TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 282, Number 2, April 1984

STRONG MARTINGALE CONVERGENCE OF GENERALIZED CONDITIONAL EXPECTATIONS ON VON NEUMANN ALGEBRAS

BY

FUMIO HIAI AND MAKOTO TSUKADA

ABSTRACT. Accardi and Cecchini generalized the concept of conditional expectations on von Neumann algebras. In this paper we give some conditions for strong convergence of increasing or decreasing martingales of Accardi and Cecchini's conditional expectations.

Introduction. The study of conditional expectations and martingale convergence theorems in the operator algebraic framework was initiated by Umegaki's earlier works [**14**, **15**]. Since then, the martingale convergence theorems on von Neumann algebras have been developed by several authors. For example, the almost sure type martingale convergence was obtained by Lance [**9**] and Dang-Ngoc [**4**]. The strong martingale convergence was completed in [**13**] by one of the authors.

The conditional expectation of a von Neumann algebra M onto a von Neumann subalgebra N does not generally exist given a faithful normal state (or semifinite weight) φ on M. Indeed its existence is equivalent to the invariance of N under the modular automorphism group associated with φ (see [**12**]). To recover this drawback of noncommutative conditional expectations, Accardi and Cecchini [**1**] generalized the concept of conditional expectations on von Neumann algebras by using the Tomita-Takesaki theory [**11**]. Accardi and Cecchini's conditional expectation ε: $M \to N$ with respect to φ always exists but is not necessarily a projection onto N and lacks a useful property that $\varepsilon(axb) = a\varepsilon(x)b$ for a, $b \in N$ and $x \in M$.

Araki [**2**] established the convergence of modular operators and modular conjugations given an increasing net of von Neumann subalgebras and a faithful normal state. By extending the arguments in [**2**], we have obtained in [**8**] the martingale type convergence of modular automorphism groups under an increasing or decreasing net of von Neumann subalgebras and a faithful normal semifinite weight. On the same lines, we investigate in this paper the strong convergence of martingales of Accardi and Cecchini's conditional expectations (called here generalized conditional expectations).

In §1 of this paper we state the definition of the generalized conditional expectation with respect to a faithful normal semifinite weight and give easy technical

Received by the editors March 15, 1983 and, in revised form, May 22, 1983.

1980 *Mathematics Subject Classification*. Primary 46L10.

Key words and phrases. Conditional expectation, martingale, Tomita-Takesaki theory, faithful normal semifinite weight, nonabelian K-flow.

lemmas. In §2 we establish necessary and sufficient conditions for the strong convergence of an increasing martingale of generalized conditional expectations. In §3 we show the strong convergence of a decreasing martingale of generalized conditional expectations with suitable assumptions. Finally we have an application to some convergence properties for nonabelian K-flows.

The authors would like to express their gratitude to Professor H. Umegaki for his advice and constant encouragement.

1. Preliminaries. Let M be a von Neumann algebra and φ be a faithful normal semifinite weight on M. We use the usual notations $\mathfrak{n}_\varphi = \{x \in M: \varphi(x^*x) < \infty\}$ and $\mathfrak{m}_\varphi = \mathfrak{n}_\varphi^*\mathfrak{n}_\varphi$. Let $(\mathcal{H}_\varphi, \pi_\varphi)$ be the GNS representation of M induced by φ where the canonical injection of $\mathfrak{n}_\varphi$ into $\mathcal{H}_\varphi$ is denoted by $x \mapsto x_\varphi$. Then $\mathfrak{A}_\varphi = \{x_\varphi$: $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*\}$ is an achieved left Hilbert algebra and $\pi_\varphi(M)$ is its left von Neumann algebra. Let Δ_φ, J_φ and σ_t^φ be the modular operator, the modular conjugation and the modular automorphism group, respectively, associated with φ.

Let N be a unital von Neumann subalgebra of M. The *conditional expectation* ε: $M \to N$ with respect to φ is a unique norm one normal projection ε of M onto N such that $\varphi(x) = \varphi(\varepsilon(x))$ for all $x \in M_+$. According to Takesaki [**12**], there exists the conditional expectation ε: $M \to N$ with respect to φ if and only if $\varphi \upharpoonright N$ is semifinite and $\sigma_t^\varphi(N) = N$ for every $t \in \mathbf{R}$. Recently the concept of conditional expectations has been extended by Accardi and Cecchini [**1**]. Now suppose only that $\psi = \varphi \upharpoonright N$ is semifinite, and let $(\mathcal{H}_\psi, \pi_\psi)$ and J_ψ be the GNS representation of N and the modular conjugation associated with ψ. Since the canonical injection of $\mathfrak{n}_\psi = \mathfrak{n}_\varphi \cap N$ into $\mathcal{H}_\psi$ is taken as the restriction of $x \mapsto x_\varphi$ to $\mathfrak{n}_\psi$, $\mathcal{H}_\psi$ is identified with a subspace of $\mathcal{H}_\varphi$. Let P be the orthogonal projection of $\mathcal{H}_\varphi$ onto $\mathcal{H}_\psi$, then $P \in \pi_\varphi(N)'$ and $\pi_\psi(x) = \pi_\varphi(x)P$ for all $x \in N$. The conditional expectation ε: $M \to N$ with respect to φ is introduced in [**1**] by

