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DISCONTINUOUS TRANSLATION INVARIANT FUNCTIONALS

BY

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Dedicated to Professor Shigeki Yano on his sixtieth birthday

ABSTRACT. Let G be an infinite σ-compact locally compact group. We shall study the existence of many discontinuous translation invariant linear functionals on a variety of translation invariant Fréchet spaces of Radon measures on G. These spaces include the convolution measure algebra $M(G)$, the Lebesgue spaces $L^p(G)$, where $1 \leqslant p \leqslant \infty$, and certain combinations thereof. Among other things, it will be shown that $C(G)$ has many discontinuous translation invariant functionals, provided that G is amenable. This solves a problem of G. H. Meisters.

Let G be a locally compact group. In the present paper, we shall consider various translation invariant spaces of functions and measures on G, and prove the existence of discontinuous invariant functionals on such spaces. Among other things, it will be shown that $C(G)$ admits "many" discontinuous left-invariant functionals, provided that G is an infinite, σ-compact, locally compact amenable group (our hypotheses are a little weaker than amenability). This resolves the problem of G. H. Meisters **[11]** as to whether there exists a discontinuous invariant functional on $C(\Pi)$, where Π denotes the circle group. As far as the compact groups are concerned, the problems for $C(G)$ and for $L^\infty(G)$ are equivalent (see Remark (IV) stated below). As some basic references on this subject, we refer to E. Granirer's papers **[3, 4]** on invariant means, the nice survey article **[14]** by Meisters, and G. S. Woodward's paper **[22]** on discontinuous invariant functionals on a variety of invariant spaces.

Let G be an arbitrary locally compact (LC) group. All of the Lebesgue spaces $L^p(G)$, $1 \leqslant p \leqslant \infty$, are taken with respect to a fixed left Haar measure λ_G on G. If G is compact, we normalize λ_G so that $\lambda_G(G) = 1$. Let $C(G)$ and $C_c(G)$ denote the Banach space of all continuous bounded (complex-valued) functions on G, and the subspace of all members of $C(G)$ with compact supports, respectively. We denote by $C_0(G)$ the uniform closure of $C_c(G)$ in $C(G)$. Thus the dual of $C_0(G)$ may be identified with the convolution measure algebra $M(G)$ of G (cf. Hewitt and Ross **[7]**).

A net (f_α) in $C_c(G)$ converges to the zero function if and only if (i) $f_\alpha \to 0$ uniformly on G and (ii) the f_α's are eventually supported by a fixed compact set (depending on the net). This topology for $C_c(G)$ is different from the uniform topology, and will be used throughout the paper unless otherwise mentioned. The continuous dual of $C_c(G)$ will be denoted $C_c'(G)$. An element of $C_c'(G)$ is often called

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a Radon measure on G (cf. [**16**]). If a linear subspace X of $C_c'(G)$ is equipped with a topological-linear-space topology which is stronger than the weak topology $\sigma(X, C_c(G))$, we call X a *topological linear space on* G. Normed, Banach, and Fréchet spaces on G are defined in a similar fashion. Notice that if X and Y are two Fréchet spaces on G such that $X \subset Y$, then the identity map of X into Y is continuous by the closed graph theorem. We shall regard all of the spaces $L^p(G) \subset L^1_{\mathrm{loc}}(G)$ and $M(G)$ as linear subspaces of $C_c'(G)$. In particular, if $f \in L^1_{\mathrm{loc}}(G)$, then

$$\langle \phi, f \rangle = \langle \phi, f\lambda_G \rangle = \int \phi(x) f(x)\, d\lambda_G(x) \qquad (\phi \in C_c(G)).$$

Similarly, $M(G) + L^p(G)$ denotes the algebraic sum of $M(G)$ and $L^p(G)$ in $C_c'(G)$.

For each $\phi \in C_c(G)$ and $a \in G$, define ${}_a\phi$ and $\phi_a \in C_c(G)$ by setting ${}_a\phi(x) = \phi(ax)$ and $\phi_a(x) = \phi(xa)$ for all $x \in G$. Let δ_a denote the unit point-measure at a. Thus we have

$$\langle \phi, \delta_a * \mu \rangle = \langle {}_a\phi, \mu \rangle \quad \text{and} \quad \langle \phi, \mu * \delta_a \rangle = \langle \phi_a, \mu \rangle$$

for $\phi \in C_c(G)$ and $\mu \in C_c'(G)$. Here follow some other definitions which we shall need later.

(a) A subset X of $C_c'(G)$ is left invariation if $\delta_a * X \subset X$ for all $a \in G$. Right- (or two-sided) invariance is defined similarly.

(b) For a subset X of $C_c'(G)$, we define

$$T_l(X) = \mathrm{sp}\{f - \delta_a * f : f \in X \,\&\, a \in G\}, \quad T_r(X) = \mathrm{sp}\{f - f * \delta_a : f \in X \,\&\, a \in G\},$$

and $T(X) = T_l(X) + T_r(X)$, where "sp" stands for linear span.

(c) If X and Y are two topological spaces such that $X \subset Y$ set-theoretically, we shall write "$-$" for the closure in the smaller space X and "$\sim$" for the closure in the larger space Y.

(d) For a topological linear space X, X' denotes the continuous dual of X. If X is a normed space and $r > 0$, we define $B_r(X) = \{f \in X: \|f\| \leq r\}$.

