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ON BORDISM GROUPS OF IMMERSIONS

BY

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ABSTRACT. The bordism group of immersions of oriented n-manifolds into $\mathbf{R}^{n+k}$ is identified with the stable homotopy group $\Pi_{n+k}^{s}(\mathrm{MSO}(k))$. We study these groups for $n-2 \leqslant k \leqslant n$, and discuss the behaviour of double points and their relation with the corresponding bordism groups of embeddings.

1. Introduction. Let $I\Omega_{n,k}$ denote the bordism group of immersions of oriented n-manifolds into $\mathbf{R}^{n+k}$. Here a bordism between two immersions $i_0: M_0 \looparrowright \mathbf{R}^{n+k}$ and $i_1: M_1 \looparrowright \mathbf{R}^{n+k}$ is an immersion of a compact oriented $(n+1)$-manifold $j: W \looparrowright \mathbf{R}^{n+k} \times I$ such that $\partial W = M_0 \cup -M_1$ and $j|_{M_0} = i_0 \times \{0\}$ and $j|_{M_1} = i_1 \times \{1\}$. In the usual manner bordism defines an equivalence relation and bordism classes form an abelian group (under disjoint union) which is identified with the stable homotopy group $\pi_{n+k}^{s}(\mathrm{MSO}(k))$, of the Thom space $\mathrm{MSO}(k)$ of the canonical oriented k-plane bundle over $\mathrm{BSO}(k)$.

The object of this paper is to study the groups $I\Omega_{n,k}$ for $n-2 \leqslant k \leqslant n$ and to discuss the behaviour of double points and the relation of these groups with the corresponding bordism groups of embeddings.

Bordism groups of immersions were studied first by Wells [**We**] who determined the unoriented groups $I\mathfrak{N}_{n,n}$ and $I\mathfrak{N}_{4n,4n-1}$. These results were extended by Koschorke and Olk who completed the computations of $I\mathfrak{N}_{n,k}$ for $n-2 \leqslant k \leqslant n$ (see [**K**, §10]). We shall make use of these computations..

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2. Some exact sequences involving bordism groups of immersions. We describe three exact sequences and compute some low-dimensional bordism groups appearing in them. The first sequence was obtained by Szücs [**Sz**] and Koschorke [**K**]. The other two sequences are due to Salomonsen [**Sa**]. We refer to these articles for a detailed description of the sequences.

Given a subgroup G of the orthogonal group $O(m)$ we will denote by Ω_j^G the bordism group of j-manifolds whose stable normal bundle admits a reduction to G. We will be mainly interested in the cases where G is

$$z(k) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} : A \in \mathrm{SO}(k) \right\},$$

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$$w(k) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : A, B \in \mathrm{SO}(k) \right\}$$

or

$$\Delta\mathrm{SO} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in \mathrm{SO}(l), l \gg j \right\}.$$

Let $f\colon I\Omega_{n,k} \to \Omega_n$ denote the forgetful homomorphism that retains the oriented bordism class of the domain of a class of immersions and let $E\Omega_{n,k}$ stand for the bordism group of classes of embeddings of oriented n-manifolds in $\mathbf{R}^{n+k}$. From now on we will assume $n < 2k - 1$. This is the metastable range and corresponds to the range in which only double points arise from self-transverse immersions.

2.1 PROPOSITION. *Let $n < 2k - 1$. Then the following sequences are exact*:

$$\cdots \to \Omega_{n-k}^{z(k)} \xrightarrow{\partial} I\Omega_{n,k} \xrightarrow{f} \Omega_n \to \Omega_{n-k-1}^{z(k)} \to \cdots, \tag{2.2}$$

$$\cdots \to \Omega_{n-k}^{\Delta\mathrm{SO}} \to I\Omega_{n,k} \xrightarrow{g} I\Omega_{n,k+1} \xrightarrow{e} \Omega_{n-k-1}^{\Delta\mathrm{SO}} \to \cdots, \tag{2.3}$$

$$\cdots \to \Omega_{n-k+1}^{w(k)} \to E\Omega_{n,k} \xrightarrow{h} I\Omega_{n,k} \xrightarrow{D} \Omega_{n-k}^{w(k)} \to \cdots. \tag{2.4}$$

