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THE NUMBER OF FACTORIZATIONS OF NUMBERS LESS THAN x INTO DIVISORS GREATER THAN y

BY

DOUGLAS HENSLEY

ABSTRACT. Let $A(x, y)$ be the number in the title. There is a function $h\colon [0, \infty) \to [0, 2]$, decreasing and convex, with $h(0) = 2$ and $\lim_{r\to\infty} h(r) = 0$, such that if $r = \log y/\sqrt{\log x}$ then as $x \to \infty$ with r fixed,

$$A(x, y) = \frac{C(r)x\exp\left(h(r)\sqrt{\log x}\right)}{(\log x)^{3/4}}\left(1 + O(\log x)^{-1/4}\right).$$

The estimate is uniform on intervals $0 < r \leqslant R_0$. As corollaries we have for $\log y = \theta(\log x)^{1/4}$,

$$\lim_{x\to\infty} \frac{A(x, y)}{A(x, 1)/y} = e^{\theta^2/2},$$

and if $\log y = o(\log x)^{1/4}$ then $A(x, y) \approx A(x, 1)/y$.

1. Introduction. In counting factorizations we make no distinction between $2 \cdot 2 \cdot 3$, $2 \cdot 3 \cdot 2$ and $3 \cdot 2 \cdot 2$, and list four factorizations of 12: 12, $6 \cdot 2$, $4 \cdot 3$ and $3 \cdot 2 \cdot 2$.

Let $a(n)$ denote the number of such factorizations of n. MacMahon observed, about 1920, that $\sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{d=2}^{\infty}(1 - d^{-s})^{-1}$. Shortly thereafter Oppenheim considered the average and maximum values of $a(n)$, $1 \leqslant n \leqslant x$. He found

$$(1.1) \qquad A(x) := \sum_{n=1}^{x} a(n) \cong x\exp\left(2\sqrt{\log x}\right)/\left(2\sqrt{\pi}\,(\log x)^{3/4}\right),$$

as did Szekeres and Turan somewhat later [**2, 3**]. Recently the question of the number of factorizations of numbers $n \leqslant x$ using only divisors $d \leqslant y$ was discussed, and estimated to within a factor of $(\log x)^{O(1)}$ [**1**]. Here we are concerned with

$$A(x, y) := \sum_{n=1}^{[x]} a_y(n),$$

where $a_y(n)$ is the number of factorizations of n into divisors $d > y$. We make the conventional assumption that $a_y(1) = 1$ for any y, so that $A(x, y) = 0$ for $x < 1$, and $A(x, y) = 1$ for $1 \leqslant x \leqslant y$.

We prove the following results.

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THEOREM 1. *There exists* $C_1 > 0$ *such that if* y *is an integer*, $0 < \varepsilon \leq 1$ *and* $1 \leq \log y \leq \varepsilon(\log x)^{1/4}$ *then*

$$yA(x, y)/A(x) \in \left(1 - C_1\varepsilon^2, 1 + C_1\varepsilon^2\right).$$

To state Theorem 2 we need some notation. For $x > y > 1$ let

$$r = r(x, y) = \log y/\sqrt{\log x}\,.$$

Let $q = q(r)$ be the unique solution in $0 < q < 1$ of

$$2\log q - \log(1 - rq) + rq/(1 - rq) = 0.$$

REMARK. In this paper, r is the reciprocal of the r used in [**1**].

Let $h(r) = 2q + rq^2/(1 - rq)$. Let $C(r) = \{4\pi(1 - rq + \frac{1}{2}r^2q^2)\}^{-1/2}$.

THEOREM 2. *Uniformly in* $(\log x)^{1/5} \leq \log y \leq R(\log x)^{1/2}$,

$$A(x, y) = C(r)x(\log x)^{-3/4}e^{h(r)\sqrt{\log x}}\left(1 + O_R\left(1/(\log x)^{1/4}\right)\right),$$

for any fixed $R > 0$.

COROLLARY. *For arbitrary real* $\theta > 0$, *with* $\log y = \theta(\log x)^{1/4}$,

$$\lim_{x\to\infty} yA(x, y)/A(x) = e^{\theta^2/2}.$$

REMARK. This explains the $1 + O(\varepsilon^2)$ when $\log y \leq \varepsilon(\log x)^{1/4}$ in Theorem 1; no better result is possible.

The proof of Theorem 1 is an exercise in complex analysis and presents no great novelty. For the most part it follows Oppenheim's original proof.

The proof of Theorem 2 uses some of the techniques of [**1**]. The novelty here is that for the case at hand we can dispense with grouping factorizations according to the number of divisors from various intervals (α^{i-1}, α^i), and we can evaluate explicitly the probabilities that arise in a temporary reformulation of the problem as a question of chance. This permits accuracy to within a factor of $1 + O(\log x)^{-1/4}$ instead of $(\log x)^{O(1)}$ which we got in [**1**]. I think the new techniques might succeed with the old $K(x, y)$ and give similar improvements in accuracy. The hard step will be to evaluate more accurately the probabilities that arise in the $K(x, y)$ reformulation. ($K(x, y)$ is the number of factorizations using divisors $\leq y$.)

2. Complex analysis. Before we get into the matter too deeply we note that for *fixed* $y \in \mathbf{Z}$,

$$\lim_{x\to\infty} A(x, y)/A(x) = 1/y$$

by an elementary inclusion-exclusion argument based on (1.1) and the fact that $\prod_{d=2}^{y}(1 - 1/d) = 1/y$.

To start the analysis now, let $f_y(s) = \prod_{n=y+1}^{\infty}(1 - n^{-s})^{-1}$, for $s = \sigma + it$, $\sigma > 1$. Then $f_y(s) = \sum_{n=1}^{\infty} a_y(n)/n^s$ for $\sigma > 1$, where $a_y(n)$ is, as in §1, the number of ways in which n is the product of integers greater than y.

Let $B(x, y) = \sum_{n\leq x}(x - n)a_y(n)$, and recall $A(x, y) = \sum_{n\leq x}a_y(n)$.

As in Oppenhiem [**2**] let $f(s) = \prod_{n=2}^{\infty}(1 - n^{-s})^{-1}$ $(\sigma > 1)$ and let

$$g_y(s) = \prod_{n=2}^{y} (1 - n^{-s}).$$

Then $f_y(s) = f(s)g_y(s)$.