$$\pi_\psi(\varepsilon(x)) = J_\psi P J_\varphi \pi_\varphi(x) J_\varphi J_\psi, \qquad x \in M,$$

which is a faithful normal completely positive map of M into N such that $\varepsilon(1) = 1$ and $\varphi(x) = \varphi(\varepsilon(x))$ for all $x \in M_+$ (cf. [**1**, Theorem 7.5]). We here call ε the *generalized conditional expectation* with respect to φ. This coincides with the conditional expectation (as a norm one projection onto N) with respect to φ whenever the latter exists.

In the following lemmas, we assume that $\psi = \varphi \upharpoonright N$ is semifinite and ε: $M \to N$ is the generalized conditional expectation with respect to φ.

LEMMA 1. *If* $x \in \mathfrak{n}_\varphi$, *then* $\varepsilon(x) \in \mathfrak{n}_\psi$ *and*

$$(\varepsilon(x))_\varphi = J_\psi P J_\varphi x_\varphi.$$

PROOF. Let $x \in \mathfrak{n}_\varphi$. Then $\varepsilon(x) \in \mathfrak{n}_\psi$ follows from

$$\psi(\varepsilon(x)^*\varepsilon(x)) \leqslant \psi(\varepsilon(x^*x)) = \varphi(x^*x) < \infty.$$

Choosing a net $\{a_j\}$ in $\mathfrak{m}_\psi \cap N_+$ with $a_j \nearrow 1$, we have

$$\begin{aligned}(\varepsilon(x))_\varphi &= \lim_j J_\psi \pi_\psi(a_j) J_\psi(\varepsilon(x))_\varphi = \lim_j \pi_\psi(\varepsilon(x)) J_\psi(a_j)_\varphi \\ &= \lim_j J_\psi P J_\varphi \pi_\varphi(x) J_\varphi(a_j)_\varphi = \lim_j J_\psi P \pi_\varphi(a_j) J_\varphi x_\varphi \\ &= J_\psi P J_\varphi x_\varphi. \quad \text{Q.E.D.}\end{aligned}$$

LEMMA 2. *If γ is a $*$-automorphism of M and $\tilde{\varepsilon}$: $M \to \gamma^{-1}(N)$ is the generalized conditional expectation with respect to $\varphi \circ \gamma$, then $\tilde{\varepsilon} = \gamma^{-1} \circ \varepsilon \circ \gamma$.*

PROOF. Let $\tilde{\varphi} = \varphi \circ \gamma$, $\tilde{\psi} = \tilde{\varphi} \restriction \gamma^{-1}(N)$ and $\tilde{P}$ be the orthogonal projection of $\mathcal{H}_{\tilde{\varphi}}$ onto $\mathcal{H}_{\tilde{\psi}}$. There is a unitary operator U of $\mathcal{H}_{\tilde{\varphi}}$ onto $\mathcal{H}_\varphi$ such that $Ux_{\tilde{\varphi}} = (\gamma(x))_\varphi$ for all $x \in \mathfrak{n}_{\tilde{\varphi}}$. Then it is readily seen that $P = U\tilde{P}U^*$, $J_\varphi = UJ_{\tilde{\varphi}}U^*$ and $\pi_\varphi(\gamma(x)) = U\pi_{\tilde{\varphi}}(x)U^*$ for all $x \in M$. Hence

$$\begin{aligned}\pi_\psi(\gamma(\tilde{\varepsilon}(x))) &= U\pi_{\tilde{\psi}}(\tilde{\varepsilon}(x))U^* = UJ_{\tilde{\psi}}\tilde{P}J_{\tilde{\varphi}}\pi_{\tilde{\varphi}}(x)J_{\tilde{\varphi}}J_{\tilde{\psi}}U^* \\ &= J_\psi P J_\varphi \pi_\varphi(\gamma(x)) J_\varphi J_\psi = \pi_\psi(\varepsilon(\gamma(x))), \qquad x \in M,\end{aligned}$$

so that we have $\gamma \circ \tilde{\varepsilon} = \varepsilon \circ \gamma$. Q.E.D.

