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THE RADIUS RATIO AND CONVEXITY PROPERTIES IN NORMED LINEAR SPACES

BY

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ABSTRACT. The supremum of the ratios of the self-radius $r_A(A)$ of a convex bounded set in a normed linear space X to its absolute radius $r_X(A)$ is related to the supremum of the relative projection constants of the maximal subspaces of X. Necessary conditions and sufficient conditions for these suprema to be smaller than 2 are given. These conditions are selfadjoint superproperties similar to B-convexity, superreflexivity and P-convexity.

0. Notation. For a normed linear space X we use the following standard notation: X^* denotes the dual space of X. $B(X) = B_X(0, 1)$ denotes the closed unit ball of X, $\{x \in X;\ \|x\| \leq 1\}$, and $S(X) = S_X(0, 1)$ the unit sphere $\{x \in X;\ \|x\| = 1\}$. $B(x_0, r) = x_0 + rB(X)$ and $S(x_0, r) = x_0 + rS(X)$ are, respectively, the ball and sphere of radius r centered at x_0. For $A \subset X$, $\operatorname{diam} A \equiv \sup_{x,y\in A}\|x - y\|$ is the diameter of A. For $A \subset X$ and $y \in X$ we set $d(y, A) \equiv \inf_{x\in A}\|x - y\|$ and $r(y, A) \equiv \sup_{x\in A}\|x - y\|$. Observe that $|r(z, A) - r(y, A)| \leq \|z - y\|$. For $A, G \subset X$, $r_G(A) \equiv \inf_{y\in G} r(y, A)$ is the *relative radius* of A with respect to G. $r_A(A)$ is the *self-radius* of A and $r_X(A)$ is the *absolute radius* of A. Clearly $r(y, A) = r(y, \overline{\operatorname{conv}} A)$ and $r_G(A) = r_G(\overline{\operatorname{conv}} A)$, where $\operatorname{conv} A$ denotes the convex hull of A. Span A denotes the linear span of A.

The normed space Y is said to be *finitely represented* in the normed space X if, for every finite-dimensional subspace G of Y and every $\varepsilon > 0$, there is a linear isomorphism T from G onto a subspace of X such that $\|T\|\|T^{-1}\| < 1 + \varepsilon$. A property (P) of normed spaces is said to be a *superproperty* if whenever X has the property and Y is finitely represented in X, Y has it too. In particular, X is *superreflexive* if every space which is finitely represented in X is reflexive. Superreflexivity is known to be an *isomorphic property*, i.e. if X has it, so does every isomorph of it, and a *self-dual property*, i.e. X has it if and only if X^* has it **[16, 17]**.

If Y is a subspace of X, the *projection constant* of Y in X is

$$\lambda(Y, X) \equiv \inf\{\|P\|;\ P \text{ is a linear projection of } X \text{ onto } Y\}.$$

In particular, if Y is a maximal closed subspace, $Y = f^{-1}0$ for some $f \in S(X^*)$, then

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the linear projections of X onto Y are the operators $P = I - f \otimes z$, i.e. $Px = x - f(x)z$, where $z \in f^{-1}1$. Thus, for such Y,

$$\lambda(Y, X) = \inf_{z \in f^{-1}1} \sup_{x \in B(X)} \|x - f(x)z\|.$$

Since for every $\varepsilon > 0$ we can choose $z \in f^{-1}1$ with $\|z\| < 1 + \varepsilon$, it follows that $\lambda(Y, X) \leqslant 2$ for every maximal subspace Y of X. If X is an inner product space, then $\lambda(Y, X) = 1$ for every closed subspace Y. This paper deals with the problem of when the normed space X is closer to a Hilbert space in the sense that the bound for $\lambda(Y, X)$ is better than the trivial one, 2. Since inner product spaces are "the most convex", this can be regarded as a convexity property of X.

1. Projections onto maximal subspaces and the self-radius of convex subsets of the unit ball. Define the *hyperplane projection constant* of the normed space X to be

$$\begin{aligned} H(X) &\equiv \sup\{\lambda(H, X);\ H \text{ a maximal subspace of } X\} \\ &= \sup_{f \in S(X^*)} \inf_{z \in f^{-1}1} \sup_{x \in B(X)} \|x - f(x)z\|. \end{aligned}$$

As observed above, $1 \leqslant H(X) \leqslant 2$ for every normed X. Call X *H-convex* if $H(X) < 2$.

Two well-known classes of normed spaces are easily seen to be H-convex:

(a) Every finite-dimensional space is H-convex. In fact, if $\dim X = n$ then $H(X) \leqslant 2(n-1)/n$ (Bohnenblust [**4**]).

(b) Every uniformly nonsquare space is H-convex. Recall that X is called uniformly nonsquare (James [**15**]) if there is an $\varepsilon > 0$ such that for every $x, y \in B(X)$ we have $\min(\|x + y\|, \|x - y\|) < 2 - \varepsilon$. Indeed, if X is such and $f \in S(X^*)$, take any $z \in f^{-1}1$ with $\|z\| < 1 + \varepsilon/3$ and consider the projection $P = I - f \otimes z$ onto $f^{-1}0$. If for some $x \in B(X)$ we had $\|x - f(x)z\| > 2 - \varepsilon/3$, then we should have $\|f(x)z\| > 1 - \varepsilon/3$, hence

$$|f(x)| > \frac{1 - \varepsilon/3}{1 + \varepsilon/3}.$$

But then $y = f(x)z/\|f(x)z\|$ would satisfy both

$$\|x - y\| \geqslant \|x - f(x)z\| - \varepsilon/3 > 2 - \varepsilon$$

and

$$\|x + y\| \geqslant |f(x + y)| = |f(x)| + 1/\|z\| > 2 - \varepsilon,$$

contradicting the uniform nonsquareness condition. Thus $H(X) \leqslant 2 - \varepsilon/3$. (An indirect proof of this was given in [**9**].)

While these two classes have strong permanence properties, it is not clear from the definition that H-convexity is hereditary to subspaces. It does pass to quotients:

1.1. PROPOSITION. *If T: $X \to Y$ is a linear operator such that, for some β, $T\beta B(X) \supset B(Y)$, then $H(Y) \leqslant \beta\|T\|H(X)$.*

PROOF. Given $g \in Y^*$, let $f = g \circ T \in X^*$. If $P = I - f \otimes z$, $z \in f^{-1}1$ projects X onto $f^{-1}0$, then $Q = I - g \otimes Tz$ projects Y onto $g^{-1}0$ (since $g(Tz) = f(z) = 1$). If $y \in B(Y)$, take $x \in T^{-1}y$ with $\|x\| \leqslant \beta$. Then

$$\begin{aligned}\|Qy\| &= \|y - g(y)Tz\| = \|Tx - f(x)Tz\| \leqslant \|T\|\|x - f(x)z\| \\ &= \|T\|\|Px\| \leqslant \beta\|T\|\|P\|. \quad \blacksquare\end{aligned}$$

In particular, if $Y = X/Z$ (Z a closed subspace), then $H(Y) \leqslant H(X)$. If Y is K-isomorphic to X, then $H(Y) \leqslant KH(X)$.

We want to replace H-convexity by a more workable property which is obviously hereditary.

