, x) = \frac{|p|^2}{2} + \frac{M}{p+1} |z|^{p+1} + h(q(x)) z$$

is convex in (p, z), the solution is unique.

The inequality $h \leqslant 0$ implies that $H_q(\max(z, 0)) \leqslant H_q(z)$; hence the minimizer z is nonnegative. Finally, since h is nonincreasing, if $z > 1$ then

$$A_z(p, z, x) = Mz^p + h(q(x)) \geqslant Mz^p + h(1) = Mz^p - M + \lambda f(1) > 0.$$

Therefore $H_q(\min(z, 1)) \leqslant H_q(z)$, so that $z \leqslant 1$. We have thus proved that the minimizer of (1.17) satisfies

$$0 \leqslant z \leqslant 1. \tag{1.18}$$

Clearly also

$$\Delta z - Mz^p = h(q(x)) \quad \text{in } \Omega, \qquad z = 1 \quad \text{on } \partial\Omega \tag{1.19}$$

and $z \in C^{2+\alpha}(\bar{\Omega})$.

We now take $w_0 \equiv 1$ and define inductively w_m as the solution of (1.17) with $q = w_{m-1}$. By (1.18),

$$w_1 \leqslant w_0. \tag{1.20}$$

Assuming inductively that $w_i \leqslant w_{i-1}$ for $i \leqslant m - 1$, we proceed to show that

$$w_m \leqslant w_{m-1}. \tag{1.21}$$

First

$$\Delta(w_m - w_{m-1}) = M(w_m^p - w_{m-1}^p) + h(w_{m-1}) - h(w_{m-2}) \quad \text{in } \Omega,$$
$$w_m - w_{m-1} = 0 \quad \text{on } \partial\Omega.$$

Since h is nonincreasing we get

$$\Delta(w_m - w_{m-1}) \geqslant M(w_m^p - w_{m-1}^p) \quad \text{in } \Omega;$$

hence $w_m - w_{m-1}$ cannot take a negative minimum in Ω, and (1.21) follows.

From (1.20), (1.21) and (1.18) we get

$$0 \leqslant w_m \leqslant w_{m-1} \leqslant 1. \tag{1.22}$$

By elliptic estimates, $|w_m|_{C^{2,\alpha}(\bar\Omega)} \leqslant C$. Hence $w_m \to u$ in $C^2(\bar\Omega)$, $u \in C^{2+\alpha}(\bar\Omega)$, and $\Delta u = Mu^p + h(u) = Mu^p + \lambda f(u) - Mu^p$ in Ω, i.e., u is a solution of (0.1).

We claim that u is a maximal solution. Indeed, if v is any solution of (0.1) then $v \leqslant 1 = w_0$. Writing

$$\Delta(w_{m+1} - v) = M(w_{m+1}^p - v^p) + h(w_m) - h(v)$$

we easily deduce (cf. the proof of (1.21)) that if $w_m \geqslant v$ in Ω then $w_{m+1} \geqslant v$ in Ω. It follows that $u = \lim w_m \geqslant v$.

Similarly, starting with $z_0 = 0$ solve (1.17) successively with $q = z_{m-1}$, $z = z_m$ and obtain a sequence $0 \leqslant z_{m-1} \leqslant z_m \leqslant 1$ with $z = \lim z_m$ a minimal solution.

Added in proof. It has been brought to our attention that Theorem 1.8 is a special case of a result of H. Amann (Indiana Univ. Math. J. **21** (1971), 123–146).

THEOREM 1.9. *If* $\lambda_1 < \lambda_2$ *then*

$$\bar u_{\lambda_2} < \bar u_{\lambda_1} \quad on\ \{\bar u_{\lambda_2} > 0\}, \qquad N_{\bar u_{\lambda_1}} \subset \operatorname{int} N_{\bar u_{\lambda_2}},$$
$$\underline{u}_{\lambda_2} < \underline{u}_{\lambda_1} \quad on\ \{\underline{u}_{\lambda_2} > 0\}, \qquad N_{\underline{u}_{\lambda_1}} \subset \operatorname{int} N_{\underline{u}_{\lambda_2}}.$$

PROOF. Let w_m be the sequence constructed in the preceding proof for $\lambda = \lambda_1$. Then $\bar u_{\lambda_2} \leqslant 1 = w_0$. We can proceed inductively to show that $w_{m+1} - \bar u_{\lambda_2} \geqslant 0$. Indeed, if $w_m - \bar u_{\lambda_2} \geqslant 0$ then, since $\lambda_1 - \lambda_2 < 0$,

$$\begin{aligned}\Delta(w_{m+1} - \bar u_{\lambda_2}) &= M\left(w_{m+1}^p - \bar u_{\lambda_2}^p\right) + h(w_m) - h(\bar u_{\lambda_2}) + (\lambda_1 - \lambda_2) f(\bar u_{\lambda_2}) \\ &\leqslant M\left(w_{m+1}^p - \bar u_{\lambda_2}^p\right),\end{aligned}$$

and the maximum principle yields $w_{m+1} - \bar u_{\lambda_2} \geqslant 0$. It follows that $\bar u_{\lambda_1} \geqslant \bar u_{\lambda_2}$.

Similarly one shows that $\underline{u}_{\lambda_1} \geqslant \underline{u}_{\lambda_2}$, and the rest follows from Theorems 1.5 and 1.6.

THEOREM 1.10. (i) *If* Ω *is a ball* $B_R(0)$ *and* u *a minimal or a maximal solution, then* $u(x) = u(|x|)$;

(ii) *if* u *is a minimal or a maximal solution of*

$$\begin{aligned}\Delta u &= \lambda f(u) \quad in\ \tilde\Omega = \{R_1 < |x| < R_2\}, \\ u &= 1 \quad if\ |x| = R_1, \\ u &= 0 \quad if\ |x| = R_2\end{aligned} \tag{1.23}$$

where $0 < R_1 < R_2$, *then* $u(x) = u(|x|)$.

Notice that the existence of minimal and maximal solutions of (1.23) is established as in Theorem 1.8.

PROOF. If T is an orthogonal transformation and w_m is the sequence which converges to a maximal solution (as in Theorem 1.8) then by uniqueness $w_m(Tx) = w_m(x)$. Hence also $u(Tx) = u(x)$ for the maximal solution, so that $u(x) = u(|x|)$.

THEOREM 1.11. *Let* $\bar{u}$, $\underline{u}$ *be the maximal and minimal solutions of* (0.1) *and* $\bar{v}$, $\underline{v}$ *the maximal and minimal solutions for a ball* $B_R(x^0)$ *in* Ω *and a shell* $S_{R_1,R_2}(y^0) = B_{R_2}(y^0)\backslash B_{R_1}(y^0)$ *with* $B_{R_1}(y^0) \subset R^n\backslash\Omega$ *and zero boundary values on the outer surface. Then*

$$\bar{u} \leqslant \bar{v} \quad \text{in } B_R(x^0), \tag{1.24}$$

$$\underline{v} \leqslant \underline{u} \quad \text{in } S_{R_1,R_2}(y^0) \cap \Omega. \tag{1.25}$$

PROOF. To prove (1.24), we take as usual the sequences $\bar{u}_m$, $\bar{v}_m$ which converge to $\bar{u}$ and $\bar{v}$, respectively, and assume inductively that $\bar{u}_m \leqslant \bar{v}_m$ on $B_R(x^0)$ (trivially $\bar{u}_0 = \bar{v}_0 = 1$). Thus

$$\begin{aligned}\Delta(\bar{u}_{m+1} - \bar{v}_{m+1}) &= M(\bar{u}^p_{m+1} - \bar{v}^p_{m+1}) + h(\bar{u}_m) - h(\bar{v}_m)\\ &\geqslant M(\bar{u}^p_{m+1} - \bar{v}^p_{m+1}) \quad \text{in } B_R(x^0)\end{aligned}$$

and $\bar{u}_{m+1} - \bar{v}_{m+1} \leqslant 0$ on $\partial B_R(x^0)$. It follows that $\bar{u}_{m+1} - \bar{v}_{m+1} \leqslant 0$ in $B_R(x^0)$. Taking $m \to \infty$, (1.24) follows. The proof of (1.25) is similar.

REMARK. Theorem 1.11 can be extended to minimizers as well as to general comparison domains (instead of just balls and shells). However, only Theorem 1.11 will be needed later on.

2. Uniqueness of solutions.

THEOREM 2.1. *If*

$$f'(t) + \frac{f(t)}{1-t} > 0 \qquad (0 < t \leqslant 1) \tag{2.1}$$

then the solution of (1.2) *is unique.*

This theorem is due to Cohen and Laetsch **[5]** whose proof depends on the concept of maximal solutions. We give here another proof which can be extended to general Dirichlet boundary conditions.

PROOF. Let u be a solution of (1.2) and define a function w by $u = g(w)$, where $g(t) = 1 - t^\beta$ ($\beta > 1$). Then $w = (1-u)^{1/\beta}$ for $0 \leqslant u \leqslant 1$. We compute

$$\Delta w + \frac{g''(w)}{g'(w)}|\nabla w|^2 - \frac{f(g(w))}{g'(w)} = 0. \tag{2.2}$$

If

$$\left(\frac{g''(t)}{g'(t)}\right)' \leqslant 0, \quad \left(\frac{f(g(t))}{g'(t)}\right)' \geqslant 0 \tag{2.3}$$

then, by Corollary 1.2, there exists at most one solution w of (2.2) with $w = 0$ on $\partial\Omega$; this gives the assertion of the theorem. To prove (2.3) we easily check that

$$\frac{g''(t)}{g'(t)} = \frac{\beta - 1}{t}, \quad \text{so that } \left(\frac{g''}{g'}\right)' \leqslant 0.$$

The second inequality in (2.3) reduces to $(f(1-t^{\beta})/t^{\beta-1})' \leqslant 0$ or, with $s = 1 - t^{\beta}$, to

$$f'(s) + \frac{\beta - 1}{\beta}\frac{f(s)}{1-s} \geqslant 0.$$

But this is a consequence of (2.1) if β is large enough.

REMARK 2.1. If $f(t)$ is concave then $f(1) - f(u) \leqslant (1-u)f'(u)$; consequently $f'(u) + f(u)/(1-u) > f(1)/(1-u)$. Thus if $f(1) > 0$ then f satisfies (2.1) and thus (1.2) has at most one solution. If $f(1) = 0$ then the solution is not unique, in general. Indeed, if $n = 1$, $\Omega = \{-a < x < a\}$ then (1.2) reduces to

$$\begin{aligned} u'' &= f(u) \quad \text{for } 0 < x < a, \\ u'(0) &= 0, \qquad u(a) = 1. \end{aligned} \tag{2.4}$$

If a is sufficiently large then the solution satisfies: $u(0) = 0$ and $u' = \sqrt{2F(u)}$ and we can construct a concave $f(u)$ such that (1.1) holds with $m = 0$ and $u(0) = 0$, $u(a) = 1$. Another solution is given by $u \equiv 1$.

