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POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS— EXISTENCE AND NONEXISTENCE OF SOLUTIONS WITH RADIAL SYMMETRY IN $L_p(\mathbf{R}^N)$

BY

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ABSTRACT. It is shown that when r is nonincreasing, radially symmetric, continuous and bounded below by a positive constant, the solution set of the nonlinear elliptic eigenvalue problem

$$-\Delta u = \lambda u + ru^{1+\sigma}, \quad u > 0 \quad \text{on } \mathbf{R}^N, \qquad u \to 0 \quad \text{as } |x| \to \infty,$$

contains a continuum $\mathcal{C}$ of nontrivial solutions which is unbounded in $\mathbf{R} \times L_p(\mathbf{R}^N)$ for all $p \geqslant 1$. Various estimates of the L_p norm of u are obtained which depend on the relative values of σ and p, and the Pohozaev and Sobolev embedding constants.

Introduction.

1.1. *The main results*. Consider the problem

$$-\Delta u(\mathbf{x}) = \lambda u(\mathbf{x}) + r(\mathbf{x})u(\mathbf{x})^{1+\sigma}, \qquad \mathbf{x} \in \Omega, \tag{1.1}$$

$$u(\mathbf{x}) > 0, \qquad \mathbf{x} \in \Omega, \tag{1.2}$$

$$u(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^N$ is a spherical domain, $\sigma > 0$, $N \geqslant 2$ and r is continuous, radially symmetric and radially nonincreasing with $r > 0$ on $\bar{\Omega}$. When $N \geqslant 3$ and $\sigma \geqslant 4/(N-2)$, the celebrated inequality of Pohozaev [**10**] implies there are no solutions $u \neq 0$ of (1.1)–(1.3) with $\lambda \leqslant 0$. However, no matter what the value of $N \geqslant 2$ and $\sigma > 0$ there always exists a connected set of solutions (λ, u) of (1.1)–(1.3), which is unbounded in $\mathbf{R} \times L_p(\Omega)$, for all $p \in [1, \infty) \cap (N\sigma/2, \infty)$. This result is a consequence of Rabinowitz's global bifurcation theorem [**11**] (see Appendix) and is given in detail in Theorems 2.4, 2.6 and the remark following Theorem 2.6. The work of Gidas, Ni and Nirenberg [**6**] implies that all solutions u of (1.1)–(1.3) are radially symmetric and radially decreasing on Ω.

Here we consider the problem

$$-\Delta u(\mathbf{x}) = \lambda u(\mathbf{x}) + r(\mathbf{x})u(\mathbf{x})^{1+\sigma}, \qquad \mathbf{x} \in \mathbf{R}^N, \tag{1.4}$$

$$u(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty, \tag{1.5}$$

(1.6) *u is continuous, radially symmetric and radially decreasing*,

and obtain the following results which may be compared and contrasted with those just mentioned for (1.1)–(1.3). To make this presentation as simple as possible, we

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suppose r *is bounded below by a positive constant on* $\mathbf{R}^N$ *and is continuous, radially symmetric and radially nonincreasing.* (A much more involved theory pertaining to (1.4)–(1.6) when $r(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ will be presented elsewhere.)

First we observe that if $\lambda > 0$, there are no solutions of (1.4)–(1.6). (Note that if u satisfies (1.6) then it cannot be zero.) Now let $\mathcal{S}$ denote the set

$$\{(\lambda, u) \in (-\infty, 0] \times C^2(\mathbf{R}^N): (1.4)\text{–}(1.6)\ \textit{hold}\} \cup \{(0,0)\}.$$

The following remarks imply the existence of solutions of (1.4)–(1.6) regardless of the values of $\sigma > 0$ and $N \geqslant 2$. Let us suppose without further qualification that $1 \leqslant p \leqslant \infty$.

There is a set $\mathcal{C} \subset \mathcal{S}$, *such that* $(0,0) \in \mathcal{C}$, *which has the following properties*:

(I) $\mathcal{C}$ *is an unbounded, connected subset of* $(-\infty, 0] \times L_p(\mathbf{R}^N)$ *if* $p > N\sigma/2$. $\mathcal{C}$ *is not connected in* $(-\infty, 0] \times L_{N\sigma/2}(\mathbf{R}^N)$.

(II) $\{\lambda\colon (\lambda, u) \in \mathcal{C}\backslash\{(0,0)\}\} \supset (-\infty, 0)$ *if and only if* $N = 2$ *or* $N \geqslant 3$ *and* $\sigma < 4/(N-2)$.

(III) $\{\lambda\colon (\lambda, u) \in \mathcal{C}\} = \{0\}$ *if* $N \geqslant 3$ *and* $\sigma \geqslant 4/(N-2)$. *However this is not a necessary condition for* $(0, u) \in \mathcal{S}$, $u \neq 0$.

In addition, we establish the following inequalities concerning elements of $\mathcal{S}$. If $(\lambda, u) \in \mathcal{S}$:

(IV) $$\|u\|^p_{L_p(\mathbf{R}^N)} \geqslant (\text{const})\, u(0)^{p - N\sigma/2}, \qquad p \geqslant 1.$$

(V) $$\|u\|^p_{L_p(\mathbf{R}^N)} \leqslant (\text{const})\, u(0)^{p - N\sigma/2}, \qquad p > N\sigma/2.$$

(VI) *If* $N = 2$, *or if* $N \geqslant 3$ *and either* $\sigma \leqslant 2/(N-2)$ *or* $1 - r(\infty)/r(0) < (4 - \sigma(N-2))/2N$, *then* $u(0) \to 0$ *as* $\lambda \to 0$, $(\lambda, u) \in \mathcal{S}$. *Indeed if* $(0, u) \in \mathcal{S}$, *then* $u = 0$.

(In (IV) we cannot exclude the possibility that the left-hand side is infinite when $p \leqslant N\sigma/2$.)

(VII) *If* $\lambda < 0$, *then for all* ε, $0 < \varepsilon < |\lambda|$,

$$u(\mathbf{x}) \exp\left\{\sqrt{|\lambda| - \varepsilon}\, |\mathbf{x}|\right\} \to 0 \quad \textit{as } |\mathbf{x}| \to \infty.$$

(VIII) *If* $\lambda = 0$, *then*

$$u(\mathbf{x}) \geqslant (\text{const}) |\mathbf{x}|^{2-N}.$$

(VII) and (VIII) are elementary consequences of the differential equation. A more complete account of the asymptotic form of u is to be found in [7].

Existence questions for solutions of (1.3)–(1.6) have been considered by various methods and in circumstances when the nonlinearity is more complicated. We cite the work of Strauss [**12**], Stuart [**13, 14**], Berestický and Lions [**3**], and Berestický, Lions and Peletier [**4**] in this regard.

However, our intention is to present a theory which unifies the known existence results for the nonsingular problem on Ω (§2) and the existence theory on $\mathbf{R}^N$, given in §3, in the limit as the radius of Ω increases. In particular we wish to highlight the role of Pohozaev's idea in the context of $\mathbf{R}^N$ ((II), (III) and (VI) above), and the $L_p(\mathbf{R}^N)$ *a priori* bounds (IV), (V) which hold for elements of $\mathcal{S}$. The connectedness of $\mathcal{C}$ seems to be a particularly significant bonus from our method.

To some extent this scheme is close to that adopted by Beresticki and Lions [**3**]. However, while they examine an autonomous nonlinearity more general than ours, in our context their results are restrictive in requiring $\sigma < 2/(N-2)$ and r to be a constant. Stuart [**13**] has given a variational treatment of (1.4), (1.5) in $L_2(\mathbf{R}^N)$ which avoids the necessity to concentrate on radially symmetric solutions or radially symmetric r. However in the L_2 setting it is necessary to be restrictive about the size of σ, and questions of connectedness of the solutions set cannot be examined by variational arguments. Nonetheless, in certain respects, Stuart's theory is the most general so far available.

Our more specific theory for radially symmetric solutions exploits the usual one-dimensional formulation (3.13)–(3.15) in the spirit of [**2, 15**], and indeed it is difficult to see how so complete a theory might be obtained when r is not monotonic and radially symmetric. It is this in particular which makes possible the proofs of (III) and (VI) based on Pohozaev's method [**10**]. We might point out in passing that (VI) implies the nonexistence of nonzero radially symmetric, radially decreasing solutions of

$$-\Delta u(\mathbf{x}) = r(\mathbf{x})u(\mathbf{x})^{1+\sigma}, \qquad \mathbf{x} \in \mathbf{R}^N,$$

when $N = 2$ or $N \geqslant 3$, and σ, N and r satisfy the hypotheses in (VI) (Corollary 3.6). This is related to a Liouville-type results of Gidas and Spruck [**8**, Theorem 4.1] in the context of radially symmetric solutions and sheds a little light on their Example 1 [**8**, §4]. There $r(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, while in our work r is bounded below by a positive constant.