Define the *radius ratio constant* of the normed space X by

$$R(X) \equiv \sup\{r_A(A)/r_X(A);\ A \subset X \text{ nontrivial bounded convex}\}.$$

This is clearly also $\sup\{r_A(A);\ A \subset B(X) \text{ convex}\}$. Since for every set A and any X we have $r_X(A) \leqslant r_A(A) \leqslant \operatorname{diam} A \leqslant 2r_X(A)$, $1 \leqslant R(X) \leqslant 2$ for any normed X. Garkavi **[10]** and Klee **[20]** proved that $R(X) = 1$ characterizes inner product spaces among all normed spaces of dimension > 2 (in fact it suffices that $r_A(A) \leqslant 1$ for $A = \operatorname{conv}(x, y, z)$, $x, y, z \in B(X)$). We call X *R-convex* if $R(X) < 2$. This convexity property seems more natural, since it is easily seen to have permanence properties:

1.2. LEMMA. *If T is a linear operator from the subspace Y of X to Z such that $T\beta B(Y) \supset B(Z)$, then $R(Z) \leqslant \beta\|T\|R(X)$.*

PROOF. Given any $A \subset B(Z)$ convex, choose, for every $z \in A$, $y_z \in \beta B(Y)$ with $Ty_z = z$. Let $W = \operatorname{conv}\{y_z;\ z \in A\}$. Then $TW = A$. If $w_0 \in W$ is any, then

$$r_A(A) \leqslant r(Tw_0, A) = r(Tw_0, TW) \leqslant \|T\|r(w_0, W).$$

Hence

$$r_A(A) \leqslant \|T\|r_W(W) = \beta\|T\|r_{\beta^{-1}W}(\beta^{-1}W).$$

But $\beta^{-1}W$ is a convex subset of $B(X)$. $\blacksquare$

In particular: If Y is a closed subspace of X, then $R(Y) \leqslant R(X)$. If Y is K-isomorphic to a subspace of X, then $R(Y) \leqslant KR(X)$. If Y is a quotient of X, then $R(Y) \leqslant R(X)$.

It is easy to verify that finite-dimensional spaces and uniformly nonsquare spaces are R-convex: Suppose $A \subset B(X)$ is convex. If $\dim X = n$, observe that for every $x_0, \ldots, x_n \in A$ we have

$$\left\| x_j - \sum_{i=0}^{n} \frac{x_i}{n+1} \right\| = \left\| \frac{n}{n+1} x_j - \frac{1}{n+1} \sum_{i \neq j} x_i \right\| \leqslant \frac{2n}{n+1},$$

so that

$$\begin{aligned}\frac{1}{n+1} \sum_{i=0}^{n} x_i &\in \bigcap_{i=0}^{n} B\left(x_i, \frac{2n}{n+1}\right) \cap \operatorname{conv}(x_0, \ldots, x_n) \\ &\subset \bigcap_{i=0}^{n} B\left(x_i, \frac{2n}{n+1}\right) \cap A,\end{aligned}$$

and apply Helly's theorem and compactness. If X is uniformly nonsquare, then either diam $A \leqslant 2-\varepsilon$, hence $r_A(A) \leqslant 2-\varepsilon$, or pick $x, y \in A$ with $\|x-y\| > 2-\varepsilon$. By uniform nonsquareness $\|x+y\| < 2-\varepsilon$, and for every $z \in A$ we then have

$$\left\|\frac{x+y}{2}-z\right\| \leqslant \left\|\frac{x+y}{2}\right\| + \|z\| < 2-\frac{\varepsilon}{2}.$$

In fact we have:

1.3. PROPOSITION. *$R(X) \leqslant H(X)$ for every normed space X.*

PROOF. If $R(X)=1$, there is nothing to prove. Consider any convex $A \subset X$ with $r_A(A) > r_X(A)+\varepsilon$, $\varepsilon>0$. Let $C \equiv \{x \in X;\ r(x,A) < r_A(A)-\varepsilon\}$, $C_\varepsilon \equiv \{y \in X;\ d(y,C)<\varepsilon\}$. C and C_ε are convex nonempty open sets. For $y \in C_\varepsilon$ we have $r(y,A) < r_A(A)$, hence $C_\varepsilon \cap A = \varnothing$. Let H be a separating hyperplane between A and C_ε. Translating, we may assume H is a maximal subspace $H = f^{-1}0$, $f \in S(X^*)$ and $f(A) \leqslant 0$ while $f(C_\varepsilon) \geqslant 0$. But then $f(C) \geqslant \varepsilon$, hence $C \cap H = \varnothing$. Let P be any linear projection of X onto H. Let $z \in X$ satisfy $r(z,A) < r_X(A)+\varepsilon$. If $x \in A$ is any, let y be the point where the segment joining the points x and z, which lie on different sides of H, intersects H. Then

$$\begin{aligned}\|x-Pz\| &\leqslant \|x-y\| + \|y-Pz\| = \|x-y\| + \|Py-Pz\| \\ &\leqslant \|x-y\| + \|P\|\|y-z\| \\ &\leqslant \|P\|(\|x-y\| + \|y-z\|) = \|P\|\|x-z\| \\ &\leqslant \|P\| r(z,A) < \|P\|(r_X(A)+\varepsilon).\end{aligned}$$

Therefore $r(Pz,A) \leqslant \|P\|(r_X(A)+\varepsilon)$. But $Pz \in H$, hence $Pz \notin C$, i.e. $r(Pz,A) \geqslant r_A(A)-\varepsilon$. This shows that

$$r_A(A)-\varepsilon \leqslant \|P\|(r_X(A)+\varepsilon), \quad \text{i.e.} \quad \|P\| \geqslant (r_A(A)-\varepsilon)/(r_X(A)+\varepsilon).$$

Since P was any projection onto H,

$$(r_A(A)-\varepsilon)/(r_X(A)+\varepsilon) \leqslant \lambda(H,X) \leqslant H(X).$$

Since ε can be taken arbitrarily small, $r_A(A)/r_X(A) \leqslant H(X)$. ■

The converse implication between H-convexity and R-convexity follows from:

1.4. LEMMA. *Let $f \in S(X^*)$, $0 \leqslant \alpha \leqslant 1$ and $B_\alpha \equiv \{x \in B(X);\ f(x)=\alpha\}$. If $r_{B_\alpha}(B_\alpha) < 1+\alpha^2$, then $\lambda(f^{-1}0, X) \leqslant (7+\alpha^2)/4$.*

PROOF. Take $y \in f^{-1}1$ with $\|y\| < 1+\frac{1}{2}(1-\alpha)^2$ and $z \in B_\alpha$ with $r(z,B_\alpha) < 1+\alpha^2$. Define $P \equiv I - f \otimes \frac{1}{2}(y+\alpha^{-1}z)$. P clearly projects X onto $f^{-1}0$. If $\|x\| \leqslant 1$ and $f(x) \geqslant \alpha$, then $\alpha x/f(x) \in B_\alpha$. The ray from z to $\alpha x/f(x)$ extends at least $1-\alpha/f(x)$ more before it hits the boundary of B_α. Therefore

$$\left\|\frac{\alpha x}{f(x)}-z\right\| < (1+\alpha^2) - \left(1-\frac{\alpha}{f(x)}\right) = \alpha^2 + \frac{\alpha}{f(x)}.$$

Thus

$$\|Px\| \leqslant \left\| \frac{1}{2}(x - f(x)y) \right\| + \frac{1}{2}\left\| x - f(x)\frac{z}{\alpha} \right\| < 1 + \frac{1}{4}(1-\alpha)^2 + \frac{f(x)}{2\alpha}\left(\alpha^2 + \frac{\alpha}{f(x)}\right)$$

$$\leqslant 1 + \frac{1}{4}(1-\alpha)^2 + \frac{\alpha}{2} + \frac{1}{2} = \frac{7+\alpha^2}{4}.$$

If $\|x\| \leqslant 1$ but $0 \leqslant f(x) \leqslant \alpha$, then

$$\|Px\| \leqslant \|x\| + \frac{\alpha}{2}\left(\|y\| + \frac{\|z\|}{\alpha}\right) < 1 + \frac{\alpha}{2}\left(1 + \frac{1}{2}(1-\alpha)^2\right) + \frac{1}{2} < \frac{7+\alpha^2}{4}. \quad ■$$

1.5. Proposition. (a) *X is H-convex if and only if it is R-convex.* (b) *X is an inner product space if* $\dim X > 2$ *and* $H(X) = 1$.

