QUARTIC SURFACES OF ELLIPTIC RULED TYPE

BY

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ABSTRACT. Let X be a normal quartic surface whose resolutions are birationally equivalent to elliptic ruled surfaces. We classify the singularities on X and then describe the global structure of X.

0. Introduction. It is known [3] that the minimal resolution of a normal quartic surface in $\mathbf{P}^3$ over an algebraically closed field k of characteristic $\neq 2, 3$ is either (i) a $K3$ surface, (ii) a rational surface, (iii) birationally equivalent to an elliptic ruled surface, or (iv) a ruled surface of genus 3. We shall investigate the structure of normal quartic surfaces of type (iii). In the sequel we call such a surface simply a quartic surface of elliptic ruled type.

Let X be a quartic surface of elliptic ruled type and let $\pi: \tilde{X} \to X$ denote the minimal resolution of X. We shall study X via the structure of $\tilde{X}$ and a linear system on $\tilde{X}$ which defines the morphism π.

Since the dualizing sheaf ω_X of X is trivial, we can apply the results in [3]. Here we restate some of them (restricting to our present situation) which will play essential roles throughout this research. We use the terms and facts cited in §1 of [3] without notice.

LEMMA 1. *For any point P on X, the geometric genus $p_g(P)$ of P is not greater than* 2.

Since $\omega_X \cong O_X$, there exists a unique effective anticanonical divisor on $\tilde{X}$ whose connected components correspond by π to singular points with $p_g \geq 1$ on X. We denote this divisor by $\tilde{D}$.

LEMMA 2. *Let* $\tilde{X} = X_n \overset{\mu_n}{\to} X_{n-1} \overset{\mu_{n-1}}{\to} \cdots \overset{\mu_1}{\to} X_0 = \bar{X}$ *be a sequence of blow-downs obtaining a relatively minimal model $\bar{X}$ of $\tilde{X}$, and let $\tilde{H}$ be the pull-back of a general hyperplane section of X to $\tilde{X}$ such that $\tilde{H}$ is irreducible and nonsingular. Put*

$$H_n = \tilde{H}, \qquad H_i = \mu_{i+1} \circ \mu_{i+2} \circ \cdots \circ \mu_n(H_n) \quad (0 \leq i \leq n-1), \qquad \bar{H} = H_0,$$

$$D_n = \tilde{D}, \qquad D_i = \mu_{i+1} \circ \mu_{i+2} \circ \cdots \circ \mu_n(D_n) \quad (0 \leq i \leq n-1), \qquad \bar{D} = D_0.$$

Then we have:

(i) $D_i \in |-K_{X_i}| \, (0 \leq i \leq n)$;

(ii) μ_i is a blow-up with center at a point on $\operatorname{supp}(D_{i-1}) \cap \operatorname{supp}(H_{i-1})$;

(iii) $D_i = \mu_i^*(D_{i-1}) - E_i$ where E_i is the exceptional curve of the first kind for μ_i $(1 \le i \le n)$.

LEMMA 3. *On the elliptic ruled surface $\overline{X} \xrightarrow{\tilde{\omega}} C$, the effective anticanonical divisor $\overline{D}$ is one of the following types*:

(i) $\overline{D} = C_0 + C_1$, *where C_0 is a minimal section of $\tilde{\omega}$ and C_1 is another section disjoint from C_0.*

(ii) $\overline{D} = 2C_0 + \Sigma f_i$, *where C_0 is as above and the f_i's are fibres of $\tilde{\omega}$.*

First, in §1, we make a list of possible singularities on X with $p_g \ge 1$. After that, restricting ourselves to the case of characteristic 0, we study the structure of X in detail using the sequence of blow-ups as in Lemma 2.

We note that M. Kato and I. Naruki are studying the singularities on normal quartic surfaces to get similar results. However, their method is the analysis of homogeneous polynomials of degree 4, which is quite different from ours.

Recently I received a manuscript from D. Epema, in reply to my letter enclosing my manuscript of the present paper, in which he also studies singular surfaces of the same kind.

1. Types of singularities on X. In this section we assume that X is a normal quartic surface of elliptic ruled type, and we list the possibility of the types of singularities on X with positive geometric genus. It will be shown later that every member in our list really appears as a singularity on a normal quartic surface of elliptic ruled type.

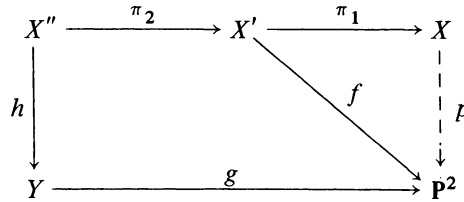LEMMA 4. *The multiplicity of each singular point on X is equal to two.*

PROOF. Let P be a singular point on X, and let $p: X \to \mathbf{P}^2$ be the projection with center at P. If $\operatorname{mult}_P X = 4$, then X is a cone over a plane quartic curve, and hence the minimal resolution of X is a ruled surface of genus 3. If $\operatorname{mult}_P X = 3$, then p turns out to be a birational map onto $\mathbf{P}^2$, so X is rational. Therefore we have $\operatorname{mult}_P X = 2$.   Q.E.D.

By the lemmas in §0, the singularities on X with $p_g \ge 1$ are either two simple elliptic singularities or a singularity with $p_g = 2$. In what follows we use the notation of §0.

PROPOSITION 1. *If X has two simple elliptic singularities, then they have the same type $\tilde{E}_7$ or $\tilde{E}_8$.*

PROOF. Let P and Q be the two simple elliptic singular points on X. We first show that the line l through P and Q in $\mathbf{P}^3$ does not lie on X. Indeed, if l is contained in X, let $\tilde{l}$ be the proper transform of l on $\tilde{X}$. Then $\tilde{l} \cap \tilde{C}_0 \ne \varnothing$, $\tilde{l} \cap \tilde{C}_1 \ne \varnothing$ and $\tilde{l} \cdot \tilde{H} = 1$, where $\sim$ means the proper transform in $\tilde{X}$. Since $\tilde{l}$ is isomorphic to $\mathbf{P}^1$, $\mu(\tilde{l})$ must be a fibre of $\overline{X}$ on which there is no center of the blow-ups $\mu: \tilde{X} \to \overline{X}$. So $\overline{H} \cdot f = 1$ for any fibre f of $\overline{X}$ and we get a contradiction because $\tilde{H}$ is a curve of genus 3.

To prove the proposition, it is enough to show that if P is of type $\tilde{E}_7$, then so is Q, since a simple elliptic singularity of multiplicity 2 is either of type $\tilde{E}_7$ or $\tilde{E}_8$. Consider the following commutative diagram:

$$
\begin{array}{ccccc}
X'' & \xrightarrow{\;\pi_2\;} & X' & \xrightarrow{\;\pi_1\;} & X \\
\downarrow{\scriptstyle h} & & \searrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle p} \\
Y & \xrightarrow{\quad g \quad} & & & \mathbf{P}^2
\end{array}
$$

where p is the projection from X with center at P, $\pi_1\colon X' \to X$ is the blow-up with center at P, $\pi_2\colon X'' \to X'$ is the normalization of X' and $g\colon Y \to \mathbf{P}^2$ is the finite morphism of degree 2 obtained by the Stein factorization of $f \circ \pi_2$. The degree of the branch locus of f is equal to 6, and X' is not normal since P is of type $\tilde{E}_7$ (Laufer [2]). Therefore, the degree of the branch locus of g is less than or equal to 4. Since $l \subsetneq X$, there is a neighbourhood U of Q in X such that U is transformed by π_1^{-1}, π_2^{-1} and h isomorphically into Y. So Y has a simple elliptic singularity which is isomorphic to Q. Thus by the classification of plane curves of degree less than or equal to 4 (cf. Hidaka and Watanabe [1]), we prove that Q is of type $\tilde{E}_7$.   Q.E.D.

PROPOSITION 2. *Suppose X has a singular point P of geometric genus equal to two. Then the exceptional set for the minimal resolution of P is one of the following four types*:

(i) $E1 + A_1$ (ii) $E1 + A_1 + A_1$ (iii) $E1 + D(3)$ (iv) $E1 + D(4)$
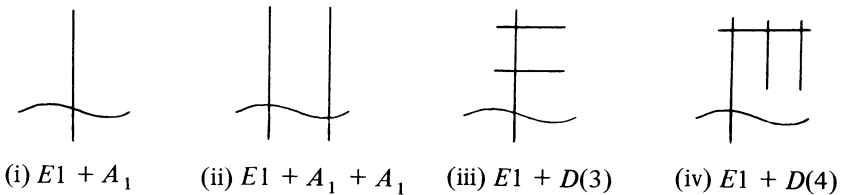
where each curved line stands for a nonsingular elliptic curve whose self-intersection number is equal to -1 in (i), -2 in (ii)–(iv), and any straight line denotes a nonsingular rational curve with self-intersection number -2.