Here g, h are the obvious forgetful homomorphisms. Let $j_k : S^k \looparrowright \mathbf{R}^{2k}$ be defined by $j_k(t, u_1, \ldots, u_k) = ((t+1)u_1, \ldots, (t+1)u_k, (1-t)u_1, \ldots, (1-t)u_k)$, where S^k is the unit sphere in $\mathbf{R}^{k+1}$ with coordinates $(t, u_1, \ldots, u_k)$. Note that j_k is an immersion with precisely one double point and that $j_k(S^k)$ is $z(k)$-invariant. If $[N]$ represents an arbitrary class in $\Omega_{n-k}^{z(k)}$ then associated to a tubular neighbourhood of an embedding $N \subset \mathbf{R}^{n+k}$ there is a fibre bundle with fibre $j_k(S^k)$. The total space of this bundle represents $\partial[N]$.

The homomorphism $e : I\Omega_{n,k+1} \to \Omega_{n-k-1}^{\Delta\mathrm{SO}}$ is defined as follows. Choose a representative immersion $M^n \looparrowright \mathbf{R}^{n+k+1}$ with normal bundle v. Consider M embedded in v via the zero section and take a section $s : M \to v$ which is transverse to M. $e([M \looparrowright R^{n+k+1}])$ is represented by the intersection manifold $M \cap s(M)$.

If $N^n \overset{i}{\looparrowright} R^{n+k}$ is a self-transverse immersion of an oriented manifold N, then $D[i]$ is represented by the double-points manifold.

The other homomorphisms appearing in these sequences can also be defined in geometric terms (see the references above).

Koschorke [**K**, 9.3] has developed a long exact sequence which is useful in computing low-dimensional bordism groups. The groups $\Omega_i^{z(k)}$, $\Omega_i^{\Delta\mathrm{SO}}$, $\Omega_i^{w(k)}$, $0 \leqslant i \leqslant 2$, can be computed using this sequence.

2.5 PROPOSITION. *The bordism groups $\Omega_i^{z(k)}$, $0 \leqslant i \leqslant 2$, $k > 2$, are given by the following table*:

	$i = 0$	$i = 1$	$i = 2$
$k \equiv 1\ (4)$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_8$
$k \equiv 2\ (4)$	$\mathbf{Z}$	$\mathbf{Z}_4$	$\mathbf{Z}_2$
$k \equiv 3\ (4)$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$k \equiv 0\ (4)$	$\mathbf{Z}$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$

2.6 PROPOSITION. *The bordism groups* $\Omega_i^{w(k)}$, $0 \leqslant i \leqslant 2$, $k > 2$, *are given by the table*:

	$i = 0$	$i = 1$	$i = 2$
$k \equiv 1\ (4)$	$\mathbf{Z}_2$	0	$\mathbf{Z}_4$
$k \equiv 2\ (4)$	$\mathbf{Z}$	$\mathbf{Z}_2$	0
$k \equiv 3\ (4)$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$k \equiv 0\ (4)$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$

2.7 PROPOSITION. *The groups* $\Omega_i^{\Delta SO}$ *are isomorphic to* $\mathbf{Z}$, $\mathbf{Z}_2$ *and* $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ *for* $i = 0, 1, 2$, *respectively.*

3. Bordism groups of immersions. We now study sequences (2.2) and (2.3). The groups $I\Omega_{n,k}$ $(n - 2 \leqslant k \leqslant n)$ are determined except for extension problems in some cases. The unoriented version of sequence (2.2) was studied by Koschorke [**K**]. The extension problems which arise from (2.2) are, in general, more difficult to solve than in the unoriented case (see [**K**, 10.4]). The unoriented analogue of (2.3), can be deduced from results of [**K** and **Sa**]. We will make use of these sequences to solve some extension problems. In particular, a detailed description of some of the homomorphisms between the bordism groups $\Omega_i^{z(k)}$, $\Omega_i^{\Delta SO}$ and the unoriented analogues will be needed. This can be achieved by comparing the corresponding long exact sequences of [**K**, 9.3].