Proof. (a) is immediate from Proposition 1.3 and Lemma 1.4. (b) is immediate from Lemma 1.4 and the Garkavi-Klee characterization (proved in [**2**]). ■

Recall the "self-Jung constant" of X, [**1**]

$$J_S(X) \equiv \sup\{2r_A(A)/\operatorname{diam} A;\ A \text{ a nontrivial convex subset of } X\}$$

which is $2/N(X)$ or $2/BS(X)$ in the notation of [**5**]. Clearly $R(X) \leqslant J_S(X)$. Inequality occurs, e.g. in Hilbert spaces, where $R(X) = H(X) = 1 < \sqrt{2} = J_S(X)$. Thus we have the immediate corollary:

1.6. Corollary. *If X has "uniformly normal structure", i.e. if $J_S(X) < 2$ (equivalently, if $N(X) = BS(X) > 1$), then X is H-convex.*

Computing the constants $H(X)$ or $R(X)$ is not easy, in general, except for the extremal values 1 and 2, e.g.:

1.7. Example. if X is a "flat" space in the sense of Harrell and Karlovitz [**14**] then there is a convex $A \subset S(X)$ with $r_A(A) = 2$. Indeed, the subsets $A = \chi(g, s)$ in [**24**, p. 203], are such.

It seems easier to deal with compact convex sets, therefore we define the *compact radius ratio constant* of X,

$$\mathrm{CR}(X) \equiv \sup\{r_K(K)/r_X(K);\ K \subset X \text{ nontrivial convex compact}\}.$$

Clearly

$$\mathrm{CR}(X) = \sup\{r_K(K);\ K \subset B(X) \text{ convex compact}\}$$
$$= \sup\{r_{K_n}(K_n);\ K_n = \operatorname{conv}(x_1,\dots,x_n);\ x_1,\dots,x_n \in B(X), n = 1,2,\dots\}.$$

(The last equality is obtained by approximating the compact K by such K_n with $\sup_{x\in K} d(x, K_n) < \varepsilon$, and then

$$r_{K_n}(K_n)/r_X(K_n) \geqslant r_K(K_n)/r_X(K) \geqslant (r_K(K) - \varepsilon)/r_X(K).)$$

Although for every compact $K \subset X$ we have $r_K(K)/r_X(K) \leqslant 2r_K(K)/\operatorname{diam} K < 2$, we still may have $\mathrm{CR}(X) = 2$. Call X *C-convex* if $\mathrm{CR}(X) < 2$.

1.8. Lemma. *If, for every finite-dimensional subspace F of Z, there is a linear operator T_F from a subspace Y_F of X to Z such that $T_F(\beta B(Y_F)) \supset B(F)$ and $\|T_F\| \leqslant \alpha$ (α, β constants), then* $\mathrm{CR}(Z) \leqslant \alpha\beta\mathrm{CR}(X)$.

PROOF. Same as Lemma 1.2, this time with finite A. ■

In particular: (a) If Z is finitely represented in X, then $\mathrm{CR}(Z) \leqslant \mathrm{CR}(X)$. (b) If Z is K-finitely represented in X, then $\mathrm{CR}(Z) \leqslant K\mathrm{CR}(X)$. (c) If $Z = X/Y$, then $\mathrm{CR}(Z) \leqslant \mathrm{CR}(X)$.

1.9. LEMMA. *If X is reflexive, then* $\mathrm{CR}(X) = R(X)$.

PROOF. Given a closed and convex bounded $A \subset B(X)$, let $\rho_0 \equiv \sup\{r_{\mathrm{conv}\, F}(F);\ F \subset A \text{ finite}\}$. For every $\rho > \rho_0$ and every finite $F \subset A$,

$$\bigcap_{y \in F} B(y, \rho) \cap A \supset \bigcap_{y \in F} B(y, \rho) \cap \mathrm{conv}\, F \neq \varnothing .$$

By weak compactness of the balls and of A, $\bigcap_{y \in A} B(y, \rho) \cap A \neq \varnothing$, which means $r_A(A) \leqslant \rho$. ■

REMARK. In nonreflexive spaces we may have A convex with

$$\sup\{r_{\mathrm{conv}\, F}(F);\ F \subset A \text{ finite}\} < r_A(A).$$

E.g. $A = \{x \in B(c_0);\ x(1) \geqslant x(2) \geqslant \cdots\}$. Clearly $r_A(A) = 1$. If $x^1, \ldots, x^m \in A$, let $k \equiv \min\{n;\ x^r(n) < \frac{1}{2}$ for some $r = 1, \ldots, m\}$, and $l \equiv \max\{n;\ x^s(n) \geqslant \frac{1}{2}$ for some $s = 1, \ldots, m\}$. Let $x = (x^r + x^s)/2$. If $n < k$, then $x^i(n), x(n) \in [\frac{1}{2}, 1]$, hence $|x^i(n) - x(n)| \leqslant \frac{1}{2}$. If $k \leqslant n \leqslant l$, then $\frac{1}{4} \leqslant x(n) < \frac{3}{4}$, hence $|x^i(n) - x(n)| \leqslant \frac{3}{4}$ $(i = 1, \ldots, m)$. If $l < n$, then $x^i(n)$, $x(n) \in [0, \frac{1}{2})$ and again $|x^i(n) - x(n)| \leqslant \frac{1}{2}$. Thus $r(x, (x^1, \ldots, x^m)) \leqslant \frac{3}{4}$.

Of course, in any non-inner-product space of dimension > 2 we have a convex A and a finite $F \subset A$ with $r_{\mathrm{conv}\, F}(F) > r_A(A)$—by the Garkavi-Klee theorem we can take A to be the unit ball and F to be a triplet.

1.10. COROLLARY. *Consider the following statement*:
(a) *C-convexity ⇒ H-convexity*;
(b) *C-convexity ⇒ reflexivity*;
(c) *C-convexity ⇒ superreflexivity*;
(d) *H-convexity is a superproperty*;
(e) *H-convexity ⇒ superreflexivity*;
(f) *H-convexity ⇒ reflexivity.*
Then (a) ⇔ (b) ⇔ (c) ⇒ (d) ⇒ (e) ⇒ (f).

PROOF. (c) ⇒ (b) trivially. (b) ⇒ (a) by Lemma 1.9. If X is not superreflexive then, by [**25**, p. 216], X is not "super nonflat", i.e. there is a flat Y finitely represented in X. By Example 1.7, Y is not H-convex. Therefore (a) ⇒ (c) (this Y, provided (a), will not be C-convex, and, by Lemma 1.8, neither can X be C-convex) and (d) ⇒ (e). (a) ⇒ (e) by Lemma 1.8 and (e) ⇒ (f) trivially. ■

REMARK. We do not know if any of (a)–(f) holds.