We have from [**2**] that there is an expansion

$$f(s) = \exp(1/(s-1))\left(1 + \sum_{n=1}^{\infty} \alpha_n (s-1)^n\right), \tag{2.1}$$

valid for $|s - 1| < 1/2$, and that $f(s) = O(|t|^{\varepsilon})$ for any positive ε, uniformly for $|t| \geqslant t_0(\varepsilon)$ and $\sigma > 1$.

We require an estimate for the growth of the coefficients $C_k(y)$ in the expansion

$$\begin{aligned} g_y\left(1 + \frac{1}{w}\right) &= \frac{1}{y}\prod_{n=2}^{y}\left(1 - (1 - n^{-1/w})/(n-1)\right) \\ &= \frac{1}{y}\left(1 + \sum_{k=1}^{\infty} C_k(y) w^{-k}\right). \end{aligned} \tag{2.2}$$

LEMMA 2.1. *For* $k \leqslant y$, $|C_k(y)| \leqslant (4\log y)^{2k}$, *while* $|C_k(y)| \leqslant (4\log y)^{2y}$ $(k \geqslant y)$.

PROOF. Expanding the product in (2.2) using $n^{-1/w} = \sum_{j=0}^{\infty}(-1)^j (j!)^{-1}(\log n/w)^j$, we get

$$C_k(y) = \sum_{\bar V \in S(k)} (-1)^k \prod_{n=2}^{y} (n-1)^{-\beta_n(\bar V)} (\log n)^{V_n}/V_n!, \tag{2.3}$$

where

$$S(k) = \left\{\bar V = (V_2, V_3, \ldots, V_y) \,\middle|\, V_2, \ldots, V_y \geqslant 0 \in Z \text{ and } \sum_{2}^{y} V_n = k\right\},$$

and where $\beta_n(\bar V) = 1$ if $V_n \geqslant 1$, else 0. Thus

$$|C_k(y)| \leqslant (\log y)^k \sum_{\bar V \in S(k)} \prod_{n=2}^{y} \left(\frac{1}{n-1}\right)^{\beta_n(\bar V)} \frac{1}{V_n!}. \tag{2.4}$$

For any fixed $T \subseteq \{2, 3, \ldots, y\}$ with $\#T = r \leqslant k$, the sum over the terms of (2.4) in which $V_n \geqslant 1$ if and only if $n \in T$ is

$$\prod_{n \in T} (n-1)^{-1} \sum_{P(k,r)} \prod_{j=1}^{r} 1/p_j!,$$

where $P(k, r)$ is the set of ordered partitions of k into positive integers $p_1, p_2, \ldots, p_r$, and this

$$= \prod_{n \in T} (n-1)^{-1} r^k/k! = (r^k/k!) \prod_{n \in T} (n-1)^{-1}.$$

Summing over all T with r elements, we get

$$(r^k/k!) \sum_{n_1 \neq n_2 \neq \cdots \neq n_r} \prod (n_i - 1)^{-1} \leq (r^k/k!)\left(\sum_2^y (n-1)^{-1}\right)^r$$
$$\leq (r^k/k!)(1 + \log y)^r.$$

Summing over the possible values of r gives $\sum_{r=1}^{\min\{y,k\}} (r^k/k!)(1+\log y)^r$. For $k \leq y$ this yields

$$|C_k(y)| \leq \frac{k^k}{(k-1)!}(1+\log y)^k (\log y)^k \quad \text{and} \quad |C_k(y)| \leq (4 \log y)^{2k}$$

while for $k > y$, $|C_k(y)| \leq (4\log y)^{2y}$.

After this we can copy [2]. We have

$$B(x, y) = \sum_{n \leq x} (x-n)a_y(n) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} f(s) g_y(s)\, ds. \tag{2.5}$$

From the definition of $g_y(s)$ clearly $|g_y(s)| \leq y$ for $\sigma = 1$. Thus in analogy with [2], (2.6)

$$B(x, y) = J(x, y) + O(x^2 y), \quad \text{where } J(x, y) = \frac{1}{2\pi i}\int_\Gamma \frac{x^{s+1}}{s(s+1)} f(s) g_y(s)\, ds$$

and Γ half-encircles 1 to the right at radius $a > 0$. Further, by Cauchy's theorem, for $b > 0$

$$J(x, y) = \frac{1}{2\pi i}\int_{b-i}^{b+i} \frac{x^{2+1/w}}{(w+1)(2w+1)} f_y(1 + 1/w)\, dw + O(x^2 y), \tag{2.7}$$

on the change of variable $s = 1 + 1/w$. Now

$$\frac{f(1+1/w)}{(w+1)(2w+1)} = \frac{1}{2} e^w \sum_{n=0}^{\infty} \delta_n w^{-2-n} \tag{2.8}$$

uniformly on $\sigma = b$, with $\delta_0 = 1$ and δ_n certain constants, with $|\delta_n| \leq K^n$ for some $K > 2$ and all n [2].

If we set $C_0(y) = 1$ we have

$$\frac{f(1+1/w) g_y(1+1/w)}{(w+1)(2w+1)} = \frac{e^w}{2y}\sum_{n=0}^{\infty}\sum_{j=0}^{n} \delta_{n-j} C_j(y) w^{-n-2}. \tag{2.9}$$

Let $d_n(y) = \sum_{j=0}^{n} \delta_{n-j} C_j(y)$. Then by Lemma 2.1, $d_0(y) = 1$, and for $\log y \geq 2K$,

$$|d_n(y)| \leq 2(4\log y)^{2n} \quad \text{for } n \leq y, \tag{2.10}$$
$$|d_n(y)| \leq 2K^{n-y}(4\log y)^{2y} \quad \text{for } n \geq y.$$

As in [2] we now have

$$J(x, y) + O(x^2 y) = \frac{1}{2}\frac{x^2}{y\sqrt{\log x}} I_1\left(2\sqrt{\log x}\right) \tag{2.11}$$
$$+ \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{1}{2}\frac{x^2}{y} e^{w + (\log x/w)} \sum_{\nu=1}^{\infty} \frac{d_\nu(y)}{w^\nu}\frac{dw}{w^2}.$$

