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# FACTORING COMPACT AND WEAKLY COMPACT OPERATORS THROUGH REFLEXIVE BANACH LATTICES[1]

BY

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ABSTRACT. When does a weakly compact operator between two Banach spaces factor through a reflexive Banach lattice?

This paper provides some answers to this question. One of the main results: If an operator between two Banach spaces factors through a Banach lattice with weakly compact factors, then it also factors through a reflexive Banach lattice. In particular, the square of a weakly compact operator on a Banach lattice factors through a reflexive Banach lattice.

Similar results hold for compact operators. For instance, the square of a compact operator on a Banach lattice factors with compact factors through a reflexive Banach lattice.

**1. Preliminaries.** For terminology concerning Riesz spaces and Banach lattices, we follow [**2** and **17**]. In this section, we briefly review a few basic results concerning Banach lattices.

Let $X$ be a Riesz space. Then the symbol $X^+$ will denote the positive cone of $X$, i.e., $X^+ = \{x \in X: x \geqslant 0\}$. The sets of the form

$$[-x, x] := \{y \in X: -x \leqslant y \leqslant x\}, \qquad x \in X^+,$$

are called the *order intervals* of $X$. A subset $A$ of $X$ is said to be *solid* whenever $|x| \leqslant |y|$ in $X$ and $y \in A$ imply $x \in A$. The *solid hull* of a set $A$ is the smallest solid set that contains $A$ and is precisely the set $\operatorname{sol}(A) := \{x \in X: \exists y \in A \text{ with } |x| \leqslant |y|\}$. A solid vector subspace of $X$ is referred to as an *ideal* of $X$.

In this paper the word "operator" is synonymous with "linear operator." An operator $T: X \to Z$ between two Riesz spaces is said to be *positive*, in symbols $0 \leqslant T$, whenever $x \in X^+$ implies $T(x) \in Z^+$.

A norm $\|\cdot\|$ on a Riesz space is said to to be a *lattice norm* whenever $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$. A *Banach lattice* is a Riesz space equipped with a lattice norm under which it is a Banach space. A Banach lattice is said to have *order continuous norm* whenever $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$. The closed unit ball of an arbitrary Banach space $Z$ will be denoted by $B_Z$; i.e., $B_Z := \{z \in Z: \|z\| \leqslant 1\}$.

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For our work, we shall need the following characterization of Banach lattices with order continuous norms. (We shall denote the norm dual of an arbitrary Banach space $Z$ by $Z'$ and its second dual by $Z''$.)

THEOREM 1.1. *For a Banach lattice $X$ the following statements are equivalent*:

1. *$X$ has order continuous norm.*
2. *Each order interval of $X$ is weakly compact.*
3. *$X$ is an ideal of $X''$.*
4. *Every relatively weakly compact subset of $X^+$ has a relatively weakly compact solid hull.*

PROOF. For the equivalence of (1), (2), and (3) see [**17**, Theorem 5.10, p. 89].

(3) $\Rightarrow$ (4) Let $A \subseteq X^+$ be relatively weakly compact, and let $x'' \in X''$ be a $w^*$-closure point of sol($A$). Pick a net $\{x_\alpha\} \subseteq \text{sol}(A)$ with $x_\alpha \xrightarrow{w^*} x''$. For each $\alpha$, choose some $y_\alpha \in A$ with $-y_\alpha \leqslant x_\alpha \leqslant y_\alpha$. By passing to a subnet, we can assume that $y_\alpha \xrightarrow{w} y$ holds in $X$. This implies $-y \leqslant x'' \leqslant y$ in $X''$, and so, $x'' \in X$. Therefore, sol($A$) is relatively weakly compact. (For an alternate proof of this implication see [**18**, Proposition 2.1, p. 241].)

(4) $\Rightarrow$ (2) Note that for each $x \in X^+$ we have $\text{sol}\{x\} = [-x, x]$ and that $[-x, x]$ is weakly compact. □

It should be noted that in a Banach lattice with order continuous norm the solid hull of a relatively weakly compact set need not be relatively weakly compact [**16**, p. 308]. However, there are some Banach lattices in which the solid hull of a relatively weakly compact set remains relatively weakly compact. These are the KB-spaces, and they will be needed for our work.

Following the Russian literature, we say that a Banach lattice $X$ is a *KB-space* (Kantorovič-Banach space) whenever every increasing norm bounded sequence of $X^+$ is norm convergent. Clearly, every KB-space has order continuous norm.