Our next step is to get lower estimates for $\mathrm{CR}(X)$ in some spaces.

1.11. LEMMA. *If X has a symmetric sequence (e_j) and we define*

$$\varphi_n(t) = \left\| (1-t)e_1 + \sum_{j=2}^{n} te_j \right\|,$$

then

$$\mathrm{CR}(X) \geqslant \sup_n \frac{\varphi_n(1/n)}{\min_t \varphi_n(t)}.$$

(Recall that a sequence (e_n) in a normed linear X is called "symmetric" if, for every n scalars, $\alpha_1, \dots, \alpha_n$, and every permutation σ of $1, \dots, n$,

$$\left(\left\| \sum_{j=1}^{n} \alpha_{\sigma(j)} e_j \right\| = \left\| \sum_{j=1}^{n} \alpha_j e_j \right\| \right).$$

PROOF. Consider $K_n = \mathrm{conv}(e_1, \dots, e_n)$. By symmetry,

$$\text{if} \quad y = \sum_{j=1}^{n} \alpha_j e_j \in K_n \quad \text{then} \quad r\left(\sum_{j=1}^{n} \alpha_{\sigma(j)} e_j,\, K_n \right) = r(y, K_n).$$

But

$$\sum_\sigma \alpha_{\sigma(j)} = \sum_{i=1}^{n} \sum_{\sigma(j)=i} \alpha_{\sigma(j)} = \sum_{i=1}^{n} (n-1)!\alpha_i = (n-1)!;$$

thus

$$\frac{1}{n!} \sum_\sigma \sum_{j=1}^{n} \alpha_{\sigma(j)} e_j = \frac{1}{n} \sum_{j=1}^{n} e_j$$

satisfies

$$\varphi_n\left(\frac{1}{n}\right) = r\left(\frac{1}{n} \sum_{j=1}^{n} e_j,\, K_n \right) \leqslant r(y, K_n),$$

i.e. $r_{K_n}(K_n) = \varphi_n(1/n)$. Clearly,

$$r_X(K_n) \leqslant r\left(\sum_{i=1}^{n} te_i,\, K_n \right) = \varphi_n(t) \quad \text{for every } t,$$

so

$$\mathrm{CR}(X) \geqslant \frac{\varphi_n(1/n)}{\varphi_n(t)} \quad \text{for all } t. \qquad \blacksquare$$

1.12. EXAMPLES. (a) $X = l_1$. Here $\varphi_n(t) = |1-t| + (n-1)|t|$.

$$\varphi_n(1/n) = 2(n-1)/n,$$

while $\min_t \varphi_n(t) = \varphi_n(0) = 1$. Therefore

$$\mathrm{CR}(l_1) \geqslant \sup_n 2\frac{n-1}{n} = 2,$$

i.e. $H(l_1) = R(l_1) = \mathrm{CR}(l_1) = J_S(l_1) = 2$.

(b) $X = c_0$. Here $\varphi_n(t) = \max(|1-t|, |t|)$, $\varphi_n(1/n) = (n-1)/n$, while $\min \varphi_n(t) = \varphi_n(\frac{1}{2}) = \frac{1}{2}$. Again, $H(c_0) = R(c_0) = \mathrm{CR}(c_0) = J_S(c_0) = 2$.

(c) $X = l_p$, $1 < p < \infty$. Here

$$\varphi_n(t) = \left(|1-t|^p + (n-1)|t|^p\right)^{1/p}, \qquad \varphi_n(1/n) = \frac{\left((n-1)^p + (n-1)\right)^{1/p}}{n},$$

$$\min_t \varphi_n(t) = \varphi_n\left(\left(1 + (n-1)^{1/p-1}\right)^{-1}\right) = \frac{\left((n-1)^{p/p-1} + (n-1)\right)^{1/p}}{1 + (n-1)^{1/p-1}}.$$

Therefore

$$R(l_p) = \mathrm{CR}(l_p) \geqslant M_p \equiv \sup_n (\psi_{n,p} \cdot \psi_{n,q}),$$

where

$$\psi_{n,p} = \left(\frac{(n-1)^{p-1} + 1}{n}\right)^{1/p} \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

When $n \to \infty$, $\psi_{n,p} \cdot \psi_{n,q} \to 1$, so that this sup is really a maximum.

(d) $X = c_0$ with Day's equivalent norm:

$$\|x\|_D = \left(\sum_{n=1}^{\infty} \left(\frac{x^*(n)}{2^n}\right)^2\right)^{1/2},$$

where $(x^*(n))_{n=1}^{\infty}$ is the nonincreasing rearrangement of $(|x(n)|)_{n=1}^{\infty}$. Here

$$\varphi_n(t) = \begin{cases} \left(\dfrac{(1-t)^2}{4} + \dfrac{t^2}{12}(1 - 4^{1-n})\right)^{1/2} & \text{if } t \leqslant \dfrac{1}{2}, \\[2ex] \left(\dfrac{t^2}{3}(1 - 4^{1-n}) + \dfrac{(1-t)^2}{4^n}\right)^{1/2} & \text{if } t \geqslant \dfrac{1}{2}, \end{cases}$$

$$\min \varphi_n(t) = \varphi_n\left(\frac{1}{2}\right), \quad \text{and} \quad \frac{\varphi_n(1/n)}{\varphi_n(1/2)} \nearrow \lim_n \frac{\varphi_n(1/n)}{\varphi_n(1/2)} = \sqrt{3}.$$

However, as follows from the next discussion, $(c_0, \|\ \|_D)$ is not C-convex.

Recall that X is called *B-convex* [**3, 11**] if l_1 is not finitely represented in X. B-convexity is a self-dual isomorphic superproperty and is equivalent to X not containing uniformly isomorphic copies of $(l_1^n)_{n=1}^{\infty}$, or to X having "a nontrivial type" in the sense of Maurey and Pisier [**22**].

1.13. COROLLARY. *If X is C-convex, then it is B-convex. More generally, if $p(X) \equiv \sup\{p;\ X$ has type $p\}$, then* $\mathrm{CR}(X) \geqslant M_p$ *(as defined in Example* 1.12(c)).

PROOF. The first part is immediate from Lemma 1.8 and Example 1.12(a). The second follows from the Maurey-Pisier theorem ($l_{p(X)}$ is finitely represented in X [**22**]) and Example 1.12(c). ∎

Since B-convexity is an isomorphic property, Day's space (Example 1.12(d)) is not B-convex, hence $\mathrm{CR}(c_0, \|\,\|_D) = 2$.

An equivalent formulation for B-convexity is **[12]**: For some n, ε,

$$\min_{\pm} \left\| \sum_{i=1}^{n} \pm x_i \right\| < n - \varepsilon$$

for every $x_1, \ldots, x_n \in B(X)$. Call this "$B(n, \varepsilon)$-convexity" or "$B(n)$-convexity". Reflexive spaces may fail to have it (e.g. $(\Sigma_{n=1}^{\infty} l_{\infty}^{n})_2$ or $(\Sigma_{n=1}^{\infty} l_n)_2$). Nonreflexive spaces may have it, e.g. James' "uniformly nonoctahedral", i.e. $B(3)$-convex, nonreflexive space **[18]** or even James' nonreflexive space of type 2 **[19]**. Uniformly nonsquare spaces are exactly the $B(2)$-convex spaces, and n-dimensional spaces are trivially $B(n+1)$-convex, and it is natural to ask whether B-convexity is not only necessary but also sufficient for H-convexity. This is discussed in the next section.