REMARK 2.2. We can also construct two solutions of (2.4) in case $f(1) > 0$, with $f(t)$ as in (1.1), monotone increasing for $0 < t < \theta_*$ and monotone decreasing for $\theta_* < t < 1$. Indeed, let $\frac{1}{2} < \theta < \theta_1 < 1$ and choose $f(t)$ concave for $0 < t < \theta$, $f(\theta) = 2\varepsilon$, $\varepsilon \leqslant f(t) \leqslant 2\varepsilon$ and $f'(t) \leqslant 0$ for $\theta < t < \theta_1$, $f(t) = \varepsilon$ if $\theta_1 < t < 1$. A solution of (2.4) with $u(0) = 0$ exists provided a is suitably chosen; a depends on ε, but $0 < \bar{\bar{a}} \leqslant a \leqslant \bar{a} < \infty$ where $\bar{a}, \bar{\bar{a}}$ are independent of ε, if ε is small enough. Another solution of (2.4) is given by

$$u(x) = 1 - \frac{\varepsilon a^2}{2} + \frac{\varepsilon x^2}{2}$$

provided ε is sufficiently small so that $1 - \varepsilon a^2/2 > \theta_1$.

REMARK 2.3. If $f(t)$ is strictly concave with $f(1) = 0$, then one can still prove uniqueness for the minimizer. Indeed, if u and v are two minimizers, set $w(t) = tu + (1-t)v$ and suppose $u \geqslant v$. We compute that

$$\frac{d}{dt}J(w(t)) = \int \{f(tu + (1-t)v) - [tf(u) + (1-t)f(v)]\}(u - v) > 0,$$

so that $J(u) > J(v)$, a contradiction.

REMARK 2.4. Theorem 2.1 can be applied to the case

$$f(t) = t^p \exp\left\{-\frac{\gamma}{\beta + 1 - \beta t}\right\} \qquad (\gamma > 0, \beta > 0) \tag{2.5}$$

which arise in reaction-diffusion [**2, 3, 7, 17**]. The condition (2.1) is satisfied for some range of the parameters β, γ.

THEOREM 2.2. *If λ is sufficiently small then there exists a unique solution u of* (0.1) *and $u > 0$ in Ω.*

PROOF. By (1.1),

$$\begin{gathered} f(t) = f_1(t) + f_2(t), \\ f_1(t) \text{ monotone increasing}, \quad |f_2'(t)| \leqslant C. \end{gathered} \tag{2.6}$$

Suppose u, v are two solutions of (0.1). Then

$$\Delta(u-v)-\lambda(f_1(u)-f_1(v))=\lambda(f_2(u)-f_2(v))$$

and $f_1(u)-f_1(v)=-cw$ where $w=u-v$ and $c\leqslant 0$. Since also $|f_2(u)-f_2(v)|\leqslant C|u-v|$, we see that

$$\Delta w+\lambda cw=H,\qquad |H|\leqslant\lambda C|w|\quad\text{in }\Omega,$$
$$w=0\quad\text{in }\partial\Omega.$$

Applying Lemma 1.1. we get $|w|_{L^\infty(\Omega)}\leqslant C\lambda|w|_{L^\infty(\Omega)}$; hence, if $C\lambda<1$ then $w\equiv 0$, which proves uniqueness.

By Lemma 1.1 we also have $u\geqslant 1-C\lambda\sup_\Omega f(u)>0$ if λ is small enough, since $f(u)\leqslant$ const if $0\leqslant u\leqslant 1$.

3. Asymptotic behavior as $\lambda\to\infty$. Let $u=u_\lambda$ be any solution of (0.1) and set $N_\lambda=N_{u_\lambda}$. We shall study the behavior of N_λ as $\lambda\to\infty$.

Set

$$\Omega_\delta=\{x\in\Omega,\operatorname{dist}(x,\partial\Omega)>\delta\},$$
$$B_R(x^0)=\{|x-x^0|<R\},\qquad B_R=B_R(0).$$

THEOREM 3.1. *There exist positive constants γ, c_1, c_2 such that for any solution u_λ*

$$\Omega_{\gamma/\sqrt{\lambda}+c_1/\lambda}\subset N_\lambda\subset\Omega_{\gamma/\sqrt{\lambda}-c_2/\lambda}\tag{3.1}$$

provided λ is large enough.

PROOF. It is easy to establish the crude estimates

$$\Omega_{C/\sqrt{\lambda}}\subset N_\lambda\subset\Omega_{c/\sqrt{\lambda}}\tag{3.2}$$

where C, c are positive constants independent of λ. Indeed, to prove the second part it suffices to show that $|Du_\lambda|\leqslant C\sqrt{\lambda}$. Since the function $w_\lambda(x)=u_\lambda(x/\sqrt{\lambda})$ satisfies $\Delta w_\lambda=f(w_\lambda)$ in $\Omega_\lambda\{x/\sqrt{\lambda}\,;\,x\in\Omega\}$ and $0\leqslant f\leqslant 1$, by elliptic estimates $|Dw_\lambda|\leqslant C$ in Ω_λ, so that, indeed, $|Du_\lambda|\leqslant C\sqrt{\lambda}$.

To prove the first inclusion in (3.2) we proceed as in [7] and consider the function

$$v(x)=A|x-x_0|^{2/(1-p)}\qquad(A>0)$$

where $x_0\in\Omega$, and set $d=\operatorname{dist}(x_0,\partial\Omega)$. Then $\Delta v\leqslant a_0A^{1-p}v^p$ in Ω and $v\geqslant 1$ on $\partial\Omega$ if $\mathrm{Ad}^{2/(1-p)}=1$, where a_0 is a positive constant independent of A. Since $\Delta u_\lambda\geqslant\lambda c(u_\lambda)^p$ $(c>0)$, if $a_0A^{1-p}=\lambda c$ then, by comparison, $u_\lambda\leqslant v$; thus

$$u_\lambda(x_0)\leqslant v(x_0)=0\quad\text{if } d=\left(\frac{a_0}{c\lambda}\right)^{1/2},$$

and the assertion follows.

The system

$$\eta'(s)=\sqrt{2F(\eta)}\quad\text{for } s<0,\qquad \eta(0)=1$$

has a unique solution as long as $\eta(s) > 0$; it determines a unique positive number γ such that $\eta(-\gamma) = 0$. Letting $\zeta(s) = \eta(\gamma + s)$ we have

$$\begin{aligned} &\zeta'(s) = \sqrt{2F(\zeta)} \quad \text{for } 0 < s < \gamma, \\ &\zeta(0) = 0, \quad \zeta(s) > 0 \quad \text{for } 0 < s < \gamma, \\ &\zeta(\gamma) = 1. \end{aligned} \tag{3.3}$$

Let $\gamma \in \partial\Omega$ and let B_R be a ball in Ω with $y \in \partial B_R$. Let U be a maximal solution for B_R. By Theorems 1.10, 1.11, $U = U(r)$, $u_\lambda \leqslant U$: Furthermore, since U is subharmonic and differentiable at $r = 0$, $U'(r) \geqslant 0$.

The function U satisfies: $U'' + (n-1)U'/r = \lambda f(U)$ and the function

$$Z(s) = U\left(R - \frac{\gamma_0}{\sqrt{\lambda}} + \frac{s}{\sqrt{\lambda}}\right) \qquad (\gamma_0 \text{ to be determined})$$

satisfies

$$Z'' + \frac{n-1}{\rho\sqrt{\lambda} + s} Z' = f(Z) \qquad \left(\rho = R - \frac{\gamma_0}{\sqrt{\lambda}}\right). \tag{3.4}$$

Since $U'(r) \geqslant 0$, the support of $Z(s)$ consists of one interval, namely $0 \leqslant s \leqslant \gamma_0$. From (3.2) (applied to U in B_R) we have

$$\gamma_0 \leqslant c, \qquad c \text{ independent of } \lambda. \tag{3.5}$$

Multiplying both sides of (3.4) by $Z'(s)$, we get

$$\frac{1}{2}(Z'^2)' + \frac{(n-1)Z'^2}{\rho\sqrt{\lambda} + s} = (F(Z))'.$$

Hence

$$(Z'^2) + \frac{C}{\sqrt{\lambda}} Z'^2 \geqslant 2(F(Z))', \qquad C > 0,$$

where C is independent of λ, by (3.5). From this we obtain

$$(Z'^2 e^{Cs/\sqrt{\lambda}})' \geqslant 2e^{Cs/\sqrt{\lambda}}(F(Z))'.$$

Integrating and using the relations $Z'(0) = 0$, $F(Z(0)) = 0$, we get

$$\begin{aligned} Z'^2(s) &\geqslant 2e^{-Cs/\sqrt{\lambda}} \int_0^s e^{ct/\sqrt{\lambda}} (F(Z(t))' \, dt \\ &= 2F(Z)(s)) - \frac{C}{\sqrt{\lambda}} \int_0^s e^{-c(s-t)/\sqrt{\lambda}} F(Z(t)) \, dt. \end{aligned}$$

Recalling that $Z'(t) \geqslant 0$ we get $Z'(s) \geqslant (1 - C/\sqrt{\lambda})^{1/2}\sqrt{2F(Z(s))}$. On the other hand, the function

$$\tilde{\zeta}(s) = \zeta\left(s\left(1 - \frac{C}{\sqrt{\lambda}}\right)^{1/2}\right) \qquad (\zeta \text{ as in } (3.3))$$

satisfies $\tilde{\zeta}'(s) = (1 - C/\sqrt{\lambda})^{1/2}\sqrt{2F(\tilde{\zeta}(s))}$. By comparison we then deduce that

$$Z(s) \geqslant \zeta\left(s\left(1 - \frac{C}{\sqrt{\lambda}}\right)^{1/2}\right). \tag{3.6}$$

Since $U(R) = 1$ means $Z(\gamma_0) = 1$, we conclude that $\gamma_0(1 - C/\sqrt{\lambda})^{1/2} \leqslant \gamma$. Recalling that $u_\lambda \leqslant U$ we deduce that

$$N_\lambda \supset \Omega_{R-\gamma_0/\sqrt{\lambda}} \supset B_{R-\gamma/\sqrt{\lambda}-c/\lambda}.$$

Thus the first part of (3.1) follows.

To prove the second part of (3.1), we introduce the following shell $\tilde{\Omega}$: The inner boundary is a sphere S_0 in $R^n \setminus \tilde{\Omega}$, with radius R, which contains a point y on $\partial\Omega$, and the outer boundary is a sphere S_1 containing $\tilde{\Omega}$.

Let V be the minimal solution for the shell $\tilde{\Omega}$ (cf. (1.23)). By Theorems 1.10, 1.11, $V = V(r)$, $V \leqslant u_\lambda$; further, $V'(r) \leqslant 0$ since V is subharmonic. The function $\bar{Z}(s) = V(R + \bar{\gamma}/\sqrt{\lambda} - s/\sqrt{\lambda})$ can be analyzed similarly to $Z(s)$. Thus we find that

$$\bar{Z}(s) \leqslant \zeta\left(s\left(1 + \frac{C}{\sqrt{\lambda}}\right)^{1/2}\right), \qquad \bar{\gamma}\left(1 + \frac{C}{\sqrt{\lambda}}\right)^{1/2} \geqslant \gamma, \tag{3.7}$$

and this yields the second part of (3.1), if we let y vary over $\partial\Omega$ in the above construction.