THEOREM 1.2. *The norm dual $X'$ of a Banach lattice $X$ is a KB-space if and only if $X'$ has order continuous norm.*

PROOF. Assume that $X'$ has order continuous norm. Let $0 \leqslant x'_n \uparrow$ hold in $X'$ with $\sup\{\|x'_n\|\} < \infty$. Then $x'(x) = \lim x'_n(x)$ exists in $\mathbf{R}$ for each $x \in X$, and moreover, this formula defines a positive linear functional on $X$. Thus, $x' \in X'$, and clearly, $x'_n \uparrow x'$ holds in $X'$. Now note that $\lim\|x'_n - x'\| = 0$. □

Recall that if $X$ is a Riesz space, then the symbol $X_n^\sim$ denotes the *order continuous dual* of $X$, i.e., the Riesz space of all linear functionals $f$ on $X$ such that $x_\alpha \downarrow 0$ in $X$ implies $\lim|f(x_\alpha)| = 0$.

The next result presents some characterizations of KB-spaces. For a proof see [**2**, p. 159; **17**, p. 95; or **16**, p. 309].

THEOREM 1.3. *For a Banach lattice $X$ the following statements are equivalent*:

1. *$X$ is a KB-space.*
2. *$X$ is a band of $X''$.*
3. *$X = (X')_n^\sim$.*

In a KB-space a relatively weakly compact set has a relatively weakly compact solid hull. This result appeared in [**1**]; see also [**17**, Exercise 30, p. 153].

THEOREM 1.4. *In a KB-space the convex solid hull of a relatively weakly compact set remains relatively weakly compact.*

PROOF. Let $A$ be a relatively weakly compact subset of $X$, and consider $A$ as a subset of $(X')_n^\sim$. By [**2**, Corollary 20.12, p. 140] we know that $A$ is relatively $\sigma((X')_n^\sim, X')$-compact if and only if its convex solid hull is also relatively $\sigma((X')_n^\sim, X')$-compact. By Theorem 1.3 we have $X = (X')_n^\sim$, and our conclusion follows. □

For constructing reflexive Banach spaces from weakly compact sets, we shall need the following basic result of [7] (see also [**8**, p. 250]).

THEOREM 1.5 (DAVIS-FIGIEL-JOHNSON -PEŁCZYŃSKI). *Let $X$ be a Banach space with closed unit ball $B = \{x \in X\colon \|x\| \leqslant 1\}$, and let $W$ be a convex, symmetric, norm bounded subset of $X$. For each $n$ put $U_n := 2^nW + 2^{-n}B$, and denote by $\|\cdot\|_n$ the Minkowski functional of $U_n$, i.e.,*

$$\|x\|_n := \inf\{\alpha > 0\colon x \in \alpha U_n\}.$$

*Set $Y = \{x \in X\colon |||x||| = (\sum_{n=1}^\infty \|x\|_n^2)^{1/2} < \infty\}$, and let $J\colon Y \to X$ denote the natural inclusion. Then:*

1. *$(Y, |||\cdot|||)$ is a Banach space and $J$ is continuous;*
2. *$W$ is a subset of the closed unit ball of $(Y, |||\cdot|||)$;*
3. *$J''\colon Y'' \to X''$ is one-to-one; and*
4. *$(Y, |||\cdot|||)$ is reflexive if and only if $W$ is a relatively weakly compact subset of $X$.*

Another important property associated with the Banach space $(Y, |||\cdot|||)$, needed for our work, is described in the next lemma.

LEMMA 1.6. *Let $W$ be a convex, symmetric, norm bounded subset of a Banach space $X$, and let $(Y, |||\cdot|||)$ be the Banach space of Theorem 1.5 determined by $W$. Then a subset of $W$ is totally bounded in $X$ if and only if it is totally bounded in $Y$.*

*In particular, if a compact operator $T\colon Z \to X$ satisfies $T(B_Z) \subseteq W$, then $T$ considered as an operator from $Z$ into $Y$ is also compact.*

PROOF. Let $D$ be a subset of $W$ which is totally bounded in $X$, and let $0 < \varepsilon < 1$. Since $x, y \in W$ implies

$$x - y \in W + W = 2W = 2^{1-n}(2^nW) \subseteq 2^{1-n}U_n,$$

we see that $\|x - y\|_n \leqslant 2^{1-n}$ holds for all $n$, and so, there exists some $k$ satisfying $\sum_{n=k}^\infty \|x - y\|_n^2 < \varepsilon$ for all $x, y \in W$. Pick $x_1, \ldots, x_m \in D$ such that

$$D \subseteq \{x_1, \ldots, x_m\} + \varepsilon 2^{-2k}B,$$

where $B$ is the closed unit ball of $X$.