2. Increasing case. In this section we discuss the martingale convergence of generalized conditional expectations for the increasing case. We fix an increasing net $\{N_\alpha\}$ of unital von Neumann subalgebras of M and let $N_\infty = \bigvee_\alpha N_\alpha$. Let φ be a faithful normal semifinite weight on M. Assume that $\psi_\alpha = \varphi \restriction N_\alpha$ is semifinite for each α and hence also $\psi_\infty = \varphi \restriction N_\infty$ is semifinite. For each α, we take $\mathfrak{n}_\alpha = \mathfrak{n}_\varphi \cap N_\alpha$, $\mathfrak{m}_\alpha = \mathfrak{n}_\alpha^* \mathfrak{n}_\alpha$, the GNS representation $(\mathcal{H}_\alpha, \pi_\alpha)$ of N_α, the left Hilbert algebra $\mathfrak{A}_\alpha = \{x_\varphi \colon x \in \mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*\}$, the modular operator Δ_α, the modular conjugation J_α and the modular automorphism group σ_t^α associated with ψ_α. then $\{\mathcal{H}_\alpha\}$ is an increasing net of subspaces of $\mathcal{H}_\varphi$. Let P_α be the orthogonal projection of $\mathcal{H}_\varphi$ onto $\mathcal{H}_\alpha$. We further take $\mathfrak{n}_\infty$, $\mathfrak{m}_\infty$, $(\mathcal{H}_\infty, \pi_\infty)$, $\mathfrak{A}_\infty$, Δ_∞, J_∞, σ_t^∞ and P_∞ analogously associated with ψ_∞. The generalized conditional expectations ε_α: $M \to N_\alpha$ and ε_∞: $M \to N_\infty$ with respect to φ are given as follows:

$$\begin{aligned}\pi_\alpha(\varepsilon_\alpha(x)) &= J_\alpha P_\alpha J_\varphi \pi_\varphi(x) J_\varphi J_\alpha, \\ \pi_\infty(\varepsilon_\infty(x)) &= J_\infty P_\infty J_\varphi \pi_\varphi(x) J_\varphi J_\infty, \qquad x \in M.\end{aligned}$$

Under the above assumptions and notations, we now state

THEOREM 3. *The following conditions are equivalent*:

(i) $\bigcup_\alpha \mathfrak{A}_\alpha$ $(\subset \mathfrak{A}_\infty)$ *is a core of* $\Delta_\infty^{1/2}$;

(ii) $\|(\varepsilon_\alpha(x))_\varphi - (\varepsilon_\infty(x))_\varphi\| \to 0$ *for every* $x \in \mathfrak{n}_\varphi$;

(iii) $s\text{-}\lim_\alpha \varepsilon_\alpha(x) = \varepsilon_\infty(x)$ *for every* $x \in M$;

(iv) $\|f \circ \varepsilon_\alpha - f \circ \varepsilon_\infty\| \to 0$ *for every* $f \in M_*$.

If φ is bounded (i.e., $\varphi(1) < \infty$), then the above conditions (i)–(iv) *are fulfilled.*

PROOF. (i) $\Rightarrow$ (ii). If $x \in \mathfrak{n}_\varphi$, then by Lemma 1 we have

$$(\varepsilon_\alpha(x))_\varphi = J_\alpha P_\alpha J_\varphi x_\varphi, \qquad (\varepsilon_\infty(x))_\varphi = J_\infty P_\infty J_\varphi x_\varphi.$$

Therefore the condition (ii) is equivalent to $s\text{-}\lim_\alpha J_\alpha P_\alpha = J_\infty P_\infty$, and so we may assume $N_\infty = M$ and show $s\text{-}\lim_\alpha J_\alpha P_\alpha = J_\varphi$. Let $f \in M_*^+$ with $f \leqslant \varphi$ be given. By [**16**, Theorem 3.2], there exists a unique $h \in \mathfrak{m}_\varphi$ with $0 \leqslant h \leqslant 1$ such that

$$f(x) = \tfrac{1}{2}\varphi(hx + xh), \qquad x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*.$$

For each α, there further exists a unique $h_\alpha \in \mathfrak{m}_\alpha$ with $0 \leqslant h_\alpha \leqslant 1$ such that

$$f(x) = \tfrac{1}{2}\varphi(h_\alpha x + xh_\alpha), \qquad x \in \mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*.$$

Under the assumption (i), the following results (1)–(3) have been proved in [**8**].

(1) $h_\varphi \in D(\Delta_\varphi)$, $(h_\alpha)_\varphi \in D(\Delta_\alpha)$, $\|(h_\alpha)_\varphi - h_\varphi\| \to 0$ and $\|\Delta_\alpha(h_\alpha)_\varphi - \Delta_\varphi h_\varphi\| \to 0$.

(2) The set $\{(1 + \Delta_\varphi)h_\varphi\}$ is total in $\mathcal{H}_\varphi$ where $h \in \mathfrak{m}_\varphi$ with $0 \leqslant h \leqslant 1$ is taken as above for any $f \in M_*^+$ with $f \leqslant \varphi$.

(3) Let $\tilde{\Delta}_\alpha = \Delta_\alpha P_\alpha + (1 - P_\alpha)$. Then $s\text{-}\lim_\alpha \tilde{\Delta}_\alpha^{it} = \Delta_\varphi^{it}$ uniformly for t in any finite interval.