2. The nonsingular problem.

2.1. *A priori bounds*. Suppose r: $\mathbf{R}^N \to [\alpha, \infty)$, $\alpha > 0$, is continuous, radially symmetric and radially nonincreasing and u is a radially symmetric solution of the boundary-value problem

$$-\Delta u(\mathbf{x}) = \lambda u(\mathbf{x}) + r(\mathbf{x})u(\mathbf{x})^{1+\sigma}, \qquad \mathbf{x} \in \Omega_R, \sigma > 0, \tag{2.1}$$

$$u(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega_R, \tag{2.2}$$

where $\Omega_R = \{|\mathbf{x}| < R: \mathbf{x} \in \mathbf{R}^N\}$. Then

$$-(x^{N-1}u'(x))' = x^{N-1}\{\lambda u(x) + r(x)u(x)^{1+\sigma}\}, \tag{2.3}$$

$$u'(0) = u(R) = 0, \tag{2.4}$$

where $u(\mathbf{x}) = u(x)$, $r(\mathbf{x}) = r(x)$ and $x = |\mathbf{x}|$ for $\mathbf{x} \in \Omega_R$. From [**6**, Theorem 1] we have that every solution of (2.1), (2.2) with $u > 0$ on Ω_R is radially symmetric and is strictly radially decreasing. Thus every such solution of (2.1), (2.2) gives rise to a solution of (2.3), (2.4) which satisfies

$$u'(x) < 0, \qquad x \in (0, R]. \tag{2.5}$$

Let ψ_R denote the unique, normalised positive eigenfunction which satisfies

$$-(x^{N-1}\psi')' = \lambda x^{N-1}\psi \quad \text{on } (0, R),$$
$$\psi'(0) = \psi(R) = 0,$$

and let $\lambda_R > 0$ denote the corresponding eigenvalue. Clearly $\lambda_R \to 0$ as $R \to \infty$.

THEOREM 2.1. *Let* (λ, u) *be a solution of* (2.3)–(2.5). *Then* (a) $\lambda < \lambda_R$; (b) *if* $N \geq 3$ *and* $\sigma \geq 4/(N-2)$, *then* $\lambda > 0$; (c) *if* $N = 2$, *or if* $N \geq 3$ *and* $\sigma < 4/(N-2)$, *then for each* $\Lambda < \lambda_R$,

$$\sup\{|u|_{L_\infty(\Omega_R)}\colon (\lambda, u) \text{ satisfies } (2.3)\text{–}(2.5), \lambda \geq \Lambda\} < \infty.$$

PROOF. After multiplying (2.3) by ψ_R and integrating by parts, the positivity of u and ψ_R yields (a).

Now to prove (b) we adapt the proof of Pohozaev's inequality as follows:

$$\begin{aligned} (2.6)\quad \frac{1}{2}R^N u'(R)^2 &= \frac{1}{2}\int_0^R \frac{d}{dx}\{x^N(u'(x)^2 + \lambda u(x)^2)\}\,dx \\ &= \int_0^R x^N u'(x)\{u''(x) + \lambda u(x)\}\,dx \\ &\quad + \int_0^R \frac{N}{2}x^{N-1}(u'(x)^2 + \lambda u(x)^2)\,dx \\ &= \int_0^R x^N u'(x)\left\{u''(x) + \frac{N-1}{x}u'(x) + \lambda u(x) + r(x)u(x)^{1+\sigma}\right\}dx \\ &\quad + \int_0^R x^{N-1}\left\{\frac{2-N}{2}u'(x)^2 + \frac{N\lambda}{2}u(x)^2 - xr(x)u(x)^{1+\sigma}u'(x)\right\}dx \\ &= \int_0^R x^{N-1}\left\{\frac{2-N}{2}u'(x)^2 + \frac{N\lambda}{2}u(x)^2 - xr(x)u(x)^{1+\sigma}u'(x)\right\}dx, \end{aligned}$$

since u satisfies (2.3). Now note that if r were continuously differentiable then

$$(2.7)\qquad \int_0^R r(x)\{Nx^{N-1}u(x)^{\sigma+2} + (\sigma+2)x^N u(x)^{\sigma+1}u'(x)\} \geq 0$$

since the left-hand side is equal to

$$r(R)R^N u(R)^{2+\sigma} - \int_0^R r'(x)u(x)^{\sigma+2}x^N\,dx \geq 0.$$

For continuous, nonincreasing r, (2.7) also holds provided $u \geq 0$. Using this in (2.6) now yields that

$$\begin{aligned} \frac{1}{2}R^N u'(R)^2 \leq \int_0^R x^{N-1}&\left\{\frac{2-N}{2}u'(x)^2 + \frac{1}{2}\lambda N u(x)^2\right\} \\ &+ \frac{N}{2+\sigma}\int_0^R r(x)x^{N-1}u(x)^{\sigma+2}\,dx. \end{aligned}$$

Now this and the fact that

$$\int_0^R x^{N-1}u'(x)^2\,dx = \int_0^R x^{N-1}(\lambda u(x)^2 + r(x)u(x)^{2+\sigma})\,dx$$

(which follows from (2.3) and (2.4)) yield

$$\frac{1}{2}R^N u'(R)^2 \leq \lambda\int_0^R x^{N-1}u(x)^2\,dx + \frac{4-\sigma(N-2)}{2(\sigma+2)}\int_0^R x^{N-1}r(x)u(x)^{2+\sigma}\,dx.$$

If $4 - \sigma(N-2) \leq 0$, (b) follows.

(c) is a consequence of the main result of Gidas and Spruck [9], but their proof simplifies considerably in this special case. For completeness we give this simplified version.

Suppose $\Lambda < \lambda_R$ is given and $\{(\lambda_n, u_n)\}$ is a sequence of solutions of (2.3), (2.4) with $\lambda_n \in [\Lambda, \lambda_R)$ and $u_n(0) \uparrow \infty$ as $n \to \infty$.

Following Gidas and Spruck, let

$$v_n(x) = u_n(0)^{-1} u_n\left(x/u_n(0)^{\sigma/2}\right), \qquad x \in \left[0, u_n(0)^{\sigma/2} R\right].$$

Then for $x \in [0, u_n(0)^{\sigma/2} R]$,

$$-\left(x^{N-1} v_n'(x)\right)' = x^{N-1}\left\{\left(\lambda_n/u_n(0)^\sigma\right) v_n(x) + r\left(x/u_n(0)^{\sigma/2}\right) v_n(x)^{1+\sigma}\right\},$$
$$v_n(x) \leqslant v_n(0) = 1, \quad \text{and} \quad v_n'(x) \leqslant 0.$$

Now by standard estimates it is easy to deduce the existence of a nonincreasing function v: $[0, \infty) \to (0, \infty)$ such that

$$v_n(x) \to v(x) \text{ uniformly on compact intervals,}$$
$$-\left(x^{N-1} v'(x)\right)' = r(0) v(x)^{1+\sigma}, \qquad x \in (0, \infty),$$
$$v(0) = 1.$$

This contradicts Corollary 3.6. Q.E.D.

THEOREM 2.2. *The following inequailties are satisfied by solutions* (λ, u) *of* (2.3)–(2.5):

$$u'(x)^2 + \lambda u(x)^2 + \frac{2}{\sigma + 2} r(x) u(x)^{2+\sigma} > 0 \quad on\ [0, R]; \tag{2.8}$$

$$u'(x)^2 \geqslant (\text{const})\left\{u(X)^\sigma - u(x)^\sigma\right\} u(x)^2, \qquad x \in (X, R); \tag{2.9}$$

$$u(x)^\sigma \leqslant (\text{const}) u(0)^\sigma / \left(1 + u(0)^\sigma x^2\right), \qquad x \in (0, X). \tag{2.10}$$

Here $X \in (0, R]$ *denotes the unique point where*

$$\lambda + \frac{2}{\sigma + 2} r(x) u(x)^\sigma \begin{cases} > 0, & x \in [0, X), \\ < 0, & x \in (X, R]. \end{cases} \tag{2.11}$$

($X = R$ *of and only if* $\lambda \geqslant 0$.*) The constants are independent of* λ, u, r *and* R *and depend only on* N, σ *and* α.