Now let $x \in D$ be fixed. Choose some $x_i$ with $\|x - x_i\| < \varepsilon 2^{-2k}$. Then for $1 \leqslant n \leqslant k$ we have

$$x - x_i \in \varepsilon 2^{-2k}B \subseteq \varepsilon 2^{-k}2^{-n}B \subseteq \varepsilon 2^{-k}U_n,$$

and so, $\|x - x_i\|_n \leqslant \varepsilon 2^{-k}$ holds for all $1 \leqslant n \leqslant k$. Therefore,

$$|||x - x_i|||^2 \leqslant \sum_{n=1}^{k} \|x - x_i\|_n^2 + \sum_{n=k}^{\infty} \|x - x_i\|_n^2 < k\varepsilon^2 2^{-2k} + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon$$

holds, from which it follows that $D$ is also a totally bounded subset of $Y$. □

Recall that a positive operator $T$: $Z \to X$ between two Riesz spaces is said to be:

(a) *interval preserving*, whenever $T[0, x] = [0, Tx]$ holds for all $x \in Z^+$; and

(b) *a lattice homomorphism*, whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in Z$.

In the sequel, the expression "an operator $T$: $Z \to X$ between two Banach spaces is positive" will mean that $Z$ and $X$ are both Banach lattices and that $T$ is a positive operator (i.e., $z \geqslant 0$ implies $T(z) \geqslant 0$).

When $X$ is a Banach lattice and $W$ is also a solid set, then the Banach space $(Y, |||\cdot|||)$ of Theorem 1.5 is itself a Banach lattice.

THEOREM 1.7. *Let $W$ be a convex, solid, and norm bounded subset of a Banach lattice $X$. If $(Y, |||\cdot|||)$ is the Banach space of Theorem* 1.5 *determined by $W$, then we have*:

1. $(Y, |||\cdot|||)$ *is a Banach lattice and $Y$ is an ideal of $X$*;
2. *the operator $J$: $Y \to X$ is an interval preserving lattice homomorphism*; *and*
3. *$J'$: $X' \to Y'$ is also an interval preserving lattice homomorphism.*

PROOF. (1) Let $|x| \leqslant |y|$ hold in $X$ with $y \in Y$. If $y \in \alpha U_n$, then (in view of the solidness of $U_n = 2^n W + 2^{-n} B$) we see that $x \in \alpha U_n$, and so, $\|x\|_n \leqslant \|y\|_n$. Therefore, $|||x||| \leqslant |||y|||$ holds. This shows that $Y$ is an ideal of $X$, and that $(Y, |||\cdot|||)$ is a Banach lattice.

(2) Obvious.

(3) It follows immediately from [**15**, Proposition 1.2, p. 89]. □

**2. Factoring weakly compact operators.** Let $T$: $Z \to X$ be a continuous operator between two Banach spaces. Recall that $T$ is said to *factor* through a Banach space $Z_1$ whenever there exist continuous operators $Z \xrightarrow{S} Z_1 \xrightarrow{R} X$ satisfying $T = RS$. (The operators $R$ and $S$ are called factors of $T$.) By [7] we know that every weakly compact operator factors through a reflexive Banach space. Also, by [**10** and **14**] we know that every compact operator factors with compact factors through a reflexive Banach space. In this section we show how a weakly compact operator can be factored through a reflexive Banach lattice.

We start our discussion with a theorem that will be the basis for our results.

THEOREM 2.1. *Let $W$ be the convex solid hull of a relatively weakly compact subset of a Banach lattice $X$, and let $(Y, |||\cdot|||)$ be the Banach lattice as defined in Theorem* 1.5. *Then $Y'$ has order continuous norm* (*and hence*, *it is a KB-space*).