(The symbols of these four types of singularities are due to S. S.-T Yau.)

PROOF. Since $p_g(P) = 2$, and P is a Gorenstein singularity, we have $p_a(P) = 1$ (Hidaka and Watanabe [1]). So we can define the elliptic sequence $Z_{B_0}, \ldots, Z_{B_{l+1}}$ according to Yau [4]. By definition, every Z_{B_i} is an effective divisor supported on the exceptional set $\mathrm{supp}(\tilde{D})$ on $\tilde{X}$. By Lemmas 2 and 3, $\mathrm{supp}(\tilde{D})$ is with simple normal crossings and has nonsingular elliptic curve $\tilde{C}_0$ as a component. Therefore, by the property of the elliptic sequence (Theorem 3.7 of [4], Proposition 2.1 of [5] and

Corollary 2.3 of [**6**]), we have:

(1) $Z_{B_{l+1}} = \tilde{C}_0$;

(2) $Z^2_{B_0} \leqslant \cdots \leqslant Z^2_{B_{l+1}} < 0$;

(3) $\Sigma^{l+1}_{i=0} Z^2_{B_i} = K^2_{\tilde{X}} = -n$ (n being the number of blow-ups in $\mu\colon \tilde{X} \to \overline{X}$);

(4) $2 \geqslant -\Sigma^l_{i=0} Z^2_{B_i}$.

(Although Yau's proof is for the case when $k = \mathbf{C}$, it is easy to check that (1)–(4) remain true in our situation for any algebraically closed field k.) From these we see that all possibilities are as follow:

	l	$Z^2_{B_0}$	$Z^2_{B_1}$	$\tilde{C}^2_0$	n
(a)	1	-1	-1	-1	3
(b)	0	-2	/	-1	3
(c)	0	-2	/	-2	4
(d)	0	-1	/	-1	2

Now we can choose a relatively minimal model $\overline{X}$ of $\tilde{X}$ so that $C^2_0 = \tilde{C}^2_0$. Indeed, if $C^2_0 > \tilde{C}^2_0$ for an $\overline{X}$, there exists a point P_0 on C_0 such that $\mu\colon \tilde{X} \to \overline{X}$ factors through the blow-up $\mu'_1\colon X'_1 \to \overline{X}$ with center at P_0. Hence, there is a morphism $\mu'\colon \tilde{X} \to \overline{X}'$, where $\overline{X}'$ is the image of the elementary transformation of $\overline{X}$ with center at P_0, so $\overline{X}'$ is another relatively minimal model of $\tilde{X}$ whose minimal section C'_0 satisfies $C'^2_0 = C^2_0 - 1$:
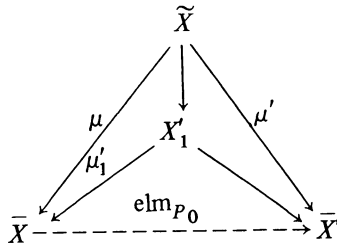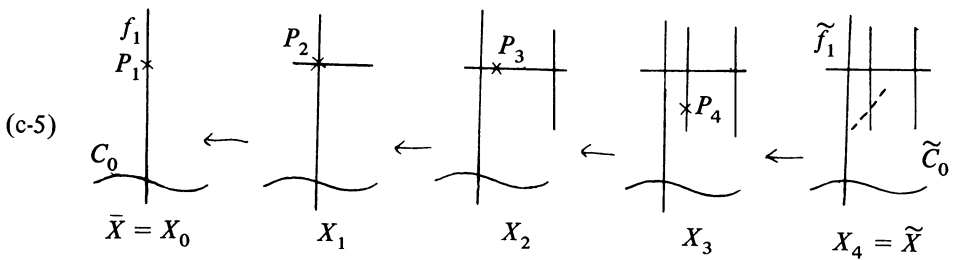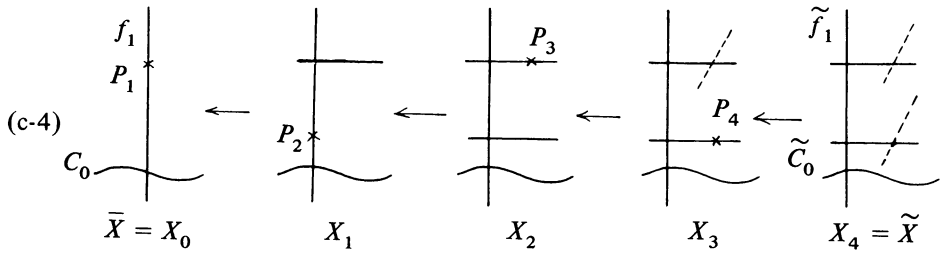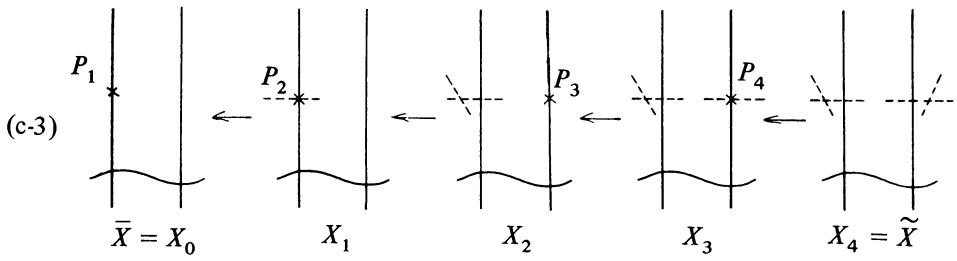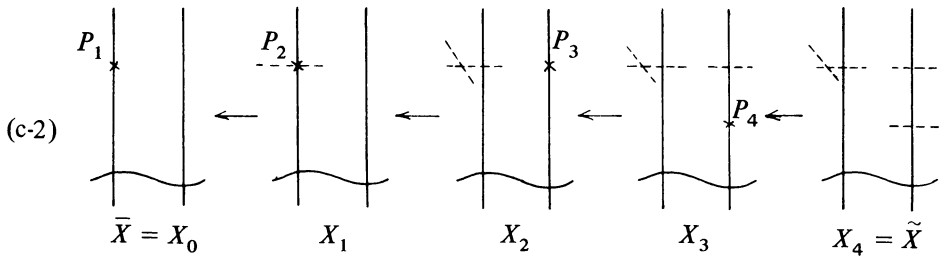

With the assumption of $C^2_0 = \tilde{C}^2_0$, and using Lemma 2, we can list the possibilities of $\overline{D}$, μ and $\tilde{D}$ corresponding to each case in the table above:

(a) and (b). $\overline{D} = 2C_0 + f_0$, f_0 is a fibre of $\tilde{\omega}\colon \overline{X} \to C$, and the centers of the blow-ups μ lie on $\mathrm{supp}(f_0)\backslash\mathrm{supp}(C_0)$. Hence $\tilde{D} = 2\tilde{C}_0 + \tilde{f}_0$. ($\tilde{\ }$ always denotes the proper transform on $\tilde{X}$.)

(c-1)–(c-3). $\overline{D} = 2C_0 + f_1 + f_2$, f_1 and f_2 are disjoint fibres of $\tilde{\omega}$.

(c-4) and (c-5). $\overline{D} = 2C_0 + 2f_1$, f_1 is a fibre of $\tilde{\omega}$.

