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PURE STATES ON SOME GROUP-INVARIANT C^*-ALGEBRAS

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ABSTRACT. Let $\mathfrak{A}$ be a UHF algebra of Glimm type n^∞, i.e., $\mathfrak{A} = \otimes^*_{k\geq 1} N_k$, where $N = N_1 = N_2 = \cdots$ are $n \times n$ matrix algebras. We define an AF-subalgebra $\mathfrak{A}^G$ of $\mathfrak{A}$, consisting of those elements of $\mathfrak{A}$ invariant under a group of automorphisms $\{\alpha_g: g \in G = \mathrm{SU}(n)\}$ of product type. $\mathfrak{A}^G$ is shown to be generated by an embedding of $S(\infty)$, the discrete group of finite permutations on countably many symbols. Let ω be a pure product state on $\mathfrak{A}$, ω^G its restriction to $\mathfrak{A}^G$. Let $e \in N$ be a one-dimensional projection with corresponding projections $e^k \in N_k$. Then if both (i) $\Sigma_{k\geq 1}\omega(e^k) = \infty$, and (ii) $0 < \Sigma_{k\geq 1}\omega(e^k)[1 - \omega(e^k)] < \infty$ hold, ω^G is not pure. ω^G is shown to be pure if there exist orthogonal one-dimensional projections $\{p_i: 1 \leq i \leq n\}$ of N with corresponding projections $p_i^k \in N_k$ such that $\omega(p_i^k) = 0$ or 1, $1 \leq i \leq n$, $k \geq 1$, and $0 < \Sigma_{k\geq 1}\omega(p_i^k) < \infty$ for at most one i.

I. Introduction. Let $\mathfrak{A}$ be a uniformly hyperfinite C^*-algebra of Glimm type n^∞, i.e., $\mathfrak{A}$ is the uniform closure of the ascending union $\bigcup_{k\geq 1}\mathfrak{A}_k$ of matrix algebras $\mathfrak{A}_k$ of type I_{n^k}. We shall call $\mathfrak{A}$ the *lattice algebra*. Motivated by a construction of Saito in [**6**], we define an AF C^*-subalgebra $\mathfrak{A}^G$ of $\mathfrak{A}$, the *invariant algebra*, formed as the closure of the ascending union $\bigcup_{k\geq 1}\mathfrak{A}_k^G$ of subalgebras $\mathfrak{A}_k^G \subset \mathfrak{A}_k$, where $\mathfrak{A}_k^G$ consists of all elements of $\mathfrak{A}_k$ invariant under an action of the group $G = \mathrm{SU}(n)$ as *-automorphisms of $\mathfrak{A}_k$ of product type. In the case $n = 2$, the union $\bigcup_{k\geq 1}\mathfrak{A}_k^G$ is the algebra of all local observables invariant under orthogonal rotations of the quantum lattice in the Heisenberg model. A theorem of Hermann Weyl [7] demonstrates a connection between the algebras $\mathfrak{A}_k^G$ and the group algebras of the permutation groups S_k on k symbols. We discuss this result and the structure of the invariant algebra in §II.

In §III we consider certain pure product states on $\mathfrak{A}$ and describe necessary and sufficient conditions for the restrictions of these states (to the invariant algebra) to be pure. In §IV we extend a method of Saito to provide conditions for a restriction of a pure product state *not* to be a factor state, hence not a pure state. In §V we provide a criterion for the unitary equivalence of certain pure states on $\mathfrak{A}^G$.

In [**10**, **11**] a similar analysis was carried out for representations of the gauge invariant subalgebra of the CAR (canonical anticommutation relations) algebra. Our results and techniques are motivated in large part by the work undertaken in these papers, as well as in [**6**].

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II. Notation and preliminaries. In this section we construct the lattice and invariant algebras, and introduce some of the notation to be used throughout this article. To construct the lattice algebra, denoted by $\mathfrak{A}$, begin by fixing an integer $n \geqslant 2$: choose a Hilbert space H of dimension n over $\mathbf{C}$, the complex numbers, and let N be the C^*-algebra of linear operators on H. Then N may be viewed as an $n \times n$ matrix algebra, with matrix units e_{kl}, $1 \leqslant k, l \leqslant n$, which satisfy

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \quad \text{and} \quad e_{ij}f_k = \delta_{jk}f_i,$$

for a fixed orthonormal basis $\{f_1,\ldots,f_n\}$. Let I be the identity operator of N; then of course, $\sum_{i=1}^n e_{ii} = I$.