PROOF. By Theorem 1.7, $J'$: $X' \to Y'$ is interval preserving, and so, $J'(X')$ is an ideal of $Y'$. On the other hand, it is also true that $J'(X')$ is weakly dense in $Y'$. Indeed, if some $y'' \in Y''$ satisfies $y''(J'x') = 0$ for all $x' \in X'$, then $J''y''(x') = 0$

holds for all $x' \in X'$, and so, $J''y'' = 0$. Since $J''\colon Y'' \to X''$ is one-to-one, we get $y'' = 0$. Thus, $J'(X')$ is weakly dense in $Y'$, and hence, norm dense in $Y'$. Now if $J'(X')$ has order continuous norm, then its norm closure $Y'$ will also have order continuous norm; see [**2**, Theorems 17.9 and 10.6]. Therefore, in order to establish that $Y'$ has order continuous norm, it is enough to show that $J'x'_n \downarrow 0$ in $J'(X')$ implies $|||J'x'_n||| \downarrow 0$.

To this end, let $J'x'_n \downarrow 0$ in $J'(X')$. Since $J'x'_n = |J'x'_n| = J'|x'_n|$ holds, we can assume that $x'_n \geqslant 0$ for each $n$. Now let $\varepsilon > 0$. Fix some $k$ with $2^{-k}\|x'_1\| < \varepsilon$. Since $W$ is the convex solid hull of a relatively weakly compact subset of $X$, it follows from [**2**, Theorems 20.9 and 20.6] that there exists some $u \geqslant 0$ in the ideal generated by $W$ in $X$ satisfying

$$x'_1(|w| - u)^+ < 2^{-k}\varepsilon \quad \text{for all } w \in W.$$

From $W \subseteq Y$ and the fact that $Y$ is an ideal of $X$, we see that $u \in Y$. Next choose some $m$ with

$$2^k J'x'_n(u) < \varepsilon \quad \text{for all } n \geqslant m.$$

Now if $y \in Y$ satisfies $|||y||| \leqslant 1$, then $\|y\|_k \leqslant 1$ also holds, and so,

$$y \in 2(2^k W + 2^{-k}B).$$

Write $y = 2(2^k w + 2^{-k}v)$ with $w \in W$ and $v \in B$. Clearly, $v \in Y$, and moreover,

$$\begin{aligned}
|J'x'_n(y)| &\leqslant 2\left[2^k J'x'_n(|w|) + 2^{-k}J'x'_n(|v|)\right] \\
&\leqslant 2\left[2^k J'x'_n(|w| - u)^+ + 2^k J'x'_n(u) + 2^{-k}x'_n(|v|)\right] \\
&\leqslant 2\left[2^k x'_1(|w| - u)^+ + 2^k J'x'_n(u) + 2^{-k}\|x'_1\|\right] \\
&\leqslant 2[2^k \cdot 2^{-k}\varepsilon + \varepsilon + \varepsilon] = 6\varepsilon
\end{aligned}$$

holds for all $n \geqslant m$. This implies $|||J'x'_n||| \leqslant 6\varepsilon$ for all $n \geqslant m$, and hence, $|||J'x'_n||| \downarrow 0$, as desired. □

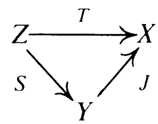
The next theorem deals with the factorization of a weakly compact operator whose range is in a Banach lattice.

THEOREM 2.2. *Let $T\colon Z \to X$ be a weakly compact operator from a Banach space $Z$ into a Banach lattice $X$. If the solid hull of $T(B_Z)$ is relatively weakly compact, then $T$ factors through a reflexive Banach lattice.*

*If, in addition, $T$ is a positive operator, then the factors can be taken to be positive operators.*

PROOF. Denote by $W$ the convex hull of the solid hull of $T(B_Z)$, i.e., $W = \operatorname{co}(\operatorname{sol}(T(B_Z)))$. By [**2**, Theorem 1.3, p. 4] the set $W$ is a solid set, and by our hypothesis, it follows that $W$ is also relatively weakly compact. Let $(Y, |||\cdot|||)$ be the reflexive Banach lattice corresponding to $W$ (Theorems 1.5 and 1.7), and let

$S\colon Z \to Y$ be the operator defined by $S(z) = T(z)$ for all $z \in Z$. Now a glance at the diagram




completes the proof of the theorem. □

The following is an immediate consequence of the preceding result and Theorem 1.4.

COROLLARY 2.3. *Every weakly compact operator $T\colon Z \to X$ from a Banach space into a KB-space factors through a reflexive Banach lattice.*

*Note.* Theorem 2.2 and Corollary 2.3 can be derived also from [**11**, Theorem 3.1, p. 403].

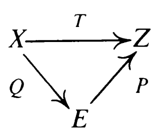
For positive operators the following version holds.