DEFINITION 3.1. The set $\Gamma_\lambda = \partial\{u_\lambda > 0\} \cap \Omega$ is called the *free boundary* of u_λ.

LEMMA 3.2. *If v is any solution of* (1.2) *then*

$$|\nabla v|^2 \leqslant 2F(v) + av \quad \text{in } \Omega \tag{3.8}$$

where $a = 2(n-1)\max_{x\in\partial\Omega}[|\nabla v(x)|\cdot K(x)]^-$ and $K(x)$ is the mean curvature of $\partial\Omega$ at x ($K \geqslant 0$ if Ω is convex).

For a proof see [**12, 13, 14, 17, 18**].

THEOREM 3.3 (a) *Suppose u is a solution of $\Delta u = f(u)$ in $B_1(0)$ with f satisfying* (1.1) *and $0 \in \partial\{u > 0\}$. There exist positive constants, α, $\beta \geqslant 1$, σ_0, τ_0 and C depending on n and the function f such that $\{x;\, x_n > \sigma\rho\} \cap B_\rho(0) \subset \{u = 0\}$ with $\sigma \leqslant \sigma_0$ and $\rho \leqslant \tau_0\sigma^\beta$ implies that $B_{\rho/4}(0) \cap \partial\{u > 0\}$ is a graph of a $C^{1+\alpha}$ function g in the direction e_n; moreover, with $x' = (x_1, \ldots, x_{n-1})$ there holds*

$$\begin{aligned} &|\nabla g(x')| \leqslant C\sigma \quad \text{for } |x'| \leqslant \frac{\rho}{8}, \\ &|\nabla g(x_1') - \nabla g(x_2')| \leqslant C\sigma\left|\frac{x_1' - x_2'}{\rho}\right|^\alpha \quad \text{for } |x_1'|, |x_2'| \leqslant \frac{\rho}{8}. \end{aligned} \tag{3.9}$$

(b) *If $n = 2$ and*

$$\limsup_{r\to 0} \frac{|\{u = 0\} \cap B_r(0)|}{|B_r(0)|} > 0$$

where $|E| =$ n-dimensional measure of E, then there exists a $\rho > 0$ such that $B_\rho(0) \cap \partial\{u > 0\}$ is a $C^{1+\alpha}$ graph.

For a proof see [**1**].

In the next theorem we improve Theorem 3.1. Let $x = h(t)$ $(t = (t_1, \ldots, t_{n-1}))$ be a $C^{1+\alpha}$ local parametrization of $\partial\Omega$ in a neighborhood of a point $x^0 = h(0)$, and denote by $\nu(t)$ the inner normal to $\partial\Omega$ at $h(t)$.

THEOREM 3.4. *There exist positive constants σ_1, σ_2 such that for any λ sufficiently large and for any solution u_λ of* (0.1), *Γ_λ is a $C^{1+\delta_1}$ surface; furthermore, in terms of local coordinates $x = h(t)$, $\Gamma_\lambda \cap B_R(h(0))$ can be represented (for small enough R) in the form $x = h(t) + k(t, \lambda)\nu(t)$ with $k(t, \lambda)$ satisfying, as a function of t,*

$$(3.10) \qquad |D_t k| \leqslant C/\lambda^{\delta_1} \qquad \left(t \in \bar{B}_R(0)\right),$$

$$(3.11) \qquad |k|_{C^{1+\delta_2}(\bar{B}_R(0))} \leqslant C.$$

Here $\bar{B}_R(0)$ is a ball of radius R in the $(n-1)$-dimensional space. Notice that, by Theorem 3.1, $|k(t, \lambda) - \gamma/\sqrt{\lambda}| \leqslant C/\lambda$.

PROOF. Consider first the case where

$$(3.12) \qquad h(t) = \left(t, \bar{h}(t)\right), \qquad \bar{h}(0) = 0, \qquad \nabla \bar{h}(0) = 0.$$

We introduce the mapping

$$(t, s) \to x = (x', x_n) = \Phi(t, s) \equiv h(t) + s\nu(t).$$

Since $\partial\Omega$ is in $C^{2+\alpha_0}$, this is a $C^{1+\alpha_0}$ diffeomorphism from $\{(t, s); |t| < 4R, 0 \leqslant s \leqslant \bar{c}\}$ onto some $\bar{\Omega}$-neighborhood V of

$$\partial\Omega \cap \left\{x = (x', x_n); x' \in \bar{B}_{3R}(0), x_n \in (-c, c)\right\}$$

for suitable $\bar{c}$, c.

From Theorem 3.2 it follows that

$$\Gamma_\lambda \cap U \subset \left\{\Phi(t, s); |t| < 4R, \left|s - \frac{\gamma}{\sqrt{\lambda}}\right| \leqslant \frac{C}{\lambda}\right\}.$$

Since $\bar{h}(0) = 0$ and $\nabla\bar{h}(0) = 0$, $x = (x', x_n) = (t, s) + O((|t| + |s|)^{1+\alpha_0})$, i.e.,

$$(3.13) \qquad x' = t + O\left((|t| + |s|)^{1+\alpha_0}\right), \quad x_n = s + O\left((|t| + |s|)^{1+\alpha_0}\right).$$

Consequently, $\Gamma_\lambda \cap [\bar{B}_{\gamma/2\sqrt{\lambda}}(0) \times (0 \leqslant s \leqslant 2\gamma/\sqrt{\lambda})]$ is contained in

$$\left\{x; |x'| < \frac{\gamma}{\sqrt{\lambda}}, \frac{\gamma}{\sqrt{\lambda}} - C\lambda^{-(1+\alpha_0)/2} < x_n < \frac{\gamma}{\sqrt{\lambda}} + C\lambda^{-(1+\alpha_0)/2}\right\}.$$

Let $\Omega_\lambda = \{x; x/\sqrt{\lambda} \in \Omega\}$, $v(x) = u(x/\sqrt{\lambda})$ $(x \in \Omega_\lambda)$ and denote the free boundary of v by $\tilde{\Gamma}_\lambda$. Thus $\Delta v = f(v)$ in Ω_λ with $\{\bar{B}_\gamma(0) \times (\gamma/2 < x_n < 3\gamma/2)\} \subset \Omega_\lambda$. Moreover

$$\tilde{\Gamma}_\gamma \cap \left\{x; |x'| < \frac{\gamma}{2}, -2\gamma < x_n < 2\gamma\right\}$$

is contained in $\{x; |x'| < \gamma, |x_n - \gamma| \leqslant C\lambda^{-\alpha_0/2}\}$.

Let $x' \in \bar{B}_{\gamma/4}(0)$. Then there is clearly at least one value $\tilde{x}_n$ with $|\tilde{x}_n - \gamma| < C\lambda^{-\alpha_0/2}$ such that $(x', \tilde{x}_n) \in \tilde{\Gamma}_\lambda$.

Now set $\sigma = \lambda^{-\alpha_0/(8\beta)}$ and $\rho = c\sigma^{\beta}$ where β is as in Theorem 3.3. We deduce that, for some $\lambda_0 > 0$, if $\lambda > \lambda_0$ then Theorem 3.3 can be applied about each point $(x', \tilde{x}_n)$. We conclude that $\tilde{x}_n$ is unique, $\tilde{x}_n = g(x')$,

$$\tilde{\Gamma}_\lambda \cap \left\{\bar{B}_{\gamma/4}(0) \times (-2\gamma, 2\gamma)\right\} = \left\{x;\, x_n = g(x'), |x'| < \frac{\gamma}{4}\right\},$$

and

$$|Dg(x')| \leqslant C\sigma \quad \text{for } x' \in \bar{B}_{\gamma/4}(0), \tag{3.14}$$

$$|Dg(x_1') - Dg(x_2')| \leqslant C\sigma^{1-\alpha\beta}|x_1' - x_2'|^{\alpha} \quad \text{if } |x_1' - x_2'| \leqslant c\sigma^{\beta}. \tag{3.15}$$

Using (3.9) we see that the last estimate remains valid if α is made smaller. Thus we may assume that $\alpha\beta < \frac{1}{2}$.

If $x_1', x_2' \in \bar{B}_{\gamma/4}(0)$ with $|x_1' - x_2'| > c\sigma^{\beta}$, then, by (3.14),

$$|Dg(x_1') - Dg(x_2')| \leqslant 2c\sigma^{1/2}|x_1' - x_2'|^{\alpha}.$$

Combining this with (3.15) and (3.14), we obtain

$$|g(x') - \gamma|_{C^{1+\alpha}(\bar{B}_{\gamma/4}(0))} \leqslant c\sigma^{1/2} = C\lambda^{-\alpha_0/16\beta}.$$

The corresponding portion of Γ_λ is given by $x_n = z(x') = \lambda^{-1/2}g(\lambda^{1/2}x')$, so that

$$|Dz(x')| \leqslant c\sigma = C\lambda^{-\delta_1} \quad \text{for } x' \in \bar{B}_{\gamma/4\sqrt{\lambda}}(0), \tag{3.16}$$

$$\left|z(x') - \frac{\gamma}{\sqrt{\lambda}}\right|_{C^{1+\alpha}(\bar{B}_{\gamma/4\sqrt{\lambda}}(0))} \leqslant C\sigma^{1/2}\lambda^{\alpha/2}. \tag{3.17}$$

Decreasing α again so that $8\beta\alpha \leqslant \alpha_0$, we obtain

$$|z(x')|_{C^{1+\alpha}(\bar{B}_{\gamma/4\sqrt{\lambda}}(0))} \leqslant C. \tag{3.18}$$

We now express z in terms of t. Write $\nu(t) = (\bar{\nu}(t), \nu_n(t))$ with $\bar{\nu}(t) = (\nu_1(t), \ldots, \nu_{n-1}(t))$. We claim that for $t \in \bar{B}_{\gamma/(8\sqrt{\lambda})}(0)$ the function $k(t, \lambda)$ is well defined. Indeed, if $(x_i', z(x_i')) = h(t) + s_i\nu(t)$ $(i = 1, 2)$ for $s_1 \neq s_2$ then

$$\frac{|z(x_2') - z(x_1')|}{|x_2' - x_1'|} = \frac{|\nu_n(t)|}{|\bar{\nu}(t)|}.$$

Since $\nu_n(t) \approx 1$ and $|\bar{\nu}(t)| \ll 1$, this contradicts (3.16).

We can now write

$$(x', z(x')) = \left(t + k(t, \lambda)\bar{\nu}(t), \bar{h}(t) + k(t, \lambda)\nu_n(t)\right),$$

so that

$$x' = t + k(t, \lambda)\bar{\nu}(t), \tag{3.19}$$

$$z(x') = \bar{h}(t) + k(t, \lambda)\nu_n(t). \tag{3.20}$$

Since $\nu_n(0) = 1$ we can substitute $k(t, \lambda)$ from (3.20) into (3.19) and thus deduce that

$$x' = t + \frac{z(x') - \bar{h}(t)}{\nu_n(t)}\bar{\nu}(t)$$

from which we conclude (since $\bar{\nu}(0) = 0$) that $x' = \phi(t)$ with $\phi \in c^{1+\alpha}$. Using this in (3.20) we can estimate the $C^{1+\alpha}$ norm of $k(t, \lambda)$, and thus deduce that

$$(3.21) \qquad |k(t, \lambda)|_{C^{1+\alpha}(\bar{B}_{\theta\gamma/\sqrt{\lambda}}(0))} \leq C \qquad (\theta > 0).$$

Moreover, using (3.16),

$$(3.22) \qquad |D_t k(t, \lambda)|_0 \leq C\lambda^{-\delta_1} \quad \text{for } t \in \bar{B}_{\theta\gamma/\sqrt{\lambda}}(0).$$

In deriving (3.21) we have assumed that (3.12) holds. The first assumption in (3.12) is not restrictive since it can be achieved by renaming the variables x_i as t_j. Next we can perform an affine transformation $t \to \tau$ in order to satisfy the restrictions on $\bar{h}$ in the new coordinates. This affine transformation is dilatation in the tangential variables but preserves distances along the normals (up to an error term). After deriving (3.21) for $k = k(\tau, \lambda)$ we return to the original coordinates t.