For any positive integer j, denote by H_j a copy of H, and by N_j a copy of N, with matrix units e^j_{kl} corresponding to e_{kl} in N. Given a finite subset Λ of $\mathbf{Z}^+$ (the positive integers) we define

(i) $H_\Lambda = \otimes_{j\in\Lambda} H_j$ and

(ii) $\mathfrak{A}_\Lambda = \otimes_{j\in\Lambda} N_j$.

Then if m is the order of Λ, $\mathfrak{A}_\Lambda$ is an $n^m \times n^m$ matrix algebra of operators on H_Λ, with matrix units of the form $\otimes_{j\in\Lambda} e^j_{k_jl_j}$, where $1 \leqslant k_j$, $l_j \leqslant n$ for each $j \in \Lambda$. Suppose now that $\Lambda \subset \Lambda_0$ are two finite subsets of $\mathbf{Z}^+$. Then we may write $\mathfrak{A}_{\Lambda_0}$ as $\mathfrak{A}_{\Lambda_0} = (\otimes_{j\in\Lambda} N_j) \otimes (\otimes_{j\in\Lambda_0\setminus\Lambda} N_j) = \mathfrak{A}_\Lambda \otimes (\otimes_{j\in\Lambda_0\setminus\Lambda} N_j)$, hence we may regard $\mathfrak{A}_\Lambda$ as embedded in $\mathfrak{A}_{\Lambda_0}$ under the *-isomorphism $x \to x \otimes (\otimes_{j\in\Lambda_0\setminus\Lambda} I_j)$, $x \in \mathfrak{A}_\Lambda$, where I_j denotes the identity operator of the matrix algebra N_j. In the sense of the embedding map given above, we will view $\mathfrak{A}_\Lambda$ as lying inside $\mathfrak{A}_{\Lambda_0}$. In particular, let us denote by Λ_m the set of the first m positive integers and by $\mathfrak{A}_m$ the algebra $\mathfrak{A}_{\Lambda_m}$. Then $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots$, and the norm completion $\mathfrak{A}$ of the union $\mathfrak{A}_0 = \bigcup_{m\geqslant 1}\mathfrak{A}_m$ is a UHF-algebra, called the *lattice algebra*. Note that if $\Lambda \subset \mathbf{Z}^+$ is finite, then $\mathfrak{A}_\Lambda \subset \mathfrak{A}_m$ for sufficiently large m. In particular, we have $N_j \subset \mathfrak{A}$, hence we shall interpret matrix units e^j_{kl} simultaneously as being elements of both N_j and of $\mathfrak{A}$. For the general theory of UHF-algebras, we refer the reader to [**2**].

Given a finite subset $\Lambda \subset \mathbf{Z}^+$, denote by S_Λ the group of permutations on the symbols in Λ. In an obvious fashion we have $S_\Lambda \subset S_{\Lambda_0}$ for $\Lambda \subset \Lambda_0$. Write $S(\infty) = \bigcup_{\Lambda\subset\mathbf{Z}^+} S_\Lambda$, where the union runs over finite subsets Λ of $\mathbf{Z}^+$. Then $S(\infty)$ is the group of permutations of $\mathbf{Z}^+$ leaving all but finitely many elements fixed. For notational purposes, we shall write S_m for the permutation group S_{Λ_m}. Recall that S_m is a group of order $m!$, and that each element $g \in S_m$ may be written uniquely, up to the order in which they appear, as a product of disjoint cycles. Furthermore, any element of S_m may be written as a product of transpositions (ij), $i \neq j$, $1 \leqslant i, j \leqslant m$.