In (c-1)–(c-5), μ and $\tilde{D}$ are described in the following figures:

(c-1)

f_1 f_2

P_1

C_0

$\overleftarrow{\mu_1}$ P_2

$\overleftarrow{\mu_2}$

P_3

$\overleftarrow{\mu_3}$

P_4 $\overleftarrow{\mu_4}$

$\widetilde{f_1}$ $\widetilde{f_2}$

$\widetilde{C_0}$

$\bar{X} = X_0$ X_1 X_2 X_3 $X_4 = \widetilde{X}$

(c-2)

P_1 P_2 P_3

P_4

$\bar{X} = X_0$ X_1 X_2 X_3 $X_4 = \widetilde{X}$

(c-3)

P_1 P_2 P_3 P_4

$\bar{X} = X_0$ X_1 X_2 X_3 $X_4 = \widetilde{X}$

(c-4)

f_1

P_1

C_0

P_2

P_3

P_4

$\widetilde{f_1}$

$\widetilde{C_0}$

$\bar{X} = X_0$ X_1 X_2 X_3 $X_4 = \widetilde{X}$

(c-5)

f_1

P_1

C_0

P_2

P_3

P_4

$\widetilde{f_1}$

$\widetilde{C_0}$

$\bar{X} = X_0$ X_1 X_2 X_3 $X_4 = \widetilde{X}$
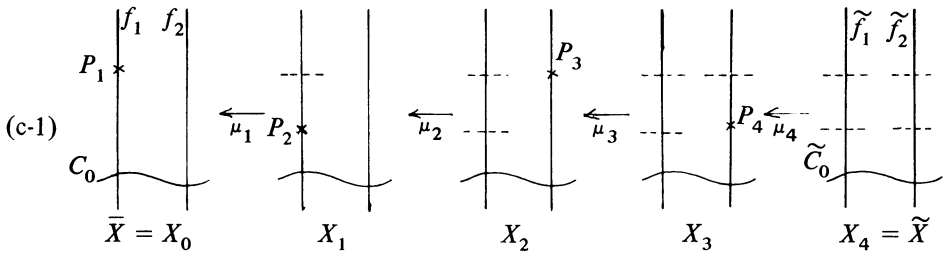
where P_i is the center of μ_i (the order of the blow-ups may be changed), the bold lines denote the support of the effective anticanonical divisor D_i on each step, while the dotted lines are the exceptional curves of the blow-ups which do not appear in D_i.

(d). $\bar{D} = 2C_0 + f_0$, f_0 is a fibre of $\tilde{\omega}$, the centers of μ lie on $\mathrm{supp}(f_0)\backslash\mathrm{supp}(C_0)$, and hence $\tilde{D} = 2\tilde{C}_0 + \tilde{f}_0$.

It remains to show that cases (a) and (b) do not occur. For (a), since $l = 1$, $\bar{D}$ must have at least three irreducible components by a property of the elliptic sequence: $\mathrm{supp}(\tilde{C}_0) = \mathrm{supp}(Z_{B_2}) \underset{\neq}{\subset} \mathrm{supp}(Z_{B_1}) \underset{\neq}{\subset} \mathrm{supp}(Z_{B_0}) = \mathrm{supp}(\tilde{D})$, a contradiction. For (b), let r_i be the multiplicity of H_i at P_i ($1 \leqslant i \leqslant 3$) and $\bar{H} = mC_0 + mf$ (f denotes a fibre and $m \in \mathbf{Z}$). Then we have

$$4 = \tilde{H}^2 = \bar{H}^2 - \sum r_i^2 = m^2 - \sum r_i^2$$

and

$$0 = \tilde{H} \cdot \tilde{f}_0 = \bar{H} \cdot f_0 - \sum r_i = m - \sum r_i.$$

Hence

$$4 = \left(\sum r_i\right)^2 - \sum r_i^2 = 2(r_1 r_2 + r_2 r_3 + r_3 r_1).$$

Since $r_i \geqslant 1$ ($1 \leqslant i \leqslant 3$), we get a contradiction.   Q.E.D.

2. Main results. In this section, we assume the ground field k is algebraically closed and has characteristic 0.

Let $Y_n \overset{\mu_n}{\to} Y_{n-1} \overset{\mu_{n-1}}{\to} \cdots \overset{\mu_1}{\to} Y_0 = Y$ be a sequence of blow-ups where Y is a nonsingular surface. Let $P_i \in Y_{i-1}$ denote the center of μ_i, and $E_i \subset Y_i$ the exceptional curve for μ_i ($1 \leqslant i \leqslant n$). Then we call $(P_1, \ldots, P_n)$ a sequence of points on Y admitting infinitely near points, and the above sequence of blow-ups is called the blow-up of $(P_1, \ldots, P_n)$. For a divisor D on Y and nonnegative integers m_i ($1 \leqslant i \leqslant n$), we denote by $|D - m_1 P_1 - \cdots - m_n P_n|$ the linear subsystem of $|D|$ consisting of elements $D' \in |D|$ such that

$$\mu_n^*\left(\mu_{n-1}^*\left(\cdots \left(\mu_1^*(D') - m_1 E_1\right)\cdots\right) - m_{n-1}E_{n-1}\right) - m_n E_n$$

remain effective on Y_n. When there is no danger of confusion, we also denote by P_i the image on Y of the point $P_i \in Y_{i-1}$.

Let C be a nonsingular curve and E a vector bundle of rank 2 on C. Assume E is the direct sum of two line bundles. Then the ruled surface $\mathbf{P}(E) \to C$ has a section which is disjoint from a minimal section. We denote such a section by C_1; we denote a minimal section by C_0, as before. In general these sections are not uniquely determined, but we fix a pair (C_0, C_1) once and for all on $\mathbf{P}(E)$ unless otherwise mentioned.

THEOREM 1. *Let C be a nonsingular elliptic curve and let L be an invertible sheaf of degree 2 on C. Let $\bar{X} = \mathbf{P}(O_C \oplus L) \overset{\bar{\omega}}{\to} C$ be the induced ruled surface. Fix an effective divisor $\bar{D} \in |-K_{\bar{X}}|$ and take a sequence of points (P_1, P_2, P_3) on $\bar{X}$ admitting infinitely*

near points. Assume the following property for (P_1, P_2, P_3):

> *If* $\overline{D} = C_0 + C_1'$, *where* C_1' *is a section of* $\tilde{\omega}$, *then each* P_i *lies on* C_1';
> *and if* $\overline{D} = 2C_0 + f_1 + f_2$, *where the* f_i's *are fibres, then the position
> of* P_1, P_2, P_3 *is one of the types* (c-1)–(c-5). *In each case,*
>   (∗) *there is no section* C' *of* $\tilde{\omega}$ *such that* $C' \sim C_1$, $C' \nleq \overline{D}$ *and* P_i,
> $P_j \in C'$ *for some* i, j $(1 \le i < j \le 3)$.

Then

(1) *there exists a unique point* P_4 *on* $\overline{X}$ (*possibly an infinitely near point*) *such that*
$|2C_1 - P_1 - P_2 - P_3| = |2C_1 - P_1 - P_2 - P_3 - P_4|$.

   Let $\mu: \tilde{X} \to \overline{X}$ *be the blow-up of* (P_1, P_2, P_3, P_4) *and* $\tilde{H}$ *the proper transform by* μ *of
a general member in* $|2C_1 - P_1 - P_2 - P_3 - P_4|$. *Then we have*

(2) $\mathrm{Bs}|\tilde{H}| = \varnothing$;

(3) $\dim|\tilde{H}| = 3$, $\tilde{H}^2 = 4$;

(4) $\tilde{H}$ *is a nonhyperelliptic curve.*

Therefore, $\tilde{H}$ *defines a birational morphism from* $\tilde{X}$ *to a normal quartic surface* $X \subset \mathbf{P}^3$
with singularities of type $2\tilde{E}_7$, $E1 + A_1 + A_1$, $E1 + D(3)$ *or* $E1 + D(4)$.

   *Conversely, any quartic surface of elliptic ruled type with at least one of these four
types of singularities is obtained by the construction above.*

REMARK. Without condition (∗), (1)–(3) of Theorem 1 hold. But in this case $\tilde{H}$
turns out to be a hyperelliptic curve so $\tilde{H}$ defines a morphism of degree 2 onto a
normal quadric surface in $\mathbf{P}^3$ (cf. Lemmas 6 and 7 in §3). Furthermore, it will be
shown in §5 that the case (c-3) occurs only if we remove condition (∗).

THEOREM 2. *Let* C *be a nonsingular elliptic curve and* L *an invertible sheaf of degree*
1 *on* C. *Let* $\overline{X} = \mathbf{P}(O_C \oplus L) \overset{\tilde{\omega}}{\to} C$ *be the induced ruled surface. Fix an effective divisor*
$\overline{D} \in |-K_{\overline{X}}|$ *and take a point* P_1 *on* $\overline{X}$ *such that*:

> *If* $\overline{D} = C_0 + C_1'$, *where* C_1' *is a section of* $\tilde{\omega}$, *then* $P_1 \in \mathrm{supp}(C_1')$,
> $O_{C_1'}(P_1) \ncong L$, *and*
>
> (∗) $\qquad\qquad\qquad\qquad O_{C_1'}(2P_1) \ncong L^2$
>
> (*identifying* C_1' *with* C *via* $\tilde{\omega}$); *and if* $\overline{D} = 2C_0 + f_0$, *where* f_0 *is a fibre
> of* $\tilde{\omega}$, *then* $P_1 \in \mathrm{supp}(f_0)\backslash\mathrm{supp}(C_0 + C_1)$.