In the sequel, G will always denote an infinite σ-compact LC group, unless otherwise mentioned. The following two lemmas are central to all of our results. Their natural "two-sided" versions are obvious and will be left to the reader.

LEMMA 1. *Let X and Y be a left-invariant Fréchet space and a left-invariant Banach space on G, respectively. Suppose that*

(i) *$X \subset Y$ and $T_l(X)^- \subset \mathrm{sp}(F) + T_l(Y)$*

for some countable subset F of Y. Then there exist a natural number N and a compact subset K of G such that $W_{N,K}^\sim$ contains a relative neighborhood of 0 *in $T_l(X)^-$ (with respect to X), where*

(ii) $W_{N,K} = \{\sum_{k=1}^N (h_k - \delta_{a_k} * h_k): h_k \in B_N(Y) \,\&\, a_k \in K \;\forall k\}$.

PROOF. Let $f_1, f_2, \ldots$ be an enumeration of the elements of F. Since G is σ-compact, we can write $G = \bigcup_{n=1}^\infty K_n$, where the K_n's are compact subsets of G with $K_n \subset K_{n+1}$ for all $n \geq 1$. Given a natural number n, define

$$V_n = \left\{ \sum_{j=1}^n c_j f_j : c_j \in \mathbf{C} \,\&\, |c_j| \leq n \;\forall_j \right\}. \tag{1}$$

Likewise, define W_n by (ii) with N and K replaced by n and K_n, respectively. Notice that V_n is compact. Since the algebraic sum of a compact set and a closed set is closed in any topological linear space, it follows that $V_n + W_n^{\sim}$ is closed in Y.

By (i), the Fréchet space $T_l(X)^-$ is covered by the sets $V_n + W_n^{\sim}$. Since $T_l(X)^-$ is contained in the Banach space Y on G, it follows from the closed graph theorem that each $(V_n + W_n^{\sim}) \cap T_l(X)^-$ is closed in $T_l(X)^-$. We, therefore, infer from the Baire category theorem that at least one of the sets $V_n + W_n^{\sim}$ contains a nonempty (relative) open subset of $T_l(X)^-$. However, we have $V_n - V_n \subset V_{2n}$ and $mV_n \subset V_{mn}$ for all m and $n \geqslant 1$ by (1); similarly for the sets $W_n^{\sim}$. Accordingly, there exists a natural number r such that $V_r + W_r^{\sim}$ contains a neighborhood U of $0 \in T_l(X)^-$.

Now we claim that $U \subset W_N^{\sim}$ for some $N \geqslant r$. To confirm this, first notice that the union of all $W_n^{\sim}$ forms a linear subspace of Y. Call this union S and put $F_p = \{f_1, \dots, f_p\}$ for $p = 1, 2, \dots, r$. If $F_r \subset S$, then $F_r \subset W_m^{\sim}$ for some m; hence $V_r \subset W_n^{\sim}$ for some $n > m$ by (1) and (ii); hence $V_r + W_r^{\sim} \subset W_n^{\sim} + W_r^{\sim} \subset W_N^{\sim}$, where $N = n + r$. Thus $F_r \subset S$ implies $U \subset W_N^{\sim}$ for some N. So assume that F_r is not contained in S. Then we may suppose that, for some $p \leqslant r$, F_p is a maximal subset of F_r which is linearly independent modulo S. After repeating a similar argument as above, we then obtain

$$U \subset V_r + W_r^{\sim} \subset \mathrm{sp}(F_p) + W_N^{\sim} \tag{2}$$

for some $N \geqslant r$. Notice that $T_l(X) \subset T_l(Y) \subset S$; hence

$$T_l(X) \cap [\mathrm{sp}(F_p) + W_N^{\sim}] = T_l(X) \cap W_N^{\sim} \tag{3}$$

by the linear independence of F_p modulo S. We infer from (2) and (3) that $U \cap T_l(X) \subset W_N^{\sim}$. Since $[T_l(X)^-] \cap W_N^{\sim}$ is closed in $T_l(X)^-$, it follows that $U \subset W_N^{\sim}$, as desired.

LEMMA 2. *Let X be a left-invariant Fréchet space on G, and let Y be a left-invariant Banach space on G such that*

(C_0) *$X \subset Y$ and $T_l(X)^- \subset \mathrm{sp}(F) + T_l(Y)$*

for some countable subset F of Y. Suppose

(C_1) *the unit closed ball $B_1(Y)$ is $\sigma(Y, C_c)$-compact;*

(C_2) *the orbit $\{\delta_x * f\colon x \in G\}$ of each $f \in X$ is bounded in X;*

and also (if G is noncompact)

(C_3) *for which $f \in X$, $\lim_{x\to\infty} \delta_x * f = 0$ in $\sigma(X, C_c)$.*

Let $W_{N,K} \subset Y$ be as in the conclusion of Lemma 1. Then:

(i) *$W_{N,K}$ is $\sigma(Y, C_c)$-compact;*

(ii) *if G is noncompact, $W_{2N,K}$ contains a neighborhood of $0 \in X$;*

(iii) *if G is compact, there exists a neighborhood V of $0 \in X$ such that $h - \lambda_G * h \in W_{2N,K}$ for all $h \in V$.*