PROOF. From (2.3) we find

$$-\int_x^R u'(x) u''(x)\, dx > \lambda \int_x^R u(x) u'(x)\, dx + \int_x^R r(x) u(x)^{1+\sigma} u'(x)\, dx,$$

whence

$$u'(x)^2 - u'(R)^2 > -\lambda u(x)^2 - \frac{2 r(x) u(x)^{2+\sigma}}{2 + \sigma}, \qquad x \in [0, R),$$

since $u' < 0$ and r is nonincreasing on $(0, R)$.

Therefore

$$u'(x)^2 + \lambda u(x)^2 + \frac{2 r(x) u(x)^{2+\sigma}}{2 + \sigma} > 0 \quad \text{on } [0, R].$$

Now since $u' < 0$ on $(0, R]$, and $u(R) = u'(0) = 0$, there exists a unique $X \in (0, R]$ satisfying (2.11). If $X \in (0, R)$, then $\lambda < 0$ and

$$|\lambda| = (2/(\sigma + 2))r(X)u(X)^{\sigma}.$$

Along with (2.8) this yields that on $(0, X)$

$$\begin{aligned} u'(x)^2 &> \frac{2}{\sigma + 2}\left(r(X)u(X)^{\sigma} - r(X)u(x)^{\sigma}\right)u(x)^2 \\ &\geq \frac{2r(x)}{\sigma + 2}\left(u(X)^{\sigma} - u(x)^{\sigma}\right)u(x)^2 \geq (\text{const})\left(u(X)^{\sigma} - u(x)^{\sigma}\right)u(x)^2, \end{aligned}$$

since $r(x) \geq \alpha$ and r is nonincreasing.

Finally, on $(0, X)$ equation (2.3) and inequality (2.11) give

$$-\left(x^{N-1}u'(x)\right)' \geq \frac{\sigma}{2 + \sigma}x^{N-1}r(x)u(x)^{1+\sigma} \geq \frac{\alpha\sigma}{2 + \sigma}x^{N-1}u(x)^{1+\sigma}.$$

Therefore

$$-x^{N-1}u'(x) \geq \frac{\alpha\sigma}{2 + \sigma}\int_0^x t^{N-1}u(t)^{1+\sigma}\,dt \geq \frac{\alpha\sigma}{N(2 + \sigma)}u(x)^{1+\sigma}x^N.$$

So

$$-u'(x)/u(x)^{1+\sigma} \geq (\alpha\sigma/N(2 + \sigma))x,$$

from which it follows that

$$\left(u(x)^{-\sigma} - u(0)^{-\sigma}\right) \geq \left(\alpha\sigma^2/2N(2 + \sigma)\right)x^2.$$

This completes the proof of the theorem. Q.E.D.

THEOREM 2.3. *If $p > \sigma/2$, there exists a constant (which depends only on σ, α and p and is independent of u, λ, r and R) such that*

$$\int_0^R u(x)^p\,dx \leq (\text{const})u(0)^{p-\sigma/2}.$$

PROOF. From (2.9) and the change of variable $t = u(x)$ we learn that

$$\begin{aligned} \int_X^R u(x)^p\,dx &\leq (\text{const})\int_0^{u(X)} \frac{t^{p-1}}{\sqrt{u(X)^{\sigma} - t^{\sigma}}}\,dt \\ &= (\text{const})\left(\int_0^1 \frac{t^{p-1}}{\sqrt{1 - t^{\sigma}}}\,dt\right)u(X)^{p-\sigma/2} \\ &= (\text{const})u(X)^{p-\sigma/2} \leq (\text{const})u(0)^{p-\sigma/2}, \qquad p \geq \frac{\sigma}{2}. \end{aligned} \tag{2.12}$$

Now from (2.10),

$$\begin{aligned}\int_0^X u(x)^p\,dx &\leq (\text{const})\int_0^X \frac{u(0)^p}{\left(1+u(0)^\sigma x^2\right)^{p/\sigma}}\,dx \\ &\leq (\text{const})\left(\int_0^{Xu(0)^{\sigma/2}} \frac{dt}{(1+t^2)^{p/\sigma}}\right)u(0)^{p-\sigma/2} \\ &\leq (\text{const})\left(\int_0^{\infty} \frac{dt}{(1+t^2)^{p/\sigma}}\right)u(0)^{p-\sigma/2} \\ &= (\text{const})u(0)^{p-\sigma/2}, \quad \text{provided } p>\sigma/2. \quad \text{Q.E.D.}\end{aligned}$$

2.2. *Existence theory*. The following theorem is a consequence of the maximum principle and the classical global bifurcation theory. Let $\mathcal{S}_R$ denote the set

$$\{(\lambda, u) \in (-\infty, \lambda_R) \times C^2([0, R])\colon (\lambda, u) \textit{ satisfies } (2.3)\text{–}(2.5)\} \cup \{(\lambda_R, 0)\},$$

and let $\mathcal{C}_R$ denote maximal subset of $\mathcal{S}_R$ connected in $\mathbf{R} \times C^2[0, R]$ and containing $(\lambda_R, 0)$.

THEOREM 2.4. *The set $\mathcal{C}_R$ is an unbounded subset of $[-\infty, \lambda_R] \times C^2[0, R]$. Moreover,*

(a) *if $N=2$, or if $N \geq 3$ and $\sigma < 4/(N-2)$, then*

$$\{\lambda\colon (\lambda, u) \in \mathcal{C}_R, u \neq 0\} = (-\infty, \lambda_R);$$

(b) *if $N \geq 3$ and $\sigma \geq 4/(N-2)$, then*

$$\{\lambda\colon (\lambda, u) \in \mathcal{C}_R, u \neq 0\} \subset (0, \lambda_R).$$

REMARK. In [**5**] Brezis reports on work with Nirenberg in which they show that in the case when $\sigma = 4/(N-2)$ and $r=$ constant the following more precise statement can be made:

$$\{\lambda\colon (\lambda, u) \in \mathcal{C}_R, u \neq 0\} = \begin{cases} (0, \lambda_R), & N \geq 4, \\ (\lambda_{R/4}, \lambda_R), & N = 3. \end{cases}$$

PROOF. The existence and unboundedness of $\mathcal{C}_R$ is established in $\mathbf{R} \times C^1[0, R]$ using the classical global bifurcation theorem of Rabinowitz [**11**] (see Appendix) and a standard application of the maximum principle. (For details of this argument in a similar situation, see [**2, 15, 16**].) The first part of the theorem is then immediate from the continuity of r and the differential equation (2.3).

Because of Theorem 2.1(a), (b), we need only concern ourselves with verification of (a). However, Theorem 2.1(c) implies $\mathcal{C}_R$ is bounded in $\mathbf{R} \times C[0, R]$ if $\{\lambda\colon (\lambda, u) \in \mathcal{C}_R\}$ is bounded in $\mathbf{R}$. This implies $\mathcal{C}_R$ is bounded in $\mathbf{R} \times C^2[0, R]$, by the differential equation, which is a contradiction. Q.E.D.

THEOREM 2.5. *The set $\mathcal{C}_R$ is unbounded and connected in $\mathbf{R} \times L_p(0, R)$ for all $p \geqslant 1$ such that $p > \sigma/2$. Indeed,*

$$\|u\|^p_{L_p(0,R)} \geqslant (\text{const}) \frac{u(0)^p}{\sqrt{\lambda_R + r(0)u(0)^\sigma}}, \qquad (\lambda, u) \in \mathcal{C}_R.$$

PROOF. Connectedness in $\mathbf{R} \times L_p(0, R)$ is implied by connectedness in $\mathbf{R} \times C^2[0, R]$. Suppose (λ, u) satisfies (2.3)–(2.5). Then

$$-\left(x^{N-1}u'(x)\right)' = x^{N-1}\left(\lambda u(x) + r(x)u(x)^{1+\sigma}\right), \qquad x \in (0, R),$$

whence

$$\begin{aligned} -x^{N-1}u'(x) &= \int_0^x t^{N-1}\left(\lambda u(t) + r(t)u(t)^{1+\sigma}\right) dt \\ &\leqslant \tilde{\beta} u(0) \int_0^x t^{N-1}\, dt = \beta^2 u(0) x^N, \end{aligned}$$

where

$$\beta^2 = \lambda_R + r(0)u(0)^\sigma \quad \text{and} \quad \tilde{\beta} = \beta^2 N.$$

Hence,

$$-u'(x) \leqslant \beta^2 u(0) x,$$

so

$$u(x) \geqslant u(0)(1 - \beta^2 x^2/2) \geqslant 0, \qquad x \in \left(0, \sqrt{2}/\beta\right). \tag{2.13}$$

Hence

$$\begin{aligned} \int_0^R u(x)^p\, dx &\geqslant u(0)^p \int_0^{\sqrt{2}/\beta} \left(1 - \frac{\beta^2 x^2}{2}\right)^p dx \\ &= \frac{\sqrt{2}\, u(0)^p}{\beta} \int_0^1 (1 - y^2)^p\, dy = (\text{const})\, \frac{u(0)^p}{\beta} \\ &= (\text{const}) \frac{u(0)^p}{\sqrt{\lambda_R + r(0)u(0)^\sigma}}. \end{aligned}$$

Since $\mathcal{C}_R$ is unbounded in $\mathbf{R} \times C^2[0, R]$, it is unbounded in $\mathbf{R} \times C[0, R]$, because elements of $\mathcal{C}_R$ are solution of the system (2.3)–(2.5). Therefore $\{(\lambda, u(0)) \colon (\lambda, u) \in \mathcal{C}_R\}$ is an unbounded subset of $\mathbf{R}^2$. If $\{\lambda \colon (\lambda, u) \in \mathcal{C}_R\}$ is unbounded, we are done. If not, then the inequality just established yields the conclusion that $\{\|u\|_{L^p(0,R)} \colon (\lambda, u) \in \mathcal{C}_R\}$ is unbounded since $p > \sigma/2$. Q.E.D.