2. Superreflexivity type conditions and radius-ratio convexity. Call X *$I(n, \varepsilon)$-convex* if for every $x_1, \ldots, x_n \in B(X)$ we have $\min_{1 \leq k \leq n} \|\Sigma_{i \neq k} x_i - x_k\| < n - \varepsilon$, and I-convex if it is $I(n, \varepsilon)$-convex for some n, ε. Clearly no space is $I(2, \varepsilon)$-convex. Uniformly nonsquare spaces are $I(3, \varepsilon)$-convex. Another trivial implication is:

2.1. LEMMA. *$B(3, \varepsilon)$-convexity $\Rightarrow$ $I(4, \varepsilon)$-convexity.*

PROOF. If $\|x_1 + x_2 + x_3 - x_4\| \geq 4 - \varepsilon$ then, necessarily, $\|x_1 + x_2 + x_3\| \geq 3 - \varepsilon$. Similarly, if also $\|x_1 + x_2 - x_3 + x_4\|$, $\|x_1 - x_2 + x_3 + x_4\|$ and $\|-x_1 + x_2 + x_3 + x_4\| \geq 4 - \varepsilon$, then $\|x_1 + x_2 - x_3\|$, $\|x_1 - x_2 + x_3\|$ and $\|-x_1 + x_2 + x_3\| \geq 3 - \varepsilon$, i.e. if X is not $I(4, \varepsilon)$-convex it cannot be $B(3, \varepsilon)$-convex. ■

2.2. COROLLARY. *I-convexity $\not\Rightarrow$ reflexivity.*

PROOF. Immediate from Lemma 2.1 and James' nonreflexive $B(3)$-convex space mentioned above. ■

A more geometric formulation of I-convexity is:

2.3. LEMMA. *X is $I(n)$-convex if and only if, for some $\varepsilon > 0$, for every $x_1, \ldots, x_n \in B(X)$, $\min_{1 \leq k \leq n} d(x_k, \mathrm{conv}(x_i)_{i \neq k}) < 2 - \varepsilon$.*

PROOF. If $\|\Sigma_{i \neq k} x_i - x_k\| < (1 - \varepsilon)n$, then

$$\left\| \frac{1}{n-1} \sum_{i \neq k} x_i - x_k \right\| = \left\| \frac{1}{n-1} \Big(\sum_{i \neq k} x_i - x_k \Big) - \frac{n-2}{n-1} x_k \right\| < \frac{n(1-\varepsilon)}{n-1} + \frac{n-2}{n-1} < 2 - \varepsilon.$$

If $\|\Sigma_{i \neq k} \alpha_i x_i - x_k\| < 2 - \varepsilon$ for some $\alpha_i \geq 0$, $\Sigma_{i \neq k} \alpha_i = 1$, then

$$\left\| \sum_{i \neq k} x_i - x_k \right\| < 2 - \varepsilon + \sum_{i \neq k} (1 - \alpha_i) = n - \varepsilon. \quad ■$$

2.4. PROPOSITION. *If X is C-convex, then it is I-convex.*

PROOF. If not, then for every n there are $x_1,\dots,x_n \in B(X)$ with $\|\Sigma_{i\neq k} x_i - x_k\| \geqslant n - 1/n$ for $k = 1,\dots,n$. Let $K_n = \operatorname{conv}(x_1,\dots,x_n)$. If $x = \Sigma_{i=1}^n \alpha_i x_i \in K_n$, $\alpha_i \geqslant 0$, $\Sigma_{i=1}^n \alpha_i = 1$, let $\alpha_k \equiv \min_{1\leqslant i\leqslant n} \alpha_i \leqslant 1/n$. Then

$$\|x - x_k\| = \left\|\sum_{i=1}^n \alpha_i x_i - x_k\right\| \geqslant \left\|\sum_{i\neq k} x_i - x_k\right\| - \left\|\sum_{i\neq k} (1-\alpha_i)x_i\right\| - \|\alpha_k x_k\|$$
$$\geqslant \left(n - \frac{1}{n}\right) - \sum_{i\neq k}(1-\alpha_1) - \alpha_k = \left(n-\frac{1}{n}\right) - (n-1) + 1 - 2\alpha_k \geqslant 2 - \frac{3}{n}.$$

Thus

$$\operatorname{CR}(X) \geqslant r_{K_n}(K_n)/r_X(K_n) \geqslant 2 - 3/n \quad \text{for all } n. \quad \blacksquare$$

2.5. PROPOSITION. *$I(n)$-convexity is a self-dual superproperty. In particular, it passes to subspaces and to quotient spaces.*

PROOF. $I(n)$ is a superproperty by its definition. The self-duality comes from the symmetry of the sign-matrix I_n representing the $I(n,\varepsilon)$-convexity condition, $(I_n)_{i,j} = 1 - 2\delta_{ij}$:

$$\begin{pmatrix} -1 & 1 & 1 & & 1 \\ 1 & -1 & 1 & & 1 \\ 1 & 1 & -1 & \cdots & 1 \\ \\ 1 & 1 & 1 & \cdots 1 & -1 \end{pmatrix}.$$

Indeed, if $x_1,\dots,x_n \in B(X)$ satisfy $\|\Sigma_{i\neq k} x_i - x_k\| > n - \varepsilon$ for $k = 1,\dots,n$, take $f_1,\dots,f_n \in B(X^*)$ with $f_k(\Sigma_{i\neq k} x_i - x_k) > n - \varepsilon$. Then $f_k(x_i) > 1 - \varepsilon$ for $i \neq k$ and $f_k(x_k) < \varepsilon - 1$, so that

$$\left\|\sum_{k\neq j} f_k - f_j\right\| \geqslant \left(\sum_{k\neq j} f_k - f_j\right)(x_j) > n - n\varepsilon.$$

Thus if X is not $I(n,\varepsilon)$-convex, X^* is not $I(n, n\varepsilon)$-convex. The other implication, X $I(n)$-convex $\Rightarrow X^*$ $I(n)$-convex, is proved similarly, or by observing that since X^{**} is finitely represented in X [**24**, p. 165], it is also $I(n)$-convex. $\blacksquare$

The I-convexity condition resembles one of the characterizations of superreflexivity, which we shall call "J-convexity". X is *$J(n,\varepsilon)$-convex* if for every $x_1,\dots,x_n \in B(X)$,

$$\min_{i\leqslant k\leqslant n}\left\|\sum_{i=1}^k x_i - \sum_{i=k+1}^n x_i\right\| < n - \varepsilon.$$

X is superreflexive if and only if it is J-convex, i.e. $J(n,\varepsilon)$-convex for some n, ε [**24**, p. 215]. $J(2,\varepsilon)$-convexity is exactly $B(2,\varepsilon)$-convexity, i.e. uniform nonsquareness. To show that J-convexity does not imply I-convexity, we need the following simple combinatorial lemma:

2.6. LEMMA. *Let J_{2n} be the sign-matrix representing the $J(2n)$-convexity condition*

$$(J_{2n})_{i,k} = \begin{cases} 1 & \text{if } i + k \leqslant 2n + 1, \\ -1 & \text{if } i + k \geqslant 2n + 2, \end{cases}$$

$i, k = 1,\ldots,n$, and let M be any matrix obtained from J_{2n} by changing the sign of a number of rows in J_{2n}. Then no two rows of M are equal, and some column of M has the sum 0.