Notice that (3.10) is a consequence of (3.16).

Finally, to prove (3.11) it suffices (in view of (3.22)) to estimate

$$|k_t(t, \lambda) - k_t(t_0, \lambda)| \quad \text{when } |t - t_0| > C\lambda^{-1/2}.$$

Using (3.10) this expression is estimated by $C\lambda^{-\delta_1} \leq C|t - t_0|^{\delta}$ with suitable $\delta > 0$. We can take δ_2 in (3.11) as the minimum of α in (3.21) and δ above.

4. Domains whose graphs have nonnegative mean curvature. For any $x \in \partial\Omega$ denote by $K(x)$ the mean curvature of $\partial\Omega$ at x. In this section we assume that

$$(4.1) \qquad K(x) \geq 0 \quad \text{for all } x \in \partial\Omega;$$

this is true, for instance, if Ω is convex. From Lemma 3.2 it then follows that

$$(4.2) \qquad |\nabla u|^2 \leq 2F(u) \quad \text{in } \Omega,$$

where u is any solution of (1.2).

The function $g(u) = \int_0^u ds/\sqrt{2F(s)}$ will play a fundamental role in the sequence. Setting $w_i = \partial w/\partial x_i$, $w_{ij} = \partial^2 w/\partial x_i\,\partial x_j$ etc., we compute, in $\{u > 0\}$,

$$(g(u(x)))_i = g'(u)u_i, \quad (g(u(x)))_{ii} = g''(u)u_i^2 + g'(u)f(u),$$

$$g'(u) = \frac{1}{\sqrt{2F(u)}}, \quad g''(u) = -f(u)(2F(u))^{-3/2}.$$

Hence

$$(4.3) \qquad \Delta g(u) = l(u)\frac{1 - |\nabla g(u)|^2}{g(u)}$$

where

$$(4.4) \qquad l(u) = \frac{f(u)g(u)}{\sqrt{2F(u)}} = \frac{f(u)}{\sqrt{2F(u)}} \int_0^u \frac{ds}{\sqrt{2F(s)}}.$$

Notice that $l(t)$ is a positive C^1 function of t away from $t = 0$, whereas near $t = 0$ $l(t) \sim \text{const} > 0$, since $f(u) \sim u^p$.

By (4.2),

$$(4.5) \qquad |\nabla g(u)| \leq 1,$$

and thus

$$\Delta g(u) \geqslant 0 \quad \text{in } \{u>0\}. \tag{4.6}$$

We shall need the Hausdorff measure estimate of [**1**] which is valid for any solution u of (1.2):

$$\Delta g(u) = d\lambda + I_{\{u>0\}} l(u) \frac{1 - |\nabla g(u)|^2}{g(u)} \tag{4.7}$$

where $d\lambda$ is absolutely continuous with respect to $dH^{n-1} \llcorner \Gamma$ (Γ is the free boundary); more precisely,

$$\begin{gathered} d\lambda = dH^{n-1} \llcorner \Gamma_{\text{red}} + \theta(x)\, dH^{n-1} \llcorner \Gamma_{\text{sing}}, \\ I_{\{u>0\}} l(u) \frac{1 - |\nabla g(u)|^2}{g(u)} \in L^1(\Omega), \qquad 0 \leqslant \theta(x) \leqslant \text{const}, \end{gathered} \tag{4.8}$$

where I_A denotes the characteristic function of a set A, and "$\llcorner$" means "restriction to."

Observation. If Γ is smooth and $x_0 \in \Gamma$ then by Green's formula,

$$\begin{aligned} \int_{\{u>0\} \cap B_R(x_0)} \Delta g(u) &= \int_{\Gamma \cap B_R(x_0)} \nabla g(u) \cdot \nu \, dH^{n-1} \\ &\quad + \int_{\{u>0\} \cap \partial B_R(x_0)} \nabla g(u) \cdot \nu \, dH^{n-1}. \end{aligned}$$

Using (4.5), (4.6) and the relation $|\nabla g(u(x))| \to 1$ if $\operatorname{dist}(x, \Gamma) \to 0$ (which is implicit in (4.8)), we obtain

$$H^{n-1}(\Gamma \cap B_R(x_0)) \leqslant H^{n-1}[\{u>0\} \cap \partial B_R(x_0)].$$

This estimate is equivalent to Γ having nonnegative curvature and, if $n = 2$, to Γ being convex. In this and in the next section we shall make this observation rigorous, exploiting (4.8) and some elementary facts from geometric measure theory, as well as Theorem 3.4 (for proving that Γ is convex if $n = 2$).

LEMMA 4.1. *Let G be any subdomain of Ω with piecewise smooth boundary $\partial G \subset \Omega$ and with $H^{n-1}(\Gamma \cap \partial G) = 0$. Then*

$$\int_G \Delta g(u) = \int_{\partial G \cap \{u>0\}} \nabla g(u) \cdot \nu \, dH^{n-1}. \tag{4.9}$$

PROOF. This is just Green's formula for a function $g(u(x))$ whose Laplacian is a measure; we shall establish it by approximation.

Let $g(u)_\varepsilon$ be a mollification of $g(u(x))$. Then, since Δg is a measure,

$$\Delta(g(u)_\varepsilon) \to \Delta g(u) \quad \text{as measures.} \tag{4.10}$$

Recalling that $(\Delta g(u))(\partial G) = 0$ (by (4.7), (4.8), and the assumption $H^{n-1}(\Gamma \cap \partial G) = 0$), we deduce from (4.10) that

$$\int_G \Delta(g(u)_\varepsilon) \to \int_G \Delta g(u). \tag{4.11}$$

By Green's formula for smooth functions we have

$$\int_G \Delta(g(u)_\varepsilon) = \int_{\partial G} \nabla(g(u)_\varepsilon) \cdot \nu \, dH^{n-1}. \tag{4.12}$$

For any small $\delta > 0$, let K be any open neighborhood of $\Gamma \cap \partial G$ with $\int_{K \cap \partial G} dH^{n-1} < \delta$. On $(\partial G \setminus K) \cap \{u > 0\}$ we have $\nabla(g(u)_\varepsilon) \to \nabla g(u)$ uniformly as $\varepsilon \to 0$. On the other hand,

$$(\partial G \setminus K) \cap \{u = 0\} \text{ is compactly contained in } \operatorname{int}\{u = 0\};$$

hence, if ε is small enough,

$$\nabla(g(u)_\varepsilon) = 0 \quad \text{on } (\partial G \setminus K) \cap \{u = 0\}.$$

From (4.5) we deduce that $|\nabla(g(u)_\varepsilon)| \leqslant 1$ and, therefore, if ε is small enough,

$$\left| \int_{K \cap \partial G} \nabla(g(u)_\varepsilon) \cdot \nu \, dH^{n-1} \right| < \delta.$$

We now break the integral on the right-hand side of (4.12) into $\partial G \cap K$ and $\partial G \setminus K$ and use the preceding remarks; we obtain, letting $\varepsilon \to 0$ and then $\delta \to 0$,

$$\int_{\partial G} \nabla(g(u)_\varepsilon) \cdot \nu \, dH^{n-1} \to \int_{\partial G \cap \{u>0\}} \nabla g(u) \cdot dH^{n-1}. \tag{4.13}$$

Taking $\varepsilon \to 0$ in (4.12) and using (4.11), (4.13), the assertion (4.9) follows.

Combining (4.9) with (4.7), (4.8) we have

COROLLARY 4.2. *If G is as in Lemma* 4.1, *then*

$$\begin{aligned} &\int_{G \cap \Gamma_{\text{red}}} dH^{n-1} + \int_{G \cap \Gamma_{\text{sing}}} \theta \, dH^{n-1} \\ &\qquad = -\int_{G \cap \{u>0\}} \Delta g(u) \, dx + \int_{\partial G \cap \{u>0\}} \nabla g(u) \cdot \nu \, dH^{n-1}. \end{aligned} \tag{4.14}$$

THEOREM 4.3. *Every C^2 portion of Γ has nonnegative mean curvature.*

PROOF. Suppose a smooth portion of Γ is given by $x_n = h(x')$, $x' = (x_1, \ldots, x_{n-1})$ varies in a ball $\bar{B}_{R'}$, and $u > 0$ if $x_n < h(x')$. Since

$$\Delta g(u) \geqslant 0 \quad \text{in } \{u > 0\}, \qquad |\nabla g(u)| \leqslant 1, \tag{4.15}$$

(4.14) gives

$$\int_{G \cap \Gamma} dH^{n-1} \leqslant \int_{\partial G \cap \{u>0\}} dH^{n-1}. \tag{4.16}$$

Take $\partial G \cap \{x_n < h(x')\}$ to be $x_n = h(x') - \varepsilon\zeta(x')$, $\zeta \geqslant 0$, $\zeta \in C_0^2(\bar{B}_R)$, $\varepsilon > 0$, where $\bar{B}_R$ is the ball $\{|x'| < R\}$. Then (4.16) yields

$$\int_{\bar{B}_R} \sqrt{1 + |\nabla h|^2} \leqslant \int_{\bar{B}_R} \sqrt{1 + |\nabla h - \varepsilon \nabla \zeta|^2}.$$

From this we deduce that $\nabla \cdot \nabla h / \sqrt{1 + |\nabla h|^2} \geqslant 0$, and the assertion follows.

Let P be an open half space with $H^{n-1}(\partial P \cap \Gamma) = 0$ and denote by $\bar{\nu}$ the outward normal along ∂P. Let G be a convex domain in P with piecewise smooth boundary, such that $\partial G \cap \partial P \neq \phi$, $u = 0$ in an open neighborhood of $\partial G \cap P$.

LEMMA 4.4. *Under the foregoing assumptions,*

$$\int_{\{u>0\}\cap\partial G\cap\partial P} dH^{n-1} \leqslant \int_{\Gamma_{\text{red}}\cap G} dH^{n-1} \tag{4.17}$$

with strict inequality if $L^n(\{u > 0\} \cap G) > 0$.