PROOF. For each open subset U of G with compact closure, let C_U denote the subspace of all $\phi \in C_c(G)$ such that $\mathrm{supp}\,\phi \subset U^-$. Then C_U forms a Banach space with respect to the uniform norm and the imbedding of C_U into $C_c(G)$ is continuous. Since Y is a Banach space contained in $C_c'(G)$, the closed graph theorem assures that there exists a finite constant $D = D_U$ such that

(4) $$|\langle \phi, h\rangle| \leq D\|\phi\|_\infty \cdot \|h\|_Y \qquad (\phi \in C_U,\ h \in Y).$$

Now we claim that the mapping

(5) $$(x, h) \to \delta_x * h\colon G \times B_1(Y) \to C_c'(G)$$

is continuous when $B_1(Y)$ is equipped with the relative weak topology $\sigma(Y, C_c)| B_1(Y)$. Suppose that (x_α, h_α) is a convergent net in $G \times B_1(Y)$ with limit (x, h). We need to prove that $\lim \delta_{x_\alpha} * h_\alpha = \delta_x * h$ in $\sigma(C_c', C_c)$. Given $\phi \in C_c(G)$, choose an open subset U of G such that U^- is compact and $\operatorname{supp}({}_x\phi) \subset U$. Since $x_\alpha \to x$ in G, it is obvious that ${}_{x_\alpha}\phi \in C_U$ for eventually all α's. It follows from (4) that

$$\begin{aligned}|\langle \phi, \delta_{x_\alpha} * h_\alpha\rangle - \langle \phi, \delta_x * h\rangle| &= |\langle {}_{x_\alpha}\phi, h_\alpha\rangle - \langle {}_x\phi, h\rangle| \\ &\leq |\langle {}_{x_\alpha}\phi - {}_x\phi, h_\alpha\rangle| + |\langle {}_x\phi, h_\alpha - h\rangle| \\ &\leq D\|{}_{x_\alpha}\phi - {}_x\phi\|_\infty + |\langle {}_x\phi, h_\alpha - h\rangle|\end{aligned}$$

for eventually all α's. Since ${}_{x_\alpha}\phi \to {}_x\phi$ uniformly and $h_\alpha \to h$ in $\sigma(Y, C_c)$, the last inequalities confirm our claim.

Put $E_K = \{h - \delta_x * h\colon h \in B_1(Y) \,\&\, x \in K\}$. From condition (C_1) and the claim just established, we infer that E_K is $\sigma(Y, C_c)$-compact whenever K is a compact subset of G. The set $W_{N,K}$ in the conclusion of Lemma 1 is the N-fold algebraic sum of NE_K, and is therefore $\sigma(Y, C_c)$-compact. This confirms (i).

Now we shall prove (ii) and (iii) assuming that

$(C_1)^*$ $$W_{N,K} = [Y + \operatorname{sp}(\lambda_G)] \cap W_{N,K}^*,$$

where $W_{N,K}^*$ denotes the $\sigma(C_c', C_c)$-closure of $W_{N,K}$ in $C_c'(G)$. Of course, $(C_1)^*$ is weaker than (C_1) but is strong enough to guarantee the norm-closedness of $W_{N,K}$ in Y. Now choose and fix a convex neighborhood V of $0 \in X$ such that $(V - V) \cap T_l(X)^- \subset W_{N,K}$. Each $f \in X$ has bounded orbit in X by (C_2). Accordingly there exists a natural number $n = n_f$ such that $n^{-1}(\delta_x * f) \in V$ for all $x \in G$. Hence

(6) $$n^{-1}(f - \delta_x * f) \in (V - V) \cap T_l(X) \subset W_{N,K} \qquad (x \in G).$$

Suppose G is noncompact. Then (C_3), combined with (6) and $(C_1)^*$, yields $n^{-1}f \in W_{N,K}$. Since $f \in X$ is arbitrary, it follows that $X \subset \bigcup_{n=1}^\infty nW_{N,K}$. But $W_{N,K}$ is norm-closed in Y by $(C_1)^*$, and X is a Fréchet space contained in the Banach space Y; hence $X \cap W_{N,K}$ is closed in X (cf. the proof of Lemma 1). It follows from the Baire category theorem that $X \cap W_{N,K}$ has nonempty interior in X. Since $W_{N,K} - W_{N,K} \subset W_{2N,K}$, we conclude that $W_{2N,K}$ contains a neighborhood of 0 in X, which established (ii).

Finally assume G is compact. Then $C_c(G) = C(G)$, $C_c'(G) = M(G)$, $\sigma(C_c', C_c)$ is nothing but the weak-* topology of $M(G)$. Choose a net (τ_α) of probability measures on G, each with finite support, such that $\tau_\alpha \to \lambda_G$ weak-*. Then $n^{-1}(f - \tau_\alpha * f) \in W_{N,K}$ for each α by (6) and the convexity of V. Passing to the weak-* limit and making use of $(C_1)^*$, we obtain $n^{-1}(f - \lambda_G * f) \in W_{N,K}$. Notice that $\lambda_G * X \subset \lambda_G * M(G) = \operatorname{sp}(\lambda_G)$. Since $f \in X$ is arbitrary, it follows at once that

(7) $$X \subset \bigcup_{n=1}^\infty n[\{c\lambda_G\colon |c| \leq 1\} + W_{N,K}].$$

Using $(\mathrm{C}_1)^*$ and (7), we repeat a similar argument as in the last paragraph to conclude that $\mathrm{sp}(\lambda_G) + W_{2N,K}$ contains a neighborhood V' of $0 \in X$. Thus each $h \in V'$ has a representation of the form $h = c\lambda_G + \mu$ for some $c \in \mathbf{C}$ and some $\mu \in W_{2N,K}$. But it is obvious that $\langle 1, \nu \rangle = 0$ for all $\nu \in W_{2N,K} \subset T_l(M(G))$. Therefore $\langle 1, h \rangle = c\langle 1, \lambda_G \rangle + \langle 1, \mu \rangle = c$, and

$$h - \lambda_G * h = h - \langle 1, h \rangle \lambda_G = \mu \in W_{2N,K},$$

as desired.