PROOF. The first claim is trivial, since the ith row of M has the only sign change at the $(n-i)$th element. For the second observe that the sums of columns are even, the first and last columns cannot have sums of the same sign (they always add to $+2$ or -2), and the sums of neighboring columns differ by 2 (since in each column there is only one sign-change of a row). ■

We also need the following lemma:

2.7. LEMMA. *Let $A \subset \{0, -1, 1\}^n$ and let (A, ε) be the following property of normed spaces X: $\forall x_1,\ldots,x_n \in B(X)$,*

$$\min_{\sigma \in A} \left\| \sum_{i=1}^{n} \sigma_i x_i \right\| < (1-\varepsilon) \sum_{i=1}^{n} |\sigma_i| .$$

Let Γ be an index set and E a "full function space on Γ" in the sense of [**6**, *p.* 35], *which is uniformly convex. Let $\{X_\gamma;\ \gamma \in \Gamma\}$ be normed spaces with the (A, ε)-property for a fixed $\varepsilon > 0$. Then the substitution space $(\Sigma_{\gamma\in\Gamma} X_\gamma)_E$ has the (A, δ)-property for some $\delta > 0$.*

In particular, l_p-sums of $I(n, \varepsilon)$-convex spaces are $I(n)$-convex and l_p-sums of $J(n, \varepsilon)$-convex spaces are $J(n)$-convex.

The proof of the J-convexity case, i.e. $|A| = 2^n$, is Lemma 16 and Theorem 17 of [**11**]. The general case follows from the same proof.

2.8. EXAMPLE. *A $J(6, \varepsilon)$-convex space which is not I-convex.* Let $E_n = \mathbf{R}^n$ with the norm

$$\|x\| \equiv \max\left(\|x\|_2,\ \max_{1\leqslant j\leqslant n} \left| \sum_{i\neq j} x_i - x_j \right| \right), \qquad X = \left(\sum_{n=1}^{\infty} E_n \right)_2 .$$

If $\{e_j^n\}_{j=1}^n$ is the unit vector basis in E_n, then $\|e_j^n\| = 1$, while $\|\Sigma_{i\neq j} e_i^n - e_j^n\| = n$ for all n, j so that E_n is not $I(n)$-convex, and X is not I-convex. We claim that E_n is $J(6, 0.02)$-convex.

Given $x^1,\ldots,x^6$ in $B(E_n)$, let $z_j = \Sigma_{i=1}^{j} x^i - \Sigma_{i=j+1}^{6} x^i$, $j = 1,\ldots,6$. For $x = \Sigma_{i=1}^{n} x_i e_i \in E_n$, set $f_k(x) = \Sigma_{i\neq k} x_i - x_k$. If $f_k(z_j) \geqslant 5.98$, then $0.98 \leqslant f_k(x^i) \leqslant 1$ for $i = 1,\ldots,j$ and $-1 \leqslant f_k(x^i) \leqslant -0.98$ for $i = j+1,\ldots,6$. Therefore if also $f_l(z_t) \geqslant 5.98$ for some l and some $t > j$, then $f_k(x^i), f_l(x^i) \in [0.98, 1]$ for $i = 1,\ldots,j$, which implies $|x_k^i - x_l^i| \leqslant 0.01$ for $i = 1,\ldots,j$. In the same way we also get $|x_k^i - x_l^i| \leqslant 0.01$ for $i = t+1,\ldots,6$. But for $j < i \leqslant t$ we get $x_k^i - x_l^i \in [0.98, 1]$. Similarly, when we have $-f_l(z_t) \geqslant 5.98$ then $|x_k^i - x_l^i| \leqslant 0.01$ for $j < i \leqslant t$ and $x_k^i - x_l^i \in [0.98, 1]$ for the other i's.

Now, from the six inequalities $\|z_j\| \geqslant 5.98$, $j = 1,\dots,6$, only one can result from the $\|\cdot\|_2$-norm (since $\|x\|_2, \|y\|_2 \leqslant 1 \Rightarrow \min\|x \pm y\|_2 \leqslant 2 - \sqrt{2}$). Therefore the other five inequalities are of the form $\varepsilon_i f_k(z_{j_i}) \geqslant 5.98$, where $j_1 < j_2 < \cdots < j_5$ and $\varepsilon_i \in \{\pm 1\}$. By Lemma 2.6, one of the corresponding columns of five signs in the matrix $(\varepsilon_i (J_6)_{ik})$ sums up to 1 or to -1. By our observations, the corresponding x^j must have at least 3 distinct coordinates which are 0.01-close one to the other, and two others which differ from them by 0.98 to 1. Therefore

$$\begin{aligned}\|x^j\|_2^2 &\geqslant \min_t \left(3(t + 0.01)^2 + 2(t + 0.98)^2\right) \\ &= \min_t (5t^2 + 3.98t + 1.9211) = 1.1991,\end{aligned}$$

which contradicts $\|x^j\| \leqslant 1$.

By Giesy's result (Lemma 2.7), X is $J(6)$-convex.

Another application of Lemma 2.7 is:

2.9. EXAMPLE. *A reflexive I-convex space which is not superreflexive.* Let X be the l_2-sum of the finite-dimensional subspaces of James' nonreflexive $B(3)$-convex space.

2.10. COROLLARY. *H-convexity, C-convexity and I-convexity are not isomorphic properties.*

PROOF. By the Enflo-James theorem [**24**, p. 163], superreflexivity is equivalent to uniform convexifiability, so that the non-I-convex space of Example 2.8 can be renormed to be uniformly convex, hence H-convex, C-convex and I-convex. ■

3. Sufficient conditions for H-convexity. A superclass of spaces containing the uniformly convex spaces, the uniformly smooth spaces and the finite-dimensional spaces, but strictly smaller than the class of superreflexive spaces, was introduced by Kottman [**21**]: Call X *$P(n, \varepsilon)$-convex* if for every $x_1,\dots,x_n \in B(X)$ we have $\min_{j\neq k}\|x_j - x_k\| < 2 - \varepsilon$, and *P-convex* if it is $P(n,\varepsilon)$-convex for some n, ε. However, uniformly nonsquare spaces are not necessarily P-convex and vice-versa, so that P-convexity does not seem to be an appropriately general sufficient condition for H-convexity. Sastry and Naidu [**23**] introduced the weaker *$O(n,\varepsilon)$-convexity*: For every $x_1,\dots,x_n \in B(X)$, $\min_{i\neq j}\min\|x_i \pm x_j\| < 2 - \varepsilon$. $O(n,\varepsilon)$-convexity is implied by $P(n,\varepsilon)$-convexity or by uniform nonsquareness and it implies $J(n,\varepsilon)$-convexity. It is a superproperty, but it is not self-dual. Sastry and Naidu also studied the dual properties, "E-convexity" and "F-convexity". None of the above is necessary for H-convexity, as shown by:

3.1. EXAMPLE. *An H-convex space which is neither O-convex nor E-convex.* This is $X = l_2$ with the equivalent norm $\max(\|x\|_2, \sup_{i\neq j}(|x_i| + |x_j|))$. It is shown in [**24**] that it is neither O-convex nor E-convex. However, since $\|x\|_2 \leqslant \|x\| \leqslant \sqrt{2}\|x\|_2$, by 1.1, $H(X) \leqslant \sqrt{2}$.