THEOREM 2.6. *When regarded as a set of radially symmetric solutions of* (2.1), (2.2), *$\mathcal{C}_R$ is unbounded in $\mathbf{R} \times L_p(\Omega_R)$ for all $p > N\sigma/2$. Indeed,*

$$\|u\|^p_{L_p(\Omega_R)} \geqslant (\text{const}) u(0)^p / \left(\lambda_R + r(0)u(0)^\sigma\right)^{N/2}.$$

PROOF. From (2.13) we have

$$\int_0^R x^{N-1}u(x)^p\,dx \geqslant u(0)^p \int_0^{\sqrt{2}/\beta} x^{N-1}\left(1-\frac{\beta^2x^2}{2}\right)^p dx$$
$$= (2^{N/2})\frac{u(0)^p}{\beta^N}\int_0^1 y^{N-1}(1-y^2)^p\,dy$$
$$= (\text{const})\frac{u(0)^p}{\left(\lambda_R + r(0)u(0)^\sigma\right)^{N/2}}.$$

The proof is now exactly as before. Q.E.D.

REMARK. The conclusion of Theorem 2.6 is false when $\sigma \geqslant 4/(N-2)$ for all p, $1 \leqslant p < N\sigma/2$. To see this note that $\lambda \in (0, \lambda_R)$ by Theorem 2.1 and, as a consequence, $X = R$ in (2.11). Therefore (2.10) yields

$$x^2 u(x)^\sigma \leqslant \text{const},$$

from which it follows that

$$\int_0^R x^{N-1}u(x)^p\,dx \leqslant \int_0^R \frac{(\text{const})x^{N-1}}{x^{2p/\sigma}}dx \leqslant (\text{const}) \quad \text{if } p < \frac{N\sigma}{2}.$$

THEOREM 2.7. *When* (λ, u) *satisfies* (2.3)–(2.5),

$$(2N-3)\int_0^Y u'(x)^2\,dx - \lambda\int_0^Y u(x)^2\,dx \leqslant \frac{2r(0)}{\sigma+2}\int_0^Y u(x)^{\sigma+2}\,dx,$$

for any $Y \in (0, R]$.

PROOF. An integration over (x, R) after multiplication of (2.3) by u'/x^{N-1} gives

$$u'(x)^2 - 2(N-1)\int_x^R \frac{u'(x)^2}{x}dx$$
$$= -\lambda u(x)^2 + 2\int_x^R r(x)u(x)^{1+\sigma}u'(x)\,dx + u'(R)^2$$
$$\geqslant -\lambda u(x)^2 - 2r(x)u(x)^{2+\sigma}/(2+\sigma).$$

Now integration over $(0, Y)$ and an integration by parts gives

$$(3-2N)\int_0^Y u'(x)^2\,dx \geqslant -\lambda\int_0^Y u(x)^2\,dx - \frac{2r(0)}{\sigma+2}\int_0^Y u(x)^{\sigma+2}\,dx. \quad \text{Q.E.D.}$$

3. The singular problem.

3.1. *Existence theory.* To obtain solutions of

$$-\Delta u(\mathbf{x}) = \lambda u(\mathbf{x}) + r(\mathbf{x})u(\mathbf{x})^{1+\sigma}, \qquad \mathbf{x} \in \mathbf{R}^N, \tag{3.1}$$

$$u(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty, \tag{3.2}$$

$$u(\mathbf{x}) > 0, \qquad \mathbf{x} \in \mathbf{R}^N, \tag{3.3}$$

we examine the limit of the sets $\mathcal{C}_R$ as $R \to \infty$. To do this, extend each u as zero on (R, ∞) when $(\lambda, u) \in \mathcal{C}_R$. Then for each R the set $\mathcal{C}_R$ can be considered as a subset of either $\mathbf{R} \times L_p(\mathbf{R}^+)$ or $\mathbf{R} \times L_p(\mathbf{R}^N)$ when the radial symmetry of positive solutions of the PDE (2.1), (2.2) is recognised. Here, as elsewhere, $\mathbf{R}^+ = [0, \infty)$. To avoid confusion we consider, for the moment, $\mathcal{C}_R$ as a subset of $\mathbf{R} \times L_p(\mathbf{R}^+)$ and later return to the $\mathbf{R} \times L_p(\mathbf{R}^N)$ context.

Let $U \subset \mathbf{R} \times L_p(\mathbf{R}^+)$ be any bounded, open set with $(0,0) \in U$, and $p > \sigma/2$.

Since $\lambda_R \to 0$ as $R \to \infty$ and $(\lambda_R, 0) \in \mathcal{C}_R$, which is unbounded in $\mathbf{R} \times L_p(\mathbf{R}^+)$ (Theorem 2.5), there exists $(\lambda^R, u_R) \in \mathcal{C}_R$ such that

$$(\lambda^R, u_R) \in \mathcal{C}_R \cap \partial U,$$
$$u_R(x) > 0, \quad u_R'(x) < 0 \quad \text{on } (0, R),$$
$$u_R'(0) = u_R(R) = 0.$$

Now without loss of generality suppose $\lambda^R \to \lambda$ as $R \to \infty$. Since $\lambda^R \leqslant \lambda_R$ for each R, and $\lambda_R \to 0$, we conclude that

$$\lambda^R \to \lambda \leqslant 0 \quad \text{as } R \to \infty. \tag{3.4}$$

By Theorem 2.5, we know that $\|u_R\|_{L_\infty(0,R)}$ is bounded, since $p > \sigma/2$, and $\|u_R\|_{L_p(0,R)}$ is bounded. From the inequality in Theorem 2.7 we know that for fixed $Y \in (0, \infty)$, $\int_0^Y u_R'(x)^2\,dx \leqslant$ const. where the constant does not depend on R sufficiently large.

Therefore there exists a sequence $\{u_{R(n)}\}$ in the set $\{u_R\}_{R \in (0,\infty)}$ and a function $u \in L_p(\mathbf{R}^+)$ such that $R(n) \to \infty$,

$$u_{R(n)} \to u \text{ uniformly on compact subsets of } [0, \infty), \tag{3.5}$$

and (λ, u) satisfies

$$-(x^{N-1}u'(x))' = x^{N-1}(\lambda u(x) + r(x)u(x)^{1+\sigma}), \qquad x \in (0, \infty); \tag{3.6}$$

$$u(x) \to 0 \quad \text{as } x \to \infty; \tag{3.7}$$

$$u'(0) = 0; \qquad u(x) \geqslant 0, \quad u'(x) \leqslant 0, \qquad x \in (0, \infty). \tag{3.8}$$

For notational convenience let (λ_n, u_n) denote the sequence of solutions of (2.3)–(2.5) on $(0, R(n))$ satisfying (3.5).

THEOREM 3.1. *$(\lambda_n, u_n) \to (\lambda, u)$ in $\mathbf{R} \times L_p(\mathbf{R}^+)$, $p > \sigma/2$.*

PROOF. First note that the function u in (3.5) is not zero. If it were, then $u_n(0) \to 0$ and $\|u_n\|_{L_p(\mathbf{R}^+)} \to 0$ by Theorem 2.3. Moreover, since (λ_n, u_n) satisfies (2.3) on $(0, R(n))$ and u_n is nonincreasing, we have

$$\begin{aligned}
0 &< \left(\frac{\pi}{2}\right)^2 \int_0^1 u_n(x)\cos\left(\frac{\pi}{2}x\right)dx - \frac{\pi}{2}u_n(1)\\
&= -\int_0^1 u_n''(x)\cos\left(\frac{\pi}{2}x\right)dx\\
&\leqslant \int_0^1 \left(\lambda_n u_n(x) + r(x)u_n(x)^{1+\sigma}\right)\cos\left(\frac{\pi}{2}x\right)dx\\
&< \left(\lambda_n + r(0)u_n(0)^\sigma\right)\int_0^1 u_n(x)\cos\left(\frac{\pi}{2}x\right)dx.
\end{aligned}$$

This gives

$$\lambda_n + r(0)u_n(0)^\sigma \geqslant 0,$$

which implies $\lambda_n \to 0$ as $n \to \infty$, by (3.4) and the assumption $u_n(0) \to 0$.