COROLLARY 2.4. *Let $T\colon Z \to X$ be a positive weakly compact operator between two Banach lattices. If $X$ has order continuous norm, then $T$ factors (with positive factors) through a reflexive Banach lattice.*

PROOF. Note that $\operatorname{sol}(T(B_Z)) = \operatorname{sol}(T(B_Z^+))$ holds. Taking into account Theorem 1.1(4), we see that $T(B_Z)$ has a relatively weakly compact solid hull in $X$. Now apply Theorem 2.2. □

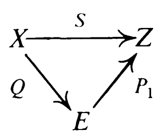
The following result describes a factorization property of weakly compact operators whose domains are Banach lattices.

THEOREM 2.5. *Let $T\colon X \to Z$ be a weakly compact operator between two Banach spaces. If $X$ is a Banach lattice such that $X'$ has order continuous norm, then there exist a reflexive Banach lattice $E$ and a factorization of $T$*



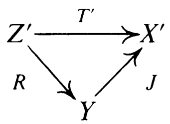

*with the factor $Q$ a lattice homomorphism and the factor $P$ positive if $T$ is positive.*

*Moreover, if another operator $X \xrightarrow{S} Z$ satisfies $0 \leqslant S \leqslant T$, then $S$ admits a factorization through the reflexive Banach lattice $E$*

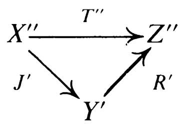



*with $0 \leqslant P_1 \leqslant P$.*

PROOF. Clearly, $T'\colon Z' \to X'$ is weakly compact, and by Theorem 1.2, the Banach lattice $X'$ is a KB-space. Thus, by Theorem 1.4, if $W$ denotes the convex solid hull of $T'(B_{Z'})$, then $W$ is a relatively weakly compact subset of $X'$. Let $(Y, ||| \cdot |||)$ be the reflexive Banach lattice of Theorems 1.5 and 1.7 determined by $W$ (where $Y$ is also an ideal of $X'$). Next, define the operator $R$: $Z' \to (Y, |||\cdot|||)$ by $R(z') = T'(z')$, and note that $R$ is continuous (and that $R$ is positive if $T$ is positive). Thus, we have the diagrams:




and




Since $J'(X'')$ is dense in $Y'$ and $T''(X'') \subseteq Z$, we see that $R'(Y') \subseteq Z$. Now consider the reflexive Banach lattice $E = Y'$ and the continuous operators $X \overset{Q}{\to} E \overset{P}{\to} Z$ where $P(u) = R'(u)$ for $u \in E$ and $Q(x) = J'(x)$ for $x \in X$. By Theorem 1.7, $Q$ is a lattice homomorphism, and clearly, $P$ is positive if $T$ is positive. Now note that $T = PQ$ holds.

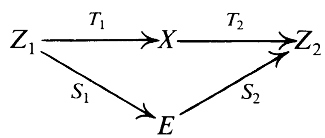
Now assume $0 \leqslant S \leqslant T$. Then $S'(Z') \subseteq Y$ holds. On the other hand, by [**4**, Theorem 7] $S$ is also weakly compact, and hence, $S''(X'') \subseteq Z$ also holds. As above, this implies that the operator $S_1$: $Z' \to Y$ defined by $S_1(z') = S'(z')$ satisfies $S_1'(Y') \subseteq Z$. Define $P_1$: $Y' \to Z$ by $P_1(y') = S_1'(y')$, and note that $0 \leqslant P_1 \leqslant P$ and $S = P_1 Q$ hold. The proof of the theorem is now complete. □

Consider two positive operators $S$, $T$: $X \to X$ on a Banach lattice satisfying $0 \leqslant S \leqslant T$. By [**4**, Example 1] we know that if $T$ is weakly compact then $S$ need not be weakly compact. However, we proved in [**4**] that if $T$ is weakly compact, then $S^2$ is necessarily weakly compact. The next theorem is a generalization of this result.