If we take $b = \sqrt{\log x}$ ($\geqslant 32K(\log y)^2$ by assumption) and let $w = b + it$ then

$$\left| \sum_{\nu=1}^{\infty} \frac{d_\nu(y)}{w^\nu} \right| = O\big((\log y)^2/|w|\big) \quad \text{on } w = b + it$$

so the absolute value of the integral in (2.11) is

$$\begin{aligned} &O\big(x^2(\log y)^2/y\big) \int_0^\infty |w|^{-3} \exp\big(b + b\log x/\,(b^2 + t^2)\big)\, dt \\ &= O\big(x^2 e^{2\sqrt{\log x}}(\log y)^2/\,(y\log x)\big) \\ &\quad \cdot \int_0^\infty (1 + h^2)^{-3/2} \exp\big(-h^2\sqrt{\log x}\,/(1 + h^2)\big)\, dh \end{aligned}$$

with $h = t/\sqrt{\log x} = O(x^2 e^{2\sqrt{\log x}}(\log y)^2/(y(\log x)^{5/4}))$. Thus

$$(2.12) \qquad J(x, y) + O(x^2 y) = \frac{1}{2}\frac{x^2}{y\sqrt{\log x}} I_1\big(2\sqrt{\log x}\big) + O\big(x^2 e^{2\sqrt{\log x}}(\log y)^2/\big(y(\log x)^{5/4}\big)\big),$$

and since $I_1(z) = (2\pi z)^{-1/2} e^z (1 + O(1/z))$,

$$(2.13) \qquad B(x, y) = \frac{x^2 e^{2\sqrt{\log x}}}{4\sqrt{\pi}\, y(\log x)^{3/4}} \big(1 + O\big((\log y)^2/\sqrt{\log x}\big)\big),$$

uniformly in $(2K)^2 \leqslant (\log y)^2 \leqslant \sqrt{\log x}\,/(32K)$.

Now

$$\frac{\partial J}{\partial x} = \frac{1}{2\pi i}\int_C \frac{x^{1+1/w}}{w(w+1)} f_y(1 + 1/w)\, dw,$$

and

$$\frac{\partial^2 J}{\partial x^2} = \frac{1}{2\pi i}\int_C \frac{x^{1/w}}{w^2} f_y(1 + 1/w)\, dw,$$

where C is the half circle to the right of O about O at radius $1/a$.

Expanding $f_y(1 + 1/w)e^{-w}/w(w+1)$ as a series in $1/w$ gives

$$\frac{\partial J}{\partial x} = \frac{x}{y}\frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}\,(\log x)^{3/4}}\left\{1 + O\left(\frac{(\log y)^2}{\sqrt{\log x}}\right)\right\} \quad \text{and} \quad \frac{\partial^2 J}{\partial x^2} = O\left(\frac{1}{y}\frac{e^{2\sqrt{\log x}}}{(\log x)^{3/4}}\right).$$

both uniformly in $(2K)^2 \leqslant (\log y)^2 \leqslant (32K)^{-1}\sqrt{\log x}$. The details parallel those of (2.7)–(2.12).

Finally for this range of y by an exact copy of the Tauberian argument in [**2**] from $B(x) \sim J(x)$ to $A(x) \sim J'(x)$ we have

$$\begin{aligned} A(x, y) &= J'(x, y) + O\big(xe^{(1+1/2K)\sqrt{\log x}}\big) \\ &= \frac{x}{y}\frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}\,(\log x)^{3/4}}\left(1 + O\left(\frac{(\log y)^2}{\sqrt{\log x}}\right)\right), \end{aligned}$$

which proves Theorem 1.

3. Elementary methods. Our proof of Theorem 2 is based on a counting principle set in the language of probability, and on the calculus that boils down the resulting estimate of $A(x, y)$ to the form $C(r)x\log^{-3/4} xe^{h(r)\sqrt{\log x}}(1 + O_R(\log^{-1/4} x))$. The devices used here seem to be considerably stronger than the standard complex-analytic approach of §2. Indeed one purpose of §2, aside from treating the case when y is too small for the approach of this section, is to highlight the advantage here of real methods.

We start with a reformulation of $A(x, y)$.

(3.1)

$$A(x, y) = \#\left\{\xi\colon \{y+1, y+2,\dots,x\} \to \{0,1,2,\dots\} \,\middle|\, \sum_{d>y} \xi_d \log d \leqslant \log x\right\}.$$

Let $u = \log x/\log y$ and $r = \log y/\sqrt{\log x} = \sqrt{\log x}/u$. Let $X_1, X_2,\dots$ be independent random variables uniformly distributed on $\{y+1,\dots,x\}$. For $n \leqslant u$, let $\bar{x}$ denote an n-tuple of integers. Then

$$(3.2)\quad \operatorname{Prob}\left(\sum_1^n \log X_i \leqslant \log x\right) = (x-y)^{-n}\#\left\{\bar{x} \,\middle|\, y < x_i \leqslant x \text{ for } 1 \leqslant i \leqslant n \text{ and } \sum_1^n \log x_i \leqslant \log x\right\}.$$

Associate with each $\bar{x}$ the $(x-y)$-tuple $\xi = (\xi_{y+1}, \xi_{y+2},\dots,\xi_x)$ with $\xi_d =$ the number of times d occurs in $\bar{x}$. Then $\sum_{y+1}^{x} \xi_d = n$.

The number of times a given ξ occurs among the set of all n-tuples $\bar{x}$ is $n!\prod_{d=y+1}^{x}(\xi_d!)^{-1}$.

Let $\|\xi\| = \prod_{y+1}^{x} d^{\xi_d}$, and $\sigma(\xi) = \sum_{y+1}^{x}\xi_d$. Then

$$(3.3)\qquad \operatorname{Prob}\left(\sum_1^n \log X_i \leqslant \log x\right) = (x-y)^{-n} n! \sum_{\|\xi\| \leqslant x,\, \sigma(\xi)=n}\ \prod_{y+1}^{x} (\xi_d!)^{-1},$$

and

$$(3.4)\qquad \sum_{\|\xi\| \leqslant x,\, \sigma(\xi)=n}\ \prod_{y+1}^{x} (\xi_d!)^{-1} = (x-y)^n (n!)^{-1} \operatorname{Prob}\left(\sum_1^n \log X_i \leqslant \log x\right).$$

Now $A(x, y) = \sum_{\|\xi\| \leqslant x} 1$, so with the notation $P_n = \operatorname{Prob}(\sum_1^n \log X_i \leqslant \log x)$,

$$(3.5)\qquad A(x, y) \geqslant \sum_{n \leqslant u} (x-y)^n (n!)^{-1} P_n.$$

The inequality is quite sharp, for, in fact, also

$$(3.6)\qquad A(x, y) \leqslant (1 - 2/y)^{-1} \sum_{n \leqslant u} (x-y)^n (n!)^{-1} P_n.$$

For the proof we need a lemma.