We now define vector-valued continuous functions $F_\alpha(z)$ on the strip $0 \leqslant \operatorname{Re} z \leqslant 1$, analytic in the interior, by

$$F_\alpha(z) = e^{z^2}\left(\Delta_\alpha^z(h_\alpha)_\varphi - \Delta_\varphi^z h_\varphi\right), \qquad 0 \leqslant \operatorname{Re} z \leqslant 1.$$

Since

$$\|F_\alpha(it)\| \leqslant e^{-t^2}\|(h_\alpha)_\varphi - h_\varphi\| + e^{-t^2}\left\|\left(\tilde{\Delta}_\alpha^{it} - \Delta_\varphi^{it}\right)h_\varphi\right\|,$$
$$\|F_\alpha(1 + it)\| \leqslant e^{1-t^2}\|\Delta_\alpha(h_\alpha)_\varphi - \Delta_\varphi h_\varphi\| + e^{1-t^2}\left\|\left(\tilde{\Delta}_\alpha^{it} - \Delta_\varphi^{it}\right)\Delta_\varphi h_\varphi\right\|, \qquad t \in \mathbf{R},$$

we have

$$\begin{aligned} e^{1/4}\|J_\alpha(h_\alpha)_\varphi - J_\varphi h_\varphi\| &= e^{1/4}\|\Delta_\alpha^{1/2}(h_\alpha)_\varphi - \Delta_\varphi^{1/2}h_\varphi\| = \|F_\alpha(1/2)\| \\ &\leqslant \sup_{t\in\mathbf{R}}\left(\|F_\alpha(it)\|, \|F_\alpha(1 + it)\|\right) \to 0 \end{aligned}$$

by using (1) and (3). Hence

$$\|J_\alpha P_\alpha h_\varphi - J_\varphi h_\varphi\| \leqslant \|(h_\alpha)_\varphi - h_\varphi\| + \|J_\alpha(h_\alpha)_\varphi - J_\varphi h_\varphi\| \to 0.$$

Moreover, the set $\{h_\varphi\}$, taken for any $f \in M_*^+$ with $f \leqslant \varphi$, is total in $\mathcal{H}_\varphi$, because (2) implies

$$\{h_\varphi\}^\perp = \left((1 + \Delta_\varphi)^{-1}\mathcal{H}_\varphi\right)^\perp = \{0\}.$$

Thus $s\text{-}\lim_\alpha J_\alpha P_\alpha = J_\varphi$ is proved.

(ii) $\Rightarrow$ (i). Suppose that $s\text{-}\lim_\alpha J_\alpha P_\alpha = J_\infty P_\infty$ and hence $s\text{-}\lim_\alpha J_\alpha P_\alpha J_\infty P_\infty = P_\infty$. Take the generalized conditional expectation $\varepsilon_{\infty,\alpha}\colon N_\infty \to N_\alpha$ with respect to ψ_∞. Given $x \in \mathfrak{n}_\infty \cap \mathfrak{n}_\infty^*$, by Lemma 1 we obtain, for each α, $\varepsilon_{\infty,\alpha}(x) \in \mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*$ and

$$\|(\varepsilon_{\infty,\alpha}(x))_\varphi - x_\varphi\| = \|J_\alpha P_\alpha J_\infty x_\varphi - x_\varphi\| \to 0,$$
$$\|\Delta_\infty^{1/2}(\varepsilon_{\infty,\alpha}(x))_\varphi - \Delta_\infty^{1/2}x_\varphi\| = \|(\varepsilon_{\infty,\alpha}(x^*))_\varphi - (x^*)_\varphi\| = \|J_\alpha P_\alpha J_\infty(x^*)_\varphi - (x^*)_\varphi\| \to 0.$$

Thus (i) is satisfied.

(i) $\Rightarrow$ (iii). Given $x \in M$ and $\xi \in \mathcal{H}_\infty$, we have

$$\begin{aligned}
&\|\pi_\infty(\varepsilon_\alpha(x))\xi - \pi_\infty(\varepsilon_\infty(x))\xi\| \\
&\quad \leq \|\pi_\infty(\varepsilon_\alpha(x))(P_\alpha\xi - \xi)\| + \|\pi_\alpha(\varepsilon_\alpha(x))P_\alpha\xi - \pi_\infty(\varepsilon_\infty(x))\xi\| \\
&\quad \leq \|x\|\,\|P_\alpha\xi - \xi\| + \|J_\alpha P_\alpha J_\varphi \pi_\varphi(x) J_\varphi J_\alpha P_\alpha \xi - J_\infty P_\infty J_\varphi \pi_\varphi(x) J_\varphi J_\infty \xi\| \\
&\quad \to 0,
\end{aligned}$$

because $s\text{-}\lim_\alpha J_\alpha P_\alpha = J_\infty P_\infty$ and hence $s\text{-}\lim_\alpha P_\alpha = P_\infty$. Thus $s\text{-}\lim_\alpha \varepsilon_\alpha(x) = \varepsilon_\infty(x)$.