We try, therefore, for a more general class. Call X *$Q(n,\varepsilon)$-convex* if for no $x_1,\dots,x_n \in B(X)$ we have $\|\sum_{i=1}^{k-1} x_i - x_k\| \geqslant k - \varepsilon$ for $k = 1,\dots,n$, and *Q-convex* if it is $Q(n,\varepsilon)$-convex for some n, ε.

3.2. PROPOSITION. *X is $Q(n)$-convex if and only if, for some $\varepsilon > 0$ and every $x_1, \ldots, x_n \in B(X)$, we have*

$$\min_{1 \leqslant k < n} d\left(x_{k+1}, \operatorname{conv}(x_i)_{i=1}^n\right) < 2 - \varepsilon.$$

PROOF. If $\|\sum_{i=1}^{k-1} \alpha_i x_i - x_k\| < 2 - \varepsilon$ for some $\alpha_i \geqslant 0$, $\sum \alpha_i = 1$, then

$$\left\| \sum_{i=1}^{k-1} x_i - x_k \right\| < 2 - \varepsilon + \sum_{i=1}^{k-1} (1 - \alpha_i) = k - \varepsilon.$$

If $\|\sum_{i=1}^{k-1} x_i - x_k\| < k - \varepsilon$, then

$$\left\| \frac{1}{k-1} \sum_{i=1}^{k-1} x_i - x_k \right\| \leqslant \frac{1}{k-1} \left\| \sum_{i=1}^{k-1} x_i - x_k \right\| + \frac{k-2}{k-1} < 2 - \frac{\varepsilon}{k-1}. \quad \blacksquare$$

3.3. PROPOSITION. *$Q(n)$-convexity is a self-dual superproperty (hence passes to subspaces and to quotients). It is strictly between $O(n-1)$-convexity and $J(n)$-convexity or $I(n)$-convexity.*

PROOF. $Q(n)$-convexity is a superproperty by its definition. Self-duality, like those of $I(n)$-convexity and $J(n)$-convexity, follows from the symmetry of the representing sign-matrix:

$$(Q_n)_{i,j} = \begin{cases} 1 & \text{if } j < i, \\ -1 & \text{if } j = i, \\ 0 & \text{if } j > i. \end{cases}$$

Indeed, if $x_1, \ldots, x_n \in B(X)$ are such that $\|\sum_{i=1}^{k-1} x_i - x_k\| \geqslant k - \varepsilon$, $k = 1, \ldots, n$, take $f_k \in S(X^*)$ with $f_k(\sum_{i=1}^{k-1} x_i - x_k) \geqslant k - \varepsilon$. Then

$$\left\| \sum_{j=1}^{k-1} f_j - f_k \right\| \geqslant \left(\sum_{j=1}^{k-1} f_j - f_k \right)(x_k) \geqslant k - n\varepsilon.$$

If X is $O(n-1, \varepsilon)$-convex and $x_1, \ldots, x_n \in B(X)$, then either $\|x_j + x_k\| < 2 - \varepsilon$ for some $j < k < n$ and then $\|\sum_{i=1}^{n-1} x_i - x_n\| < n - \varepsilon$, or $\|x_j - x_k\| < 2 - \varepsilon$ for some $j < k$ and then $\|\sum_{i=1}^{k-1} x_i - x_k\| < k - \varepsilon$. Thus $O(n-1, \varepsilon)$-convexity implies $Q(n, \varepsilon)$-convexity. Since $Q(n)$-convexity is self-dual while $O(n)$-convexity and O-convexity are not, Q-convexity does not imply O-convexity.

If $\|\sum_{i=1}^{k-1} x_i - x_k\| < k - \varepsilon$, then clearly

$$\left\| \sum_{i=1}^{k-1} x_i - \sum_{i=k}^{n} x_i \right\| < n - \varepsilon \quad \text{and} \quad \left\| \sum_{i \neq k} x_i - x_k \right\| < n - \varepsilon,$$

so that $Q(n, \varepsilon)$-convexity implies both $J(n, \varepsilon)$-convexity and $I(n, \varepsilon)$-convexity. However, since J-convexity and I-convexity are not comparable, Q-convexity is not equivalent to any of them.

3.4. PROPOSITION. *If X is $Q(n, \varepsilon)$-convex, then $R(X) \leqslant 2 - \varepsilon/n$.*

PROOF. If $A \subset B(X)$ satisfies $r_A(A) > 2 - \varepsilon/n$, take $x_1 \in A$ any and, inductively, $x_{k+1} \in A$ with

$$\left\| x_{k+1} - \sum_{i=1}^{k} \frac{x_i}{k} \right\| > 2 - \frac{\varepsilon}{n}, \qquad k = 1,\dots,n-1.$$

If $y = \sum_{i=1}^{k} \alpha_i x_i$, $\alpha_i \geqslant 0$, $\Sigma\alpha_i = 1$, let $\alpha_j \equiv \max_{1 \leqslant i \leqslant k} \alpha_i \geqslant 1/k$. Then

$$\frac{1}{k} \sum_{i=1}^{k} x_i = \sum_{i \neq j} \beta_i x_i + \beta_j y,$$

where $\beta_j = 1/k\alpha_j$, $\beta_i = k^{-1}(1 - \alpha_i/\alpha_j)$ for $i \neq j$, $\beta_i \geqslant 0$ and $\sum_{i=1}^{k} \beta_i = 1$. Therefore

$$2 - \frac{\varepsilon}{n} < \left\| x_{k+1} - \sum_{i=1}^{k} x_i \right\| \leqslant \sum_{i \neq j} \beta_i \|x_{k+1} - x_i\| + \beta_j \|x_{k+1} - y\|$$
$$< 2(1 - \beta_j) + \beta_j \|x_{k+1} - y\|,$$

i.e.

$$\|x_{k+1} - y\| > 2 - \varepsilon/\beta_j n = 2 - \varepsilon k/n - \alpha_j \geqslant 2 - \varepsilon.$$

Thus X cannot be $Q(n, \varepsilon)$-convex. ∎

Observe that by Lemma 2.7, l_p-sums of $Q(n, \varepsilon)$-convex spaces are $Q(n)$-convex. Also, as in Corollary 2.10, Q-convexity (and also: P-convexity, O-convexity, E-convexity and F-convexity) is not an isomorphic property. Since Q-convexity is an intermediate property between uniform convexity and superreflexivity, which are isomorphically equivalent, Q-convexifiability is equivalent to superreflexivity (as are P-, O-, E- and F-convexities).