Therefore $(\lambda_n, u_n) \to 0$ in $\mathbf{R} \times L_p(\mathbf{R}^+)$, which contradicts the fact that $(0,0) \in U$, an open subset of $\mathbf{R} \times L_p(\mathbf{R}^+)$, and $(\lambda_n, u_n) \in \partial U$. Hence the function u in (3.5) is not zero on $(0, \infty)$.

Let $X_n \in (0, R(n)]$ satisfy (2.11) when λ, u are R are replaced by λ_n, u_n and $R(n)$. Since $\|u_n\|_{L_p(\mathbf{R}^+)} \leqslant$ const., and u_n is nonincreasing, it follows that

$$xu_n(x)^p \leqslant \int_0^x u_n(t)^p\,dt \leqslant \text{const.},$$

whence

$$u_n(x) \leqslant (\text{const})/x^{1/p}, \tag{3.9}$$

where the constant is independent of n and of x.

Theorem 2.2 and the proof of Theorem 2.3 yield the following inequalities:

$$\int_y^{R(n)} u_n(x)^p\,dx \leqslant (\text{const})u_n(y)^{p-\sigma/2}, \qquad y \geqslant X_n, \tag{3.10}$$

and

$$x^2 u_n(x)^\sigma \leqslant \text{const.}, \qquad x \in (0, X_n). \tag{3.11}$$

If $X_n \to \infty$, then (3.5), (3.11) and the dominated convergence theorem yield

$$\int_0^{X_n} u_n(x)^p\,dx \to \int_0^\infty u(x)^p\,dx, \qquad p > \sigma/2. \tag{3.12}$$

Moreover, by (3.11),

$$u_n(X_n) \to 0 \quad \text{since } X_n \to \infty \text{ as } n \to \infty.$$

Therefore, by (3.10), with $y = X_n$, and (3.12),

$$\int_0^{R(n)} u_n(x)^p\,dx = \int_0^\infty u_n(x)^p\,dx \to \int_0^\infty u(x)^p\,dx$$

if $X_n \to \infty$ as $n \to \infty$.

Now suppose $\{X_n\}$ is bounded. Then for any $\varepsilon > 0$ there exists $Y > 0$ such that

$$\int_Y^\infty u_n(x)^p\,dx \leqslant \varepsilon, \quad \text{for all } n,$$

by (3.9) and (3.10). Since $u_n \to u$ uniformly on $[0, Y]$, it follows in this case also that

$$\int_0^\infty u_n(x)^p\,dx \to \int_0^\infty u(x)^p\,dx.$$

Since $\|u_n\|_{L_p(\mathbf{R}^+)} \to \|u\|_{L_p(\mathbf{R}^+)}$, and (3.5) holds, we have $u_n \to u$ in $L_p(\mathbf{R}^+)$. Q.E.D.

Now let $\mathbb{S}$ denote the set

$$\{(\lambda, u) \in \mathbf{R} \times C^2(\mathbf{R}^+)\colon (3.6)\text{–}(3.8)\ \textit{holds},\ u \neq 0\} \cup \{(0,0)\},$$

Clearly $\mathbb{S}$ also represents a set of radially symmetric solutions of (3.1)–(3.3).

THEOREM 3.2. *For* $p \geqslant 1$ *let* $\mathcal{C}(p)$ *denote the maximal subset of* $\mathcal{S}$ *connected in* $\mathbf{R} \times L_p(\mathbf{R}^+)$ *and containing* $(0,0)$. *Then* $\mathcal{C}(p)$ *is unbounded if* $p > \sigma/2$ *and* $\mathcal{C}(\sigma/2) = \{(0,0)\}$.

REMARK. This result requires no assumptions about $\sigma > 0$.

PROOF. If U is any bounded, open subset of $\mathbf{R} \times L_p(\mathbf{R}^+)$, $p > \sigma/2$, and $(0,0) \in U$, then Theorem 3.1 implies $\partial U \cap \mathcal{S} \neq \varnothing$. To show that $\mathcal{C}(p)$ is unbounded, $p > \sigma/2$, it is sufficient to show that bounded subsets of $\mathcal{S}$ are relatively compact in $\mathbf{R} \times L_p(\mathbf{R}^+)$, $p > \sigma/2$. (That this is sufficient for our purposes is the essential observation in **[11]**, and it is proved explicitly in [**1**, Appendix].) To obtain the relative compactness of bounded subsets of $\mathcal{S}$ essentially requires a proof identical to that of Theorem 3.1. Therefore, we will not repeat it.

Now suppose $1 \leqslant \sigma/2$. If (λ, u) satisfies (3.13)–(3.15), $u \neq 0$, Theorem 3.3 implies $\lambda \leqslant 0$ and $\|u\|_{L_{\sigma/2}(\mathbf{R}^+)} \geqslant$ const. > 0. Therefore, if $p = \sigma/2$, $\mathcal{C}(p)$ consists of $\{(0,0)\}$. Q.E.D.

REMARK. The a priori bounds of the next section ensure that for all $p, q > \sigma/2$, $\mathcal{C}(p) = \mathcal{C}(q)$ and we can denote the set by $\mathcal{C}$. When $\mathcal{C}$ is considered as a set of radially symmetric solutions of (3.1)–(3.3), it will be seen to be unbounded and connected in $\mathbf{R} \times L_p(\mathbf{R}^N)$ if $p > N\sigma/2$ and $\mathcal{C}$ is not connected in $\mathbf{R} \times L_{N\sigma/2}(\mathbf{R}^N)$, since $\|u\|_{L_{N\sigma/2}(\mathbf{R}^N)} \geqslant a > 0$ when $(\lambda, u) \in \mathcal{C}$, $u \neq 0$, and $(0,0) \in \mathcal{C}$ (Theorem 3.3(d)).

3.2. *A priori bounds and some nonexistence results*. Throughout this section suppose $(\lambda, u) \in \mathbf{R} \times C^2[0,\infty)$ is a solution of

$$(3.13)\qquad -\left(x^{N-1}u'(x)\right)' = x^{N-1}\left(\lambda u(x) + r(x)u(x)^{1+\sigma}\right), \qquad x \in (0,\infty),$$

$$(3.14)\qquad u'(0) = 0, \qquad u(x) \to 0 \quad \text{as } x \to \infty,$$

$$(3.15)\qquad u'(x) < 0, \qquad x \in (0,\infty).$$

We begin by giving a result analogous to Theorems 2.3, 2.5 and 2.6.

THEOREM 3.3. (a) $\lambda \leqslant 0$.

(b) $\|u\|^p_{L_p(\mathbf{R}^+)} \geqslant (\text{const})u(0)^{p-\sigma/2}$, $p \geqslant 1$.

(c) *If, in addition*, $p > \sigma/2$, *then*

$$\|u\|^p_{L_p(\mathbf{R}^+)} \leqslant (\text{const})u(0)^{p-\sigma/2}.$$

(d) *When considered as solutions of* (3.1)–(3.3),

$$\|u\|^p_{L_p(\mathbf{R}^N)} \geqslant (\text{const})u(0)^{p-N\sigma/2}, \qquad p \geqslant 1,$$
$$\|u\|^p_{L_p(\mathbf{R}^N)} \leqslant (\text{const})u(0)^{p-N\sigma/2}, \qquad p > N\sigma/2.$$

The constants depend on α, β, $0 < \alpha \leqslant r(x) \leqslant \beta$, *and* $p \geqslant 1$, $\sigma > 0$.

PROOF. (a) If ψ_R denotes the eigenfunction, corresponding to λ_R, defined just before Theorem 2.1, then

$$\int_0^R \psi_R(x)x^{N-1}\left(\lambda u(x) + r(x)u(x)^{1+\sigma}\right)dx = \int_0^R -\left(x^{N-1}u'(x)\right)'\psi_R(x)\,dx$$
$$< -\int_0^R \left(x^{N-1}\psi_R'(x)\right)'u(x)\,dx = \lambda_R\int_0^R x^{N-1}\psi_R(x)u(x)\,dx.$$

Since u, ψ_R are nonnegative it follows that $\lambda < \lambda_R$ for each R, so $\lambda \leqslant 0$ since $\lambda_R \to 0$ as $R \to \infty$.