3.1 THEOREM. *For* $n > 0$, $I\Omega_{n,n} \cong \Omega_n \oplus \mathbf{Z}$ *if* n *is even and* $I\Omega_{n,n} \cong \Omega_n \oplus \mathbf{Z}_2$ *if* n *is odd. The* $\mathbf{Z}$ *or* $\mathbf{Z}_2$ *factor is generated by the class of the immersion* $j_n : S^n \looparrowright \mathbf{R}^{2n}$.

PROOF. From the Whitney immersion theorem and sequence (2.2) we get the commutative diagram with horizontal exact sequence:

$$\begin{array}{ccccccc} 0 & \to & \Omega_0^{z(k)} & \to I\Omega_{n,n} & \to & \Omega_n & \to & 0 \\ & & \cong\searrow & D & & & & \\ & & \Omega_0^{w(k)} & & & & & \end{array}$$

□

3.2 THEOREM. *For* $n > 3$ *the groups* $I\Omega_{n,n-1}$ *are given by*

$$I\Omega_{n,n-1} \cong \begin{cases} \Omega_n & \text{for } n \equiv 0\ (4), \\ \Omega_n \oplus \mathbf{Z}_2 & \text{for } n \equiv 2\ (4) \text{ or } n+1 \text{ a power of } 2, \\ \Omega_n \oplus \mathbf{Z}_4 & \text{for } n \equiv 3\ (4),\ n+1 \text{ not a power of } 2. \end{cases}$$

If $n \equiv 1$ (4) *then* $I\Omega_{n,n-1}$ *is an extension of* $\Omega_n \oplus \mathbf{Z}_2$ *by* $\mathbf{Z}_2$.

PROOF. If $n + 1$ is not a power of 2 then every orientable $(n + 1)$-manifold immerses in $\mathbf{R}^{2n}$ [**Mah-P**]. Thus sequence (2.2) takes the form

$$0 \to \Omega_1^{z(n-1)} \to I\Omega_{n,n-1} \to \Omega_n \to 0.$$

The case $n \equiv 0$ follows immediately from 2.5. The splitting of this sequence for $n \equiv 2$ (4) follows by comparison with the corresponding unoriented sequence.

If $n \equiv 1$ or 3 (4) then $|\Omega_1^{z(n-1)}| = 4$ and hence $|I\Omega_{n,n-1}| = 4|\Omega_n|$, provided $n+1$ is not a power of 2. Sequence (2.3) reduces then to

$$0 \to \mathbf{Z}_2 \to I\Omega_{n,n-1} \to \Omega_n \oplus \mathbf{Z}_2 \to 0.$$

If $n \equiv 3$ (4) this extension is nontrivial, as every element in Ω_n has order 2 and $\mathbf{Z}_4 \cong \Omega_1^{z(n-1)}$ injects into $I\Omega_{n,n-1}$.

Finally assume $n+1$ is a power of 2. By [**Mah-P**, 4.2.1] $\mathbf{C}P^{(n+1)/2}$ does not immerse up to cobordism in $\mathbf{R}^{2n}$. Hence $\Omega_1^{z(n-1)}$ does not inject into $I\Omega_{n,n-1}$ and sequence (2.3) becomes

$$\mathbf{Z}_2 \xrightarrow{0} I\Omega_{n,n-1} \to \Omega_n \oplus \mathbf{Z}_2 \to 0. \quad \square$$

We now study the groups $I\Omega_{n,n-2}$, $n > 5$. Let $\alpha(k)$ denote the number of ones in the binary expansion of an integer k.

If $\alpha(n+1) > 2$ then the forgetful homomorphism $I\Omega_{n+1,n-2} \to \Omega_{n+1}$ is onto. This follows either by Cohen's immersion theorem [**C**] or by showing that each $(n+1)$-dimensional multiplicative generator has a representative that immerses in $\mathbf{R}^{2n-1}$ (see [**Wa, 0**]). If $n+1 \equiv 2$ (4) then $I\Omega_{n+1,n-2} \to \Omega_{n+1}$ is always onto, as there is a system of generators of Ω_* with no elements in these dimensions [**Wa**]. Hence if either $\alpha(n+1) > 2$ or $n+1 \equiv 2$ (4) then we get an exact sequence

$$0 \to \Omega_2^{z(n-2)} \to I\Omega_{n,n-2} \to \Omega_n \to 0.$$

3.3 THEOREM. *Let $n > 5$.*

(i) *If $n \equiv 0$ (4) then $I\Omega_{n,n-2} \cong \Omega_n \oplus \mathbf{Z}_2$ if $\alpha(n) > 1$. If $\alpha(n) = 1$ then $I\Omega_{n,n-2}$ is isomorphic to the subgroup of Ω_n consisting of classes* [**M**] *with Stiefel number $w_2 \cdot \bar{w}_{n-2}(M) = 0$.*