We abbreviate *left-* [*two-sided*] *invariant linear functional* as LILF [TILF]. It is evident that Lemma 2 as well as Lemma 1 has a natural "two-sided" analogue. All that follows are corollaries to Lemma 2.

THEOREM 1. *There exist uncountably many TILF's on $M(G)$ whose restrictions to one of $M_a(G)$ or $M_d(G)$ are linearly independent modulo the continuous functionals on $M_a(G)$ or $M_d(G)$.*

PROOF. The symbols $M_a(G)$ and $M_d(G)$ denote the absolutely continuous and discrete measures on G, respectively. Notice that $M_a(G)$ is a closed two-sided ideal while $M_d(G)$ is a closed two-sided invariant subalgebra of $M(G)$.

By Zorn's lemma, $T(M_a(G))^-$ contains a subset F_a which is maximal with respect to the linear independence modulo $T(M_a(G))$. Similarly there exists a subset F_d of $T(M_d(G))^-$ which is maximal with respect to the linear independence modulo $T(M_d(G))$. If G is discrete, then $M_a(G) = M_d(G) = M(G)$, so we shall choose $F_a = F_d$. If G is nondiscrete, it is obvious that $F_a \cap F_d = \varnothing$ and that $F_a \cup F_d$ is linearly independent modulo $T(M(G))$. Notice that

$$T(M_a(G))^- = \mathrm{sp}(F_a) + T(M_a(G)) \subset \mathrm{sp}(F_a) + T(M(G)),$$

and similarly for $M_d(G)$ and F_d.

Now suppose that X is any two-sided invariant, closed subspace of $M(G)$ and that there exists a countable subset F of $M(G)$ such that $T(X)^- \subset \mathrm{sp}(F) + T(M(G))$. We then claim that the unit ball $B_1(X)$ of X is not weak-* dense in $B_1(M(G))$. Indeed, by a natural "two-sided" version of Lemma 2, there exists a weak-* compact subset W of $T(M(G))$ such that: if G is noncompact, then $B_1(X) \subset W$; and if G is compact, then $\mu - \lambda_G * \mu \in W$ for all $\mu \in B_1(X)$. First assume G is noncompact. Since W is weak-* compact, we then have $B_1(X)^* \subset W$, where $B_1(X)^*$ denotes the weak-* closure of $B_1(X)$ in $M(G)$. Since $W \subset T(M(G))$, we conclude that $\langle 1, \mu \rangle = 0$ for all $\mu \in B_1(X)^*$; hence $B_1(X)$ is not weak-* dense in $B_1(M(G))$. Next assume that G is compact (and infinite). Then $\mu - \lambda_G * \mu \in W$ for all $\mu \in B_1(X)^*$, again by the weak-* compactness of W. Taking discrete parts, we obtain $\mu_d \in (W)_d \subset T(M_d(G))$ for all $\mu \in B_1(X)^*$. A similar argument as above therefore shows that $B_1(X)$ is not weak-* dense in $B_1(M(G))$.

By the Hahn-Banach convexity theorem, both $B_1(M_a(G))$ and $B_1(M_d(G))$ are weak-* dense in $B_1(M(G))$. It follows from the above paragraph that neither F_a nor F_d is countable. In order to obtain uncountably many TILF's on $M(G)$ with the desired properties, it is sufficient to extend $\{\mu + T(M(G)) \colon \mu \in F_a \cup F_d\}$ to a Hamel base of the quotient space $M(G)/T(M(G))$. (Notice here that the dimension of $T(M)^-/T(M)$ plays an important role.) This completes the proof.

For two normed spaces X and Y on G, we define a norm on $X \cap Y$ by setting $\|f\| = \|f\|_X + \|f\|_Y$ for $f \in X \cap Y$. Notice that if both X and Y are Banach spaces on G, then so is $X \cap Y$.

THEOREM 1*. *Suppose G is noncompact (and σ-compact). Let $X \subset L^1(G)$ be a left-invariant Fréchet space on G whose topology is induced by countably many, left-invariant seminorms [e.g., $X = L^1 \cap C_0$]. If X contains an element with nonzero Haar integral, then there exist uncountably many TILF's on $L^1(G)$ whose restrictions to X are linearly independent modulo X'.*

PROOF. Unfortunately the closed unit ball of $L^1(G)$ is not $\sigma(C_c', C_c)$-compact, unless G is discrete. So we replace $L^1(G)$ by $Y = M(G)$. Since the topology of X is induced by left-invariant seminorms, each element of X has bounded (left) orbit in X. Since $X \subset L^1(G) \subset M(G)$ and G is noncompact, X also satisfies condition (C_3) in Lemma 2.