We define a unitary representation π_m of S_m into $\mathfrak{A}_m$ as follows: for $g \in S_m$, let $\pi_m(g) \in \mathfrak{A}_m$ be the operator defined on vectors $\phi_1 \otimes \cdots \otimes \phi_m$ in H_{Λ_m} by

$$\pi_m(g)[\phi_1 \otimes \cdots \otimes \phi_m] = \phi_{g^{-1}(1)} \otimes \cdots \otimes \phi_{g^{-1}(m)},$$

and extended to all of H_{Λ_m} by linearity. It is not difficult to verify that π_m is well defined, and that $\pi_m(gh) = \pi_m(g) \cdot \pi_m(h)$ for any group elements $g, h \in S_m$. That $\pi_m(g)$ is a unitary operator is also clear: this may be verified by noting that the image $\pi_m(ij)$ of a transposition (ij) has the explicit form

$$\pi_m(ij) = \sum_{k,l=1}^{n} e^i_{kl} \otimes e^j_{lk}.$$

This operator is selfadjoint and has square deal to the identity operator, hence is unitary. In general, since any $g \in S_m$ is a product of transpositions $g = t_1 \cdots t_k$, we have that $\pi_m(g) = \prod_{i=1}^k \pi_m(t_i)$ is a product of unitaries, hence is itself unitary, and furthermore

$$\pi_j(g)^* = \pi_m(t_k)^* \cdots \pi_m(t_1)^* = \pi_m(t_k) \cdots \pi_m(t_1) = \pi_m(t_k \cdots t_1) = \pi_m(g^{-1}).$$

The group representation π_m of S_m may be extended in an obvious fashion to a representation (also denoted π_m) of the group algebra $A(S_m)$ (over the complex numbers) of S_m into $\mathfrak{A}_m$. This representation need not be faithful (in fact, for $m \geqslant n$ one verifies that π_m maps the operator $\Sigma_{g \in S_m}(\operatorname{sgn} g)g$ to the 0 operator in $\mathfrak{A}_m$) but the image is closed under multiplication and the adjoint operation, hence forms a C^*-subalgebra of $\mathfrak{A}_m$, which we shall denote by $\mathfrak{A}^G_m$. In an analogous fashion, one may construct a *-representation π_Λ of $A(S_\Lambda)$ into $\mathfrak{A}_\Lambda$: we shall denote the image of this map by $\mathfrak{A}^G_\Lambda$. The inclusions $S_1 \subset S_2 \subset \cdots$ imply $\mathfrak{A}^G_1 \subset \mathfrak{A}^G_2 \subset \cdots$; hence the norm closure $\mathfrak{A}^G$ of the union $\mathfrak{A}^G_0 = \bigcup_{m \geqslant 1} \mathfrak{A}^G_m$ is an AF-algebra (see [1]). We shall call the C^*-subalgebra $\mathfrak{A}^G$ of $\mathfrak{A}$ the *invariant algebra*.

Remark 1. Since $\mathfrak{A}_m \subset \mathfrak{A}$, for $m = 1, 2, \ldots$, we may regard the representations π_m of S_m as unitary representations of the groups S_m into $\mathfrak{A}$. With this convention the mappings $\{\pi_m\}$ are seen to be consistent, i.e., given $g \in S_m \cap S_r$ for some m and r, we have $\pi_m(g) = \pi_r(g)$. This permits us to make the following definition.

Definition 2.1. Let π: $S(\infty) \to \mathfrak{A}$ be the unitary representation of $S(\infty)$ given by $\pi(g) = \pi_m(g)$, for any m such that $g \in S_m$. Furthermore, π may be extended by linearity to the set of finite linear combinations of elements in $S(\infty)$. The image of this set is $\mathfrak{A}^G_0$.

Whenever there is no danger of confusion, we will omit the notation π for the representation mapping $S(\infty)$ into $\mathfrak{A}$. Thus the symbol g will play the two roles of denoting an element of $S(\infty)$ and of $\mathfrak{A}$. Similarly, finite sums of the form $\Sigma_{g \in S(\infty)} \alpha_g g$, $\alpha_g \in \mathbf{C}$, will often be interpreted as lying in $\mathfrak{A}$.