LEMMA 3.1. *For any integer $t \geqslant 1$ and any real $s \geqslant t$, $2A(s, t) \leqslant A(2s, t)$.*

PROOF. For $t > 1$, $A(s, t-1) = \sum_{m=0}^{\infty} A(s/t^m, t)$. Also $A(s, t) = 0$ for $s < 1$, 1 for $1 \leqslant s \leqslant t$, and $s - t + 1$ for $\sqrt{s} \leqslant t \leqslant s$. Fix $T > 1$ and $s \leqslant T$, and make the inductive assumption that $2A(p, q) \leqslant A(2p, q)$ for $p \leqslant T$ and $q \geqslant s$. This is clear if $s = T$. Let $M = [\log T/\log s]$. Then for $p < s$, evidently

$$A(2p, s-1) - 2A(p, s-1) \geqslant 0,$$

while for $T \geqslant p \geqslant s$,

$$\begin{aligned} A(2p, s-1) - 2A(p, s-1) &\geqslant \sum_{m=0}^{M} A(2p/s^m, s) - 2A(p/s^m, s) \\ &\geqslant (1-2) + \left((2p/s^{M-1} - s + 1) - 2(p/s^{M-1} - s + 1)\right) \\ &\quad + \sum_{m=0}^{M-2} \left(A(2p/s^m, s) - 2A(p/s^m, s)\right) \\ &\geqslant -1 + (s-1) + 0 \end{aligned}$$

by the inductive assumption, $\geqslant 0$ for $s - 1 \geqslant 1$. □

We now continue with the proof of (3.6). Let

$$A_1(x, y) = \#\{\xi \,|\, \|\xi\| \leqslant x \text{ and } \xi_d = 0 \text{ or } 1 \text{ for all } d\},$$

and let $A_2(x, y) = A(x, y) - A_1(x, y)$. Then $A_2(x, y) \leqslant \Sigma_{d>y} A(x/d^2, y)$. From Lemma 3.1 applied repeatedly, $A(x/d^2, y) \leqslant (2/d^2)A(x, y)$. Thus

$$\sum_{d>y} A(x/d^2, y) \leqslant 2A(x, y) \sum_{d>y} 1/d^2 < (2/y)A(x, y). \tag{3.7}$$

Let $H(x, y) = \Sigma_{\|\xi\| \leqslant x} \Pi_{y+1}^{x} (\xi_d!)^{-1}$ and let $H_1(x, y)$ be the same sum taken only over ξ counted in $A_1(x, y)$. Clearly

$$1 \geqslant H_1(x, y)/H(x, y) \geqslant A_1(x, y)/A(x, y).$$

Also $H_1(x, y) = A_1(x, y)$. Since $A_1(x, y)/A(x, y) > (1 - 2/y)$ by (3.7) we conclude that

$$A(x, y) \leqslant (1 - 2/y)^{-1} H(x, y) = (1 - 2/y)^{-1} \sum_{n \leqslant u} (x - y)^n (n!)^{-1} P_n,$$

which proves (3.6).

In view of (3.5) and (3.6) any estimate of $\Sigma_{n \leqslant u} (x - y)^n (n!)^{-1} P_n$ gives an estimate of $A(x, y)$.

For an upper bound we replace X_i in $P_n = \operatorname{Prob}(\Sigma_1^n \log X_i \leqslant \log x)$ with random variables Y_i independent and uniformly distributed on the real interval (y, x), and set $P_n^+ = \operatorname{Prob}(\Sigma_1^n \log Y_i \leqslant \log x) \geqslant P_n$. The probability density function for $\log Y_i$ has the form $e^{Ct+D} I(\log y, \log x)$ and the convolution of n copies has the form $e^{Ct+D'} S(t)$, where $S(t)$ is a spline which is zero for $t \leqslant n \log y$ and is $D''(t - n\log y)^n$ for $n \log y < t \leqslant \log x$. Explicit calculation now gives

$$P_n^+ = \left(\frac{\log x - \log y}{x - y}\right)^n \frac{x}{(n-1)!} \int_0^{1-(n-1)/(u-1)} h^+(t)\, dt, \tag{3.8}$$

where

$$h^+(t) = t^{n-1} \exp(-C(1 - (n-1)/(u-1) - t))$$

and

$$C = (\log x - n \log y)/(1 - (n-1)/(u-1)).$$

Let I_n^+ be the definite integral in (3.8). Now with random variables Z_i independent and uniformly distributed on $(y+1, x)$, we put

$$P_n^- = \left(1-(x-y)^{-1}\right)^n \operatorname{Prob}\left(\sum_1^n \log Z_i \leqslant \log x\right) \leqslant P_n,$$

and calculate in the same way
(3.9)

$$P_n^- = (\log x - \log(y+1))^n (x-y-1)^{-n}\left(x/(n-1)!\left(1-(x-y)^{-1}\right)^n\right) I_n^-,$$

where

$$I_n^- = \int_0^{1-(n-1)/(v-1)} h^-(t), \qquad v = \log x/\log(y+1),$$

$$K = (\log x - n\log(y+1))(1-(n-1)/(v-1))^{-1}$$

and

$$h^-(t) = t^{n-1}\exp(-K(1-(n-1)/(v-1)-t)).$$

This simplifies to

$$P_n^- = \left(\frac{\log x - \log(y+1)}{x-y}\right)^n \frac{x}{(n-1)!} I_n^-. \tag{3.10}$$

Next we have a lemma about I_n^+ and I_n^-.