Let F be a subset of $T_l(X)^-$ which is maximal with respect to the linear independence modulo $T_l(X)$. Let F_0 be a subset of F which is maximal with respect to the linear independence modulo $T(M(G))$. Then we have

$$T_l(X)^- = \mathrm{sp}(F) + T_l(X) \subset \mathrm{sp}(F_0) + T(M(G)).$$

Therefore the proofs of Lemmas 1 and 2 apply to the present situation *mutatis mutandis*. Since X contains an element with nonzero Haar integral by the hypotheses, it follows from Lemma 2 that the set F_0 cannot be countable. This completes the proof.

REMARKS. (I) Let G be an arbitrary LC group. If there exists a closed normal subgroup H of G such that G/H is infinite and σ-compact, then $M(G)$ has uncountably many TILF's whose restrictions to $M_a(G)$ are linearly independent modulo $[M_a(G)]'$. The proof is routine.

(II) It is possible to prove that if G is a noncompact LCA group, then every translation invariant linear operator on $L^1(G)$ is continuous. (We omit the proof although it is nontrivial.) Thus the study of invariant functionals and the study of invariant operators are somewhat different (cf. [**9**]).

THEOREM 2. *Suppose $1 < p \leq \infty$. Then the following assertions are equivalent.*

(i) *Each LILF on $L^p(G)$ is either a constant multiple of the Haar integral (if G is compact) or the zero functional (if G is noncompact).*

(ii) *The dimension of the quotient space $L^p(G)/T_l(L^p(G))$ is at most countable.*

(iii) *There exist a natural number N, a finite constant C, and a compact subset K of G such that each $f \in L^p(G)$ has a representation of the form*

$$f = a + \sum_{k=1}^{N} (f_k - \delta_{x_k} * f_k),$$

where $a \in \mathbf{C}$ [$a = 0$ if G is noncompact], $x_k \in K$, $f_k \in L^p(G)$, and $\|f_k\|_p \leq C\|f\|_p$ for all k's.

PROOF. The space $X = Y = L^p(G)$ is a left-invariant Banach space on G and possesses the three properties (C_1), (C_2) and (C_3) in Lemma 2. The only exceptional case arises when G is noncompact and $p = \infty$, because then $L^\infty(G)$ does not satisfy (C_3). However, this difficulty may be circumvented as follows.

Suppose that G is noncompact (and σ-compact) and that $T_l(C_0(G))^- \subset \mathrm{sp}(F) + T_l(L^\infty(G))$ for some countable subset F of $L^\infty(G)$. It is obvious that $X = C_0(G)$ satisfies conditions (C_2) and (C_3) in Lemma 2. It follows from Lemma 2 with $Y = L^\infty(G)$ that there exists a natural number N and a compact subset K of G such that $B_1(C_0(G))$ is contained in the weak-* compact subset $W_{N,K}$ of $T_l(L^\infty(G))$. But then $B_1(L^\infty(G)) \subset W_{N,K}$, since $B_1(C_0(G))$ is weak-* dense in $B_1(L^\infty(G))$. We have thus proved that (ii) $\Rightarrow$ (iii) holds even in the exceptional case under discussion. This completes the proof.

THEOREM 2*. *Let G be a σ-compact, noncompact, amenable LC group, and let $1 < p \leqslant \infty$. Then there exist uncountably many LILF's on $L^1(G) + L^p(G)$ whose restrictions to $L^p \cap C_0(G)$ are linearly independent modulo the continuous linear functionals on $L^p \cap C_0(G)$.*

PROOF. The space $Y = M(G) + L^p(G)$ forms a Banach space on G with respect to the intermediate norm defined by

$$\|f\| = \inf\{\|g\|_M + \|h\|_p : g \in M(G), h \in L^p(G), \& \, g + h = f\}.$$

It is easy to check that $X = L^p \cap C_0(G)$ and Y satisfy conditions (C_1)–(C_3) in Lemma 2.

Suppose, by way of contradiction, that $T_l(X)^- \subset \mathrm{sp}(F) + T_l(Y)$ for some countable subset F of Y. Then Lemma 2 yields a $\sigma(Y, C_c)$-compact subset $W_{N,K}$ of $T_l(Y)$ such that $B_1(L^p \cap C_0) \subset W_{N,K}$. By using an appropriate approximate identity, one checks that $B_1(L^p \cap C_0)$ is $\sigma(C_c', C_c)$-dense in $B_1(L^p \cap L^\infty)$; hence $B_1(L^p \cap L^\infty) \subset W_{N,K}$. The remainder of the proof may be accomplished by modifying Woodward's method in [**22**, Theorem 1], as follows:

Since $B_1(L^p \cap L^\infty) \subset W_{N,K}$, each element f of $L^p \cap L^\infty$ can be written in the form

$$f = \mu + \sum_{k=1}^{N} (h_k - \delta_{x_k} * h_k), \tag{8}$$

where $\mu \in M(G)$, $h_k \in L^p(G)$ and $x_k \in K$ for all k's. Since G is assumed to be amenable, it satisfies the Følner condition [**5**]. We can, therefore, find a sequence (E_n) of compact subsets of G such that

$$\lambda_G(E_n \Delta x^{-1} E_n) < n^{-1} \lambda_G(E_n) \qquad (x \in K, n \in \mathbf{N}), \tag{9}$$

where Δ stands for symmetric difference of sets. There is no loss of generality in assuming that $\lambda_G(E_n) > n^2$ for all $n \geqslant 1$. Writing $q = p/(p-1)$, we infer from (9) and Hölder's inequality that $h \in L^p(G)$ and $x \in K$ imply