Remark 2. Let $c \in S(\infty)$ be a cycle. Then c is of the form $(i_1 i_2 \cdots i_r) = (i_1 i_2)(i_2 i_3) \cdots (i_{r-1} i_r)$ for distinct integers $i_1, \ldots, i_r \in \mathbf{Z}^+$. By induction, $\pi(c)$ has the form $\Sigma^n_{k_1, \ldots, k_r} e^{i_1}_{k_1 k_2} \otimes \cdots \otimes e^{i_r}_{k_r k_1}$, hence $\pi(c) \in \mathfrak{A}_{\Lambda'}$, where $\Lambda' = \{i_1, \ldots, i_r\}$. These observations lead to the following

Definition 2.2. For a cycle $c = (i_1 \cdots i_r) \in S(\infty)$, let $s(c)$ (the symbols of c) be the set $\{i_1, \ldots, i_r\}$. More generally, for $g \in S(\infty)$, if we write g in its unique form as a product of disjoint cycles $g = c_1 \cdots c_k$, define $s(g) = \bigcup_{i=1}^k s(c_i)$. (Note that $\pi(g)$ lies in $\mathfrak{A}_{s(g)}$.)

Although we make no appeal to the following classical result of H. Weyl in this paper, we include it for completeness and to explain the terminology "invariant algebra."

THEOREM 2.1 [7]. *Let $\mathcal{C}_m$ be the C^*-subalgebra of $\mathfrak{A}_m$ generated by all operators of the form $u \otimes u \otimes \cdots \otimes u$ (m terms), where $u \in \mathrm{SU}(n) = G$. Given a set $X \subset \mathfrak{A}_m$, let X' denote the commutant (relative to $\mathfrak{A}_m$) of X. Then $[\mathcal{C}_m]' = \mathfrak{A}_m^G$, $[\mathfrak{A}_m^G]' = \mathcal{C}_m$.*

III. Pure states on the invariant algebra. We mention a few properties of the lattice algebra $\mathfrak{A}$ before stating the main result of this section. First we make the following definition.

DEFINITION 3.1. Given a subset $X \subset \mathfrak{A}$, let X^C, the commutant of X in $\mathfrak{A}$, denote the set of elements $y \in \mathfrak{A}$ satisfying $xy = yx$ for all $x \in X$.

Note that from the preceding section, $\mathfrak{A}$ is generated as a C^*-algebra by the $n \times n$ matrix algebras N_j, $j = 1, 2, \ldots$; furthermore, for any $k \neq l$ we have $N_k \subset N_l^c$, i.e., the algebras N_j are mutually commutative. We then say that the set $\{N_j : j = 1, 2, \ldots\}$ is a factorization of $\mathfrak{A}$. Furthermore, we shall refer to a state ω as a product state if for any $k \neq l$, $x \in N_k$, $y \in N_l$ implies $\omega(xy) = \omega(x)\omega(y)$, and we signify that ω is a product state by writing $\omega = \omega|_{N_1} \otimes \omega|_{N_2} \otimes \cdots$. Conversely, given states $\omega_1, \omega_2, \ldots$ on the algebras $N_1, N_2, \ldots$, respectively, one may form a product state $\omega = \omega|_{N_1} \otimes \omega|_{N_2} \otimes \cdots$, where $\omega|_{N_j} = \omega_j$. (For a general treatment of product states, see [3].) That product states are factor states (i.e., the GNS representation π_ω associated with ω is a factor representation) follows immediately from Powers' primary decomposition theorem, which we state below.

THEOREM 3.1 [4, THEOREM 2.5]. *Suppose $\mathfrak{A}$ is a UHF algebra and $\{\mathfrak{A}_i : i = 1, 2, \ldots\}$ is an increasing sequence of $(n_i \times n_i)$ matrtix algebras which generate $\mathfrak{A}$. Suppose ω is a state of $\mathfrak{A}$. Then the following conditions are equivalent.*

(i) *The state ω induces a factor representation of $\mathfrak{A}$.*

(ii) *For every $x \in \mathfrak{A}$ there is an integer $r > 0$ depending only on x, such that $|\omega(xy) - \omega(x)\omega(y)| \leq \|y\|$ for all $y \in \mathfrak{A}_r^c$.*

(iii) *For every $x \in \mathfrak{A}$ there is an $(m \times m)$ matrix algebra $\mathfrak{M}$ such that $|\omega(xy) - \omega(x)\omega(y)| \leq \|y\|$ for all $y \in \mathfrak{M}^c$.*