(i) $\Rightarrow$ (iv). Since $\pi_\infty(N_\infty)$ on $\mathcal{H}_\infty$ is a standard form of N_∞, given $f \in M_*^+$ there is a $\xi \in \mathcal{H}_\infty$ such that $f(x) = \langle \pi_\infty(x)\xi, \xi\rangle$ for all $x \in N_\infty$ (cf. [7, Lemma 2.10]). For every $x \in M$, we have

$$\begin{aligned}
|f(\varepsilon_\alpha(x)) - f(\varepsilon_\infty(x))| &\leq |\langle \pi_\infty(\varepsilon_\alpha(x))(\xi - P_\alpha\xi), \xi\rangle| \\
&\quad + |\langle \pi_\alpha(\varepsilon_\alpha(x))P_\alpha\xi, \xi\rangle - \langle \pi_\infty(\varepsilon_\infty(x))\xi, \xi\rangle| \\
&\leq \|x\|\,\|P_\alpha\xi - \xi\|\,\|\xi\| \\
&\quad + |\langle \pi_\varphi(x) J_\varphi J_\alpha P_\alpha \xi, J_\varphi J_\alpha P_\alpha \xi\rangle - \langle \pi_\varphi(x) J_\varphi J_\infty P_\infty \xi, J_\varphi J_\infty P_\infty \xi\rangle| \\
&\leq \|x\|\,\|\xi\|\big(\|P_\alpha\xi - \xi\| + 2\|J_\alpha P_\alpha \xi - J_\infty P_\infty \xi\|\big).
\end{aligned}$$

This shows that (i) implies (iv).

(iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii). If either (iii) or (iv) holds, then $w\text{-}\lim_\alpha \varepsilon_\alpha(x) = \varepsilon_\infty(x)$ for all $x \in M$. For every $x \in \mathfrak{n}_\varphi$, noting that $\{(\varepsilon_\alpha(x))_\varphi\}$ is bounded in $\mathcal{H}_\infty$, let ξ be any weak accumulation point of $\{(\varepsilon_\alpha(x))_\varphi\}$ and choose a subnet $\{(\varepsilon_{\alpha'}(x))_\varphi\}$ of $\{(\varepsilon_\alpha(x))_\varphi\}$ such that $(\varepsilon_{\alpha'}(x))_\varphi \to \xi$ weakly. Since $w\text{-}\lim_{\alpha'} \varepsilon_{\alpha'}(x) = \varepsilon_\infty(x)$, we then have $\xi = (\varepsilon_\infty(x))_\varphi$ (cf. [**10**, p. 28]). Hence it follows that $(\varepsilon_\alpha(x))_\varphi \to (\varepsilon_\infty(x))_\varphi$ weakly. On the other hand, $\|(\varepsilon_\alpha(x))_\varphi\| \leq \|(\varepsilon_\infty(x))_\varphi\|$ holds for each α by Lemma 1. Thus $\|(\varepsilon_\alpha(x))_\varphi - (\varepsilon_\infty(x))_\varphi\| \to 0$.

Finally let φ be bounded. For every $x \in N_\infty$, there is a net $\{x_j\}$ in $\cup_\alpha N_\alpha$ such that $s\text{-}\lim_j x_j = x$ and $s\text{-}\lim_j x_j^* = x^*$. We then have $\|(x_j)_\varphi - x_\varphi\| \to 0$ and $\|\Delta_\infty^{1/2}(x_j)_\varphi - \Delta_\infty^{1/2} x_\varphi\| = \|(x_j^*)_\varphi - (x^*)_\varphi\| \to 0$. Hence the condition (i) is satisfied. Q.E.D.

REMARKS. (1) When each ε_α is the conditional expectation (as a norm one projection) with respect to φ, the martingale convergence properties (iii) and (iv) are satisfied (see [**13**]).

(2) Under the condition (i), it can be proved (see [**8**]) that $s\text{-}\lim_\alpha \sigma_t^\alpha(x) = \sigma_t^\infty(x)$ uniformly for t in any finite interval for every $x \in \cup_\alpha N_\alpha$. In this case, it further holds that $s\text{-}\lim_\alpha \sigma_t^\alpha(\varepsilon_\alpha(x)) = \sigma_t^\infty(\varepsilon_\infty(x))$ uniformly for t in any finite interval for every $x \in M$.