Then

(1) *there exists a unique point* P_2 *on* $\overline{X}$ (*possibly an infinitely near point*) *such that*
$|3C_1 - 2P_1| = |3C_1 - 2P_1 - P_2|$.

   Let $\mu: \tilde{X} \to \overline{X}$ *be the blow-up of* (P_1, P_2) *and* $\tilde{H}$ *the proper transform by* μ *of a
general member in* $|3C_1 - 2P_1 - P_2|$. *Then we have*:

(2) $\mathrm{Bs}|\tilde{H}| = \varnothing$;

(3) $\dim|\tilde{H}| = 3$, $\tilde{H}^2 = 4$;

(4) $\tilde{H}$ *is a nonhyperelliptic curve.*

Therefore, $\tilde{H}$ *defines a birational morphism from* $\tilde{X}$ *to a normal quartic surface* $X \subset \mathbf{P}^3$
with singularities of type $2\tilde{E}_8$ *or* $E1 + A_1$.

Conversely, any quartic surface of elliptic ruled type with at least one of these two types of singularities is obtained by the construction above.

REMARK. Even if we take $\overline{D} = C_0 + C_1'$ and $P_1 \in \text{supp}(C_1')$ such that $O_{C_1'}(P_1) \not\cong L$ and $O_{C_1'}(2P_1) \cong L^2$, (1)–(3) of Theorem 2 hold. But then $\tilde{H}$ is hyperelliptic and the image of the morphism defined by $\tilde{H}$ is a nonsingular quadric surface in $\mathbf{P}^3$ (cf. Lemma 9 and the end of §4).

COROLLARY. *Let X be a quartic surface of elliptic ruled type. Then the singular set of X is one of the following types, and each of them really occurs:*

$$\{2\tilde{E}_7, \text{ a subgraph of } A_3\}, \qquad \{E1 + A_1 + A_1, \text{ a subgraph of } A_1\},$$

$$\{E1 + D(3)\}, \quad \{E1 + D(4)\}, \quad \{2\tilde{E}_8\}, \quad \{E1 + A_1\}.$$

3. Proof of Theorem 1. $\overline{X}$, $\overline{D}$, P_1, P_2 and P_3 are as in the theorem, but at first we do not assume condition $(*)$.

PROOFS OF (1)–(3). Set $\Lambda = |2C_1 - P_1 - P_2 - P_3|$. Since $\Lambda \subsetneq |2C_1 - P_1 - P_2| \subsetneq |2C_1 - P_1| \subsetneq |2C_1|$, as is easily shown in each case, and since $\dim |2C_1| = \dim H^0(C, O_C \oplus L \oplus L^2) - 1 = 6$, we have $\dim \Lambda = 3$. First we take $\overline{D} = C_0 + C_1'$, where C_1' is a section. By $C_1' \sim C_1$, we may assume $\overline{D} = C_0 + C_1$. Since $2C_1 \cdot C_1 = 4$ and C_1 is an elliptic curve, we can define a unique point P_4 lying on the proper transform of C_1 on the blow-up of (P_1, P_2, P_3) of $\overline{X}$ such that

$$\Lambda = |2C_1 - P_1 - P_2 - P_3 - P_4|.$$

Let f_i denote the fibre on $\overline{X}$ through P_i $(1 \leq i \leq 4)$. Since

$$O_{C_1}(P_1 + P_2 + P_3 + P_4) \cong O_{C_1}(2C_1) \cong L^2,$$

identifying C_1 and C via $\tilde{\omega}$, we find the following four elements of Λ: $2C_0 + \Sigma_{i=1}^4 f_i$, $C_0 + f_5 + f_6 + C_1$, $C_0 + f_7 + f_8 + C_1$ and $2C_1$, where the f_j's $(5 \leq j \leq 8)$ are fibres disjoint from each other and from f_i $(1 \leq i \leq 4)$ such that $f_5 + f_6, f_7 + f_8 \in |\tilde{\omega}^*L|$. Let $s_0, \ldots, s_3 \in H^0(\overline{X}, O_{\overline{X}}(2C_1))$ denote the defining equations of the above four divisors. Obviously the s_i's are linearly independent, and hence form a basis of Λ. Therefore, letting $\overline{H}$ be a general member of Λ we define $\mu: \tilde{X} \to \overline{X}$ and $\tilde{H}$ as in the theorem and deduce that:

the base points of Λ are exactly $P_1, \ldots, P_4$;

$\text{Bs}|\tilde{H}| = \varnothing$;

$\dim |\tilde{H}| = \dim \Lambda = 3$ and $\overline{H}$ is nonsingular at $P_1, \ldots, P_4$.

Thus (1)–(3) follow. Next suppose $\overline{D} = 2C_0 + f_1 + f_2$, where f_1 and f_2 are fibres. (f_2 may coincide with f_1.) Let $\overline{H}$ be a general member of Λ so $2C_0 + 2f_1 + 2f_2$, $2C_0 + f_1 + f_2 + f_3 + f_4$, $C_0 + f_1 + f_2 + C_1'$ and $\overline{H}$ correspond to a basis of Λ, where f_3 and f_4 are fibres disjoint from f_1 and f_2, with $f_3 + f_4 \in |\tilde{\omega}^*L|$, and C_1' is a section disjoint from P_i $(1 \leq i \leq 4)$ and C_0. In particular, $\overline{H}$ has no component in common with $\overline{D}$. Therefore we can define P_4 uniquely as the intersection of the proper transform of $\overline{H}$ in X_3 with D_3 in the notation of the figures (c-1)–(c-5) in §1. Let $\mu: \tilde{X} \to \overline{X}$ and $\tilde{H}$ be as in the theorem. Then we can show that $|\tilde{H}|$ has no basepoint. For example, in the case of (c-1), $\overline{H}$ and $C_0 + f_1 + f_2 + f_3 + f_4$ must have distinct

tangents at P_i so $|\tilde{H}|$ has no basepoint over P_i. After this, we trace the proof of the above case and obtain (1)–(3).

Let X denote the image of $\tilde{X}$ under the morphism $\Phi_{|\tilde{H}|}$ associated with $\tilde{H}$. Then X is a hypersurface in $\mathbf{P}^3$ of degree 4 or 2 by (3), and X is normal because the pull-back of a general hyperplane section of X is a nonsingular curve $\tilde{H}$. It is clear by our construction that there is a unique effective anticanonical divisor $\tilde{D}$ on $\tilde{X}$, and the configuration of $\tilde{D}$ is $2\tilde{E}_7$, $E1 + A_1 + A_1$, $E1 + D(3)$ or $E1 + D(4)$ according to our choice of $\bar{D}$, P_1, P_2 and P_3. Moreover, by a direct computation, we see that supp($\tilde{D}$) consists of one or two of the connected components of the exceptional set for $\Phi_{|\tilde{H}|}$. By $\tilde{H} \cdot K_{\tilde{X}} = 0$, the restriction of $\Phi_{|\tilde{H}|}$ to $\tilde{H}$ is the canonical map of $\tilde{H}$; therefore, X is a quartic surface if and only if $\tilde{H}$ is nonhyperelliptic. We will show a criterion for this property in a generalized situation.

Let Σ_0 and Σ_2^0 denote a nonsingular quadric surface in $\mathbf{P}^3$ and a cone over a nonsingular conic in $\mathbf{P}^3$, respectively, and let v denote the vertex of Σ_2^0. Let Y be a nonsingular surface birationally equivalent to an elliptic ruled surface. Suppose there is an effective anticanonical divisor D of Y such that every irreducible component of D has self-intersection number greater than -4. Moreover, suppose there exists on Y a nonsingular curve H of genus 3 such that $\mathrm{Bs}|H| = \varnothing$, $\dim|H| = 3$ and $H \cdot D = 0$, so we have a morphism $\Phi = \Phi_{|H|}$ from Y onto a normal quartic surface in $\mathbf{P}^3$ or a Σ_0 or a Σ_2^0. Since $H \cdot D = 0$, $\Phi(D)$ consists of finite points.

LEMMA 5. *Assume* $\Phi(Y) = \Sigma_2^0$. *Then* $v \notin \Phi(D)$.