In [3, Corollary 2.2] Guichardet proved that a product state $\omega = \omega|_{N_1} \otimes \omega|_{N_2} \otimes \cdots$ is pure on $\mathfrak{A}$ if and only if each state $\omega|_{N_j}$ is pure. Recall that a state ρ on a matrix algebra N is pure if and only if $\rho(x) = \mathrm{Tr}(Ex)$, $x \in N$, where E is a rank-one projection on N, and Tr is the trace mapping on the $n \times n$ matrix algebra N with $\mathrm{Tr}(I) = n$. Furthermore, if $\phi \in H$ is a unit vector satisfying $E\phi = \phi$, then $\rho(x) = (x\phi, \phi)$: denote this state by ω_ϕ. Hence if ω is a pure product state on $\mathfrak{A}$ there exist unit vectors $\phi_j \in H$ (under the identification $N_j = N = B(H)$) such that $\omega|_{N_j} = \omega_{\phi_j}$: by an abuse of notation we shall write $\omega = \omega_{\phi_1} \otimes \omega_{\phi_2} \otimes \cdots$.

The following theorem, the main result of this section, provides a sufficient condition for the restriction $\omega|_{\mathfrak{A}^G}$ of a pure product state to be pure.

THEOREM 3.2. *Suppose $\omega = \omega_{\phi_1} \otimes \omega_{\phi_2} \otimes \cdots$ is a product state on $\mathfrak{A}$, $\{f_1, \ldots, f_n\}$ an orthonormal basis of H, and r a mapping from $\mathbf{Z}^+$ to $\{1, 2, \ldots, n\}$ such that*

$|(f_{r(k)}, \phi_k)| = 1$, $k = 1, 2, \ldots$. *For* $j = 1, 2, \ldots, n$, *let* c_j *denote the cardinality of the set* $\gamma_j = \{k \in \mathbf{Z}^+ : r(k) = j\}$. *Then* $\omega|_{\mathfrak{A}^G} = \omega^G$ *is pure if and only if at most one of the* c_j *satisfies* $0 < c_j < \infty$.

We first prove the necessity of the condition. The following lemmas will be of use.

LEMMA 3.3. *Let* Λ *be a finite subset of* $\mathbf{Z}^+$, $\omega_\Lambda = \omega|_{\mathfrak{A}_\Lambda}$. *Then for* x *in* $\mathfrak{A}_\Lambda$, $\omega_\Lambda(x) = (xF_\Lambda, F_\Lambda)$, *where* $F_\Lambda \in H_\Lambda$ *is given by* $F_\Lambda = \otimes_{k \in \Lambda} \phi_k$.

PROOF. Any $x \in \mathfrak{A}_\Lambda$ may be written as a linear combination of operators of the form $e^{k_1}_{i_1 j_1} \otimes \cdots \otimes e^{k_m}_{i_m j_m}$, where $\{k_1, \ldots, k_m\} = \Lambda$, hence using linearity and the equality

$$\left(e^{k_1}_{i_1 j_1} \otimes \cdots \otimes e^{k_m}_{i_m j_m} F_\Lambda, F_\Lambda\right) = \prod_{s=1}^{m} \left(e^{k_s}_{i_s j_s} \phi_{k_s}, \phi_{k_s}\right)$$

$$= \omega\left(e^{k_1}_{i_1 j_1} \cdots e^{k_m}_{i_m j_m}\right) = \omega_\Lambda\left(e^{k_1}_{i_1 j_1} \cdots e^{k_m}_{i_m j_m}\right),$$

we have the result.

COROLLARY. $\omega = \omega' = \omega_{f_{r(1)}} \otimes \omega_{f_{r(2)}} \otimes \cdots$.

PROOF. Combining the proof above with the equation $(e^k_{ij}\phi_k, \phi_k) = (e^k_{ij} f_{r(k)}, f_{r(k)})$, we see that ω, ω' agree on the norm dense subset $\mathfrak{A}_0$ of $\mathfrak{A}$, hence $\omega = \omega'$ by continuity of ω and ω'.