(b) Repeating the argument used to prove Theorem 2.5 gives

$$u(x) \geqslant u(0)(1-\gamma^2 x^2), \qquad x \in (0,\infty),$$

since $\lambda \leqslant 0$ by (a), where $\gamma^2 = r(0)u(0)^\sigma/2$. Hence

$$\int_0^\infty u(x)^p\,dx \geqslant u(0)^p \int_0^{1/\gamma} (1-\gamma^2x^2)^p\,dx = (\text{const})\frac{u(0)^p}{\gamma},$$

which gives (b).

The proof of (c) is identical to that of Theorem 2.3 once it has been verified that an analogue of (2.8) is valid, namely

$$u'(x)^2 + \lambda u(x)^2 + (2/(\sigma+2))r(x)u(x)^{2+\sigma} > 0 \quad \text{on } (0,\infty).$$

To this end we need only verify the existence of a sequence $\{\mathbf{Z}_n\}$ such that $u'(\mathbf{Z}_n) \to 0$ and $\mathbf{Z}_n \to \infty$ as $n \to \infty$. Existence is obvious, since $u(x) \to 0$ as $x \to \infty$. Now (3.13) gives

$$\begin{aligned} u'(x)^2 - u'(\mathbf{Z}_n)^2 &\geqslant \lambda u(\mathbf{Z}_n)^2 - \lambda u(x)^2 + 2\int_x^{\mathbf{Z}_n} r(t)u(t)^{1+\sigma}u'(t)\,dt \\ &\geqslant \lambda u(\mathbf{Z}_n)^2 - \lambda u(x)^2 + \frac{2r(x)}{2+\sigma}\left(u(\mathbf{Z}_n)^{2+\sigma} - u(x)^{2+\sigma}\right). \end{aligned}$$

Since $u(\mathbf{Z}_n)$ and $u'(\mathbf{Z}_n)$ tend to 0, the result follows.

(d) Using the estimates obtained in the proofs of (b) and (c), this is now immediate. Q.E.D.

The next result gives circumstances when no solutions exist except for $\lambda = 0$. The example following shows these conditions are not necessary for existence when $\lambda = 0$, and the situation is then further clarified in Theorem 3.5.

THEOREM 3.4. *If $N \geqslant 3$ and $\sigma \geqslant 4/(N-2)$, then $\lambda = 0$ when $(\lambda, u) \in \mathbb{S}\backslash\{(0,0)\}$.*

PROOF. The argument which gives (2.6) yields for $y > 0$.

$$\begin{aligned} \frac{1}{2}\left\{y^N\left(u'(y)^2 + \lambda u(y)^2\right)\right\} &= \frac{2-N}{2}\int_0^y x^{N-1}u'(x)^2\,dx + \frac{\lambda N}{2}\int_0^y x^{N-1}u(x)^2\,dx \\ &\quad - \int_0^y x^N r(x)u(x)^{1+\sigma}u'(x)\,dx \\ &\leqslant \frac{2-N}{2}\int_0^y x^{N-1}u'(x)^2\,dx + \frac{\lambda N}{2}\int_0^y x^{N-1}u(x)^2\,dx \\ &\quad + \frac{N}{2+\sigma}\int_0^y r(x)x^{N-1}u(x)^{\sigma+2}\,dx, \end{aligned}$$

(where we have used the observation that (2.7) remains valid when R is replaced by y and u satisfies (3.13)–(3.15))

$$\begin{aligned} &= \lambda\int_0^y x^{N-1}u(x)^2\,dx + \frac{4-\sigma(N-2)}{2(2+\sigma)}\int_0^y x^{N-1}r(x)u(x)^{2+\sigma}\,dx \\ &\quad + \frac{2-N}{2}y^{N-1}u(y)u'(y), \qquad y > 0. \end{aligned}$$

If $4-\sigma(N-2) \leqslant 0$, it follows that for $y>0$,

(3.16)

$$\frac{1}{2}\left\{y^N\left(u'(y)^2+\lambda u(y)^2\right)\right\} \leqslant \lambda \int_0^y x^{N-1} u(x)^2\,dx + \frac{2-N}{2} y^{N-1} u(y) u'(y).$$

However, if $\lambda<0$, (3.13) yields

$$\begin{aligned} -u''(x) &\leqslant \lambda u(x) + r(x) u(x)^{1+\sigma} \\ &\leqslant (\lambda/2) u(x) \quad \text{for all } x \text{ sufficiently large,} \end{aligned}$$

since $u(x) \to 0$ as $x \to \infty$. Therefore,

$$u'(x)^2 - u'(\mathbf{Z}_n)^2 \geqslant (\lambda/2)\left(u(\mathbf{Z}_n)^2 - u(x)^2\right),$$

which gives

$$u'(x)^2 \geqslant |\lambda/2| u(x)^2 \quad \text{for all } x \text{ sufficiently large,}$$

where $\{\mathbf{Z}_n\}$ is the sequence introduced in the proof of Theorem 3.3(c). Therefore

$$-u'(x) \geqslant \sqrt{|\lambda/2|}\, u(x), \quad \text{for all } x \geqslant K, \text{ say,}$$

whence

$$u(x) \leqslant u(K) \exp\left\{-\sqrt{|\lambda|/2}\,(x-K)\right\}, \qquad x \geqslant K.$$

Therefore u decays exponentially to zero, and by (3.13) it follows that u' also decays exponentially to zero. But this is impossible by (3.16) because

$$\lambda \int_0^y x^{N-1} u(x)^2\,dx \leqslant \delta < 0$$

for all y sufficiently large. Hence $\lambda=0$ when $\sigma \geqslant 4/(N-2)$ and $N \geqslant 3$. Q.E.D.

EXAMPLE. Let $N \geqslant 3$, $0<\gamma<(N-2)/2$, $\sigma = 1/\gamma$, and

$$r_\gamma(x) = 2\gamma\{N - 2(1+\gamma)x^2/(1+x^2)\}, \qquad x \in [0,\infty).$$

Then $0 < 2\gamma(N-2-2\gamma) \leqslant r_\gamma(x) \leqslant 2N\gamma$, and r_γ is nonincreasing. However the boundary value problem

$$\text{(3.17)} \quad -u''(x) - (N-1)\frac{u'(x)}{x} = \lambda u(x) + r_\gamma u(x)^{1+\sigma}, \qquad x \in (0,\infty),$$

$$\text{(3.18)} \quad u'(0)=0, \quad u(x) \to 0 \quad \text{as } x \to \infty,$$

has a solution

$$\lambda = 0, \qquad u(x) = 1/(1+x^2)^\gamma.$$

We note, however, that $\sigma = 1/\gamma > 2/(N-2)$ in this example. The next result shows that if this inequality is reversed there are no solutions with $\lambda=0$.

Note also that if $\sigma \geqslant 4/(N-2)$ then, by the previous result, $\lambda = 0$. In particular, $u(x) = 1/(1+x^2)^{(N-2)/2}$ is a solution of (3.17), (3.18) with $\lambda=0$ and

$$r_\gamma(x) = 2\gamma N, \quad \gamma = (N-2)/2, \quad \text{and} \quad \sigma = 2/\gamma.$$

Therefore the second part of the next result is best possible, in a certain sense.

THEOREM 3.5. (a) *If* $N = 2$, *or* $N \geqslant 3$ *and* $0 < \sigma \leqslant 2/(N-2)$, *then* $\lambda < 0$.

(b) *If* $N \geqslant 3$, $0 < \sigma < 4/(N-2)$, *and* $1 - r(\infty)/r(0) < (4 - \sigma(N-2))/2N$, *then* $\lambda < 0$.

(c) *If* $r =$ *constant and* $0 < \sigma < 4/(N-2)$, *then* $\lambda < 0$.

REMARK. In (b) some restriction is placed on the ratio of the supremum to infimum of the nonincreasing function r. It is worth noting the obvious fact that, in the example just given, r_γ violates this restriction for all $\gamma < (N-2)/2$ and $\sigma = 1/\gamma > 2/(N-2)$.