THEOREM 2.6. *Consider the scheme of weakly compact operators*

$$Z_1 \overset{T_1}{\to} X \overset{T_2}{\to} Z_2$$

*between Banach spaces. If $X$ is a Banach lattice, then there exist a reflexive Banach lattice $E$ and a factorization of $T_2T_1$*

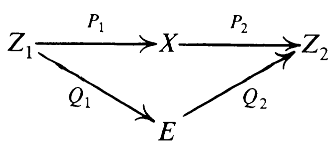



*such that $S_i$ $(i = 1, 2)$ is positive if $T_i$ is positive.*

*Moreover, if another scheme of operators*

$$Z_1 \overset{P_1}{\to} X \overset{P_2}{\to} Z_2$$

*satisfies $0 \leqslant P_i \leqslant T_i$ $(i = 1, 2)$, then there exists a factorization of $P_2P_1$ through the reflexive Banach lattice $E$*



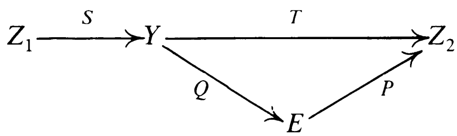

*such that $0 \leqslant Q_i \leqslant S_i$ holds for each $i$.*

PROOF. Consider two weakly compact operators $Z_1 \xrightarrow{T_1} X \xrightarrow{T_2} Z_2$ between Banach spaces with $X$ a Banach lattice. Let $W$ be the convex solid hull of the relatively weakly compact set $T_1(B_{Z_1})$ and consider the Banach lattice $(Y, |||\cdot|||)$ of Theorems 1.5 and 1.7. By Theorem 2.1 the Banach lattice $Y'$ has order continuous norm.

Now if $S\colon Z_1 \to Y$ is defined by $S(z) = T_1(z)$, then $S$ is continuous. On the other hand, if $T$ denotes the restriction of $T_2$ to $Y$, then (since every norm bounded subset of $Y$ is also norm bounded in $X$) the operator $T\colon Y \to Z_2$ is weakly compact. Clearly, the operators $Z_1 \xrightarrow{S} Y \xrightarrow{T} Z_2$ satisfy $T_2T_1 = TS$.

If $0 \leqslant P_i \leqslant T_i$ $(i = 1, 2)$ holds, then clearly $P_1(Z_1) \subseteq Y$, and if $Z_1 \xrightarrow{R_1} Y \xrightarrow{R_2} Z_2$ are defined by $R_1(z) = P_1(z)$ and $R_2(z) = P_2(z)$, then $0 \leqslant R_1 \leqslant S$, $0 \leqslant R_2 \leqslant T$, and $P_2P_1 = R_2R_1$ hold.

By Theorem 2.5 there exists a factorization of $T$ through a reflexive Banach lattice $E$




such that $Q$ is positive and with $P$ positive if $T$ is positive. Moreover, in case $0 \leqslant P_2 \leqslant T_2$ holds, there exists an operator $Q_2\colon E \to Z_2$ satisfying $0 \leqslant Q_2 \leqslant P$ and $R_2 = Q_2Q$. To finish the proof, take $S_1 = QS$, $S_2 = P$, and $Q_1 = QR_1$. □

An important special case of Theorem 2.6 is the following

COROLLARY 2.7. *If $T$: $X \to X$ is a weakly compact operator on a Banach lattice, then $T^2$ factors (with positive factors if $T$ is positive) through a reflexive Banach lattice.*

*Note.* After submitting this paper, we learnt of the work of N. Ghoussoub and W. B. Johnson [**12**] which has some overlapping with ours.

We close this section with two problems related to the question in the abstract.

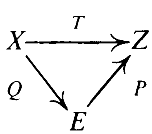
*Problem* 1. Does a weakly compact operator between two Banach lattices factor through a reflexive Banach lattice?

*Problem* 2. Does a positive weakly compact operator between two Banach lattices factor (if possible, with positive factors) through a reflexive Banach lattice?

**3. Factoring compact operators.** In this section we shall study when a compact operator has a factorization with compact factors through a reflexive Banach lattice. Some results of this type were obtained in [**6**].

We start the section with the "compact operator" analogue of Theorem 2.5.

THEOREM 3.1. *Let $T$: $X \to Z$ be a compact operator from a Banach lattice $X$ into a Banach lattice space $Z$. If $X'$ has order continuous norm, then $T$ admits a factorization*



*through a reflexive Banach lattice E with the factor Q a lattice homomorphism and the factor P compact. In case T is also positive, then P can be chosen to be a positive operator.*

*Moreover, if another operator* $S: X \to Z$ *satisfies* $0 \leqslant S \leqslant T$, *then S admits a factorization through the reflexive Banach lattice E*

$$\begin{array}{ccc} X & \xrightarrow{\quad S \quad} & Z \\ & {\scriptstyle Q}\searrow \quad \nearrow{\scriptstyle P_1} & \\ & E & \end{array}$$

*with* $0 \leqslant P_1 \leqslant P$.