LEMMA 3.2. *Uniformly in integers $n \geqslant 2$ and in real $\theta \geqslant 0$,*

$$\int_0^1 s^{n-1} e^{n\theta s - n\theta}\, ds = \frac{1}{n+n\theta}\left(1 + O(\log^2 n/n)\right).$$

PROOF. Since $e^{s-1} - \frac{1}{2}(s-1)^2 \leqslant s \leqslant e^{s-1}$ for $0 \leqslant s \leqslant 1$,

$$\int_0^1 s^{n-1} e^{n\theta(s-1)}\, ds \leqslant \int_0^1 e^{(n-1)(s-1)+n\theta(s-1)}\, ds \leqslant \int_{-\infty}^1 e^{(n+n\theta-1)(s-1)}\, ds$$

$$= \frac{1}{n(1+\theta)-1} = \frac{1}{n+n\theta}\left(1+O(\log^2 n/n)\right).$$

Also

$$\int_0^1 s^{n-1} e^{n\theta(s-1)}\, ds \geqslant \int_{1-\log n/n}^1 \exp(s-1-\log^2 n/2n^2)(n-1) + \theta n(s-1)\, ds$$

$$\geqslant \int_{-\log n/n}^0 \exp\left(n(1+\theta)r - r - \frac{1}{2}(\log^2 n/n)\right) dr$$

$$\geqslant \left(1 - \frac{1}{2}\log^2 n/n\right)\left(\frac{1}{n+n\theta-1}\right)\left(1 - e^{-(1+\theta)\log n}\right)$$

$$= \frac{1}{n+n\theta}\left(1+O(\log^2 n/n)\right). \quad \square$$

Now

$$I_n^+ = \left(1 - \frac{n-1}{u-1}\right)^n \left(\frac{1}{n+\log x - n\log y}\right)\left(1 + O(\log^2 n/n)\right)$$

by Lemma 3.2, while

$$I_n^- = \left(1 - \frac{n-1}{v-1}\right)^n \left(\frac{1}{n + \log x - n\log(y+1)}\right)\left(1 + O(\log^2 n/n)\right).$$

Thus

$$P_n^+ = \left(\frac{\log x - \log y}{x - y}\right)^n \frac{x}{(n-1)!}\left(1 - \frac{n-1}{u-1}\right)^n$$
$$\cdot \left(\frac{1}{n + \log x - n\log y}\right)\left(1 + O(\log^2 n/n)\right)$$

and on simplification,

(3.11)
$$P_n^+ = \frac{(\log x)^n}{(x-y)^n}\frac{x}{(n-1)!}\left(1 - \frac{n}{u}\right)^n \left(\frac{1}{n + \log x - n\log y}\right)\left(1 + O(\log^2 n/n)\right).$$

Similarly

(3.12)
$$P_n^- = \frac{(\log x)^n}{(x-y)^n}\frac{x}{(n-1)!}\left(1 - \frac{n}{v}\right)^n \left(\frac{1}{n + \log x - n\log(y+1)}\right)\left(1 + O(\log^2 n/n)\right).$$

We use (3.11) in (3.6) and (3.12) in (3.5) and conclude

(3.13)
$$A(x, y) \leqslant \left(1 - \frac{2}{y}\right)^{-1} x \sum_{n<u} \frac{\log^{n-1} x}{(n-1)!n!}\left(1 - \frac{n}{u}\right)^{n-1}\left(1 - \frac{n}{\log x - n\log y + n}\right)$$
$$\cdot \left(1 + O(\log^2 n/n)\right)$$

and

(3.14)
$$A(x, y) \geqslant x \sum_{n<v} \frac{\log^{n-1} x}{(n-1)!n!}\left(1 - \frac{n}{v}\right)^{n-1}$$
$$\cdot \left(1 - \frac{n}{\log x - n\log(y+1) + n}\right)\left(1 + O(\log^2 n/n)\right).$$

We now claim that for $\sqrt{u} < n < u - \sqrt{u}$, the corresponding terms of (3.13) and (3.14) are in a ratio of $1 + O(1/\sqrt{\log x})$.

For

$$(1 - n/v)^{n-1}(1 - n/u)^{1-n} \geqslant \left(1 - u^{3/2}(1/v - 1/u)\right)^{n-1} \geqslant \left(1 - u^{1/2}/(y\log y)\right)^u.$$

By assumption $\log y \geqslant \log^{1/5} x$, so

$$\left(1 - u^{1/2}/(y\log y)\right)^u \geqslant \left(1 - \log^{1/5} x \exp(-\log^{1/5} x)\right)^u \geqslant 1 + O\left(1/\sqrt{\log x}\right).$$

Also if $n < u - \sqrt{u}$, $\log x - n\log y \geqslant u^{-1/2}\log x$, so

$$\left(1 - n(\log x - n\log(y+1) + n)^{-1}\right)\left(1 - n(\log x - n\log y + n)^{-1}\right)^{-1}$$
$$= 1 + O(u^{1/2}/y\log x) = 1 + O\left(1/\sqrt{\log x}\right).$$

This proves the claim.

To dispose of the terms in (3.13) and (3.14) with $n \leqslant \sqrt{u}$ or $n \geqslant u - \sqrt{u}$, and for further use, we need another lemma.

LEMMA 3.3. *Suppose* (C_n) *is a log-concave sequence of positive numbers, with* C_0 *the largest. For each* $\varepsilon > 0$ *let* $N(\varepsilon)$ *denote the largest n with* $C_n > \varepsilon C_0$, *and let* $m(\varepsilon)$ *be the least m with* $C_m > \varepsilon C_0$. *Then for* $\varepsilon < 1$, $\Sigma_{m(\varepsilon)}^{n(\varepsilon)} C_k \geqslant (1 - 2\varepsilon)\Sigma_{-\infty}^{\infty} C_k$.

COROLLARY. *For log-concave* $c(t)$: $(-\infty, \infty) \to [0, \infty)$ *with* $c(0)$ *largest, with the same notation except that now* $n(\varepsilon)$ *and* $m(\varepsilon)$ *are the larger and smaller solutions of* $c(t) = \varepsilon c(0)$,

$$\int_{m(\varepsilon)}^{n(\varepsilon)} c(t)\, dt \geqslant (1 - 2\varepsilon) \int_{-\infty}^{\infty} c(t)\, dt.$$

REMARK. In fact the Corollary holds with ε in place of 2ε on the right.