$$\left| \int_{E_n} (h - \delta_x * h)\, dx \right| \leqslant \int_{E_n \Delta x^{-1} E_n} |h|\, dx \leqslant \lambda_G(E_n \Delta x^{-1} E_n)^{1/q} \|h\|_p$$
$$\leqslant n^{-1/q} \lambda_G(E_n)^{1/q} \|h\|_p$$

for all $n \geqslant 1$. It follows from (8) that $f \in L^p \cap L^\infty$ implies

$$\left|\int_{E_n} f\,dx\right| \leqslant \|\mu\| + n^{-1/q}\lambda_G(E_n)^{1/q} \sum_{k=1}^{N} \|h_k\|_p$$
$$\leqslant C_f \cdot n^{-1/q}\lambda_G(E_n)^{1/q} \qquad (n = 1,2,\ldots),$$

where C_f is a finite constant depending only on f. Therefore the Banach-Steinhaus theorem yields a finite constant C such that

$$\left|\int_{E_n} f\,dx\right| \leqslant Cn^{-1/q}\lambda_G(E_n)^{1/q}(\|f\|_p + \|f\|_\infty) \tag{10}$$

for all $f \in L^p \cap L^\infty$ and all n's. In particular, choosing f to be the indicator function of E_n, we obtain

$$\lambda_G(E_n) \leqslant Cn^{-1/q}\lambda_G(E_n)^{1/q}\left[\lambda_G(E_n)^{1/p} + 1\right]$$
$$\leqslant 2Cn^{-1/q}\lambda_G(E_n) \qquad (n = 1,2,\ldots).$$

Since q is a finite positive number, the last inequalities give us the desired contradiction.

THEOREM 3. *Let X be a left-invariant closed subspace of $L^\infty(G)$ such that $B_1(X)$ is weak-* dense in $B_1(L^\infty)$ [e.g., $C_0(G)$, $C_u(G)$, $C(G)$, etc.]. Suppose either:*
(i) *G is compact and $L^\infty(G)$ has two linearly independent LILF's,*
(ii) *G is noncompact and $L^\infty(G)$ has a nonzero LILF, or*
(iii) *$l^\infty(G)$ has a LILF which does not annihilate $C(G)$.*
Then there exist uncountably many LILF's on $L^\infty(G)$ whose restrictions to X are linearly independent modulo X'.

PROOF. A moment's glance at the proof of Theorem 2 shows that the desired conclusion is certainly true if either (i) or (ii) holds. So it will suffice to show that (iii) implies (i) if G is compact, and (ii) if G is noncompact. If G is discrete, then there is nothing to prove. So assume G is nondiscrete.

Passing to a nondiscrete metrizable quotient of G (cf. [7, p. 71]), we can find a compact subset E of G such that $\lambda_G(E) > 0$ and E has empty interior. Then no finitely many translates of E cover G (a.e.) by the Baire category theorem. Since the maximal ideal space of $L^\infty(G)$ is compact, it follows that there exists a nonzero complex homomorphism Φ of $L^\infty(G)$ such that $\Phi(\delta_x * \xi_E) = 0$ for all $x \in G$. (ξ_E denotes the indicator function of E.) It is easy to see that such a Φ can be chosen so that $\Phi(f) = f(e)$ for all $f \in C(G)$, where e is the identity element of G. Define a mapping Φ' from $L^\infty(G)$ into $l^\infty(G)$ by setting

$$(\Phi' f)(x) = \Phi(\delta_{x^{-1}} * f) \qquad (f \in L^\infty(G) \text{ and } x \in G). \tag{11}$$

Thus $\Phi'\xi_E = 0$ and $(\Phi' f)(x) = f(x)$ for all $f \in C(G)$. Moreover, $f \in L^\infty(G)$, $a \in G$, and $x \in G$ imply

$$\Phi'(\delta_a * f)(x) = \Phi(\delta_{x^{-1}} * \delta_a * f) = (\Phi' f)(a^{-1}x)$$

by (11). It follows immediately that $\Phi'[T_l(L^\infty(G))] \subset T_l(l^\infty(G))$.

Now suppose that either (i) or (ii) fails to hold. Then $L^\infty(G) = \mathrm{sp}(\xi_E) + T_l(L^\infty(G))$ by Theorem 2 with $p = \infty$. Since $C(G) \subset L^\infty(G)$ and $\Phi'\xi_E = 0$, it follows that $C(G) = \Phi'[C(G)] \subset T_l(l^\infty(G))$. This contradicts (iii) and the proof is complete.

COROLLARY 4. *Let G be a σ-compact, infinite, LC group. If G is amenable as a discrete group, a fortiori, if G is abelian, then there exist uncountably many LILF's on $L^\infty(G)$ whose restrictions to $C_0(G)$ are linearly independent modulo $[C_0(G)]'$.*

PROOF. Obvious from the definition of amenability.

REMARKS. (III) Whenever G is a nondiscrete σ-compact LC group, there exist uncountably many TILF's on $M(G) + L^\infty(G)$ whose restrictions to $L^1(G)$ are linearly independent modulo $[L^1(G)]'$. This may be proved by modifying the proof of Theorem 1.