PROOF. By Green's formula [**8**, 4.5.6]

$$0 = \int_{\{u>0\}\cap G} \operatorname{div} \bar{\nu}\, dx = \int_{\Gamma_{\text{red}}\cap G} \bar{\nu} \cdot \nu \, dH^{n-1} + \int_{\{u>0\}\cap\partial G\cap\partial P} \bar{\nu} \cdot \nu.$$

Hence

$$\int_{\{u>0\}\cap\partial G\cap\partial P} dH^{n-1} = -\bar{\nu} \cdot \int_{\Gamma_{\text{red}}\cap G} \nu \, dH^{n-1} \equiv -\bar{\nu} \cdot \nu^*.$$

This gives (4.17). Further, if $L^n(\{u > 0\} \cap G) > 0$, then by an isoperimetric inequality [**9**] $\nu^* \neq 0$ and then the normal at each reduced boundary point is in the direction $\bar{\nu}$. It follows by [**9**, Theorem 4.8] that $\Gamma \cap G$ is a plane normal to $\bar{\nu}$, which is impossible.

5. Two-dimensional convex domains. In this section we shall study the dead core N_u of a solution u, assuming that

$$\Omega \text{ is a two-dimensional convex domain.} \tag{5.1}$$

By a component of N_u we mean a maximal connected subset of N_u; it is necessarily a closed set.

THEOREM 5.1. *If T is a component of N_u with nonempty interior, then T is a closed convex domain with $C^{1+\beta}$ boundary, and*

$$\operatorname{dist}(T, N_u \setminus T) > 0. \tag{5.2}$$

PROOF. Let O be an interior point of T and let l_1, l_2 be rays initiating at O and forming an angle $< \pi$. These rays intersect ∂T for the first time, say, at P_1, P_2. We form the triangle $G = \{P_1, P_2, O\}$ and claim that

$$\{u > 0\} \cap G = \varnothing. \tag{5.3}$$

Indeed, otherwise Lemma 4.4 gives (assuming first that $H^{n-1}(\overline{P_1P_2} \cap \Gamma) = 0$)

$$\int_{\overline{P_1P_2}\cap\{u>0\}} dH^1 < \int_{\Gamma_{\text{red}}\cap G} dH^1.$$

On the other hand, from Corollary 4.2 and (4.15), we obtain $\int_{G\cap\Gamma_{\text{red}}} dH^1 \leqslant \int_{\partial G\cap\{u>0\}} dH^1$, a contradiction.

If $H^{n-1}(\overline{P_1P_2} \cap \Gamma) > 0$ then since $H^{n-1}_{\text{loc}}(\Gamma) < \infty$ we can find $\tilde{P}_1, \tilde{P}_2$ with $\tilde{P}_i \in \overline{OP_i}$ and $|\tilde{P}_i - P_i|$ arbitrarily small so that $H^{n-1}(\overline{\tilde{P}_1\tilde{P}_2} \cap \Gamma) = 0$ and $\{\tilde{P}_1, \tilde{P}_2, O\}$ still violates (5.3) (since $\{u > 0\}$ is open). The previous argument can then be applied to $\{\tilde{P}_1, \tilde{P}_2, O\}$ in order to derive a contradiction.

Having proved (5.3), we now denote by T_0 the union of segments $\overline{OP_1}$ when l_1 varies over all possible directions. From (5.3) it follows that T_0 is convex; in particular, ∂T_0 is Lipschitz continuous. Since $n = 2$ we can apply Theorem 3.3(b). It then follows that ∂T_0 is in $C^{1+\beta}$ and $u > 0$ in some $(\Omega \setminus T_0)$-neighborhood of ∂T_0. Hence $T = T_0$ and (5.2) holds.

COROLLARY 5.2. *If λ is sufficiently large then N_λ is a closed convex domain with $C^{1+\beta}$ boundary.*

Indeed, by Theorem 3.4, N_λ is a closed domain with $C^{1+\beta}$ boundary; now apply Theorem 5.1.

THEOREM 5.3. *Let* (2.1) *and* (5.1) *hold. Then the null set N_u of the solution of* (1.2) *is either a closed convex domain with $C^{1+\beta}$ boundary, or a single point, or empty.*

PROOF. The proof is by continuity with respect to the parameter λ. For any λ there is a unique solution u_λ (Theorem 2.1) and, by Theorem 1.7, u_λ increases and

$$N_\lambda \text{ decreases as } \lambda \text{ decreases.} \tag{5.4}$$

Further, if $\lambda_n \to \lambda$ then $u_{\lambda_n}(x) \to u_\lambda(x)$ uniformly with respect to $x \in \Omega$.

By Corollary 5.2,

$$\begin{aligned}&N_\lambda \text{ is a closed convex domain with } C^{1+\beta} \text{ boundary,}\\ &\text{provided } \lambda \text{ is large enough.}\end{aligned} \tag{5.5}$$

By (5.4) and Theorem 2.2 there is a number $\lambda_* > 0$ such that

$$\operatorname{int} N_\lambda \neq \varnothing \quad \text{if } \lambda > \lambda_*, \qquad \operatorname{int} N_\lambda = \varnothing \quad \text{if } \lambda < \lambda_*. \tag{5.6}$$

The continuity $\lambda \to u_\lambda(x)$ implies that $N_\lambda \supset \limsup_{\lambda_n \to \lambda} N_{\lambda_n}$. Recalling (5.4) we then deduce that, whenever $\lambda_n \downarrow \lambda$,

$$N_\lambda = \lim_{\lambda_n \downarrow \lambda} N_{\lambda_n} = \bigcap_{\lambda_n} N_{\lambda_n}; \tag{5.7}$$

notice that if each N_{λ_n} is a closed convex set then the same is true of $\bigcap N_{\lambda_n}$, i.e., N_λ is also a closed convex set.

We next prove that

$$\begin{aligned}&\text{if } N_{\lambda_0} \text{ is a closed convex domain then } N_\lambda \text{ is also a closed}\\ &\text{convex domain for all } \lambda_0 - \eta < \lambda < \lambda_0, \text{ provided } \eta \text{ is small}\\ &\text{enough.}\end{aligned} \tag{5.8}$$

Once this is proved, we conclude, upon recalling (5.5) and (5.7), (5.6) that N_λ is a closed convex domain with $C^{1+\beta}$ boundary if and only if $\lambda > \lambda_*$.

To prove (5.8), let $N_{\lambda_0,\delta} = \{x \in N_{\lambda_0}; \operatorname{dist}(x, \partial N_{\lambda_0}) > \delta\}$. If $u_\lambda(x^0) > 0$ for some $x^0 \in N_{\lambda_0}$, then, by nondegeneracy **[15, 16]** $\sup_{B_\delta(x^0)} u_\lambda \geq c\delta^{2/(1-p)}$ provided $\delta \leq r_0$, where r_0, c are small positive constants independent of λ, x^0. Hence, if $\lambda_0 - \eta < \lambda < \lambda_0$ and η is small enough depending on δ, say $\eta \leq \eta(\delta)$, then

$$\sup_{B_\delta(x^0)} u_{\lambda_0} \geq \frac{c}{2}\delta^{2/(1-p)} > 0,$$

and consequently $x^0 \notin N_{\lambda_0,\delta}$. We have thus proved that $N_\lambda \supset N_{\lambda_0,\delta}$ if $\lambda_0 - \eta(\delta) < \lambda < \lambda_0$. This implies that Γ_λ lies in the region $N_{\lambda_0} \backslash N_{\lambda_0,\delta}$ of "width" δ and, consequently (recalling that ∂N_{λ_0} is in $C^{1+\beta}$, by Theorem 3.3), the flatness condition holds along Γ_λ provided δ is small enough. Consequently, by Theorem 3.3, Γ_λ is a locally $C^{1+\beta}$ graph that can be parametrized by Γ_{λ_0} and, then, by Theorem 5.1 N_λ is a closed convex domain. We have thus completed the proof of (5.8).

It remains to show that N_{λ_*} consists of a single point; Theorem 1.6 then implies that $N_\lambda = \varnothing$ if $\lambda < \lambda_*$. Since $N_{\lambda_*} = \bigcap_{\lambda > \lambda_*} N_\lambda$, N_{λ_*} is closed convex set. Thus it consists either of a single point or of an interval I. We shall assume the latter and derive a contradiction. For simplicity we take $I = \{(x_1, 0); -a < x_1 < a\}$. Set

$$\tilde{u} = \begin{cases} u & \text{in } B_a^+ = B_a \cap \{x_2 \geqslant 0\}, \\ 0 & \text{in } B_a^-, \end{cases}$$

and notice that $\tilde{u} \in C^2$. The free boundary of $\tilde{u}$ is I. By [**1**], the flatness of the free boundary implies that $g(\tilde{u})$ is in $C^{1+\beta}$ in B_a^+, and $|\nabla g(\tilde{u})| = 1$ on $x_2 = 0+$.

Consider now the subharmonic function $w = g(\tilde{u}) - x_2$ in B_a^+. Since $w(x_1, 0) = 0$ and $|\nabla g| \leqslant 1$, $w \leqslant 0$ in B_a^+. Hence, by the maximum principle, either (i) $w \equiv 0$ or (ii) $\partial w(x_1, 0)/\partial x_2 < 0$. The second case implies $\partial g(\tilde{u})/\partial x_2 < 1$ at $(x_1, 0)$, a contradiction; hence $g(\tilde{u}) - x_2 \equiv 0$ in B_a^+.

In $\Omega_+ = \Omega \cap \{x_2 > 0\}$ we can write (4.3) in the form $\Delta g = k(g)(1 - |\nabla g|^2)$. Since also $\Delta x_2 = 0 = k(x_2)(1 - |\nabla x_2|^2)$, we can write $\Delta w = A \cdot \nabla w + ew$ in Ω_+, with smooth coefficients A, e. Hence, by unique continuation, $w \equiv 0$ in Ω_+, which is a contradiction since $w = g(1) - x_2$ on $\partial\Omega \cap \{x_2 > 0\}$.

REMARK 5.1. Consider $\Delta w = h(w)$ in $\Omega \backslash K$, $w = 0$ on $\partial\Omega$, $w = 1$ on ∂K, where Ω and K are convex domains in R^n and $\bar{K} \subset \Omega$. Caffarelli and Spruck [**4**] proved that if

$$h(t) \geqslant 0, \qquad h'(t) \geqslant 0 \quad \forall t, h(0) = 0,$$

then the sets $\bar{K} \cup \{w > c\}$ are convex for all $c \in (0, 1)$. If we set $w = 1 - u$ in (1.2), then (1.2) reduces to

$$\Delta w = h(w) \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega,$$

with $w = 1$ in the dead core, where $h(t) = -f(1 - t)$. The assumptions on f made in Theorem 5.1 are, in terms of h:

$$h(t) < 0, \qquad h'(t) + |h(t)|/t > 0 \quad \text{for } 0 \leqslant t < 1,$$
$$h(t) \sim -c(1 - t)^p \quad \text{as } t \uparrow 1 \ (c > 0), \qquad h(t) = 0 \quad \text{if } t > 1.$$

The basic difference in the respective assumptions on h is in the signature of h.