PROOF OF LEMMA 3.3. The worst case is when $C_k = \varepsilon^{k/(1+n(\varepsilon))} C_0$ for $k > 0$ and $C_k = \varepsilon^{-k/(1+m(\varepsilon))} C_0$ for $k < 0$. In that case

$$\sum_{-\infty}^{\infty} C_k = C_0 \left(\frac{1}{1 - C_1} + \frac{1}{1 - C_{-1}} - 1 \right),$$

while

$$\sum_{m(\varepsilon)}^{n(\varepsilon)} C_k = \sum_{-\infty}^{\infty} C_k - \frac{\varepsilon C_0}{1 - C_1} - \frac{\varepsilon C_0}{1 - C_{-1}},$$

and the smaller sum is clearly greater than $(1 - 2\varepsilon)\Sigma_{-\infty}^{\infty} C_k$.

Now the terms of (3.13) and (3.14) are, apart from the $(1 + O(\log^2 n/n))$ factor, a log-concave sequence. The ratio of consecutive terms for n near $\sqrt{u}$ is $\sim \log y$, while for n near $u - \sqrt{u}$ it is less than $\log x/e^{\sqrt{u}}$. Since $\log y \geqslant \log^{1/5} x$, and since $u \geqslant \sqrt{\log x}/R$, in both (3.13) and (3.14) the contribution of the tails is $O(1/\sqrt{\log x})$ of the whole.

Now let

$$j_n(x, y) = \frac{x \log^{n-1} x}{(n-1)!n!} \left(1 - \frac{n}{u}\right)^{n-1} \left(1 - \frac{n}{\log x - n \log y + n}\right),$$

and let

$$L(x, y) = \sum_{n=1}^{[u]} j_n(x, y), \qquad M(x, y) = \sum_{[\sqrt{u}]}^{[u - \sqrt{u}]} j_n(x, y).$$

As above, and again using Lemma 3.3, $M(x, y) = L(x, y)(1 + O(1/\sqrt{\log x}))$. Also, $M(x, y) = A(x, y)(1 + O(1/\sqrt{\log x}))$ since between $\sqrt{u}$ and $u - \sqrt{u}$ the terms of $M(x, y)$ and the terms in (3.13) and (3.14) all agree to within a factor of $1 + O(1/\sqrt{\log x})$, and since these terms comprise all but $O(1/\sqrt{\log x})$ of the whole in each case.

To estimate $L(x, y)$ we introduce a function $f(s)$ so that $xe^{f(u)} \sim j_n(x, y)$, and consider $x\int_0^u e^{f(s)}\, ds$. We take this $f(s)$:

$$(3.15)\quad f(s) = -\log(2\pi) - 2s \log s + 2s + (s - 1)(\log\log x + \log(1 - s/u)).$$

Then for $\varepsilon > 0$, and $\sqrt{u} \leqslant n \leqslant (1-\varepsilon)u$,

$$e^{f(n)} = \left(1 + O_\varepsilon\left(1/\sqrt{\log x}\right)\right) j_n(x, y). \tag{3.16}$$

Now $f(s)$ is concave. We shall later show that for each R there exists $\varepsilon = \varepsilon(R)$ such that for $r \leqslant R$, with $c(t) = e^{f(t)}$,

$$\sqrt{u} \leqslant m\left(\log^{-1/2} x\right) < n\left(\log^{-1/2} x\right) \leqslant (1 - \varepsilon(R))u,$$

in the notation of Lemma 3.3. Consequently,

(3.17) All of $L(x, y)$, $M(x, y)$, $\Sigma_{\sqrt{u}<n<u(1-\varepsilon)} j_n(x, y)$, $\Sigma_{\sqrt{u}<n<u(1-\varepsilon)} e^{f(n)}$ and $\Sigma_{n<u} e^{f(n)}$ are equal to within factors of $1 + O(1/\sqrt{\log x})$.

We then show that $x\Sigma_{n<u} e^{f(n)} = x\int_0^u e^{f(s)}\, ds(1 + O(\log^{-1/4} x))$, and evaluate this integral to an accuracy of $1 + O(\log^{-1/4} x)$ to obtain Theorem 2 and the Corollaries.

None of this to come is all that deep or complex; it is standard calculus and some lengthy but routine parts are left to the reader.

4. Calculus. Although x and y do not appear explicitly in '$f(s)$', strictly speaking we should write $f(s, x, y)$. This said, we have

$$f'(s) = -2\log s + \log\log x + \log(1 - s/u) - \frac{s-1}{u-s} \tag{4.1}$$

and

$$f''(s) = -\left(\frac{2}{s} + \frac{1}{u-s} + \frac{u-1}{(u-s)^2}\right). \tag{4.2}$$

Let $S = S(x, y)$ be the unique solution of

(4.3)

$$-2\log S + \log\log x + \log(1 - S/u) - (S-1)/(u-S) = 0, \qquad 0 < S < u.$$

Now recall that $r = \sqrt{\log x}/u$ and that $r \leqslant R$.

LEMMA 4.1. *For all $R > 0$ there exists $C(R) > 0$ such that if $0 < r \leqslant R$ and $u \geqslant C(R)$ then* $(3\log(e + r))(u - S) \geqslant u$.

PROOF. For $s < S$, $f' > 0$ while for $s > S$, $f' < 0$. Let $s_1 = u(1 - (3\log(e + r))^{-1})$. Then

$$\begin{aligned} f'(s_1) &= -2\log u - 2\log\left(1 - (3\log(e+r))^{-1}\right) + \log\log x - \log 3 \\ &\quad - \log\log(e+r) + 3\log(e+r)/u - 3\log(e+r) + 1 \\ &< -\log(e+r) - 2\log(2/3) - \log 3 + 3\log(e+r)/u + 1 \\ &< \log 3 - 2\log 2 + 1 - (1 - 3/u) = \log(3/4) + 3/u \\ &< 0 \quad \text{for } u > 3/\log(4/3). \end{aligned}$$

Thus $f'(s_1) < 0$ so $s_1 > S$, which proves the lemma with $C = 3/\log(4/3)$.

LEMMA 4.2. *For fixed x, $S = S(r)$ is decreasing as a function of r for $r \geqslant 0$, and* $\lim_{r\to 0} S(r) = \sqrt{\log x}$.