(IV) $L^\infty(G)$ has a discontinuous LILF if and only if so does $C_{lu}(G)$, the Banach space of all left uniformly continuous bounded functions on G. The "only if" part is a special case of Theorem 3. The proof of the "if" part requires Cohen's factorization theorem [**2**]; see also [**6**].

(V) Let $\mathrm{LIM}(L^\infty(G))$ denote the set of all left-invariant means on $L^\infty(G)$. As is well known, if G is an infinite LC group which is amenable as a discrete group, then the dimension of [the linear space spanned by] $\mathrm{LIM}(L^\infty(G))$ is quite "huge"; see [**1, 3–5, 18–21**]. This can also happen for some nonamenable groups.

Suppose that G and H are two infinite compact groups, and that G is amenable as a discrete group. For each $M \in \mathrm{LIM}(L^\infty(G))$, define

$$\langle f, M \times \lambda_H \rangle = \left\langle \int f(x, y)\, d\lambda_H(y), M_x \right\rangle \qquad (f \in L^\infty(G \times H)).$$

It is obvious that this is well defined and that the correspondence $M \to M \times \lambda_H$ is an isomorphism of $\mathrm{LIM}(L^\infty(G))$ into $\mathrm{LIM}(L^\infty(G \times H))$. We do not know whether condition (iii) in Theorem 3 characterizes the amenable groups.

(VI) Let G be a free group with two generators (cf. [7]). It is a well-known fact that $l^\infty(G)$ has no left-invariant mean. In fact, the zero functional is the only one LILF on $l^\infty(G)$. Therefore the conclusion of Theorem 3 fails to hold for some groups.

Let a and b be the free generators of G, and let $f \in l^\infty(G)$ be given. We claim that f has a representation of the form

$$f(x) = g(ax) - g(x) + h(bx) - h(x) \qquad (x \in G)$$

for some g and $h \in l^\infty(G)$. To confirm this, notice that G is the union of the sets $\{e\}$, A and B, where $A[B]$ is the set of all reduced words starting with a nonzero power of $a[b]$. Therefore we can write $f = f_1 + f_2$, where $f_1, f_2 \in l^\infty(G)$, $\mathrm{supp}\, f_1 \subset A \cup \{e\}$, and $\mathrm{supp}\, f_2 \subset B \cup \{e\}$. Using the fact that G is the union of the disjoint sets $b^n(A \cup \{e\})$ for $n = 0, \pm 1, \ldots$, we define

$$h(b^n x) = \begin{cases} f_1(x) & \text{if } x \in A \cup \{e\} \text{ and } n > 0, \\ 0 & \text{if } x \in A \cup \{e\} \text{ and } n \leqslant 0. \end{cases}$$

Similarly, define

$$g(a^n x) = \begin{cases} f_2(x) & \text{if } x \in B \cup \{e\} \text{ and } n > 0, \\ 0 & \text{if } x \in B \cup \{e\} \text{ and } n \leqslant 0. \end{cases}$$

One checks that $f_1(x) = h(bx) - h(x)$ and $f_2(x) = g(ax) - g(x)$ for all $x \in G$. Hence $f = f_1 + f_2$ admits a decomposition of the desired form. It follows immediately that the only LILF on $l^\infty(G)$ is the trivial functional.

(VII) Let G be an arbitrary LC group. Suppose G contains a closed normal subgroup H such that G/H is compact and $C(G/H)$ admits a discontinuous LILF. Then there exist uncountably many LILF's on $C_c(G)$ which are linearly independent modulo $C_c'(G)$. This may be proved by using Theorem (15.21) of [7] and applying Lemma 2 (cf. [17]).

EXAMPLES. (a) Let $A_1(\mathbf{R})$ denote the space of all $f \in L^1 \cap C^\infty(\mathbf{R})$ such that $\operatorname{supp} \hat{f} \subset [-1, 1]$ and $f^{(n)} \in L^1(\mathbf{R})$ for all $n \in \mathbf{N}$. Here $\hat{f}$ denotes the Fourier transform of f. It is evident that $A_1(\mathbf{R})$ forms an invariant Fréchet space on $\mathbf{R}$ with respect to the seminorms

$$p_n(f) = \|f\|_1 + \|f'\|_1 + \cdots + \|f^{(n)}\|_1 \qquad (n = 1, 2, \ldots).$$

Let Y denote the linear span in $C_c'(\mathbf{R})$ of all $M(\mathbf{R}) + L^p(G)$ with $1 \leqslant p < \infty$. Then Y has uncountably many TILF's whose restrictions to $A_1(\mathbf{R})$ are linearly independent modulo $[A_1(\mathbf{R})]'$.

To prove this, put $Y_m = M(\mathbf{R}) + L^m(\mathbf{R})$ for $m \geqslant 1$. Then $Y_m \subset Y_{m+1}$ and Y is the union of all Y_m with $m \in \mathbf{N}$. Accordingly, Y may be regarded as the strict inductive limit of the invariant Banach spaces Y_m ($m \in \mathbf{N}$). Suppose by way of contradiction that the above-stated result is false. Then we can modify the proofs of Lemmas 1 and 2 to obtain three natural numbers N, q and n with the following property: each $f \in A_1(\mathbf{R})$ can be written in the form

$$f = \sum_{j=1}^{N} \left(\mu_j - \delta_{x_j} * \mu_j\right) + \sum_{j=1}^{N} \left(h_j - \delta_{x_j} * h_j\right),$$

where $\mu_j \in M(\mathbf{R})$, $h_j \in L^q(\mathbf{R})$, $x_j \in [-N, N]$, $\|\mu_j\|_M \leqslant N p_n(f)$, and $\|h_j\|_q \leqslant N p_n(f)$ for all j's.