PROOF. Let $\pi_1: \Sigma_2 = \mathbf{P}(O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(-2)) \to \Sigma_2^0$ be the blow-up with center at v, and let Γ_0 be the exceptional rational curve for π_1. Let Γ be a general hyperplane section of Σ_2^0 and let Γ_1 denote the proper transform of Γ on Σ_2. We may assume that $\Phi(H) = \Gamma$. Then we get the following commutative diagram:

where $\pi_2: \tilde{Y} \to Y$ is a minimal sequence of blow-ups such that the induced map $\tilde{Y} \to \Sigma_2$ becomes a morphism, Z is the Stein factorization of $\tilde{Y} \to \Sigma_2$, and $\tilde{H}$ is the proper transform of H. Let B denote the branch locus of g. Then $B \sim 2(n\Gamma_0 + mf)$ for some $n, m \in \mathbf{Z}$ where f denotes a fibre of the ruling on Σ_2. Since $\tilde{H} \to \Gamma_1$ is a morphism of degree 2 between nonsingular curves of genus 3 and 0, respectively, we have $B \cdot \Gamma_1 = 8$ and hence $B \sim 2n\Gamma_0 + 8f$. Furthermore, since B is reduced,

$$-2 = \Gamma_0^2 \leqslant \Gamma_0 \cdot B = -4n + 8.$$

If $n = 0$, then Z is a ruled surface of genus 3, and hence a contradiction. Therefore we have $n = 1$ or 2.

Assume $\Phi(D) \ni v$. Then there exists a nonsingular elliptic curve $C \leqslant D$ such that $\Phi(C) = v$ by Lemmas 2 and 3. Let $\tilde{C}$ denote the proper transform of C on $\tilde{Y}$. If $f(\tilde{C})$ is a curve, then clearly $\tilde{Y} = Y$, and $g^{-1}(\Gamma_0) = f(\tilde{C}) \cong \tilde{C} = C$ (consider the configuration of D). Hence, we get a contradiction because $C^2 = 2 \cdot \Gamma_0^2 = -4$. Thus $f(\tilde{C})$ is a point, and so Z has a singular point on $g^{-1}(\Gamma_0)$ which is not a rational double point.

Suppose $n = 1$. Then $B \cdot f = 2$. From this we easily see that Z has at worst rational double points. Suppose $n = 2$; then $B \sim 4\Gamma_0 + 8f$. If $\Gamma_0 \nleqslant B$, then Z is smooth over Γ_0 since $\Gamma_0 \cdot B = 0$. Hence $B = \Gamma_0 + B_0$ for some reduced effective divisor B_0 not containing Γ_0. But since $B_0 \cdot \Gamma_0 = 2$, in this case also Z has only rational double points over Γ_0, a contradiction.   Q.E.D.

LEMMA 6. $\Phi(Y)$ is a normal quartic surface if and only if there is no curve C on Y satisfying

$(**)$ C is nonsingular, elliptic, $C \cdot H = 2$ and $\mathrm{supp}(C) \cap \mathrm{supp}(D) = \varnothing$.

PROOF. Suppose $\Phi(Y)$ is a quartic surface and there is a curve C on Y satisfying $(**)$. Then $\Phi(C)$ is a curve and is birationally equivalent to C, because $C \cdot H > 0$ and Φ is a birational morphism. But by $\Phi(C) \cdot h = C \cdot H = 2$ for a hyperplane section h of $\Phi(Y)$, $\Phi(C)$ must be a rational curve, and hence a contradiction. Next suppose $\Phi(Y) = \Sigma_0$. Let us take a general fibre l of one of the two rulings on Σ_0. Then it is easy to verify that the proper transform of l on Y satisfies $(**)$. Finally suppose $\Phi(Y) = \Sigma_2^0$ and let l be a general generating line of Σ_2^0. Then, by Lemma 5, l and $\Phi(D)$ are disjoint. It follows that the proper transform of l on Y satisfies $(**)$. Q.E.D.

Now we return to the proof of Theorem 1. To prove that X is really a quartic surface under the assumption of $(*)$, it suffices to apply Lemma 6 to the case of $Y = \tilde{X}$ (and so $H = \tilde{H}$, $D = \tilde{D}$) and combine it with the following

LEMMA 7. The nonexistence of curves on $\tilde{X}$ satisfying $(**)$ is equivalent to condition $(*)$.

PROOF. If there is a section C' on $\overline{X}$ such that $C' \sim C_1$, $C' \nleqslant \overline{D}$ and $C' \ni P_i$, P_j for some i, j $(1 \leqslant i < j \leqslant 3)$, then clearly the proper transform of C' on $\tilde{X}$ satisfies $(**)$.

Conversely, suppose there exists on $\tilde{X}$ a curve $\tilde{C}$ satisfying $(**)$. Put $C' = \mu(\tilde{C})$, and $r_i = \mathrm{mult}_{P_i} C'$ $(1 \leqslant i \leqslant 4)$. Since $\tilde{C} \cdot \tilde{C}_0 = 0$, we have $C' \cdot C_0 = 0$, so $C' = mC_1$ for some $m \in \mathbf{Z}$. Therefore we have

$$2 = \tilde{C} \cdot \tilde{H} = C' \cdot \overline{H} - \sum r_i = 4m - \sum r_i,$$

$$0 = \tilde{C} \cdot \tilde{D} = C' \cdot \overline{D} - \sum r_i = 2m - \sum r_i.$$

So we get $m = 1$ and $\sum r_i = 2$. We may assume $r_1 = 1$ and $r_2 = 0$. If $r_3 = 1$ and $r_4 = 0$, there is nothing to prove. Assume $r_3 = 0$ and $r_4 = 1$. This is the case of $\overline{D} = C_0 + C_1'$ or (c-1) or (c-2). Let C'' be a section which is disjoint from C_0 and passes through P_2, and let $\overline{D}|_{C''} = P_2 + P_3'$. Since $P_3 \neq P_4$ as points on $\overline{X}$, we can define $\Lambda' = |2C_1 - P_1 - P_2 - P_4|$ and we have $C' + C'' \in \Lambda'$. But, in fact, Λ'

coincides with $\Lambda = |2C_1 - P_1 - P_2 - P_3 - P_4|$. Therefore, $P_3' = P_3$ and hence C'' is a desired section.   Q.E.D.

Finally, we prove the second part of the theorem. Let X be a quartic surface of elliptic ruled type with at least $2\tilde{E}_7$, $E1 + A_1 + A_1$, $E1 + D(3)$ or $E1 + D(4)$ as singularities. Let $\pi: \tilde{X} \to X$ be the minimal resolution of X. Then from the proof of Proposition 2, we see there is a ruled surface $\overline{X} = \mathbf{P}(E) \overset{\tilde{\omega}}{\to} C$ over an elliptic curve C such that $C_0^2 = -2$, in particular, E splits (the notation is the same as before), and there exists a sequence of points (P_1, P_2, P_3, P_4) on $\overline{X}$ such that $\tilde{X}$ is the blowing-up of $\overline{X}$ at the P_i's. This blowing-up $\tilde{X} \overset{\mu}{\to} \overline{X}$ is the sequence in Lemma 2. We have already shown that (P_1, P_2, P_3) must satisfy the conditions described in the theorem, and hence, we only have to show the following

LEMMA 8. *Let $\overline{H} \subset \overline{X}$ be as in Lemma 2. Then $\overline{H} \in |2C_1|$.*

PROOF. In the notation of Lemma 2, $\overline{H} \equiv mC_1$ ($m \in \mathbf{Z}$). Indeed, since $\tilde{H} \cdot \tilde{C}_0 = 0$, $\overline{H}$ is numerically equivalent to some multiple mC_1. We show that $m = 2$.

Case 1. X has a singularity of $p_g = 2$. We may assume the order of P_1, P_2, P_3, P_4 is the same as one of the figures (c-1)–(c-5) in §1, and we use the notation there. Put $r_i = \text{mult}_{P_i} H_i$ for $1 \le i \le 4$. Then we have

$$4 = \tilde{H}^2 = \overline{H}^2 - \sum r_i^2 = 2m^2 - \sum r_i^2.$$

On the other hand, since the proper transform of $\overline{H}$ and f_1 must be disjoint after blowing-up (P_1, P_2), we get

$$r_1 + r_2 = \overline{H} \cdot f_1 = m.$$

Replacing $\{P_1, P_2, f_1\}$ with $\{P_3, P_4, f_2\}$, if necessary, in the case of (c-1)–(c-3), we can assume $r_1^2 + r_2^2 \ge r_3^2 + r_4^2$. Therefore, we have

$$4 = 2(r_1 + r_2)^2 - \sum r_i^2 = \left(r_1^2 + r_2^2 - r_3^2 - r_4^2\right) + 4r_1 r_2 \ge 4r_1 r_2.$$

It follows that $r_1 = r_2 = 1$, so $m = 2$.