PROOF. Repeat the proof of Theorem 2.5, and use Lemma 1.6 to see that $R: Z' \to Y$ is a compact operator which implies that $P$ is a compact operator. □

The next theorem extends Theorem 2.6 and is the major factorization result of the paper.

THEOREM 3.2. *Consider the scheme of operators*

$$Z_1 \overset{T_1}{\to} X \overset{T_2}{\to} Z_2$$

*between Banach spaces with each* $T_i$ $(i = 1, 2)$ *compact or weakly compact. If X is a Banach lattice, then there exists a reflexive Banach lattice E and a factorization of* $T_2 T_1$

$$\begin{array}{ccccc} Z_1 & \xrightarrow{\quad T_1 \quad} & X & \xrightarrow{\quad T_2 \quad} & Z_2 \\ & {\scriptstyle S_1}\searrow & & \nearrow{\scriptstyle S_2} & \\ & & E & & \end{array}$$

*such that* $S_i$ $(i = 1, 2)$ *has the same compactness property as* $T_i$ *and is positive if* $T_i$ *is positive.*

*Moreover, if another scheme of operators*

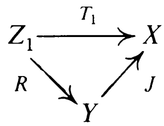
$$Z_1 \overset{P_1}{\to} X \overset{P_2}{\to} Z_2$$

*satisfies* $0 \leqslant P_i \leqslant T_i$ $(i = 1, 2)$, *then there exists a factorization of* $P_2 P_1$ *through the reflexive Banach lattice E*

$$\begin{array}{ccccc} Z_1 & \xrightarrow{\quad P_1 \quad} & X & \xrightarrow{\quad P_2 \quad} & Z_2 \\ & {\scriptstyle Q_1}\searrow & & \nearrow{\scriptstyle Q_2} & \\ & & E & & \end{array}$$

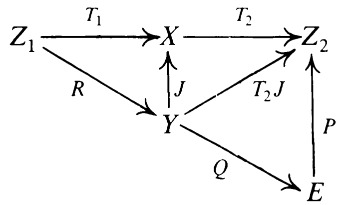
*such that* $0 \leqslant Q_i \leqslant S_i$ *holds for each i.*

PROOF. Assume that $T_1$ is compact or weakly compact, and let $W$ denote the convex solid hull of the relatively weakly compact set $T_1(B_{Z_1})$. If $Y$ denotes the Banach lattice determined by $W$, then (by Theorem 2.1) $Y'$ has order continuous norm, and $T_1$ admits the factorization




where $R(z) = T_1(z)$ for all $z \in Z_1$. Note that $J$ is a positive operator and that $R$ is positive if $T_1$ is positive.

The proof will be based upon the following diagram:




The factorization $T_2 J = PQ$ will be through an appropriate reflexive Banach lattice $E$ that will be explained below. In all cases the desired factors will be $S_1 = QR$ and $S_2 = P$. As we shall see, $Q$ always will be a positive operator, and $P$ will be positive if $T_2$ is positive.

*Case* 1. *$T_1$ and $T_2$ are both weakly compact.*

The factorization of $T_2 J$ through a reflexive Banach lattice $E$ follows from Theorem 2.5.

*Case* 2. *$T_1$ is weakly compact and $T_2$ is compact.* By Theorem 3.1 the compact operator $T_2 J$ factors through a reflexive Banach lattice $E$ with $P$ compact.

*Case* 3. *$T_1$ is compact and $T_2$ is weakly compact.* By Lemma 1.6 the operator $R$ is compact, and by Theorem 2.5 the weakly compact operator $T_2 J$ factors through a reflexive Banach lattice $E$.

*Case* 4. *$T_1$ and $T_2$ are both compact.* As above, $R$ is compact, and by Theorem 3.1 the compact operator $T_2 J$ factors through a reflexive Banach lattice $E$ with $P$ compact. Therefore, $S_1$ and $S_2$ are both compact.

Finally, note that $S_i$ is positive if $T_i$ is positive, and the rest of the proof follows from Theorem 2.6. □

An immediate consequence of the preceding theorem is the following.

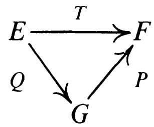
COROLLARY 3.3. *If $T: X \to X$ is a compact operator on a Banach lattice, then $T^2$ factors with compact factors through a reflexive Banach lattice.*

*If $T$ is also positive, then the factors can be taken to be positive compact operators.*

It was shown in [**9**] that if $E'$ and $F$ have order continuous norms, then every positive compact operator from $E$ into $F$ dominated by a compact operator is itself compact. (Recall that an operator $S$ is dominated by another operator $T$ whenever $S \leqslant T$ holds.) Next, we shall present a proof of this result using factorization.