Now put $I_m = [-m, m]$ for $m = 1, 2, \ldots$. Then notice that the symmetric difference $\{[-N, N] + I_m\} \Delta I_m$ has Lebesgue measure $2N$ and is disjoint from I_m. If $f \in A_1(\mathbf{R})$ is as above, we infer from Hölder's inequality that

$$\begin{aligned} \left| \int_{-m}^{m} f\,dx \right| &\leqslant \sum_{j=1}^{N} \left|\mu\left(\left[I_m - x_j\right] \Delta I_m\right)\right| + \sum_{j=1}^{N} \int_{[I_m - x_j] \Delta I_m} |h_j|\,dx \\ &\leqslant o(1) + \sum_{j=1}^{N} \left\{ \int_{[I_m - x_j] \Delta I_m} |h_j|^q\,dx \right\}^{1/q} (2N)^{(q-1)/q} \\ &= o(1) + o(1) = o(1) \quad \text{as } m \to \infty. \end{aligned}$$

Therefore, $\int_{\mathbf{R}} f\,dx = 0$ for all $f \in A_1(\mathbf{R})$, which is of course absurd.

(b) As was proved by Meisters [10], each TILF on $C^{\infty}(\Pi)$ is continuous and is therefore a constant multiple of the Lebesgue integral. However, this is not true for any $C^{(m)}(\Pi) = \{f \in C(\Pi): f^{(m)} \in C(\Pi)\}$, where $m \in \mathbf{N}$.

To prove this, fix $m \in \mathbf{N}$ and let Y_m denote the subspace of all $f \in C^{(m-1)}(\Pi)$ such that $f^{(m-1)}$ is absolutely continuous and $f^{(m)} \in L^{\infty}(\Pi)$. Then Y_m forms an invariant Banach space on Π with respect to the norm

$$\|f\|_{(m)} = \|f\|_{\infty} + \|f'\|_{\infty} + \cdots + \|f^{(m)}\|_{\infty} \qquad (f \in Y_m).$$

One checks that the imbedding of $C^{(m)}(\Pi)$ into Y_m is isometric, that $B_1(Y_m)$ is $\sigma(Y_m, C(\Pi))$-compact, and that the unit ball of $C^{(m)}(\Pi)$ is $\sigma(Y_m, C(\Pi))$-dense in $B_1(Y_m)$. Suppose that $X = C^{(m)}(\Pi)$ has no discontinuous TILF. Then Lemmas 1 and 2 assure that each $f \in Y_m$ can be written in the form

$$f = \langle 1, f \rangle + \sum_{j=1}^{N} \left(h_j - \delta_{x_j} * h_j\right),$$

where $h_j \in Y_m$ and $x_j \in \Pi$ for all j's. Taking the mth derivatives of both sides of the last equation, we obtain

$$f^{(m)} = \sum_{j=1}^{N} \left[h_j^{(m)} - \delta_{x_j} * h_j^{(m)}\right].$$

However, it is easy to see that $(d/dt)^m Y_m$ consists of all $g \in L^{\infty}(\Pi)$ with $\langle 1, g \rangle = 0$. It follows from Theorem 3 that the last expression is impossible for some $f \in Y_m$. This *reductio ad absurdum* establishes the desired result. Notice also that each TILF on $C^{(m)}(\Pi)$ which extends to a TILF on $L^2(\Pi)$ is necessarily continuous by [15].

(c) The following example is included here because of its contrast with the last example. Let G be a compact abelian group with dual Γ, and let (ψ_n) be a sequence of elements of $l^{\infty}(\Gamma)$ such that $1 \leqslant \psi_1 \leqslant \psi_2 \leqslant \cdots$. Let X consist of all $f \in C(G)$ such that

$$p_n(f) = \sum \left\{|\hat{f}(\gamma)| \cdot \psi_n(\gamma): \gamma \in \Gamma\right\} < \infty \qquad (n = 1, 2, \ldots).$$

It is obvious that X forms an invariant Fréchet space on G with respect to the seminorms $p_n(\cdot)$. Suppose that there exists a compact subgroup H of G such that G/H is an infinite torsion group. Then it is easy to construct an element $f_0 \in X$ with the following property: whenever F is a finite subset of G, then $\lambda_{H(F)} * f_0$ is nonconstant, where $H(F)$ is the compact subgroup of G generated by $H \cup F$ and $\lambda_{H(F)}$ is the normalized Haar measure of $H(F)$.

Suppose by way of contradiction that there is no TILF on $L^1(G)$ whose restriction to X is discontinuous. Then, by Lemmas 1, 2 and the Lebesgue-Radon-Nikodým theorem, we can write f_0 in the form

$$f_0 = C + \sum_{j=1}^{N} \left(g_j - \delta_{x_j} * g_j\right),$$

where $C = \langle 1, f_0 \rangle$, $g_j \in L^1(G)$ and $x_j \in G$ for all j's. Letting $F = \{x_1, \ldots, x_N\}$, we then have $\lambda_{H(F)} * f_0 = C$ a.e. on G. Since f_0 is continuous, it follows that $\lambda_{H(F)} * f_0 = C$ everywhere on G, which contradicts our choice of f_0.

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