REMARK 5.2. Denote by Σ the set of λ's for which the variational problem $(J_{\lambda f})$ does not have a unique solution. We claim that

$$\Sigma \text{ is countable.} \tag{5.9}$$

Indeed, for any large $A > 0$ choose $x^0 \in \Omega$, x^0 near $\partial\Omega$ such that $u_\lambda(x^0) > 0$ if $0 < \lambda < A$, where u_λ is a minimizer of $(J_{\lambda f})$. By Theorem 1.7, if $\lambda < \lambda' < A$ then $u_\lambda(x^0) > u_{\lambda'}(x^0)$. By Theorem 1.4, if u_λ, v_λ are two distinct minimizers for $J_{\lambda f}$ then $u_\lambda(x^0) \neq v_\lambda(x^0)$. We conclude that there is a monotone graph $\lambda \to \Phi(\lambda)$ $(0 < \lambda < A)$

such that $\Phi(\lambda)$ is a real number in (0, 1) if $\lambda \notin \Sigma$ and $\Phi(\lambda)$ is a nonzero interval if $\lambda \in \Sigma$. The assertion (5.9) now readily follows.

In the next result we drop the condition (2.1).

THEOREM 5.4. *Let* $0 < \lambda_0 < \lambda_1 < \infty$ *and suppose that for* $\lambda \in [\lambda_0, \lambda_1)$, $\lambda \to u_\lambda$ *is a family of solutions of* (0.1) *which varies continuously with* λ (*i.e*, $\lambda \to u_\lambda(x)$ *is continuous for each* $x \in \Omega$) *and* $u_\lambda \leqslant u_{\lambda'}$ *if* $\lambda \geqslant \lambda'$. *Then* (i) *if* $K_{\lambda_0,1}$ *and* $K_{\lambda_0,2}$ *are distinct components of* N_{λ_0} *then they are contained in distinct components of* N_λ *for* $\lambda_0 < \lambda < \lambda_1$; (ii) *each component of* N_{λ_0} *is either a point or a closed convex domain*; (iii) *the number of components of* N_{λ_0} *is finite*.

If for any $\lambda \in (\lambda_0, \lambda_1)$ there is a unique minimizer u_λ, then u_λ satisfies the assumptions of Theorem 5.4. Indeed the monotonicity follows from Theorem 1.7, and the continuity in λ of u_λ is assured by the uniqueness. Another natural candidate is the family of minimal (or maximal) solutions; here one must first establish their continuous dependence upon λ.

PROOF. From the monotonicity assumptions and Theorem 1.6, $K_{\lambda_0,1} \subset \operatorname{int} N_\lambda$ for $\lambda_0 < \lambda$. Hence $K_{\lambda_0,1} \subset \operatorname{int} K_\lambda$ where K_λ is a component of N_λ. Since $\operatorname{int} K_\lambda \neq \varnothing$ it follows from Theorem 5.1 that K_λ is a closed convex domain. Using the monotonicity and continuity assumptions on u_λ we can argue as in the proof of Theorem 5.3 to conclude that $N_{\lambda_0} \cap K_\lambda$ is either a point or a closed convex domain. In particular, it is connected. Hence $(N_{\lambda_0} \backslash K_{\lambda_{0,1}}) \cap K_\lambda = \varnothing$, which proves the assertion (i). Thus we have $N_{\lambda_0} \cap K_\lambda = K_{\lambda_{0,1}}$ and from the above it follows that this is either a point or a closed convex domain. Hence we have the second assertion.

Finally suppose that N_{λ_0} has an infinite number of components K_i. Then there exists a sequence of points $X_i \in K_i$ such that $X_i \to X_0 \in K_0$, a component of N_{λ_0}.

Since $K_0 \subset \operatorname{int} N_\lambda$ for any $\lambda > \lambda_0$, there is a component K_λ of N_λ with $X_0 \in K_0 \subset \operatorname{int} K_\lambda$. But then we have $K_i \cap K_\lambda \neq \varnothing$ for i sufficiently large, which contradicts (i).

From Theorem 5.4 we deduce

COROLLARY 5.5. *For each* $\lambda \in [\lambda_0, \lambda_1)$, N_λ *has a finite number of components, where each component is either a point or a closed convex domain. If one such component is a point then*

(α) *it disappears from* $N_{\lambda'}$ *for* $\lambda' < \lambda$,

(β) *it develops into a closed convex domain of* $N_{\lambda'}$ *for* $\lambda' > \lambda$.

Finally, the number of components is finite and increases as λ *increases.*

6. The coincidence sets for variational inequalities. Consider the variational inequality

$$\left.\begin{aligned} -\Delta u &\geqslant 0 \\ u - \phi &\geqslant 0 \\ (u-\phi)\Delta u &= 0 \end{aligned}\right\} \quad \text{a.e. in } \Omega, \tag{6.1}$$

$$u - \phi = 1 \quad \text{on } \partial\Omega. \tag{6.2}$$

THEOREM 6.1. *If* Ω *is a convex set in* R^2 *with* $C^{2+\alpha}$ *boundary* $(0 < \alpha < 1)$ *and if* $\Delta\phi \equiv -c$ *for some positive constant c, then the coincidence set* $\{u = \phi\}$ *is convex.*

A related result was proved by Lewy and Stampacchia [**11**] (by an entirely different method) namely: If (6.2) is replaced by

$$u > \phi \quad \text{on } \partial\Omega \tag{6.3}$$

and $\Delta\phi \equiv -c$ is replaced by

(6.4) ϕ is strictly concave and smooth,

then

(6.5) $\{u = \phi\}$ is a simply connected closed domain with Jordan boundary.

Let us show that

(6.6) the conditions (6.3), (6.4) (or even (6.2), (6.4)) do not yield, in general, the assertion that $\{u = \phi\}$ is convex.

PROOF OF (6.6). Consider the case where ϕ describes a pyramid whose base is a square having diagonals on the x- and y-axes. If $\phi = 0$ on $\partial\Omega$ and $u = 1$ on $\partial\Omega$ then $\{u = \phi\}$ is nonempty provided $\sup_\Omega \phi > 1$ (otherwise $\Delta u = 0$ and $u \equiv 1$ in Ω, so that $u < \phi$ at some point in Ω).

The set $\{u = \phi\}$ cannot include points of faces of ϕ. Indeed, if $u(x_0) = \phi(x_0)$ and $(x_0, \phi(x_0))$ lies on a face of the pyramid, then $-\Delta(u - \phi) = -\Delta u \geqslant 0$ in a neighborhood N of x_0 and, since $u - \phi \geqslant 0$, the strong maximum princple gives $u - \phi > 0$ in N.

We conclude that

(6.7) $\{u = \phi\}$ consists of points lying on the edges of the pyramid.

This gives the counterexample asserted in (6.6) in case of a ϕ which is not smooth. To get a counterexample for a smooth obstacle we approximate ϕ by strictly concave mollifiers ϕ_j and denote the corresponding solutions by u_j. By [**11**], the set $C_j = \{u_j = \phi_j\}$ is a simply connected closed domain. We assume that

$$C_j \text{ is convex for each } j \tag{6.8}$$

and derive a contradiction.

By (6.7) $u - \phi > 0$ on $\{x = \pm y, \, y \neq 0\} \cap \overline{\Omega}$. Hence, for any $\varepsilon > 0$, $u_j - \phi_j > 0$ on $\{x = \pm y, |y| \geqslant \varepsilon\} \cap \overline{\Omega}$ provided $j \geqslant j(\varepsilon)$. If (6.8) holds then we conclude that the diameter of C_j is $\leqslant \sqrt{2}\,\varepsilon$. Hence

$$\Delta(u - \phi) = \lim_j \Delta(u_j - \phi_j) = 0 \quad \text{(weak limit)}$$

if $x^2 + y^2 > 4\varepsilon^2$, i.e., $\Delta(u - \phi) = 0$ a.e. which contradicts the facts that $u \in C^{1,1}$, $\phi \notin C^{1,1}$.

PROOF OF THEOREM 6.1. Theorem 5.3 holds for $f(t) = ct^p$ with $0 < p < 1$. Going over the details of the proof we find that the proof extends to the (limiting) case $f(t) = c\chi_{[t>0]}$; here the functional J is $J(v) = \int_\Omega [\frac{1}{2} |\nabla v|^2 + cv^+]\,dx$. Thus the coincidence set for the variational inequality

$$\left.\begin{aligned} -\Delta U &\geqslant -c \\ U &\geqslant 0 \\ U(-\Delta U + c) &= 0 \end{aligned}\right\} \quad \text{a.e. in } \Omega, \tag{6.9}$$

$$U = 1 \quad \text{on } \partial\Omega \tag{6.10}$$

is convex. Setting $u = U + \phi$, we have

$$-\Delta u = -\Delta U - \Delta\phi \geqslant -c - \Delta\phi = 0 \quad \text{if } U = 0,$$
$$-\Delta u = -\Delta U - \Delta\phi = -c - \Delta\phi = 0 \quad \text{if } U > 0,$$

and (6.1), (6.2) follow.

COROLLARY 6.2. *Let Ω be a bounded convex domain in R^2 with $C^{2+\alpha}$ boundary and denote by ψ the solution of*

$$\Delta\psi = -1 \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega.$$

Then there exists a unique point $X^0 \in \Omega$ such that $\psi(X^0) = \max_{\bar\Omega} \psi$.

PROOF. Set $M = \max_{\bar\Omega} \psi$, $U = M - \psi$. Then U satisfies (6.9) with $c = 1$ and $U = M$ on $\partial\Omega$. It follows that the set $K = \{U = 0\} = \{\psi = M\}$ is convex. By analyticity of ψ, K has no interior and thus it consists of either one point (as asserted), or an interval I. Suppose the second case occurs and take for simplicity $I \subset \{y = 0\}$. Applying the Cauchy-Kowalewsky theorem to $U - y^2/2$ we find that $U - y^2/2 \equiv 0$, a contradiction.

7. The third boundary value problem. In this section we shall extend the main results of the previous sections to the third boundary value problem. Thus we consider

$$\Delta u = f(u) \quad \text{in } \Omega, \tag{7.1}$$

$$\frac{\partial u}{\partial \nu} + \mu(u - 1) = 0 \quad \text{on } \partial\Omega \tag{7.2}$$

where $f(t)$ is a function satisfying (1.1), ν is the outward normal, and μ is a given positive number. We shall imbed this problem in the family of elliptic problems

$$\begin{aligned} &\Delta u_\lambda = \lambda f(u_\lambda) \quad \text{in } \Omega, \\ &\frac{\partial u_\lambda}{\partial \nu} + \mu\sqrt{\lambda}\,(u_\lambda - 1) = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{7.3}$$

where $0 < \lambda < \infty$.

The existence of a solution of (7.1), (7.2) follows by minimizing the functional

$$J_\mu(v) = \int_\Omega \left(\frac{1}{2} |\nabla v|^2 + F(v) \right) + \mu \int_{\partial\Omega} \left(\frac{1}{2} v^2 - v \right), \qquad v \in H^1(\Omega).$$

One can also construct minimal solutions and maximal solutions as in §1.

THEOREM 7.1. *For any solution of* (7.1), (7.2) *there holds*

$$0 \leqslant u < 1 \quad \text{in } \Omega, \tag{7.4}$$

$$\frac{\partial u}{\partial \nu} > 0 \quad \text{on } \partial\Omega. \tag{7.5}$$

PROOF. If the set $K = \{u > 0\}$ is nonempty then $\partial K \cap \partial\Omega$ contains an open portion of $\partial\Omega$ (otherwise $u \equiv 0$ in K, since $\Delta u = 0$ in K) and u takes in $\bar K$ negative minimum at an interior point X of $\partial K \cap \partial\Omega$. But then $\partial u/\partial\nu < 0$, $u - 1 < 0$ at X, and (7.2) is contradicted.