Case 2. X has $2\tilde{E}_7$. Let $r_i = \text{mult}_{P_i} H_i$ ($1 \le i \le 4$). Then we have

$$4 = \tilde{H}^2 = \overline{H}^2 - \sum r_i^2 = 2m^2 - \sum r_i^2, \qquad \sum r_i = \overline{H} \cdot C_1 = 2m.$$

Therefore we get $m \ne 3$, because the above two equations have no positive integral solution (r_1, r_2, r_3, r_4) for $m = 3$. Next, since $P_1, \ldots, P_4$ lie on $\text{supp}(C_1)$, we can find another relatively minimal model $\tilde{X} \overset{\mu'}{\to} \overline{X}'$ such that $\overline{X}' = \mathbf{P}(O_C \oplus L')$, where L' is an invertible sheaf of degree 0 on C. Then there are two disjoint sections C_0', C_1' of $\tilde{\omega}'$: $\overline{X}' \to C$ and four points $P_1', P_2' \in C_0'$, $P_3', P_4' \in C_1'$ admitting infinitely near points such that μ' is the blow-up of (P_1', P_2', P_3', P_4') and $\mu'(\tilde{C}_0 + \tilde{C}_1) = C_0' + C_1'$. We may assume $O_{\overline{X}'}(C_1')|_{C_1'} \cong L'$ and $O_{\overline{X}'}(C_0')|_{C_0'} \cong L'^{-1}$. Set $\overline{H}' = \mu'(\tilde{H})$ and $r_i' = \text{mult}_{P_i'} \overline{H}'$. Then $\overline{H}' \equiv mC_0' + nf$ for some $n \in \mathbf{Z}$ where f is a fibre of $\tilde{\omega}'$. By $\tilde{H} \cdot \tilde{C}_0 = \tilde{H} \cdot \tilde{C}_1 = 0$, we get

$$r_1' + r_2' = \overline{H}' \cdot C_0' = n, \qquad r_3' + r_4' = \overline{H}' \cdot C_1' = n.$$

We may assume $r_1' \geqslant r_3' \geqslant r_4' \geqslant r_2'$ so $r_1'^2 + r_2'^2 \geqslant r_3'^2 + r_4'^2$, and equality holds if and only if $r_1' = r_3'$. Then we obtain

$$4 = \tilde{H}^2 = \overline{H}'^2 - \sum r_i'^2 \geqslant 2m(r_1' + r_2') - 2(r_1'^2 + r_2'^2),$$

i.e.

$$(***) \qquad\qquad 0 \geqslant r_1'(m - r_1') + r_2'(m - r_2') - 2,$$

and equality holds if and only if $r_1' = r_3'$. Since $m \geqslant \overline{H}' \cdot f \geqslant r_1' \geqslant r_2' \geqslant 1$, the right-hand side of $(***)$ is either (i) -2, (ii) -1 or (iii) 0. In the case of (i), we have $m - r_1' = m - r_2' = 0$ so $r_1' = r_3'$, which implies a contradiction. In case (ii), we get $m - r_1' = 0$, $r_2' = m - r_2' = 1$ and hence $m = 2$. (But it follows that $r_1' = r_3'$ because $r_1' = 2$ and $r_2' = 1$, hence this case is impossible.) In (iii), there are three possibilities: (a) $r_1' = m - r_1' = r_2' = m - r_2' = 1$; (b) $m - r_1' = 0$, $r_2' = 1$, $m - r_2' = 2$; and (c) $m - r_1' = 0$, $r_2' = 2$, $m - r_2' = 1$. In (a) we get $m = 2$, and so we are done. In (b) and (c), we get $m = 3$, but this value has already been excluded.

Therefore, we conclude that $\overline{H} \equiv 2C_1$, and hence, $\overline{H} \sim 2C_1 + \tilde{\omega}^*(D)$ for some divisor D on C of degree 0. Since $\overline{X} = \mathbf{P}(O_C \oplus L)$ for some invertible sheaf L of degree 2 on C, we have

$$\dim H^0(\overline{X}, O_{\overline{X}}(\overline{H})) = \dim H^0(C, (O_C \oplus L \oplus L^2) \otimes O_C(D))$$
$$= \begin{cases} 7 & \text{if } D \sim 0, \\ 6 & \text{if } D \not\sim 0, \end{cases}$$

and

$$\dim H^0(\overline{X}, O_{\overline{X}}(\overline{H} - C_0)) = \dim H^0(C, (L \oplus L^2) \otimes O_C(D)) = 6.$$

Since $\overline{H}$ is irreducible, $|\overline{H}|$ has no fixed component, hence we prove that $D \sim 0$, i.e. $\overline{H} \sim 2C_1$.   Q.E.D.

4. Proof of Theorem 2. We proceed in several steps which are almost parallel to the proof of Theorem 1.

First we prove (1)–(3) without assuming condition $(*)$. Let Λ denote the linear system $|3C_1 - 2P_1|$ on $\overline{X}$. Then we get $\dim \Lambda = 3$ because $\dim|3C_1| = 6$, $\dim|3C_1 - P_1| = 5$, and $|3C_1 - P_1|$ contains two elements whose tangents at P_1 are different from each other. If $\overline{D} = C_0 + C_1'$ where C_1' is a section of $\tilde{\omega}$, we may assume $C_1' = C_1$. Consider the following elements in Λ: $3C_1$, $2C_1 + C_0 + f_0$, $C_1 + 2C_0 + f_1 + f_1'$ and $3C_0 + 2f_1 + f_2$, where the f_i's and f_1' are fibres of $\tilde{\omega}$ such that $f_0 \in |\tilde{\omega}^*L|$, f_1 passes through P_1, $f_1 + f_1' \in |\tilde{\omega}^*L^2|$ and $2f_1 + f_2 \in |\tilde{\omega}^*L^3|$. Clearly the defining equations of these four divisors are linearly independent in $H^0(\overline{X}, O_{\overline{X}}(3C_1))$ and so form a basis of Λ. Let P_2 denote the intersection of f_2 with C_1, by the identification of P_2 with a point on the proper transform of C_1 in the blow-up of $\overline{X}$ with center at P_1 if $f_1 = f_2$. Then the basepoints of Λ are exactly $2P_1 + P_2$ in the infinitely near sense and we deduce (1)–(3) as in the proof of Theorem 1. If $\overline{D} = 2C_0 + f_0$, let $\overline{H}$ denote a general member of Λ such that $3C_0 + 3f_0$, $2C_0 + 2f_0 + C_1$, $C_0 + f_0 + C_2$ and $\overline{H}$ correspond to a basis of Λ, where

C_2 is an element of $|2C_1|$ passing through P_1 and not containing C_0. Let P_0 [resp. P_2, P_3, P_∞] denote the point on $\bar{X}$ such that the intersection of f_0 with C_0 [resp. $\bar{H}, C_2, C_1$] is P_0 [resp. $2P_1 + P_2$, $P_1 + P_3$, P_∞]. Then we will show that P_0, P_1, P_2, P_3 and P_∞ are different from each other. By our hypothesis there is an affine coordinate t on $f_0 \cong \mathbf{P}^1$ such that $t = 0, 1, \infty$ at P_0, P_1, P_∞, respectively. Let $t = \alpha$ at P_2 and $t = \beta$ at P_3 for some $\alpha, \beta \in k^{\sqcup}\{\infty\}$. Set $\lambda = |3C_1|_{|f_0}$. From

$$0 \to O_{\bar{X}}^-(3C_1 - f_0) \to O_{\bar{X}}^-(3C_1) \to O_{f_0}(3C_1) \to 0,$$

we have dim $\lambda = 2$. Let f_i $(1 \le i \le 5)$ be fibres disjoint from f_0 such that $f_1 + f_2 \in |\tilde{\omega}^*L^2|$ and $f_3 + f_4 + f_5 \in |\tilde{\omega}^*L^3|$. Then $3C_1$, $C_1 + 2C_0 + f_1 + f_2$ and $3C_0 + f_3 + f_4 + f_5$ cut on f_0 three elements in λ: $3P_\infty$, $P_\infty + 2P_0$ and $3P_0$. Therefore $1, t^2, t^3$ form a basis of λ. Since $2P_1 + P_2$, $P_\infty + P_1 + P_3 \in \lambda$, defined by $\bar{H}$, $C_1 + C_2 \in |3C_1|$, $(t-1)^2(t-\alpha)$ and $(t-1)(t-\beta)$ must be written as linear combinations of $1, t^2$ and t^3 with coefficients in k. Hence $\alpha = -1/2$, $\beta = -1$, so P_0, P_1, P_2, P_3 and P_∞ are different from each other. Then from the configuration of our basis of Λ, we obtain (1)–(3) as before.