3. Decreasing case. In this section we deal with the convergence of generalized conditional expectations for the decreasing case. Let $\{N_\alpha\}$ be a decreasing net of unital von Neumann subalgebras of M with $N_\infty = \cap_\alpha N_\alpha$. Let φ be a faithful normal semifinite weight on M such that $\psi_\infty = \varphi \restriction N_\infty$ is semifinite and hence each $\psi_\alpha = \varphi \restriction N_\alpha$ is semifinite. We use the notations $\mathfrak{n}_\alpha$, $\mathfrak{m}_\alpha$, $(\mathcal{H}_\alpha, \pi_\alpha)$, Δ_α, J_α, P_α and $\mathfrak{n}_\infty$, $\mathfrak{m}_\infty$, $(\mathcal{H}_\infty, \pi_\infty)$, Δ_∞, J_∞, P_∞ as in §2. We take the generalized conditional expectations ε_α: $M \to N_\alpha$ and ε_∞: $M \to N_\infty$ with respect to φ.

THEOREM 4. *Under the above assumptions, consider the following conditions:*
(i) $s\text{-}\lim_\alpha P_\alpha = P_\infty$ (*i.e.*, $\bigcap_\alpha \mathcal{H}_\alpha = \mathcal{H}_\infty$);
(ii) $\|(\varepsilon_\alpha(x))_\varphi - (\varepsilon_\infty(x))_\varphi\| \to 0$ *for every* $x \in \mathfrak{n}_\varphi$;
(iii) $s\text{-}\lim_\alpha \varepsilon_\alpha(x) = \varepsilon_\infty(x)$ *for every* $x \in M$;
(iv) $\|f \circ \varepsilon_\alpha - f \circ \varepsilon_\infty\| \to 0$ *for every* $f \in M_*$.
Then the conditions (i) *and* (ii) *are equivalent and the condition* (i) *implies* (iii) *and* (iv). *If* φ *is bounded, then the conditions* (i)–(iii) *are equivalent.*

PROOF. (i) $\Rightarrow$ (ii). By Lemma 1, the condition (ii) is equivalent to $s\text{-}\lim_\alpha J_\alpha P_\alpha = J_\infty P_\infty$. For each $f \in M_*^+$ with $f \leqslant \varphi$, there exist, by [**16**, Theorem 3.2], unique $h_\alpha \in \mathfrak{m}_\alpha$ and $h_\infty \in \mathfrak{m}_\infty$ with $0 \leqslant h_\alpha \leqslant 1$, $0 \leqslant h_\infty \leqslant 1$ such that

$$f(x) = \tfrac{1}{2}\varphi(h_\alpha x + x h_\alpha), \qquad x \in \mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*,$$

$$f(x) = \tfrac{1}{2}\varphi(h_\infty x + x h_\infty), \qquad x \in \mathfrak{n}_\infty \cap \mathfrak{n}_\infty^*.$$

Under the assumption (i), the following results (1)–(3) hold (see [**8**]).
(1) $(h_\alpha)_\varphi \in D(\Delta_\alpha)$, $(h_\infty)_\varphi \in D(\Delta_\infty)$, $\|(h_\alpha)_\varphi - (h_\infty)_\varphi\| \to 0$ and $\|\Delta_\alpha(h_\alpha)_\varphi - \Delta_\infty(h_\infty)_\varphi\| \to 0$.
(2) The set $\{(1 + \Delta_\infty)(h_\infty)_\varphi\}$ is total in $\mathcal{H}_\infty$ where $h_\infty \in \mathfrak{m}_\infty$ with $0 \leqslant h_\infty \leqslant 1$ is taken as above for any $f \in M_*^+$ with $f \leqslant \varphi$.
(3) Let $\tilde{\Delta}_\alpha = \Delta_\alpha P_\alpha + (1 - P_\alpha)$ and $\tilde{\Delta}_\infty = \Delta_\infty P_\infty + (1 - P_\infty)$. Then $s\text{-}\lim_\alpha \tilde{\Delta}_\alpha^{it} = \tilde{\Delta}_\infty^{it}$ uniformly for t in any finite interval.
Defining

$$F_\alpha(z) = e^{z^2}\left(\Delta_\alpha^z (h_\alpha)_\varphi - \Delta_\infty^z (h_\infty)_\varphi\right), \qquad 0 \leqslant \operatorname{Re} z \leqslant 1,$$

and using (1) and (3), we have

$$\left\|F_\alpha\left(\tfrac{1}{2}\right)\right\| = e^{1/4}\left\|J_\alpha(h_\alpha)_\varphi - J_\infty(h_\infty)_\varphi\right\| \to 0$$

as in the proof of Theorem 3. Hence

$$\left\|J_\alpha(h_\infty)_\varphi - J_\infty(h_\infty)_\varphi\right\| \leqslant \left\|(h_\infty)_\varphi - (h_\alpha)_\varphi\right\| + \left\|J_\alpha(h_\alpha)_\varphi - J_\infty(h_\infty)_\varphi\right\| \to 0.$$