If u takes maximum in $\bar{\Omega}$ at Y, and $u(Y) \geqslant 1$ then $Y \in \partial\Omega$ (since $\Delta u \geqslant 0$ in Ω) and $\partial u/\partial \nu > 0$, $u - 1 \geqslant 0$ at Y, again contradicting (7.2). We have thus completed the proof of (7.4); (7.5) follows from (7.4) and (7.2).

THEOREM 7.2. *If f satisfies* (2.1) *then the solution of* (7.1), (7.2) *is unique.*

PROOF. Setting $u = 1 - w^{\beta}$ ($\beta > 0$) we find that w satisfies (2.2), (2.3), and $\partial w/\partial \nu + \mu/\beta w = 0$ on $\partial\Omega$. The uniqueness result of Corollary 1.2 easily extends to the present case.

From now on we assume that

(7.6) $\qquad \Omega$ is a convex domain in R^2 with $\partial\Omega$ in $C^{3+\alpha}$.

We shall denote points in R^2 by $X = (x, y)$.

LEMMA 7.3. *Any solution of* (7.1), (7.2) *satisfies*

(7.7) $$|\nabla u|^2 \leqslant 2F(u) \quad \textit{in } \Omega.$$

PROOF. Set $\Phi = |\nabla u|^2 - 2F(u)$. As in **[12]** Φ cannot take its maximum in $\bar{\Omega}$ at an interior point, since

$$\Delta\Phi - \frac{f}{|\nabla u|^2} \nabla u \cdot \nabla\Phi \geqslant 0 \quad \text{in } \{\nabla u \neq 0\}.$$

Thus, if (7.7) is not true then Φ takes positive maximum at a point $X^0 \in \partial\Omega$. Take for definiteness $X^0 = (R, 0)$ with $x = R - \alpha y^2 + O(y^3) \equiv g(y)$ describing $\partial\Omega$ near X^0, and the segment $\{(x, 0), 0 \leqslant x < R\}$ in Ω; since Ω is convex, $\alpha \geqslant 0$.

By (7.5), $u_x > 0$ at X^0; further, by the maximum principle, $\Phi_x(X^0) > 0$, i.e.,

(7.8) $$u_x u_{xx} + u_y u_{xy} - f u_x > 0 \quad \text{at } X^0.$$

The boundary condition (7.2) can be written in the form

(7.9) $$u_x(g(y), y) - g'(y) u_y(g(y), y) + \mu(u - 1)\sqrt{1 + (g'(y))^2} = 0.$$

Since $\Phi(g(y), y)$ takes maximum at $y = 0$, $d\Phi/dy = 0$, $d^2\Phi/dy^2 \leqslant 0$ at $y = 0$, i.e,

(7.10) $$\Phi_y = 0 \quad \text{at } X^0,$$

(7.11) $$-2\alpha\Phi_x + \Phi_{yy} \leqslant 0 \quad \text{at } X^0.$$

Differentiating (7.9) and taking $y = 0$, we get $u_{xy} + 2\alpha u_y + \mu u_y = 0$, that is,

(7.12) $$u_{xy} = -(2\alpha + \mu) u_y.$$

Next, (7.10) can be written in the form $u_{xy} u_x + u_{yy} u_y - f u_y = 0$, so that, after using (7.12) and (7.1),

(7.13) $$u_y[(2\alpha + \mu) u_x + u_{xx}] = 0 \quad \text{at } X^0.$$

Consider first the case $u_y(X^0) \neq 0$. Then we get from (7.13), $u_{xx} = -(2\alpha + \mu) u_x < 0$; hence $u_x u_{xx} < 0$. Since, further $u_y u_{xy} < 0$ by (7.12), the left-hand side of (7.8) is negative, a contradiction.

Consider next the case $u_y(X^0) = 0$. Then, by (7.12), also $u_{xy}(X^0) = 0$ and, by (7.8), $u_{xx} - f > 0$, $u_{yy} < 0$ at X^0.

Differentiating (7.9) twice and substituting $y = 0$ we then obtain

$$-2\alpha u_{xx} + u_{xyy} + 4\alpha u_{yy} + 4\alpha^2\mu(u-1) = 0 \quad \text{at } X^0. \tag{7.14}$$

From (7.11) we also obtain

$$-2\alpha(u_{xx}u_x - fu_x) + u_x u_{xyy} + u_{yy}^2 - fu_{yy} \leqslant 0$$

at X^0. Substituting u_{xyy} from (7.14) into the last inequality, we get

$$-2\alpha(u_{xx} - f) + \frac{u_{yy}^2}{u_x} - \frac{fu_{yy}}{u_x} - 2\alpha u_{xx} - 4\alpha u_{yy} + 4\alpha^2\mu(1-u) \leqslant 0,$$

or

$$2\alpha f + \frac{u_{yy}^2}{u_x} - \frac{fu_{yy}}{u_x} - 4\alpha u_{yy} + 4\alpha^2\mu(1-u) \leqslant 0$$

which is a contradiction since all the terms on the left are nonnegative (recall that $\alpha \geqslant 0$) and the second and third ones are positive.

We shall now study the null sets N_λ. Using Lemma 7.3 we can establish as in §5

LEMMA 7.4. *If a component of N_λ has nonempty interior then it is a convex set.*

LEMMA 7.5. *There exists a positive number γ such that*

$$\Omega_{(\gamma+o(1))/\sqrt{\lambda}} \subset N_\lambda \subset \Omega_{(\gamma+o(1))/\sqrt{\lambda}} \tag{7.15}$$

where $o(1) \to 0$ uniformly as $\lambda \to \infty$.

PROOF. We begin with the crude estimates

$$\Omega_{c_0/\sqrt{\lambda}} \subset N_\lambda \subset \Omega_{c/\sqrt{\lambda}} \qquad (c_0 > 0, c > 0). \tag{7.16}$$

To prove the left-hand side we can use the same supersolution

$$v(X) = A\,|X - X_0|^{2/(1-p)}$$

as in Theorem 3.1, since $v(X) \geqslant 1 > u_\lambda$ on $\partial\Omega$. To prove the right-hand side of (7.16) we set $w_\lambda(X) = u_\lambda(X/\sqrt{\lambda})$ and $\Omega_\lambda = \{X/\sqrt{\lambda};\ X \in \Omega\}$. Since $\partial w_\lambda/\partial\nu + \mu(w_\lambda - 1) = 0$ on $\partial\Omega_\lambda$, elliptic estimates give $|Dw_\lambda| \leqslant C$, i.e., $|Du_\lambda| \leqslant C\sqrt{\lambda}$. Thus it suffices to establish a uniform lower bound

$$u_\lambda \geqslant c \quad \text{on } \partial\Omega, \qquad c > 0. \tag{7.17}$$

If this is not true when we can find sequences $\lambda_m \to \infty$, $X^m \in \partial\Omega$ such that $u_{\lambda_m}(X^m) \to 0$, $X^m \to \bar{X}$ as $m \to \infty$. Suppose for simplicity that the tangent to $\partial\Omega$ at $\bar{X}$ is horizontal and set $w_m(X) = u_{\lambda_m}((X - X^m))/\sqrt{\lambda_m}$. Then

$$\begin{aligned} &\Delta w_m = f(w_m) \quad \text{in } \tilde{\Omega}_{m,R} = \{-C < y < \phi_m(x), -R < x < R\}, \\ &\frac{\partial w_m}{\partial\nu} + \mu(w_m - 1) = 0 \quad \text{on } y = \phi_m(x), \qquad \phi_m(0) = 0, \\ &0 \leqslant w_m \leqslant 1 \quad \text{in } \tilde{\Omega}_{m,R}, \end{aligned}$$

for any $C > 0$, $R > 0$ and $m \geqslant m_0(C, R)$, and $w_m(0,0) \to 0$ as $m \to \infty$. Notice that $y = \phi_m(x)$ is a translate of $\partial\Omega_{\lambda_m}$, and, $|D^j\phi_m(x)| \leqslant \eta(\lambda_m) \to 0$ if $m \to \infty$ $(0 \leqslant j \leqslant 3)$, uniformly with respect to $x \in (-R, R)$.

By elliptic estimates we may assume that $w_m \to w$ uniformly in compact sets and

$$(7.18)\qquad \begin{aligned} &\Delta w = f(w) \quad \text{in } T = \left\{-\frac{C}{2} < y < 0, -\frac{R}{2} < x < \frac{R}{2}\right\},\\ &\frac{\partial w}{\partial y} + \mu(w-1) = 0 \quad \text{on } y = 0,\\ &0 \leqslant w \leqslant 1 \quad \text{in } T. \end{aligned}$$

Since $w(0,0) = 0$, if follows that $w_y + \mu(w-1) \leqslant -\mu < 0$ at $(0,0)$, a contradiction to the second equation in (7.18).

Having proved (7.16) we now proceed to establish (7.15). For any point $\bar{X} \in \partial\Omega$ we consider

$$w_\lambda(X) = u_\lambda\left(\frac{X-\bar{X}}{\sqrt{\lambda}}\right) \quad \text{in } \tilde{\Omega}_\lambda = \left\{\frac{X-\bar{X}}{\sqrt{\lambda}};\ X \in \Omega\right\}.$$

We shall assume for simplicity that the tangent to $\partial\Omega$ at $\bar{X}$ is horizontal. Notice that the $\partial\Omega$-neighborhood of $\bar{X}$ is mapped into $y = \phi_\lambda(x)$ with $\phi_\lambda(0) = 0$ and for any $R > 0$

$$(7.19)\qquad |D^j\phi_\lambda(x)| \leqslant \eta(\lambda) \quad \text{if } |x| < 2R, \qquad 0 \leqslant j \leqslant 3,$$

where $\eta(\lambda) \to 0$ if $\lambda \to \infty$. From (7.16) and its proof we have, for $\lambda \geqslant \lambda_0(R)$,

$$(7.20)\qquad \begin{aligned} &w_\lambda \geqslant c > 0 \quad \text{if } -c_1 \leqslant y \leqslant 0, \quad |x| < 2R,\\ &w_\lambda = 0 \qquad \text{if } -c_3 \leqslant y \leqslant -c_2, \quad |x| < 2R, \end{aligned}$$

where $0 < c_1 < c_2 < c_3$, and c_i are independent of λ, R. By Lemma 7.4 the component $\tilde{N}_\lambda$ of $\{w_\lambda = 0\}$ which contains the point $(0, -c_2)$ is convex. Setting

$$\tilde{\Omega}_{\lambda,R} = \{-c_3 < y < \phi_\lambda(x), -R < x < R\},$$

the set $\tilde{\Omega}_{\lambda,R} \cap \tilde{N}_\lambda$ is bounded by the lines $\{y = -c_3\}$, $\{x = \pm R\}$ and a curve $y = k_\lambda(x)$, where $k_\lambda(x)$ is concave for $|x| < 2R$. Since $-c_2 < k_\lambda(x) < -c_1$, we deduce that $|k_\lambda'(x)| \leqslant C$ a.e. for $|x| < R$. Hence for any subsequence of λ's converging to ∞ there is a subsequence for which

$$(7.21)\qquad k_\lambda(x) \to k(x), \qquad w_\lambda(X) \to w(X)$$

uniformly in compact subsets in R^2, and $k(x)$ is concave and $-c_2 \leqslant k(x) \leqslant -c_1$. It follows that

$$(7.22)\qquad k(x) \equiv \text{const} = -\gamma_0.$$

Recall also that

$$(7.23)\qquad \Delta w_\lambda = f(w_\lambda) \quad \text{in } \tilde{\Omega}_\lambda.$$

From (7.21)–(7.23) and nondegeneracy [**1**] it follows that $\{y = -\gamma_0\}$ lies in the free boundary of w. Applying Lemma 7.6 below we deduce that w is a 1-dimensional solution $w(x)$. To compute it, we need to solve $\zeta'(t) = \sqrt{2F(\zeta(t))}$ for $t < 0$, $\zeta'(0) + \mu(\zeta(0) - 1) = 0$. $\zeta(0)$ is determined by $-\mu(\zeta(0) - 1) = \sqrt{2F(\zeta(0))}$, i.e., $\sigma \equiv \zeta(0)$ is the solution of $\chi(\sigma) = 0$ where $\chi(x) = \sqrt{2F(s)} + \mu(s-1)$. Noting that

$\chi(0) < 0$, $\chi(1) > 0$, $\chi'(s) > 0$, there is indeed a unique solution σ. Denote by γ the smallest positive number such that $\zeta(-\gamma) = 0$. Then $w = \zeta(z - \gamma + \gamma_0)$ and $\gamma_0 = \gamma$.