(ii) *If $n \equiv 3$ (4) $I\Omega_{n,n-2}$ is isomorphic to $\Omega_n \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_8$ if $\alpha(n+1) \geqslant 3$ and to $\Omega_n \oplus \mathbf{Z}_4$ if $\alpha(n+1) = 1$. If $\alpha(n+1) = 2$ then $I\Omega_{n,n-2}$ is an extension of $\Omega_n \oplus \mathbf{Z}_4$ by $\mathbf{Z}_2$.*

(iii) *If $n \equiv 2$ (4) then $I\Omega_{n,n-2}$ is an extension of $\Omega_n \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ by $\mathbf{Z}_2$.*

(iv) *If $n \equiv 1$ (4) then $I\Omega_{n,n-2}$ is an extension of $\Omega_n \oplus \mathbf{Z}_2$ by $\mathbf{Z}_2$.*

PROOF. Let $n \equiv 0$ (4), $\alpha(n) \geqslant 2$. We have commutative diagrams:

$$\alpha(n) > 2 \qquad \begin{array}{ccccccccc} 0 & \to & \mathbf{Z}_2 & \to & I\Omega_{n,n-2} & \to & \Omega_n & \to & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}_2 & \to & \mathfrak{N}_n \oplus \mathbf{Z}_2 & \to & \mathfrak{N}_n & \to & 0 \end{array}$$

$$\alpha(n) = 2 \qquad \begin{array}{ccccccccc} 0 & \to & \mathbf{Z}_2 & \to & I\Omega_{n,n-2} & \to & \Omega_n & \to & 0 \\ & & \cong\downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}_2 & \to & \mathfrak{N}_{n/\mathbf{Z}_2} \oplus \mathbf{Z}_4 & \to & \mathfrak{N}_n & \to & 0 \end{array}$$

The upper diagram shows $I\Omega_{n,n-2} \cong \Omega_n \oplus \mathbf{Z}_2$ if $\alpha(n) > 2$. If $n = 2^m + 2^l$ then the $\mathbf{Z}_4$ factor of $I\mathfrak{N}_{n,n-2}$ is generated by an immersion of $\mathbf{R}P^{2^m} \times \mathbf{R}P^{2^l}$ [**K**]. But $\mathbf{R}P^{2^m} \times \mathbf{R}P^{2^l}$ is not cobordant to an oriented manifold. Therefore the top sequence in the lower diagram also splits.

If $\alpha(n) = 1$ then the Dold manifold $P(1, n/2)$ does not immerse up to cobordism in $\mathbf{R}^{2n-1}$ as its number $\bar{w}_2 \cdot \bar{w}_{n-1}$ is nonzero. Then (2.2) gives

$$I\Omega_{n+1,n-2} \to \Omega_{n+1} \to \mathbf{Z}_2 \xrightarrow{0} I\Omega_{n,n-2} \to \Omega_n \to \mathbf{Z}_2 \to 0.$$

Now assume $n \equiv 3$ (4). If $\alpha(n+1) \geq 3$ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}_2 \oplus \mathbf{Z}_8 & \to & I\Omega_{n,n-2} & \to & \Omega_n & \to & 0 \\ & & \psi \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \to & \mathfrak{N}_n \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \to & \mathfrak{N}_n & \to & 0 \end{array}$$

where $\psi(1,0) = (1,0,0)$ and $\psi(0,1) = (0,0,1)$. This implies that $I\Omega_{n,n-2} \cong \Omega_n \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_8$. If $\alpha(n+1) = 2$ then $\mathbf{C}P^{n+1/2}$ does not immerse up to oriented cobordism in $\mathbf{R}^{2n-1}$ and coker $f\colon I\Omega_{n+1,n-2} \to \Omega_{n+1}$ is $\mathbf{Z}_2$. By (2.2) $|I\Omega_{n,n-2}| = 8|\Omega_n|$ and sequence (2.3) reduces to

$$0 \to \mathbf{Z}_2 \to I\Omega_{n,n-2} \to \Omega_n \oplus \mathbf{Z}_4 \to 0.$$