Let $\mu\colon \tilde{X} \to \bar{X}$ and $\tilde{H}$ be as in the theorem and let $X = \Phi_{|\tilde{H}|}(\tilde{X})$. Then X is a normal hypersurface in $\mathbf{P}^3$ of degree 2 or 4 according as $\tilde{H}$ is hyperelliptic or not. It is easy to show that the irreducible curves on $\tilde{X}$ which are contracted by $\Phi_{|\tilde{H}|}$ are exactly the components of a unique effective anticanonical divisor $\tilde{D}$ of $\tilde{X}$. Moreover, the configuration of $\tilde{D}$ is $2\tilde{E}_8$ or $E1 + A_1$ according to our choice of $\bar{D}$. Therefore, for the first part of the theorem it is enough to show the following

LEMMA 9. *X is a normal quadric surface if and only if $\bar{D} = C_0 + C_1'$ and $O_{C_1'}(2P_1) \cong L^2$.*

PROOF. Assume X is a normal quadric surface. Then, by Lemma 6, there is a curve $\tilde{C}$ on $\tilde{X}$ such that

(∗∗) $\tilde{C}$ is nonsingular elliptic, $\tilde{C} \cdot \tilde{H} = 2$ and supp$(\tilde{C}) \cap$ supp$(\tilde{D}) = \varnothing$.

Put $C' = \mu(\tilde{C})$. Then we have $C' \sim mC_1$ for some $m \in \mathbf{Z}$ because C' is disjoint from C_0. Set $r_i = \text{mult}_{P_i} C'$ $(i = 1, 2)$. Then we have

$$2 = \tilde{C} \cdot \tilde{H} = C' \cdot \bar{H} - 2r_1 - r_2 = 3m - 2r_1 - r_2,$$
$$0 = \tilde{C} \cdot \tilde{D} = C' \cdot \bar{D} - r_1 - r_2 = m - r_1 - r_2.$$

Hence we get $r_1 = 2$, $r_2 = 0$, $m = 2$ or $r_1 = 0$, $r_2 = 1$, $m = 1$. As was shown in the proof of (1)–(3), if such a curve C' exists on $\bar{X}$, then $\bar{D} = C_0 + C_1'$ with C_1' a section. We have $C' \sim 2C_1'$, $C'|_{C_1'} = 2P_1$ in the former case, and $C' \sim C_1'$, $C'|_{C_1'} = P_2$ in the latter case. Anyhow we get $O_{C_1'}(2P_1) \cong L^2$ since $O_{C_1'}(2P_1 + P_2) \cong L^3$.

Conversely, let $\bar{D} = C_0 + C_1'$ and $O_{C_1'}(2P_1) \cong L^2$. Then there exists on $\bar{X}$ an irreducible curve C' with a double point P_1 such that $C' \sim 2C_1$. The proper transform of C' on $\tilde{X}$ satisfies (∗∗) in Lemma 6 and hence X is a normal quadric surface.   Q.E.D.

Suppose X is a quartic surface of elliptic ruled type with at least $2\tilde{E}_8$ or $E1 + A_1$ as singularities. Let $\pi\colon \tilde{X} \to X$ be the minimal resolution of X, and let $\tilde{H}$ be the proper transform on $\tilde{X}$ of a general hyperplane section of X. Then from Lemmas 2, 3

and the proof of Proposition 2, we have that there exist an elliptic ruled surface $\overline{X} = \mathbf{P}(E) \overset{\tilde{\omega}}{\to} C$ with $C_0^2 = -1$ so we may assume $E = O_C \oplus L$ with $\deg L = 1$, $\overline{D} \in |-K_{\overline{X}}|$ (hence $\overline{D} = C_0 + C_1'$ with C_1' a section or $\overline{D} = 2C_0 + f_0$ with f_0 a fibre such that $f_0|_{C_0} \cong L$) and a sequence of two points (P_1, P_2) on $\mathrm{supp}(\overline{D}) \backslash \mathrm{supp}(C_0)$ such that $\mu: \tilde{X} \to \overline{X}$ is the blowing-up at (P_1, P_2). Put $\overline{H} = \mu(\tilde{H})$. By Lemma 2, P_1 and P_2 must lie on $\overline{H}$.

LEMMA 10. $\overline{H} \in |3C_1 - 2P_1|$.

PROOF. Let $\overline{H} \sim mC_1$ and let $r_i = \mathrm{mult}_{P_i} \overline{H}$ $(i = 1, 2)$. Then we have

$$4 = \tilde{H}^2 = \overline{H}^2 - r_1^2 - r_2^2 = m^2 - r_1^2 - r_2^2,$$

$$0 = \overline{H} \cdot \overline{D} - r_1 - r_2 = m - r_1 - r_2.$$

Hence $m = r_1 + r_2 = 3$. Since we may assume $r_1 \geq r_2$, we get $r_1 = 2$.   Q.E.D.

Finally assume $\overline{D} = C_0 + C_1'$ and $O_{C_1'}(P_1) \cong L$, or $\overline{D} = 2C_0 + f_0$ and $P_1 \in \mathrm{supp}(C_1)$, i.e. P_1 is the intersection of f_0 with C_1. Then, by the basis of $\Lambda = |3C_1 - 2P_1|$ taken in the proof of (1)–(3), we see that the proper transform of a general member of Λ on $\tilde{X}$ must have a fixed component, a contradiction. The necessity of condition (∗) follows from Lemma 9.

Therefore we complete the proof of Theorem 2.

Suppose $\overline{D} = C_0 + C_1'$, $O_{C_1'}(P_1) \ncong L$ but $O_{C_1'}(2P_1) \cong L^2$. Let $s_0, \ldots, s_3$ denote the basis of Λ in the first part of this section. Since $f_1 = f_1'$ in this case, we get $s_0 s_3 = s_1 s_2$. Therefore the induced hypersurface X in $\mathbf{P}^3$ is defined by the equation $X_0 X_3 = X_1 X_2$ for some homogeneous coordinates (X_0, X_1, X_2, X_3) of $\mathbf{P}^3$. Hence, X is a nonsingular quadric surface.

5. Proof of the Corollary. Let $\pi: \tilde{X} \to X$ denote the minimal resolution of X. Looking at the construction of $\tilde{X}$ described in Theorems 1 and 2, but at first without minding condition (∗), we can list the possible configuration of curves on $\tilde{X}$ which are disjoint from $\tilde{H}$ (notation is the same as in Theorems 1 and 2), i.e. the possible configuration of the exceptional curves for π, as fillows:

$$\{2\tilde{E}_7, \text{ a subgraph of } A_3\}, \qquad \{E1 + A_1 + A_1, \text{ a subgraph of } A_1 + A_1\},$$

$$\{E1 + D(3)\}, \quad \{E1 + D(4)\}, \quad \{2\tilde{E}_8\}, \quad \{E1 + A_1\}.$$

The possibility of $\{E1 + A_1 + A_1, A_1, A_1\}$ arises when we take (P_1, P_2, P_3) in Theorem 1 so that (P_1, P_2, P_3, P_4) are in the position of (c-3). But in this case the sections of $\tilde{\omega}$ through P_1 must pass through P_3. Indeed, suppose $\overline{H} \in |2C_1 - P_1 - P_2 - P_3|$ cuts $2P_1$ on f_1 and $2P_3$ on f_2. Let C' be a section through P_1 and disjoint from C_0, and let P_3' and P_0 denote the intersections of f_2 with C' and C_0, respectively. Set $\lambda = |2C_1 - P_1 - P_2|_{|f_2}$. By

$$0 \to O_{\overline{X}}(2C_1 - f_1 - f_2) \to O_{\overline{X}}(2C_1) \to O_{f_1 + f_2}(2C_1) \to 0,$$

we get $\dim |2C_1|_{|f_1 + f_2} = 3$, and hence

$$\dim \lambda \leq \dim |2C_1 - P_1 - P_2|_{|f_1 + f_2} = 1.$$

Therefore, from $2P_0$, $2P_3$, $2P_3' \in \lambda$ and $P_0 \neq P_3$, P_3', we deduce that $P_3 = P_3'$. So condition (∗) does not hold in this case.