Since w is uniquely determined, we conclude that as $\lambda \to \infty$.

$$w_\lambda(X) \to \zeta(x). \tag{7.24}$$

Hence for any $\varepsilon > 0$ the free boundary of w_λ in $\tilde{\Omega}_{\lambda,R}$ lies in the strip $|y + \gamma| < \varepsilon$ provided $\lambda \geqslant \lambda(\varepsilon, R)$. Finally, since $\lambda(\varepsilon, R)$ can be taken uniformly with respect to the initial point $\bar{X}$ on $\partial\Omega$ (otherwise we construct sequences $\bar{X}_m \in \partial\Omega$ and $\lambda_m \to \infty$ and derive a contradiction by working with $w_{\lambda_m}(X) = u_\lambda((X - \bar{X}_m)/\sqrt{\lambda_m})$), (7.15) follows.

LEMMA 7.6. *Let*

$$T_{\mathfrak{b},R} = \{-\mathfrak{b} < y < 0; -R < x < R\} \qquad (\mathfrak{b} > 0,\ R > 0)$$

and let u be a solution of

$$\begin{aligned}
&0 \leqslant u \leqslant 1, \qquad \Delta u = f(u) \quad \text{in } T_{\mathfrak{b},R},\\
&u(x, y) = 0 \quad \text{if } -R < x < R, \qquad -\mathfrak{b} \leqslant y \leqslant -\gamma_0 \quad (\gamma_0 \in (0, \mathfrak{b})),\\
&\{(x, -\gamma_0); -R < x < R\} \subset \partial\{u > 0\},
\end{aligned}$$

satisfying (7.7) *in* $T_{\mathfrak{b},R}$. *Then* $u(x, y) \equiv u(y)$ *and* $u(y) > 0$ *if* $-\gamma_0 < \gamma < 0$.

PROOF. By [**8**, 2.10.19(4)], for any $h \in L^1(R^n)$, $s > 0$,

$$\lim_{r\to 0} r^{s-n} \int_{B_r(y)} h(x)\,dx = 0 \tag{7.25}$$

except for y in an H^{n-s}-null set. Recalling (4.8) and $l(u) \to \text{const} > 0$ as $u \to 0$, we see that $[(1 - |\nabla g(u)|^2)/g(u)]I_{\{u>0\}} \in L^1$, so that

$$\lim_{t\to 0} \frac{1}{t} \int_{B_t(\xi,-\gamma_0)\cap\{u>0\}} \frac{1 - |\nabla g(u)|^2}{g(u)}\,dY = 0 \quad \text{for a.a. } \xi \in (-R, R). \tag{7.26}$$

Fix such ξ and consider, for any small $\tau > 0$,

$$v_\tau(X) = \frac{1}{\tau} g(u(\tau X + (\xi, -\gamma_0))) \quad \text{in } B_2(0).$$

Notice that

$$v_\tau(x, y) = 0 \quad \text{if } y \leqslant 0, \tag{7.27}$$

$$\frac{1 - |\nabla v_\tau(X)|^2}{v_\tau(X)} = \tau \frac{1 - |\nabla g(u(Y))|^2}{g(u(Y))} \tag{7.28}$$

where $Y = \tau X + (\xi, -\gamma_0)$. By (7.26),

$$\begin{aligned}
0 &= \lim_{\tau\to 0} t^{-1} \int_{B_{2\tau}(\xi,-\gamma_0)\cap\{u>0\}} \frac{1 - |\nabla g(u(Y))|^2}{g(u(Y))}\,dY \\
&= \lim_{\tau\to 0} \int_{B_2(0)\cap\{v_\tau>0\}} \frac{1 - |\nabla v_\tau(X)|^2}{v_\tau(X)}\,dX.
\end{aligned} \tag{7.29}$$

Since, by (4.5), $|\nabla v_\tau|\leq 1$, there is a sequence $\tau_i\downarrow 0$ with $v_{\tau_i}(X)\to v_0(X)$ uniformly in $B_1(0)$ and (7.29) gives (using Fatou's lemma) $|\nabla v_0|\equiv 1$ in $B_2(0)\cap\{v_0>0\}$. Since, by (4.3), (7.28), $|\Delta v_\tau|\leq C(1-|\nabla v_\tau|^2)/v_\tau$, we also have $\Delta v_0=0$ in $B_2(0)\cap\{v_0>0\}$. By (7.27) $v_0(x,y)=0$ if $-2<x<2, y\leq 0$, and, by nondegeneracy **[1]**,

$$(0,0)\in\partial\{v_0>0\}.$$

It now easily follows that $v_0(x,y)=y^+$ in $B_2(0)$.

Consider the function $z(X)=g(u(X))-(y+\gamma_0)^+$ in $T_{\mathfrak{b},R}$. From (4.5), (4.6),

$$\Delta z\geq 0 \quad \text{in } \mathcal{D}'(T_{\mathfrak{b},R}), \tag{7.30}$$

and

$$\frac{\partial z}{\partial y}=\frac{\partial g(u)}{\partial y}-1\leq 0 \quad \text{in } T_{\mathfrak{b},R}\cap\{y>-\gamma_0\}.$$

Also $z(x,-\gamma_0)=0$, so that

$$z\leq 0 \quad \text{in } T_{\mathfrak{b},R}, \qquad z=0 \quad \text{in } T_{\mathfrak{b},R}\cap\{y\leq-\gamma_0\}.$$

Next for any ξ as in (7.26),

$$\begin{aligned}\lim_{\tau_i\to 0}\frac{z(\xi,-\gamma_0+\tau_i)}{\tau_i}&=\lim_{\tau_i\to 0}\frac{g(u(\xi,-\gamma_0+\tau_i))}{\tau_i}-1\\&=\lim_{\tau_i\to 0}v_{\tau_i}(0,1)-1=v_0(0,1)-1=0.\end{aligned}$$

Thus $\partial z/\partial y=0$ at $(\xi,-\gamma_0)$. Recalling (7.30) and using the strong maximum principle, we deduce that $z\equiv 0$, i.e., $g(u(x,y))=(y+\gamma_0)^+$, and the assertion follows.

We shall need the following comparison result.

LEMMA 7.7. *If* (2.1) *holds and*

$$f'(t)+\frac{f(t)}{t}>0 \quad \textit{for all } 0<t\leq 1, \tag{7.31}$$

then, for any $\lambda_0>0$, *if* $\lambda>\lambda_0$ *and* $\lambda-\lambda_0$ *is small enough*

$$\frac{\sqrt{\lambda_0}}{\sqrt{\lambda}}u_\lambda\leq u_{\lambda_0}, \tag{7.32}$$

$$\operatorname{int} N_\lambda\supset N_{\lambda_0}. \tag{7.33}$$

PROOF. Set $\theta=(\lambda_0/\lambda)^{1/2}$. Then $\Delta(\theta u_\lambda)=h(u_\lambda)\lambda_0 f(\theta u_\lambda)$ where

$$h(t)=\frac{\theta\lambda}{\lambda_0}\frac{f(t)}{f(\theta t)}=\frac{1}{\theta}\frac{f(t)}{f(\theta t)}>1$$

by (7.31), provided $\theta<1$, $1-\theta$ is small enough. Thus

$$\Delta(\theta u_\lambda)>\lambda_0 f(\theta u_\lambda), \qquad \Delta u_{\lambda_0}=\lambda_0 f(u_{\lambda_0}).$$

By Corollary 1.2 (working with $(1-u_{\lambda_0})^{1/\beta}$, $(1-\theta u_\lambda)^{1/\beta}$) we find that $\theta u_\lambda-u_{\lambda_0}$ cannot take positive maximum in Ω. Thus, if (7.32) is not true then there is a point

$X^0 \in \partial\Omega$ where $\theta u_\lambda - u_{\lambda_0}$ attains its positive maximum in $\bar{\Omega}$, and

$$\theta u_\lambda > u_{\lambda_0}, \qquad \theta \frac{\partial u_\lambda}{\partial \nu} \geqslant \frac{\partial u_{\lambda_0}}{\partial \nu} \quad \text{at } X^0.$$

But then, at X^0,

$$0 = \frac{1}{\sqrt{\lambda_0}} \frac{\partial u_{\lambda_0}}{\partial \nu} + \mu(u_{\lambda_0} - 1) < \frac{\theta}{\sqrt{\lambda_0}} \frac{\partial u_\lambda}{\partial \nu} + \mu(\theta u_\lambda - 1)$$

$$< \frac{1}{\sqrt{\lambda}} \frac{\partial u_\lambda}{\partial \nu} + \mu(u_\lambda - 1) = 0, \quad \text{a contradiction.}$$

Having proved (7.32), the proof of (7.33) is then similar to the proof of Theorem 1.6.

We can now state the main result of this section.

THEOREM 7.8. *Assume that* (7.6) *and* (2.1), (7.31) *hold. Then there exists a* $\lambda_* > 0$ *such that* N_λ *is a closed convex domain with* $C^{1+\beta}$ *boundary for any* $\lambda > \lambda_*$, N_{λ_*} *consists of a single point, and* $N_\lambda = \varnothing$ *if* $\lambda < \lambda_*$.

PROOF. For λ large enough Lemma 7.5 implies the flatness condition and thus, by [**1**], ∂N_λ is a $C^{1+\beta}$ curve which can be parametrized by $\partial\Omega$. Lemma 7.4 then shows that N_λ is a convex domain. We can now use the monotonicity of N_λ (in λ) established in Lemma 7.7 in order to proceed as in the proof of Theorem 5.3. Notice finally that $N_0 = \varnothing$, by the maximum principle; hence $N_\lambda = \varnothing$ if λ is small enough, that is, $\lambda_* > 0$.

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