Using this result, henceforth we shall assume, without loss of generality, that $\phi_k = f_{r(k)}$, for $k = 1, 2, \ldots$.

LEMMA 3.4. *Let* $g \in S(\infty)$ *be written as a product of disjoint cycles*, $g = d_1 d_2 \cdots d_k$. *Then* $\omega(g) = \prod_{j=1}^{k} \omega(d_j)$.

PROOF. We have $d_j \in \mathfrak{A}_{s(d_j)}$. Since the cycles d_j are disjoint it follows that the sets $s(d_j)$ are mutually disjoint, and thus from the product state property of ω,

$$\omega(d_1 \cdots d_k) = \omega(d_1) \cdots \omega(d_k).$$

LEMMA 3.5. *Let* ω *be as above and* g *a cycle in* $S(\infty)$, *then*

$$\omega(g) = \begin{cases} 1, & \text{if } s(g) \subseteq \gamma_j, \text{ some } j \in \{1, 2, \ldots, n\}, \\ 0, & \text{otherwise}. \end{cases}$$

PROOF. Suppose $g = (k_1 \cdots k_m)$. Then writing $g = \sum^n_{j_1, \ldots, j_m = 1} e^{k_1}_{j_1 j_2} \cdots e^{k_m}_{j_m j_1}$, we have (with the convention $j_{m+1} = j_1$)

$$\omega(g) = \sum_{j_1, \ldots, j_m = 1}^{n} \left[\prod_{i=1}^{m} \omega\left(e^{k_i}_{j_i j_{i+1}}\right)\right]$$

$$= \sum_{j_1, \ldots, j_m = 1}^{n} \left[\prod_{i=1}^{m} \left(e^{k_i}_{j_i j_{i+1}} f_{r(k_i)}, f_{r(k_i)}\right)\right]$$

$$= \sum_{j_1, \ldots, j_m = 1}^{n} \left[\prod_{i=1}^{m} \delta_{j_i j_{i+1}} \cdot \delta_{j_i r(k_i)}\right] = \sum_{j=1}^{n} \left[\prod_{i=1}^{m} \delta_{j r(k_i)}\right].$$

Thus we have $\omega(g) = 0$ unless $r(k_i) = r(k_j)$ for all i, j, and if the $r(k_i)$ coincide we have $\omega(g) = 1$, as asserted.

COROLLARY. *Let $g = d_1 \cdots d_k$, where $d_1, \ldots, d_k$ are disjoint cycles. Then*

$$\omega(g) = \begin{cases} 1, & \text{if } s(d_i) \subseteq \gamma_{j_i}, \text{ some } j_i,\ i = 1,2,\ldots,k, \\ 0, & \text{otherwise}. \end{cases}$$

PROOF. Combine the results of the two preceding lemmas.

REMARK. Viewing $g \in S(\infty)$ as a permutation on the symbols $\{1,2,\ldots\}$, our Corollary implies that

$$\omega(g) = \begin{cases} 1, & \text{if } g\colon \gamma_i \to \gamma_i,\ i = 1,2,\ldots,n, \\ 0, & \text{otherwise}, \end{cases}$$

where $g\colon \gamma_i \to \gamma_i$ means that g permutes the members of γ_i into themselves.

PROPOSITION 3.6. *Preserve the notation of Theorem* 3.2. *Suppose that* $0 < c_v < \infty$, $0 < c_u < \infty$, *for some* $u, v \in \{1,2,\ldots,n\}$, $u \neq v$. *Then* ω^G *is not pure.*

PROOF. For convenience of notation, let us assume $u = 1$, $v = 2$, and let Λ be any finite subset of $\mathbf{Z}^+$ containing both γ_1 and γ_2. Let Γ denote the set of all subsets γ of $\gamma_1 \cup \gamma_2$ of order $c_1 = \text{card}(\gamma_1)$ (in particular, $\gamma_1 \in \Gamma$), and let J be the number of subsets lying in Γ. For each γ we define a tensor product vector $F_{\Lambda,\gamma}$ in H_Λ as follows:

$$F_{\Lambda,\gamma} = \bigotimes_{j \in \Lambda} \chi_j, \quad \text{where } \psi_j = \begin{cases} \phi_j, & j \notin \gamma_1 \cup \gamma_2, \\ f_1, & j \in \gamma, \\ f_2, & j \in (\gamma_1 \cup \gamma_2)\backslash\gamma. \end{cases}$$