For the other cases, we can choose (P_1, P_2, P_3) or P_1 in Theorems 1 or 2 corresponding to them so that condition (∗) is satisfied. This can be proved directly in each case by choosing P_i avoiding the finite points which do not satisfy the required conditions. Examples in §6 also show the existence of quartic surfaces with these types of singularities.

6. Defining equations. Let us find defining equations in suitable homogeneous coordinates of $\mathbf{P}^3$ of quartic surfaces X of elliptic ruled type with at least two simple elliptic singularities.

Let $\overline{X}$, L, P_i $(1 \leq i \leq 4)$ [resp. $i = 1, 2$], etc. be as in Theorem 1 [resp. Theorem 2], where we have seen the correspondence between them and X. Set $\Lambda = |2C_1 - P_1 - \cdots - P_4|$ [resp. $|3C_1 - 2P_1 - P_2|$]. To find the defining equation of X it is enough to find a relation of degree 4 of a basis of Λ.

Case 1. Suppose X has $2\tilde{E}_7$. As before, let f_i $(1 \leq i \leq 8)$ denote the fibre on $\overline{X}$ such that f_i passes through P_i $(1 \leq i \leq 4)$, f_j $(5 \leq i \leq 8)$ is disjoint from P_i's, and $f_5 + f_6$, $f_7 + f_8 \in |\tilde{\omega}^*L|$, $f_5 + f_6 \neq f_7 + f_8$. Then $A_0 = 2C_0 + \Sigma_{i=1}^4 f_i$, $A_1 = C_0 + f_5 + f_6 + C_1'$, $A_2 = C_0 + f_7 + f_8 + C_1'$ and $A_3 = 2C_1'$ correspond to a basis s_0, s_1, s_2, s_3 of Λ. Since $2A_0 + 2A_3$, $A_0 + 2A_1 + A_3$, $A_0 + A_1 + A_2 + A_3$, $A_0 + 2A_2 + A_3$, $4A_1$, $3A_1 + A_2$, $2A_1 + 2A_2$, $A_1 + 3A_2$ and $4A_2$ are greater than $4C_0 + 4C_1'$, and since

$$\dim H^0(\overline{X}, O_{\overline{X}}(4 \cdot 2C_1 - 4C_0 - 4C_1'))$$
$$= \dim H^0(\overline{X}, O_{\overline{X}}(2(f_1 + f_2 + f_3 + f_4))) = 8,$$

there exists a nontrivial relation

$$a_1 s_0^2 s_3^2 + s_0 s_3 \left(a_2 s_1^2 + a_3 s_1 s_2 + a_4 s_2^2\right)$$
$$+ a_5 s_1^4 + a_6 s_1^3 s_2 + a_7 s_1^2 s_2^2 + a_8 s_1 s_2^3 + a_9 s_2^4 = 0$$

for some $a_1, \ldots, a_9 \in k$. Hence, if (X_0, X_1, X_2, X_3) are the homogeneous coordinates of $\mathbf{P}^3$ corresponding to (s_0, s_1, s_2, s_3), then X is defined by

$$F(X_0, X_1, X_2, X_3) = a_1 X_0^2 X_3^2 + X_0 X_3 F_2(X_1, X_2) + F_4(X_1, X_2) = 0,$$

where F_i is a homogeneous polynomial in X_1, X_2 of degree i $(i = 2, 4)$. Since X has double points at $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$, we have $a_1 \neq 0$ so we may assume $a_1 = 1$. Moreover, the discriminant of F at $(1, 0, 0, 0)$,

$$F_2^2 - 4F_4 = 0,$$

is the sum of distinct four lines meeting at $(0, 0, 1)$ in $\mathbf{P}^2$ with homogeneous coordinates (X_1, X_2, X_3) (cf. the proof of Proposition 1). Therefore, after a suitable change of coordinates, we may assume

(1) $F = \left(X_0 X_3 + aX_1^2 + bX_1 X_2 + cX_2^2\right)^2 + X_1 X_2 (X_1 - X_2)(X_1 - dX_2) = 0$

for some $a, b, c, d \in k$, $d \neq 0, 1$.

Conversely, the quartic surface X defined by an equation of form (1) is normal for any $a, b, c, d \in k$ if $d \neq 0, 1$. Moreover, X has singularities of type $\tilde{E}_7$ at $(1, 0, 0, 0)$

and $(0, 0, 0, 1)$, for the dehomogenizations of (1) at these points are quasi-homogeneous polynomials of weights $(2, 4, 4)$. Their j-invariants are both equal to

$$2^8(d^2 - d + 1)^3/d^2(d - 1)^2.$$

If α is an $(l + 1)$ple root of

$$c^2 y^4 + (2bc + d)y^3 + (2ac + b^2 + d + 1)y^2 + (2ab + 1)y + a^2 = 0,$$

then X also has a rational double point of type A_l at $(0, 1, \alpha, 0)$. Except for these points X has no singular point.

Case 2. Suppose X has $2\tilde{E}_8$. Let f_i $(0 \leqslant i \leqslant 2)$ and f_1' denote the fibre on $\overline{X}$ such that f_i passes through P_i $(i = 1, 2)$, $f_0 \in |\tilde{\omega}^*L|$ and $f_1 + f_1' \in |\tilde{\omega}^*L^2|$. Let s_0, s_1, s_2, s_3 be the basis of Λ corresponding to $3C_0 + 2f_1 + f_2$, $2C_0 + f_1 + f_1' + C_1'$, $C_0 + f_0 + 2C_1'$ and $3C_1'$, respectively. Then, in a similar way as in the above case, we have a nontrivial relation

$$a_1 s_0^2 s_3^2 + a_2 s_0 s_1 s_2 s_3 + a_3 s_1^3 s_3 + a_4 s_0 s_2^3 + a_5 s_1^2 s_2^2 = 0$$

for some $a_1, \ldots, a_5 \in k$. So X is defined by

$$F = a_1 X_0^2 X_3^2 + a_2 X_0 X_1 X_2 X_3 + a_3 X_1^3 X_3 + a_4 X_0 X_2^3 + a_5 X_1^2 X_2^2 = 0.$$

Moreover, since X has double points at $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$, and since the discriminant of F at $(0, 0, 0, 1)$,

$$\left(a_2 X_1 X_2 + a_4 X_2^3\right)^2 - 4a_1\left(a_3 X_1^3 + a_5 X_1^2 X_2^2\right) = 0,$$

is reduced, we may assume $a_1 = a_3 = a_4 = 1$, so

$$(2) \qquad F = X_0^2 X_3^2 + X_1^3 X_3 + X_0 X_2^3 + aX_0 X_1 X_2 X_3 + bX_1^2 X_2^2 = 0$$

for some $a, b \in k$.

Conversely, any quartic surface defined by an equation of form (2) has, if it is normal, singularities of type $\tilde{E}_8$ at $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$. This surface is normal if and only if

$$\Delta = a^4 b - a^3 - 8a^2 b^2 + 36ab + 16b^3 - 27 \neq 0.$$

In this case, the j-invariants of the two singularities are both equal to

$$\left((a^2 - 4b)^2 + 24a\right)^3/\Delta.$$

For quartic surfaces of elliptic ruled type with a singularity of $p_g = 2$, we give only typical examples of their defining equations:

$$(3) \quad E1 + A_1 + A_1 : \left(X_0 X_1 + X_2^2\right)^2 + X_1^4 + X_3^4 = 0,$$

$$(4) \quad E1 + D(3) : \left(X_0 X_1 + X_2^2 + X_3^2\right)^2 + X_1^4 + X_1 X_3^3 = 0,$$

$$(5) \quad E1 + D(4) : \left(X_0 X_1 + X_2^2\right)^2 + X_1^4 + X_1 X_3^3 = 0,$$

$$(6) \quad E1 + A_1 : X_0^2 X_1^2 + X_0 X_2^3 - 3X_0 X_1 X_2 X_3 + 16X_1^4 + 2X_1 X_3^3 - \tfrac{3}{4}X_2^2 X_3^2 = 0.$$

All of these surfaces have unique singularity at $(1, 0, 0, 0)$. On the other hand,

$$(7) \qquad \left(X_0 X_1 + X_2^2 + \sqrt{-1}\, X_3^2 \right)^2 + X_1^4 + X_3^4 = 0$$

has a singularity of type $E1 + A_1 + A_1$ at $(1, 0, 0, 0)$ and of type A_1 at $(0, 0, 0, 1)$.

Equations (3), (4) and (6) were pointed out to the author by K. Watanabe, Y. Koyama and M. Kato, respectively.

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