PROOF. By definition

$$-2\log S + \log\log x + \log(1 - S/u) - (S-1)/(u-S) = 0.$$

Let $t(r) = S(r)/\sqrt{\log x}$. Then

$$0 = -2\log t + \log(1 - rt) - \frac{rt}{1-rt}\left(1 - 1/\sqrt{\log x}\right). \tag{4.4}$$

Thus $dt/dr = t'$ satisfies

$$-\left(2/t + 2r/(1-rt) + (r/(1-rt))^2\left(t - 1/\sqrt{\log x}\right)\right)t'$$
$$= t/(1-rt) + (1-rt)^{-2}\left(t - 1/\sqrt{\log x}\right),$$

so $t' < 0$. Inspection of (4.4) shows $\lim_{r\to 0} t(r) = 1$. □

With the same notation we also have

$$t(r) \geqslant 1/(3+3r) \quad \text{for } r > 0. \tag{4.5}$$

PROOF. The right side of (4.4) is decreasing in t for fixed r, and is positive when $t = (3+3r)^{-1}$. Thus $t(r) > 1/(3+3r)$.

In particular, $t(r) \geqslant 1/(3+3R)$ for $0 < r \leqslant R$, and so $S(r) \geqslant K\sqrt{\log x}$ for $0 < r \leqslant R$, with $K = 1/(3+3R)$ constant. From Lemma 4.1 and from $0 < S(r) < u$, it follows that $1/S$, $1/(u-S)$ and $u/(u-S)^2$ are all $O(1/\sqrt{\log x})$ uniformly in $r \leqslant R$ as $x \to \infty$. Thus

$$|f''(S)| = O\left(1/\sqrt{\log x}\right) \quad \text{uniformly in } r \leqslant R. \tag{4.6}$$

LEMMA 4.3. *Uniformly in* $0 < r \leqslant R$ *and in* h, $|h| \leqslant (S(r))^{2/3}$,

$$|f''(S) - f''(S+h)| = O(|h|/\log x)$$

and

$$f(S+h) = f(S) + \frac{1}{2}h^2 f''(S) + O\left(|h|^3/\log x\right).$$

PROOF. The second assertion is a direct consequence of the first. The first follows from the calculation

$$f''(S+h) - f''(S) = \left(\frac{2}{S+h} - \frac{2}{S}\right) + \left(\frac{1}{u-S-h} - \frac{1}{u-S}\right)$$
$$+ (u-1)\left(\frac{1}{(u-S-h)^2} - \frac{1}{(u-S)^2}\right)$$
$$= -\frac{2h}{S(S+h)} + \frac{h}{(u-S)(u-S-h)} + \frac{2(u-1)h}{(u-S)(u-S-h)^2}$$
$$-\frac{h^2(u-1)}{(u-s)(u-S-h)^2}.$$

Now from (4.5) and Lemma 4.1, respectively, $1/S = O(1/u)$ and $1/(u-S) = O(1/u)$. Thus the last expression for $f''(S+h) - f''(S)$ is

$$h\left\{O(1/u^2) + O(1/u^2) + O(1/u^2) + O(u^{2/3}/u^3)\right\} = h \cdot O(1/u^2).$$

Since $u = \log x/\log y$ and $r = \log y/\sqrt{\log x} \leqslant R$, $1/u^2 = O(1/\log x)$. Thus

$$|f''(S) - f''(S+h)| = O(|h|/\log x)$$

as claimed.

Now let $g(s) = e^{f(s)}$. Then $g(s)$ is log-concave with its maximum at S. Heuristically, in $\int_0^u g(s)\,ds$ most of the mass is found near S. Near S though, $g(s)$ is approximately normal, and the approximation has a well-known integral.

To prove the claims of (3.17) and the following paragraph, we first note that

$$\left| \sum_{n<u} e^{f(n)} - \int_0^u e^{f(s)}\,ds \right| \leqslant 2e^{f(S)}.$$

Next, let $\bar{m}$ and $\bar{n}$ be the (real) solutions less and more than r, respectively, of $f(\bar{m}) = f(\bar{n}) = f(S) - \frac{1}{2}\log\log x$.

By Lemma 3.3,

$$\int_0^u g(s)\,ds = \int_{\bar{m}}^{\bar{n}} g(s)\,ds\big(1 + O(\log^{-1/2} x)\big). \tag{4.7}$$

We now claim that $|\bar{m} - S| \leqslant S^{3/5}$, $|\bar{n} - S| \leqslant S^{3/5}$. For if $h = \pm S^{3/5}$ then

$$\begin{aligned} f(S+h) &= f(S) + \frac{1}{2}h^2 f''(S) + O\big(|h|^3/\log x\big) \\ &\leqslant f(S) - h^2/S + O(\log^{-1/10} x) \\ &\leqslant f(S) - S^{1/5} + O(\log^{-1/10} x) \\ &\leqslant f(S) - \log\log x \quad \text{for large } s, \text{ since } S \geqslant K\sqrt{\log x}\,. \end{aligned}$$

Now a little calculation (left to the reader) from Lemma 4.1 shows $\bar{n} \leqslant (1 - \frac{1}{2}K)u$, and another from $\log y \geqslant \log^{1/5} x$, $r = \log y/\sqrt{\log x}$, $S \geqslant \sqrt{\log x}/(3 + 3r)$ and $u = \sqrt{\log x}/r$ shows $\bar{m} \geqslant S - S^{3/5} \geqslant \sqrt{u}$ for large x. This completes the proof of (3.17).

Now

$$\int_{\bar{m}}^{\bar{n}} g(s)\,ds = e^{f(s)} \int_{\bar{m}}^{\bar{n}} \exp\left\{ \frac{1}{2}(s - S)^2 f''(S) + O\big((s - S)^3/\log x\big) \right\} ds,$$

and

$$\begin{aligned} &\int_{\bar{m}}^{\bar{n}} \left| g(s) - \exp\left\{ f(S) + \frac{1}{2}(s - S)^2 f''(S) \right\} \right| ds \\ &\qquad = O\int_{\bar{m}}^{\bar{n}} \exp\left(f(S) + \frac{1}{2}(s - S)^2 f''(S)\big((s - S)^3/\log x\big) \right) ds \end{aligned}$$

because $|(s - S)^3/\log x| \leqslant \log^{-1/10} x$ for $\bar{m} \leqslant s \leqslant \bar{n}$. We calculate this last integral as

$$\begin{aligned} &O(\log^{-1} x) \int_{-\infty}^{\infty} \exp\left(f(S) + \frac{1}{2}v^2 f''(S) \right) v^3\,dv \\ &\qquad = O\big(e^{f(S)}/\log x\big) \int_{-\infty}^{\infty} v^3 \exp\left(\frac{1}{2}v^2 f''(S) \right) dv \\ &\qquad = O\big(e^{f(S)}\big) \quad \text{since } f''(S) < -\frac{2}{S} < -\frac{\text{const}}{\sqrt{\log x}}. \end{aligned}$$

On the other hand,

$$\int_{\bar m}^{\bar n} \exp\left(f(S) + \frac{1}{2}(s-S)^2 f''(S)\right) ds$$
$$= \int_{-\infty}^{\infty} \exp\left(f(S) + \frac{1}{2}v^2 f''(S)\right) dv\left(1 + O\left(\log^{-1/2} x\right)\right)$$

by Lemma 3.3, and this equals $\sqrt{2\pi}\,|f''(S)|^{-1/2} e^{f(S)}$.