Since (2) shows that the set $\{(h_\infty)_\varphi\}$ is total in $\mathcal{H}_\infty$, $J_\alpha P_\alpha = J_\alpha P_\infty + J_\alpha(P_\alpha - P_\infty)$ converges strongly to $J_\infty P_\infty$.
(ii) $\Rightarrow$ (i). If $s\text{-}\lim_\alpha J_\alpha P_\alpha = J_\infty P_\infty$, then for any $\xi \in \bigcap_\alpha \mathcal{H}_\alpha$ we have

$$\|P_\infty \xi\| = \|J_\infty P_\infty \xi\| = \lim_\alpha \|J_\alpha P_\alpha \xi\| = \|\xi\|,$$

so that $\xi \in \mathcal{H}_\infty$. Hence (i) holds.
(i) $\Rightarrow$ (iii). Suppose that $s\text{-}\lim_\alpha J_\alpha P_\alpha = J_\infty P_\infty$. Given $x \in M$ and $\xi \in \mathcal{H}_\infty$, we have

$$\left\|\pi_\varphi(\varepsilon_\alpha(x))\xi - \pi_\varphi(\varepsilon_\infty(x))\xi\right\| = \left\|J_\alpha P_\alpha J_\varphi \pi_\varphi(x) J_\varphi J_\alpha P_\alpha \xi - J_\infty P_\infty J_\varphi \pi_\varphi(x) J_\varphi J_\infty P_\infty \xi\right\| \to 0.$$

Choosing a net $\{a_j\}$ in $\mathfrak{m}_\infty \cap (N_\infty)_+$ with $a_j \nearrow 1$, we have

$$J_\varphi x_\varphi = \lim_j \pi_\varphi(a_j) J_\varphi x_\varphi = \lim_j J_\varphi \pi_\varphi(x) J_\varphi(a_j)_\varphi, \qquad x \in \mathfrak{n}_\varphi.$$

This shows that $\pi_\varphi(M)'\mathcal{H}_\infty$ is dense in $\mathcal{H}_\varphi$. Hence (iii) holds.

(i) $\Rightarrow$ (iv). Since $\pi_\varphi(M)'\mathcal{H}_\infty$ is dense in $\mathcal{H}_\varphi$, it suffices to prove (iv) for $f \in M_*^+$ such that

$$f(x) = \langle \pi_\varphi(x)a'\xi, a'\xi \rangle, \qquad x \in M,$$

where $a' \in \pi_\varphi(M)'$ and $\xi \in \mathcal{H}_\infty$. For this f, we have

$$\begin{aligned} |f(\varepsilon_\alpha(x)) - f(\varepsilon_\infty(x))| &= \left| \langle \pi_\varphi(\varepsilon_\alpha(x))\xi, a'^*a'\xi \rangle - \langle \pi_\varphi(\varepsilon_\infty(x))\xi, a'^*a'\xi \rangle \right| \\ &= \left| \langle \pi_\varphi(x)J_\varphi J_\alpha \xi, J_\varphi J_\alpha P_\alpha a'^*a'\xi \rangle - \langle \pi_\varphi(x)J_\varphi J_\infty \xi, J_\varphi J_\infty P_\infty a'^*a'\xi \rangle \right| \\ &\leq \|x\| \left(\|J_\alpha\xi - J_\infty\xi\| \, \|a'^*a'\xi\| + \|\xi\| \, \|(J_\alpha P_\alpha - J_\infty P_\infty)a'^*a'\xi\| \right). \end{aligned}$$

If (i) is satisfied, then it follows from the proof of (i) $\Rightarrow$ (ii) that $\|J_\alpha\xi - J_\infty\xi\| \to 0$ for all $\xi \in \mathcal{H}_\infty$. Therefore $\|f \circ \varepsilon_\alpha - f \circ \varepsilon_\infty\| \to 0$.

Finally if φ is bounded, then it is clear that (iii) implies (ii). Q.E.D.

REMARKS. (1) If each ε_α is the conditional expectation (as a norm one projection) with respect to φ, then the martingale convergence properties (iii) and (iv) hold (see [**13**]). In this case, $(\varepsilon_\alpha(x))_\varphi = P_\alpha x_\varphi$ and $(\varepsilon_\infty(x))_\varphi = P_\infty x_\varphi$ for all $x \in \mathfrak{n}_\varphi$. Hence the conditions (i) and (ii) are also satisfied (see the proof of (iii) $\Rightarrow$ (ii) in Theorem 3).

(2) We can find a von Neumann algebra M with a cyclic and separating vector ξ and a decreasing net $\{N_\alpha\}$ of von Neumann subalgebras of M such that ξ is cyclic for each N_α and $N_\infty = \bigcap_\alpha N_\alpha = \mathbf{C}1$ (cf. [**3**, Example 1]). Let $\varphi(x) = \langle x\xi, \xi \rangle$ for $x \in M$, then $\mathcal{H}_\alpha = \mathcal{H}_\varphi$ for each α and $\mathcal{H}_\infty = \mathbf{C}\xi$. Thus the conditions (i)–(iii) in Theorem 4 are not satisfied for this case.