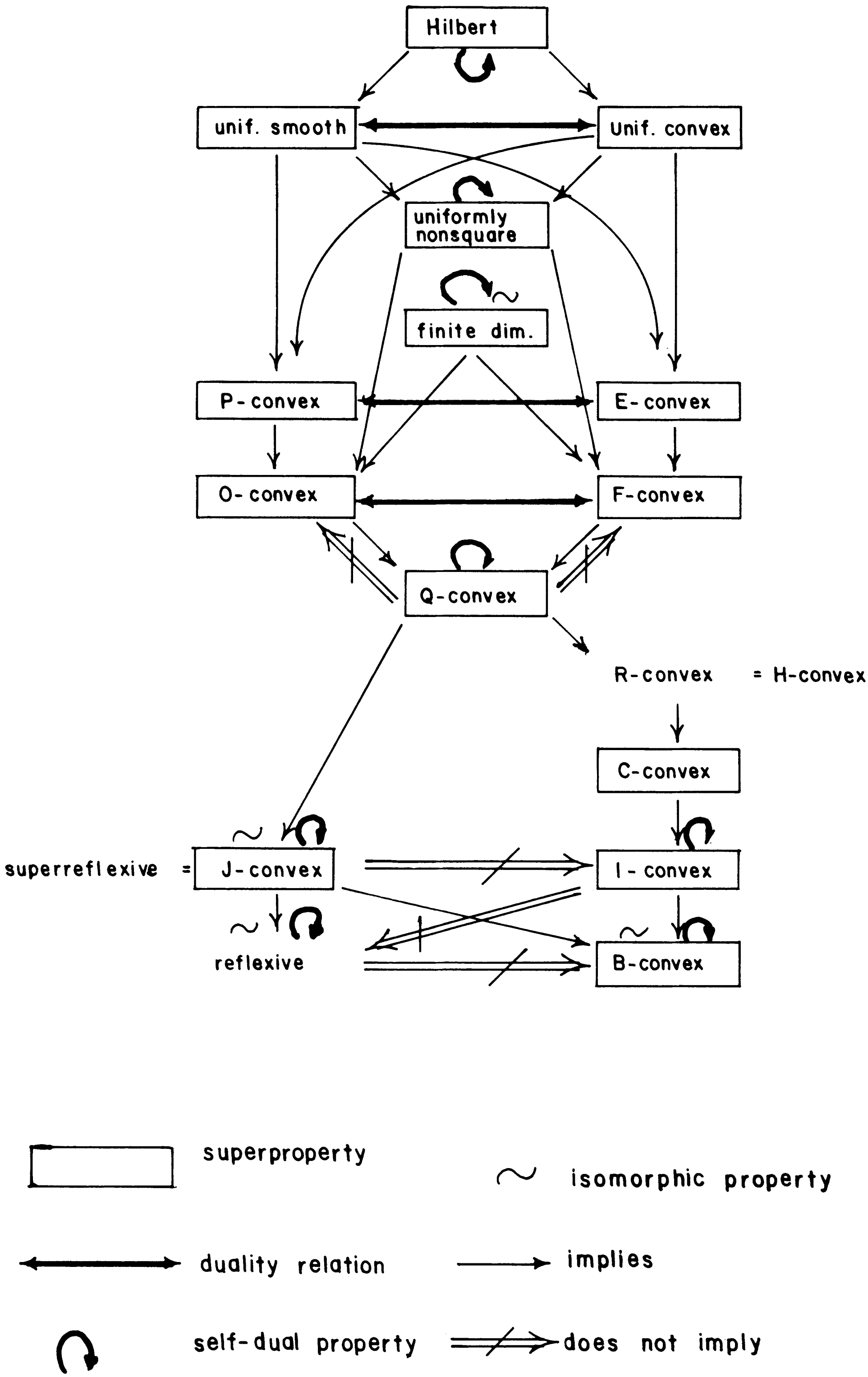
3.5. EXAMPLE. *The direct sum of two Q-convex spaces need not be even I-convex.* Let $X = (\sum_{n=1}^{\infty} \oplus E_n)_2$ be the space constructed in Example 2.8. Define on each E_n the projections:

$$P_n\left(\sum_{i=1}^{n} x_i^n e_i^n \right) \equiv \left(\sum_{i=1}^{n} x_i^n \right) e_n^n \quad \text{and} \quad Q_n \equiv I - P_n.$$

Let $P \equiv \sum_{n=1}^{\infty} \oplus P_n$, $Q \equiv \sum_{n=1}^{\infty} \oplus Q_n = I - P$. Since $\|P_1\| = 1$, $\|P_2\| = 2$ and for $n \geqslant 3$,

$$\|P_n x^n\| = \left| \sum_{i=1}^{n} x_i^n \right| = \frac{1}{n-2} \left| \sum_{k=1}^{n} \left(\sum_{i=1}^{n} x_i^n - 2x_k^n \right) \right| \leqslant \frac{n}{n-2} \|x^n\|,$$

we have $\|P\| \leqslant 3$. PX is clearly isometric to the Hilbert space l_2. In $Q_n E_n$ we have $\|x^n\| = \max(\|x\|_2, 2\|x\|_\infty)$, and $Q_n E_n$ is $2\sqrt{(n-1)/n}$-isomorphic to the Euclidean space l_2^n. Thus QX is 2-isomorphic to l_2, and X is isomorphic to $l_2 \oplus l_2 \sim l_2$. 2-isomorphism is just not enough to guarantee Q-convexity of QX, so that we have to prove it directly. We show that each $Q_n E_n$ is $Q(7, 0.1)$-convex. Indeed, assume $y^1, \dots, y^7 \in B(E_n)$ and $\|\sum_{i=1}^{k} y^i - y^{k+1}\| > k - 0.1$ for $k = 1, 2, \dots, 6$. At most one of these inequalities can result from the $\|\cdot\|_2$-norm. Each of the others is of the type $\varepsilon_k(\sum_{i=1}^{k} y_{j(k)}^i - y_{j(k)}^{k+1}) > k/2 - 0.05$ for some $j(k)$ and some $\varepsilon_k \in \{\pm 1\}$, and it implies $0.45 < \varepsilon_k y_{j(k)}^i \leqslant 0.5$ for $1 \leqslant i \leqslant k$ and $-0.5 \leqslant \varepsilon_k y_{j(k)}^{k+1} < -0.45$. We cannot


Hilbert
unif. smooth
Unif. convex
uniformly nonsquare
finite dim.
P-convex
E-convex
O-convex
F-convex
Q-convex
R-convex = H-convex
C-convex
superreflexive = J-convex
I-convex
reflexive
B-convex
superproperty
isomorphic property
duality relation
implies
self-dual property
does not imply

have $j(k)=j(l)$ for $1 \leqslant k < l$ and therefore we get five inequalities $|y^1_{j_t}| > 0.45$, $j_1 < j_2 < \cdots < j_5$, so that $\|y^1\|_2 > \sqrt{5} \cdot 0.45 > 1$, contradicting the choice $y^1 \in B(Q_n E_n)$.

REMARK. The first part of the discussion of this example could be used to show, without the combinatorial arguments, that Example 2.8 is superreflexive, and our convexity properties are not preserved under isomorphisms.

4. Concluding remarks and open problems. There is no comparability between the new convexity properties introduced here and the other hierarchies of convexity and smoothness properties weaker than uniform convexity or uniform smoothness, e.g. local uniform convexity, weak uniform convexity, uniform convexity in every direction, strict convexity, or Frechét differentiability, uniform Gateaux differentiability, and smoothness. None of the above is sufficient for B-convexity (e.g. c_0 can be equivalently renormed to have any of them; cf. [**6, 7, 25**]). None of them is necessary for Q-convexity (e.g. l^n_∞ fails them all). Similar examples can be given for near uniform convexity, normal structure, etc.

The implications between the properties we have discussed can be summarized in the diagram above: (see diagram). Relations between projection constants and the Jung constant $J(X) \equiv 2\sup\{r(A); A \subset X, \operatorname{diam} A = 1\}$ were established in [**13**] and [**8**].

We conclude with some open problems:

(1) Which of the statements in Corollary 1.10 are valid?

(2) What spaces can be equivalently renormed to become I-convex, C-convex or H-convex?

(3) Are C-convexity and H-convexity self-dual properties?

(4) Does I-convexity imply C-convexity? Does H-convexity imply Q-convexity?

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