If $n+1$ is a power of 2 then $I\Omega_{n,n-2}$ fits into the exact sequence (2.3)

$$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \to I\Omega_{n,n-2} \to \Omega_n \oplus \mathbf{Z}_2 \to 0$$

and therefore has no elements of order 8. Sequence (2.2) takes the form

$$I\Omega_{n+1,n-2} \xrightarrow{f} \Omega_{n+1} \xrightarrow{S} \mathbf{Z}_2 \oplus \mathbf{Z}_8 \to I\Omega_{n,n-2} \to \Omega_n \to 0$$

where $S[\mathbf{C}P^{(n+1)/2}] = (1,2) \in \mathbf{Z}_2 \oplus \mathbf{Z}_8$ [**0**, I.30]. It follows then that $I\Omega_{n,n-2} \cong \Omega_n \oplus \mathbf{Z}_4$.

If $n \equiv 2$ (4) the result follows by comparing (2.2) with its unoriented analogue. The case $n \equiv 1$ is treated in the next section. □

4. Double points and embeddings. The monomorphism $z(k) \to w(k)$ induces a homomorphism of the bordism groups $\Omega_{n-k}^{z(k)} \to \Omega_{n-k}^{w(k)}$. Moreover, there is a commutative diagram

$$\begin{array}{ccc} \Omega_{n-k}^{z(k)} & \to & \Omega_{n-k}^{w(k)} \\ \partial \searrow & & \swarrow D \\ & I\Omega_{n,k} & \end{array}$$

where ∂ and D are described in §2.

4.1 PROPOSITION. *Let $k > 2$. The natural homomorphism $\Omega_i^{z(k)} \to \Omega_i^{w(k)}$ fits into the following exact sequences*:

$$0 \to \Omega_1^{fr} \to \Omega_1^{z(k)} \to \Omega_1^{w(k)} \to 0 \qquad \textit{for } k \textit{ even},$$
$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to \Omega_2^{z(k)} \to \Omega_2^{w(k)} \to 0 \qquad \textit{for } k \equiv 0 \ (4),$$
$$0 \to \mathbf{Z}_4 \to \Omega_2^{z(k)} \to \Omega_2^{w(k)} \to 0 \qquad \textit{for } k \equiv 1 \ (4),$$
$$0 \to \mathbf{Z}_2 \to \Omega_2^{z(k)} \to \Omega_2^{w(k)} \to 0 \qquad \textit{for } k \equiv 3 \ (4).$$

The proof of 4.1 follows by comparing the corresponding long exact sequences of [**K**, 9.3]. This result enables us to study the following diagram of exact sequences:

$$\begin{array}{ccccccccccc} & & \Omega_2^{z(n-2)} & & & & & & \Omega_1^{z(n-1)} & & \\ & & \partial\downarrow & & & & & & \downarrow & & \\ E\Omega_{n,n-2} & \to & I\Omega_{n,n-2} & \to & \Omega_2^{w(n-2)} & \to & E\Omega_{n-1,n-2} & \to & I\Omega_{n-1,n-2} & \to & \Omega_1^{w(n-1)} \to 0 \\ & & \downarrow & & & & & & \downarrow & & \\ & & \Omega_n & & & & & & \Omega_{n-1} & & \\ & & & & & & & & \downarrow & & \\ & & & & & & & & 0 & & \end{array}$$

4.2 THEOREM. *For $n > 0$, $E\Omega_{n,n} \cong \Omega_n$.*

4.3 THEOREM. *Let $n > 3$. If n or $n + 1$ is a power of 2 then $E\Omega_{n,n-1} \cong \Omega_n$; otherwise there is a short exact sequence*

$$0 \to \mathbf{Z}_2 \to E\Omega_{n,n-1} \to \Omega_n \to 0.$$

This sequence splits if $n \equiv 2$ or 3 (4).