PROOF. Suppose $\lambda = 0$ in (3.13), $N \geqslant 3$ and $0 < \sigma \leqslant 2/(N-2)$. Then

$$-x^{N-1}u'(x) = \int_0^x r(t)u(t)^{1+\sigma}t^{N-1}\,dt \geqslant \frac{r(x)u(x)^{1+\sigma}x^N}{N} \tag{3.19}$$

by (3.15) and the monotonicity of r. Hence

$$-u'(x)/u(x)^{1+\sigma} \geqslant \alpha x/N, \quad \text{since } r(x) \geqslant \alpha > 0,$$

from which it follows that

$$u(x)^\sigma \leqslant \frac{u(0)^\sigma}{(\alpha\sigma u(0)^\sigma/2N)x^2 + 1}, \qquad x \in (0,\infty). \tag{3.20}$$

However (3.13) also yields

$$-u''(x) - (N-1)u'(x)/x = r(x)u(x)^{1+\sigma} > 0.$$

Hence

$$\frac{d}{dx}\{-xu'(x) - (N-2)u(x)\} = xr(x)u(x)^{1+\sigma} > 0. \tag{3.21}$$

However from (3.19) and (3.20),

$$0 < -xu'(x) \leqslant \frac{\text{const}}{x^{N-2}}\int_0^x \frac{t^{N-1}\,dt}{(1+t^2)^{1+1/\sigma}} \to 0 \quad \text{as } x \to \infty.$$

Since $u(x) \to 0$ as $x \to \infty$, by (3.20), it follows from (3.21) that

$$-xu'(x) - (N-2)u(x) < 0, \qquad x \in (0,\infty). \tag{3.22}$$

Hence $x^{N-2}u(x)$ is increasing on $(0,\infty)$ and, in particular,

$$u(x) \geqslant (\text{const})/x^{N-2}$$

for all x sufficiently large. If $0 < \sigma < 2/(N-2)$, this contradicts (3.20).

Suppose that $\sigma = 2/(N-2)$. Then by (3.19) it follows that

$$\begin{aligned} u(x) &= \int_x^\infty \frac{1}{t^{N-1}}\left\{\int_0^t r(w)u(w)^{1+\sigma}w^{N-1}\,dw\right\}dt \\ &= \frac{1}{N-2}\left\{\frac{1}{x^{N-2}}\int_0^x r(w)u(w)^{1+\sigma}w^{N-1}\,dw + \int_x^\infty r(w)u(w)^{1+\sigma}w\,dw\right\}. \end{aligned}$$

Hence

$$x^{N-2}u(x) \geqslant \frac{1}{N-2}\int_1^x r(w)u(w)^{1+\sigma}w^{N-1}\,dw$$

$$\geqslant (\text{const})\int_1^x \frac{w^{N-1}}{w^{(1+\sigma)(N-2)}}\,dw = (\text{const})\ln x$$

for x sufficiently large since $\sigma(N-2)=2$. But this contradicts the earlier observation that

$$\left(u(x)x^{(N-2)}\right)^{2/(N-2)} = u(x)^{\sigma}x^2 \leqslant \text{const}.$$

Now suppose $\lambda = 0$ and $N = 2$, or $N \geqslant 3$ and $\sigma < 4/(N-2)$. Then the argument leading to (2.6) now gives

$$(3.23)\quad \frac{1}{2}y^N u'(y)^2 = \int_0^y x^{N-1}\left\{\left(\frac{2-N}{2}\right)u'(x)^2 - xr(x)u(x)^{1+\sigma}u'(x)\right\}dx$$

$$= \frac{2-N}{2}\int_0^y x^{N-1}r(x)u(x)^{2+\sigma}\,dx$$

$$-\int_0^y x^N r(x)u(x)^{1+\sigma}u'(x)\,dx$$

$$+\frac{2-N}{2}y^{N-1}u(y)u'(y), \qquad y \in (0,\infty).$$

However,

$$(N-2)y^{N-1}u(y)u'(y) + y^N u'(y)^2$$

$$= -y^{N-1}u'(y)\{-(N-2)u(y) - yu'(y)\} < 0$$

by (3.22). Hence, by (3.23),

$$0 > \frac{2-N}{2}\int_0^y x^{N-1}r(x)u(x)^{2+\sigma}\,dx - \int_0^y x^N r(x)u(x)^{1+\sigma}u'(x)\,dx$$

$$\geqslant \frac{2-N}{2}\int_0^y r(0)x^{N-1}u(x)^{2+\sigma}\,dx - \int_0^y r(\infty)x^N u(x)^{1+\sigma}u'(x)\,dx,$$

since $2 - N \leqslant 0$ and $-u' > 0$,

$$= \frac{2-N}{2}r(0)\int_0^y x^{N-1}u(x)^{2+\sigma}\,dx$$

$$+r(\infty)\left\{\frac{N}{2+\sigma}\int_0^y x^{N-1}u(x)^{2+\sigma}\,dx - \frac{y^N u(y)^{2+\sigma}}{2+\sigma}\right\}$$

$$= \frac{r(0)}{2(2+\sigma)}\left(4 - \sigma(N-2) - 2N\left(1 - \frac{r(\infty)}{r(0)}\right)\right)\int_0^y x^{N-1}u(x)^{2+\sigma}\,dx$$

$$-\frac{r(\infty)y^N u(y)^{2+\sigma}}{2+\sigma}$$

$$= a\int_0^y x^{N-1}u(x)^{2+\sigma}\,dx - \frac{r(\infty)y^N u(y)^{2+\sigma}}{2+\sigma},$$

where $a > 0$. However, because of (3.20),

$$y^N u(y)^{2+\sigma} \leqslant y^N / y^{2+4/\sigma} = y^{-(4-\sigma(N-2))/\sigma} \to 0 \quad \text{as } y \to \infty.$$

This is a contradiction.

(c) This is a special case of (b), since here $1 - r(\infty)/r(0) = 0$. Q.E.D.

COROLLARY 3.6. *Suppose* $u'(0) = 0$,

$$-(x^{N-1}u'(x))' = x^{N-1}r(x)u(x)^{1+\sigma}, \qquad x \in (0, \infty),$$
$$u(x) \geqslant 0, \quad u'(x) \leqslant 0,$$

and either

(3.24) $$N = 2,$$

(3.25) $$0 < \sigma \leqslant 2/(N-2),$$

or

(3.26) $$1 - r(\infty)/r(0) < (4 - \sigma(N-2))/2N.$$

Then $u = 0$.

PROOF. It suffices to note that the argument for (3.20) remains valid so Theorem 3.5 applies. Q.E.D.

COROLLARY 3.7. *The function u decays exponentially to zero if and only if* $\lambda < 0$. *To be more precise*:

(a) *if* $\lambda = 0$, $x^{N-2}u(x)$ *is increasing*,

(b) *if* $\lambda < 0$,

$$u(x)\exp\left\{\sqrt{|\lambda| - \varepsilon}\, x\right\} \to 0 \quad \text{as } x \to \infty,\ 0 < \varepsilon < |\lambda|.$$

(See the remark following (VIII) in the Introduction.)

PROOF. (a) This is already proved in the argument leading to (3.22).

(b) From (3.13) and (3.15) it follows that

$$\begin{aligned} -u'' &\leqslant u(x)\big(\lambda + r(x)u(x)^\sigma\big) \\ &\leqslant (\lambda + \varepsilon)u(x) \quad \text{for all } x \text{ sufficiently large} \end{aligned}$$

by (3.14). Hence

$$-u'(x)u''(x) \geqslant (\lambda + \varepsilon)u(x)u'(x)$$

and, therefore,

$$u'(x)^2 \geqslant -(\lambda + \varepsilon)u(x)^2$$

for all x sufficiently large. Since $\lambda + \varepsilon < 0$, it follows that

$$-u'(x)/u(x) \geqslant \sqrt{|\lambda| - \varepsilon}$$

and

$$u(x) \leqslant u(X)\exp\left\{-\sqrt{|\lambda| - \varepsilon}\, x\right\}, \qquad x \geqslant X,$$

for X sufficiently large. Since this holds for all ε, $0 < \varepsilon < |\lambda|$, the result follows. Q.E.D.

THEOREM 3.8. *Suppose* $N = 2$, *or* $N \geqslant 3$ *and* $0 < \sigma < 4/(N-2)$. *Then if* $\Lambda < 0$ *there exists* $M(\Lambda) > 0$ *such that*:

(a) $u(0) < M(\Lambda)$ *when* $\lambda \geqslant \Lambda$ *and* (λ, u) *satisifes* (3.13)–(3.15); *and*

(b) $M(\Lambda) \to 0$ *as* $\lambda \to 0$ *if, in addition, one of* (3.24)–(3.26) *holds*.