(3) Under the condition (i), it follows (see [**8**]) that $s\text{-}\lim_\alpha \sigma_t^\alpha(x) = \sigma_t^\infty(x)$ uniformly for t in any finite interval for every $x \in M_\infty$, and further $s\text{-}\lim_\alpha \sigma_t^\alpha(\varepsilon_\alpha(x)) = \sigma_t^\infty(\varepsilon_\infty(x))$ uniformly for t in any finite interval for every $x \in M$.

As an example we lastly consider nonabelian K-flows studied by Emch [**5, 6**]. A nonabelian K-flow is described by $(M, \varphi, \gamma_t, N_0)$ where M is a von Neumann algebra, φ is a faithful normal state on M, γ_t is a one-parameter automorphism group of M, and N_0 is a von Neumann subalgebra of M such that φ is γ_t-invariant and

$$N_0 \subset \gamma_t(N_0), \qquad t \geq 0,$$
$$\bigcap_{t \in \mathbf{R}} \overline{\gamma_t(N_0)\xi} = \mathbf{C}\xi, \qquad \bigvee_{t \in \mathbf{R}} \gamma_t(N_0) = M.$$

Here it is not necessarily assumed that $\sigma_t^\varphi(N_0) = N_0$ for all $t \in \mathbf{R}$. Let $N_t = \gamma_t(N_0)$ and ε_t: $M \to N_t$ be the generalized conditional expectation with respect to φ. Then $\{N_t: t \in \mathbf{R}\}$ is an increasing net of unital von Neumann subalgebras of M with $\bigvee_t N_t = M$ and $\bigcap_t N_t = \mathbf{C}1$, and the condition (i) in Theorem 3 (resp. Theorem 4) is satisfied for $t \to \infty$ (resp. $t \to -\infty$). Moreover, it follows from Lemma 2 that $\varepsilon_t = \gamma_t \circ \varepsilon_0 \circ \gamma_{-t}$ for all $t \in \mathbf{R}$. Therefore we conclude by Theorems 3 and 4 that for every $x \in M$ and $f \in M_*$,

$$s\text{-}\lim_{t \to \infty} \varepsilon_t(x) = x, \qquad s\text{-}\lim_{t \to -\infty} \varepsilon_t(x) = \varepsilon_{-\infty}(x) = \varphi(x)1,$$
$$\|f \circ \gamma_t \circ \varepsilon_0 - f \circ \gamma_t\| = \|f \circ \varepsilon_t - f\| \to 0 \quad (\text{as } t \to \infty),$$
$$\|f \circ \gamma_t \circ \varepsilon_0 - f(1)\varphi\| = \|f \circ \varepsilon_t - f \circ \varepsilon_{-\infty}\| \to 0 \quad (\text{as } t \to -\infty).$$

References

1. L. Accardi and C. Cecchini, *Conditional expectations in von Neumann algebras and a theorem of Takesaki*, J. Funct. Anal. **45** (1982), 245–273.

2. H. Araki, *On the equivalence of the KMS condition and the variational principle for quantum lattice systems*, Comm. Math. Phys. **38** (1974), 1–10.

3. H. Araki and A. Kishimoto, *On clustering property*, Rep. Math. Phys. **10** (1976), 275–281.

4. N. Dang-Ngoc, *Pointwise convergence of martingales in von Neumann algebras*, Israel J. Math. **34** (1979), 273–280.

5. G. G. Emch, *Nonabelian special K-flows*, J. Funct. Anal. **19** (1975), 1–12.

6. ______, *Generalized K-flows*, Comm. Math. Phys. **49** (1976), 191–215.

7. U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37** (1975), 271–283.

8. F. Hiai, *Martingale type convergence of modular automorphism groups on von Neumann algebras*, preprint 1982.

9. E. C. Lance, *Martingale convergence in von Neumann algebras*, Math. Proc. Cambridge Philos. Soc. **84** (1978), 47–56.

10. Ş. Strătilă, *Modular theory in operator algebras*, Editura Academiei & Abacus Press, Tunbridge Wells, 1981.

11. M. Takesaki, *Tomita's theory of modular hilbert algebras and its applications*, Lecture Notes in Math., vol. 128, Springer-Verlag, Berlin and New York, 1970.

12. ______, *Conditional expectations in von Neumann algebras*, J. Funct. Anal. **9** (1972), 306–321.

13. M. Tsukada, *Strong convergence of martingales in von Neumann algebras*, Proc. Amer. Math. Soc. **88** (1983), 537–540.

14. H. Umegaki, *Conditional expectation in an operator algebra*, Tôhoku Math. J. **6** (1954), 177–181.

15. ______, *Conditional expectation in an operator algebra*, II, Tôhoku Math. J. **8** (1956), 86–100.

16. A. Van Daele, *A Radon Nikodym theorem for weights on von Neumann algebras*, Pacific J. Math. **61** (1975), 527–542.

Department of Information Sciences, Science University of Tokyo, Noda City, Chiba 278, Japan