THEOREM 3.4 (DODDS - FREMLIN). *Let $E$ and $F$ be two Banach lattices such that $E'$ and $F$ have order continuous norms. If a positive operator $T: E \to F$ is dominated by a compact operator, then $T$ is a compact operator.*

PROOF. By Theorem 3.1 there exists a reflexive Banach lattice $G$ and a factorization of $T$




such that $P$ is positive dominated by a compact operator. By [**5**, Theorem 2.6, p. 234], $P$ is a Dunford-Pettis operator, and hence, a compact operator. Thus, $T$ is a compact operator. □

The next result generalizes [**3**, Theorem 2.5, p. 296].

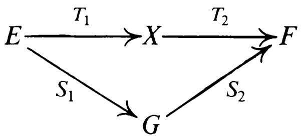
THEOREM 3.5. *Consider the scheme of positive operators*

$$E \xrightarrow{T_1} X \xrightarrow{T_2} F$$

*between Banach lattices. If $F$ has order continuous norm, $T_1$ is dominated by a weakly compact operator, and $T_2$ is dominated by a compact operator, then $T_2T_1$ is a compact operator.*

*Dually, if $E'$ has order continuous norm, $T_1$ is dominated by a compact operator, and $T_2$ is dominated by a weakly compact operator, then $T_2T_1$ is a compact operator.*

PROOF. By Theorem 3.2 there exist a reflexive Banach lattice $G$ and a factorization of $T_2T_1$




with $S_2$ positive dominated by a compact operator. By Theorem 3.4, $S_2$ is compact, and hence, $T_2T_1$ is also compact. □

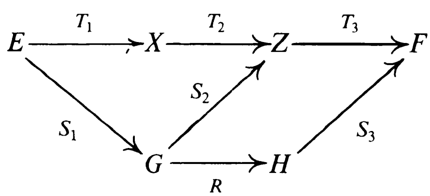
The following theorem generalizes [**3**, Theorem 2.4, p. 296].

THEOREM 3.6. *Consider the scheme of positive operators*

$$E \xrightarrow{T_1} X \xrightarrow{T_2} Z \xrightarrow{T_3} F$$

*between Banach lattices such that $T_1$ is dominated by a weakly compact operator and $T_2$ by a compact operator. If $T_3$ is dominated either by a weakly compact or by a Dunford-Pettis operator, then $T_3T_2T_1$ is a compact operator.*

PROOF. The proof is based upon the following diagram:



According to Theorem 3.2 the scheme $E \overset{T_1}{\rightarrow} X \overset{T_2}{\rightarrow} Z$ factors through a reflexive Banach lattice $G$ with $S_2$ positive dominated by a compact operator.

Assume first that $T_3$ is dominated by a Dunford-Pettis operator. Then by [**5**, Theorem 3.3, p. 236] the operator $T_3S_2$ is Dunford-Pettis, and so, $T_3T_2T_1 = (T_3S_2)S_1$ is a compact operator.

Now assume that $T_3$ is dominated by a weakly compact operator. Then by Theorem 3.2 the scheme $G \overset{S_2}{\rightarrow} Z \overset{T_3}{\rightarrow} F$ factors through a reflexive Banach lattice $H$ with the factor $R$ positive dominated by a compact operator. By Theorem 3.4, $R$ is compact, and so, $T_3T_2T_1 = S_3RS_1$ is also compact. □

In the preceding theorem the case when $T_1$ and $T_3$ were dominated by weakly compact operators was established in [**13**] by a different method.

Finally, it should be mentioned that W. B. Johnson has pointed out to us that his techniques in [**14**] yield also the following result: *If $T$: $Z \rightarrow X$ is a compact operator and $X$ is a Banach lattice with the approximation property, then $T$ factors with compact factors (which can be taken positive if $T$ is positive) through a reflexive Banach space with an unconditional basis.*

We close the paper with the corresponding open problems for compact operators.

*Problem* 3. Does a compact operator between two Banach lattices factor with compact factors through a reflexive Banach lattice?

*Problem* 4. Does a positive compact operator between two Banach lattices factor with compact factors (if possible, with positive compact factors) through a reflexive Banach lattice?

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