Theorem 4.2 follows immediately from sequence (2.4). Applying 4.1 to the previous diagram proves 4.3 for $n \equiv 0$, 2 or 3 (4). If $n \equiv 1$ (4) we obtain the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & \mathbf{Z}_2 \oplus \mathbf{Z}_2 & & \\ & & \partial\downarrow & & \\ E\Omega_{n,n-2} & \to & I\Omega_{n,n-2} & \xrightarrow{D} & \mathbf{Z}_2 \oplus \mathbf{Z}_2 \\ & & \downarrow \nearrow\varphi & & \\ & & \Omega_n & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

where the composite $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{\partial} I\Omega_{n,n-2} \xrightarrow{D} \mathbf{Z}_2 \oplus \mathbf{Z}_2$ has kernel and cokernel $\mathbf{Z}_2$. This implies that $I\Omega_{n,n-2}$ is an extension of $\Omega_n \oplus \mathbf{Z}_2$ by $\mathbf{Z}_2$. We need to compute φ above to know whether D is onto. It is equivalent to study the problem of embedding oriented manifolds up to oriented cobordism in $\mathbf{R}^{2n-2}$. Every orientable m-manifold embeds in $\mathbf{R}^{2m-1}$ [**Mas-P**], thus we only need to investigate whether generators of Ω_* in dimension n embed up to oriented cobordism in $\mathbf{R}^{2n-2}$, $n \equiv 1$ (4). Using results of R. Brown [**B**, 2.1 and 5.1] and E. Thomas [**Th**, 1.1] it can be shown that all the n-dimensional generators of Ω_* given in [**Wa**] embed in $\mathbf{R}^{2n-2}$ with the sole exception of the Dold manifolds $P(1, 2^l)$, $l > 0$. Hence the double points homomorphism $D\colon I\Omega_{n,n-2} \to \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is onto if and only if $n - 1$ is a power of 2. This completes the proof of 4.3.

4.4 COROLLARY. *If M^n is orientable then M embeds up to oriented cobordism in $\mathbf{R}^{2n-2}$ if and only if the Stiefel number $w_2 \cdot \bar{w}_{n-2}(M) = 0$. If neither n nor $n - 1$ is a power of 2 this condition is always satisfied.*

D. Ellis [E] has recently proved 4.2–4.4 using different techniques. The following theorem extends one of his results.

4.5 THEOREM. *If* $k \equiv 0$ (4) *there is short exact sequence*

$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to E\Omega_{k+2,k} \to \Omega_{k+2} \to 0.$$

PROOF. The exact sequences

$$E\Omega_{k+2,k} \to I\Omega_{k+2,k} \to \mathbf{Z}_2 \to 0, \tag{2.4}$$

$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to I\Omega_{k+2,k} \to \Omega_{k+2} \to 0 \tag{2.2}$$

show that $|E\Omega_{k+2,k}| \geqslant 4|\Omega_{k+2}|$. On the other hand, Ellis [E] has proved that $E\Omega_{k+2,k}$ fits into the exact sequence

$$E\Omega_{k+3,k} \to \Omega_{k+3} \xrightarrow{\partial} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to E\Omega_{k+2,k} \to \Omega_{k+2} \to 0,$$

implying $|E\Omega_{k+2,k}| \leqslant 4|\Omega_{k+2}|$. □

4.6 COROLLARY. *If* $n \equiv 3$ (4) *every oriented n-manifold embeds up to oriented cobordism in* $\mathbf{R}^{2n-3}$.

One final observation should perhaps be made. If $k \equiv 1$ (4) and Λ denotes the kernel of $E\Omega_{k+2,k} \to \Omega_{k+2}$ then, as in 4.5, it can be shown that

$$\Lambda = \begin{cases} \mathbf{Z}_4 & \text{if } \alpha(k+3) > 2, \\ \mathbf{Z}_2 & \text{if } \alpha(k+3) = 2 \\ 0 & \text{if } \alpha(k+3) = 1. \end{cases}$$

Therefore, if $k+3$ is a power of 2, then all codimension-k isolated singularities in $2k+3$ manifolds are orientably smoothable.

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