Since $|f''(S)| > 1/S > 1/\sqrt{\log x}$,

$$e^{f(S)} = O\left(\log^{1/4} x\right) \int_{\bar m}^{\bar n} \exp\left(f(S) + \frac{1}{2}(s-S)^2 f''(S)\right) ds.$$

Putting this together, we get

$$\int_{\bar m}^{\bar n} g(s)\, ds = \sqrt{2\pi}\,|f''(S)|^{-1/2} e^{f(S)}\left(1 + O\left(\log^{-1/4} x\right)\right) \tag{4.8}$$

and

$$A(x, y) = \sqrt{2\pi}\,|f''(S)|^{-1/2} e^{f(S)}\left(1 + O\left(\log^{-1/4} x\right)\right). \tag{4.9}$$

This is essentially Theorem 2. It remains only to simplify $|f''(S)|^{-1/2} e^{f(S)}$. We first note that if $\bar S$ is the solution in $0 \leqslant s \leqslant u$ of

$$2\log s - \log(1 - s/u) + s/(u-s) = \log\log x \tag{4.10}$$

(s instead of $s-1$ in $(s-1)/(u-s)$ of (4.3)) then

$$|\bar S - S| = O(1) \tag{4.11}$$

(left to reader). Since $f''(S) = O(1/\sqrt{\log x})$, it follows by Lemma 4.3 that

$$|f(\bar S) - f(S)| = O\left(\log^{-1/2} x\right).$$

Let $q = q(r) = \bar S(r)/\sqrt{\log x}$ and let $w = w(r) = rq(r)$. Recall $u = \sqrt{\log x}/r$. The original hypotheses for Theorem 2 were that $u \geqslant \log^{1/5} x$ and $r \leqslant R$.

Let

$$C(r) = \frac{1}{2\sqrt{\pi}}\left(1 - w + \frac{1}{2}w^2\right)^{-1/2} \quad \text{and} \quad h(r) = 2q(r) + \frac{w(r)q(r)}{1 - w(r)}.$$

These definitions agree with those in the introduction, the verification is left to the reader.

A short table of calculus facts for $q(r)$, $w(r)$ and $h(r)$ may be helpful. In every case the derivation is a routine calculation, which we omit.

$$dq/dr = -q^2(2-w)/(2 - 2w + w^2) < 0, \tag{4.12}$$
$$dw/dr = 2q(1-w)^2/(2 - 2w + w^2) > 0,$$
$$d^2q/dr^2 = 2q^3(3w^2 - 8w + 6)/(2 - 2w + w^2)^3 > 0;$$

$$q = 1 - r + \frac{3}{4}r^2 + O(r^3), \quad w = r - r^2 + O(r^3), \tag{4.13}$$
$$h = q\left(1 + (1-w)^{-1}\right) = 2 - r + \tfrac{1}{2}r^2 + O(r^3);$$

(4.14)
$$dh/dr = -q^2/(1-w) < 0,$$
$$d^2h/dr^2 = q^3w/\left((1-w)(2-2w+w^2)\right) > 0;$$

(4.15)

r	$h(r)$	r	$h(r)$
0	2		
0.1	1.90483	0.6	1.54523
0.2	1.81865	0.7	1.49077
0.3	1.74048	0.8	1.44062
0.4	1.66939	0.9	1.39428
0.5	1.60456	1.0	1.35136
2	1.04750	1000	0.01238
5	0.66283	10000	0.00167
10	0.43586	100000	0.000210
20	0.27363	1000000	0.0000254
50	0.14035		
100	0.08224		

Now since $|f(\bar{S}) - f(S)| = O(\log^{-1/2} x)$ and since $|f''(\bar{S}) - f''(S)| = O(1/\log x)$ while $|f''(S)| > 1/\sqrt{\log x}$, the right side of (4.9) is disturbed only by a factor of $1 + O(1/\sqrt{\log x})$ when S is replaced with $\bar{S}$. If we now express the modified right side in terms of r, $q(r)$, $w(r)$, $C(r)$ and $h(r)$ it simplifies to

(4.16) $$A(x, y) = C(r)x\log^{-3/4} x \exp\left(h(r)\sqrt{\log x}\right)\left(1 + O(\log^{-1/4} x)\right),$$

uniformly in y such that $\log x/\log y > \log^{1/5} x$ and $\log y/\sqrt{\log x} \leqslant R$. This proves Theorem 2.

For the corollaries let $\theta = \log y/\log^{1/4} x$ and assume $\theta = O(1)$, $x \to \infty$. Then $r = \theta \log^{-1/4} x$, $C(r) = (1 + O(\log^{-1/4} x))/2\sqrt{\pi}$, and by the Taylor series expansion of $h(r)$,

$$\exp\left(h(r)\sqrt{\log x}\right) = \exp\left(2\sqrt{\log x} - \theta\log^{1/4} x + \frac{1}{2}\theta^2 + o(1)\right)$$
$$\sim y^{-1}\exp\left(2\sqrt{\log x} + \frac{1}{2}\theta^2\right),$$

so that $A(x, y) \sim y^{-1}e^{\theta^2/2}A(x, 1)$. □

REMARK. We have not considered u fixed ($u = \log x/\log y$) as $x \to \infty$. Ad hoc calculations suggest that for this case, if n is an integer $\geqslant 1$ and if $n < u \leqslant n + 1$ then

$$A(x, x^{1/u}) \sim \frac{(u-n)^{n-1}}{n((n-1)!)^2} x(\log y)^{n-1}$$

(conjecture).

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DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843