PROOF. (a) We adapt the method of [**9**] to this simpler context. Suppose $\Lambda < 0$ and $\{(\lambda_n, u_n)\}$ is a sequence of solutions of (3.13)–(3.15) with $\lambda_n \in [\Lambda, 0]$ and $u_n(0) = m_n \to \infty$ as $n \to \infty$. Now put

$$v_n(x) = m_n^{-1} u_n\left(x/m_n^{\sigma/2}\right), \qquad x \in (0, \infty).$$

Then v_n satisfies

$$-\left(x^{N-1} v_n'(x)\right)' = x^{N-1}\left\{(\lambda_n/m_n^{\sigma}) v_n(x) + r\left(x/m_n^{\sigma/2}\right) v_n(x)^{1+\sigma}\right\},$$
$$v_n(x) \leqslant v_n(0) = 1,$$
$$v_n'(x) < 0, \qquad x \in (0, \infty).$$

Now $r(x/m_n^{\sigma/2}) \to r(0) > 0$ uniformly on compact subsets of $\mathbf{R}^+$, and $\lambda_n/m_n^{\sigma} \to 0$ as $n \to \infty$. Now a standard argument, in the spirit of that for Theorem 3.1, shows there exists a v such that $v_n \to v$ uniformly on compact subsets of $[0, \infty)$, and

$$-\left(x^{N-1} v'(x)\right)' = r(0) x^{N-1} v(x)^{1+\sigma},$$
$$v(0) = 1, \quad v'(0) = 0, \quad v'(x) \leqslant 0 \qquad \text{on } (0, \infty).$$

Since $N = 2$ or $N \geqslant 3$ and $0 < \sigma < 4/(N-2)$, this contradicts Corollary 3.6.

To prove (b) we again suppose the contrary and seek a contradiction. Let (λ_n, u_n) satisfy (3.13)–(3.15), $\lambda_n \uparrow 0$ and $u_n(0) \to a > 0$ as $n \to \infty$. Then by an argument, as in the proof of Theorem 3.1, we can show that $u_n \to u$ uniformly on compact intervals in $[0, \infty)$ where

$$-\left(x^{N-1} u'(x)\right)' = r(x) u(x)^{1+\sigma},$$
$$u(x) \geqslant 0, \quad u'(x) \leqslant 0, \qquad x \in (0, \infty),$$
$$u(0) = a.$$

This once more contradicts Corollary 3.6, since one of (3.24)–(3.26) holds. Q.E.D.

COROLLARY 3.9. $\{\lambda\colon (\lambda, u) \in \mathcal{C}(p)\} \supset (-\infty, 0)$ *if* $0 < \sigma < 4/(N-2)$, $p > \sigma/2$.

PROOF. Since $\mathcal{C}(p)$ is unbounded in $\mathbf{R} \times L_p(\mathbf{R}^+)$ the result follows from Theorems 3.8 and Theorem 3.2. Q.E.D.

REMARK. For $(N-2)/4 < \gamma < (N-2)/2$, there is an interesting situation in the earlier example. There are solutions (λ, u) with $u \neq 0$ for all $\lambda < 0$ and also for $\lambda = 0$, since in that case $2/(N-2) < \sigma < 4/(N-2)$ (recall $\gamma = 1/\sigma$).

Finally we justify the remark, following the proof of Theorem 3.2, that $\mathcal{C}(p) = \mathcal{C}(q)$, $p, q > \sigma/2$. By Theorem 3.3, $\mathcal{C}(p) \subset (-\infty, 0] \times L_q(\mathbf{R}^+)$ and $\mathcal{C}(q) \subset (-\infty, 0] \times L_p(\mathbf{R}^+)$. To show that $\mathcal{C}(p) = \mathcal{C}(q)$, it suffices to show that $\mathcal{C}(p)$ is connected in $\mathbf{R} \times L_q(\mathbf{R}^+)$ since $\mathcal{C}(p)$ and $\mathcal{C}(q)$ are maximal connected subsets of $\mathcal{S}$.

Suppose $\{(\lambda_n, u_n)\} \subset \mathcal{C}(p)$ and $(\lambda_n, u_n) \to (\lambda, u)$ in $\mathbf{R} \times L_p(\mathbf{R}^+)$. Then the argument of Theorem 3.1 gives $(\lambda_n, u_n) \to (\lambda, u)$ in $\mathbf{R} \times L_q(\mathbf{R}^+)$. Hence $\mathcal{C}(p)$ is connected in $\mathbf{R} \times L_q(\mathbf{R}^+)$, $\mathcal{C}(p) \subset \mathcal{C}(q)$. Similarly $\mathcal{C}(q) \subset \mathcal{C}(p)$, so $\mathcal{C}(p) = \mathcal{C}(q)$.

Appendix.

Rabinowitz's global theory. The following theorem was proved by P. H. Rabinowitz [**11**, Theorem 1.27]. Suppose X is a real Banach space, G: $\mathbf{R} \times X \to X$ is completely continuous,

$$G(\lambda, x) = \lambda Lx + R(\lambda, x), \qquad (\lambda, x) \in \mathbf{R} \times X,$$

where L: $X \to X$ is a compact linear operator, and R: $\mathbf{R} \times X \to X$ is continuous and such that

$$\|R(\lambda, x)\| / \|x\| \to 0 \quad \text{as } \|x\| \to 0$$

uniformly for λ in bounded intervals. Suppose μ_0 is a characteristic value of L of multiplicity one and R: $\mathbf{R} \times X \to X$ is continuously Fréchet differentiable in a neighborhood of $(\mu_0, 0)$. Let ψ_0, $\|\psi_0\| = 1$, denote an element of X such that $\psi_0 = \mu_0 L \psi_0$, and let

$$S = \{(\lambda, x) \in \mathbf{R} \times X: x \neq 0, x = G(\lambda, x)\} \cup \{(\mu_i, 0)\},$$

where $\{\mu_i\}_{i=0}^{\infty}$ denotes the set of all characteristic values of L. Then:

(a) *There exists* $\varepsilon_0 > 0$ *and continuous mappings* Λ: $(-1, 1) \to \mathbf{R}$, w: $(-1, 1) \to X$ *such that* $\Lambda(0) = \mu_0$, $w(0) = 0 \in X$ *and*

$$S \cap B_{\varepsilon_0} = \{(\Lambda(t), t(\psi_0 + w(t))): t \in (-1, 1)\},$$

where B_{ε_0} *denotes a ball of radius* ε_0 *and centre* $(\mu_0, 0)$ *in* $\mathbf{R} \times X$.

(b) *Let* $\mathcal{C}^{\pm}$ *denote the sets* $\mathcal{C}_*^{\pm} \cup \{(\mu_0, 0)\}$, *where* $\mathcal{C}_*^{\pm}$ *is the maximal connected subset of* $S \setminus \{(\mu_0, 0)\}$ *containing*

$$\{(\Lambda(t), t(\psi_0 + w(t))): t \in \mathbf{R}^{\pm} \cap ((-1, 0) \cup (0, 1))\}.$$

Then either

(i) $\mathcal{C}^{\pm}$ *is unbounded in* $\mathbf{R} \times X$, *or*

(ii) $\mathcal{C}^{\pm}$ *contains* $(\mu_i, 0)$, $i \neq 0$, *or*

(iii) $\exists u \neq 0$ *such that* $(\lambda, u) \in \mathcal{C}^{\pm}$ *and* $(\lambda, -u) \in \mathcal{C}^{\pm}$.

REMARK. Alternative (i)–(iii) hold for $\mathcal{C}^+$ and for $\mathcal{C}^-$, separately. In other words if (i) holds for $\mathcal{C}^+$ and (ii) holds for $\mathcal{C}^-$, then the theorem is vindicated.

In order to prove global existence theorems for operator equations as was done in this paper, we use a device where global bifurcation results are described in terms of open subsets of $\mathbf{R} \times X$.

Now suppose

$$S = \{(\lambda, x): x = G(\lambda, x), x \neq 0\} \cup \{(\mu^*, 0)\}$$

but G: $\mathbf{R} \times X \to X$ need not be compact. However, suppose that bounded subsets of S are relatively compact (this would follow from the complete continuity of G, but is true in many other circumstances as well; see [**1**, **2**, **15**, **16**] and §3). Let $\mathcal{C}$ denote the maximal subset of S containing $(\mu^*, 0)$.

Then $\mathcal{C}$ *is unbounded if and only* $S \cap \partial U \neq \varnothing$ *for all bounded open subsets* U *of* $\mathbf{R} \times X$ *with* $(\mu^*, 0) \in U$.

This result is implicit in [**11**] and is given explicitly in [**1**, Appendix].

Note added in proof. Recently C. A. Stuart has found a very elegant variational procedure for obtaining existence results similar to ours in the case when $p \geqslant 2$ and $\lambda < 0$ (Math. Ann. **263** (1983), 51–59). However connectedness of the solution set does not follow by variational arguments.

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