We observe that $F_{\Lambda,\gamma_1} = F_\Lambda$, as defined in Lemma 3.3, and furthermore, observe that the vectors $F_{\Lambda,\gamma}$ are orthonormal in H_Λ. Fix $\alpha > J - 2$, and define unit vectors F_Λ^-, F_Λ^+ as follows:

$$F_\Lambda^- = \left(\sqrt{\alpha^2 + J - 1}\right)^{-1}\left[\alpha F_{\Lambda,\gamma_1} - \sum_{\gamma_1 \neq \gamma \in \Gamma} F_{\Lambda,\gamma}\right]$$

and

$$F_\Lambda^+ = \left(\sqrt{\alpha^2 + J - 1}\right)^{-1}\left[\alpha F_{\Lambda,\gamma_1} + \sum_{\gamma_1 \neq \gamma \in \Gamma} F_{\Lambda,\gamma}\right].$$

We define states ω_Λ^+ and ω_Λ^- on $\mathfrak{A}_\Lambda$ by $\omega_\Lambda^+(x) = (xF_\Lambda^+, F_\Lambda^+)$ and $\omega_\Lambda^-(x) = (xF_\Lambda^-, F_\Lambda^-)$, $x \in \mathfrak{A}_\Lambda$. We wish to evaluate these states on elements g of S_Λ. Recalling that for product vectors $F = \otimes_{j \in \Lambda}\xi_j$ in H_Λ, $gF = \otimes_{j \in \Lambda}\xi_{g^{-1}(j)}$, we divide the argument into cases, according to the action of g on the sets γ_j.

Case (i). $g\colon \gamma_j \to \gamma_j$ for $j = 1,2,\ldots,n$.

Then $g(F_\Lambda^{\mp}) = F_\Lambda^{\mp}$, and by an application of the Corollary to Lemma 3.3, we have $1 = \omega_\Lambda^+(g) = \omega_\Lambda^-(g) = \omega(g)$.

Case (ii). $g\colon \gamma_j \nrightarrow \gamma_j$, some $j \geqslant 3$.

We see that $gF_{\Lambda,\gamma}$ is orthogonal to $F_{\Lambda,\gamma'}$ for all γ, γ' in Γ, hence $0 = (gF_\Lambda, F_\Lambda) = \omega_\Lambda(g) = \omega(g)$, and $0 = (gF_\Lambda^{\mp}, F_\Lambda^{\mp}) = \omega_\Lambda^{\mp}(g)$.

Case (iii). $g\colon \gamma_1 \cup \gamma_2 \to \gamma_1 \cup \gamma_2$, $g\colon \gamma_j \to \gamma_j$ for $j \geqslant 3$, but $g\colon \gamma_1 \nrightarrow \gamma_1$.

Then $\omega(g) = \omega_\Lambda(g) = (gF_\Lambda, F_\Lambda) = 0$, but noting that $gF_{\Lambda,\gamma_1} = F_{\Lambda,\gamma'}$ for some $\gamma' \in \Gamma$, $\gamma \neq \gamma'$, we have

$$gF_\Lambda^{\mp} = \left(\sqrt{\alpha^2 + J - 1}\right)^{-1}\left[\alpha F_{\Lambda,\gamma'} \mp \sum_{\gamma' \neq \gamma \in \Gamma} F_{\Lambda,\gamma}\right].$$

Hence $\omega_\Lambda^+(g) = [2\alpha + (J-2)]/[\alpha^2 + (J-1)] > 0$, and $\omega_\Lambda^-(g)$ has the value $[-2\alpha + (J-2)]/[\alpha^2 + (J-1)] < 0$.

Letting $\lambda = \frac{1}{2} - [(J-2)/4\alpha]$ (note $0 < \lambda < 1$), a straightforward calculation now gives $\lambda\omega_\Lambda^+(g) + (1-\lambda)\omega_\Lambda^-(g) = \omega(g)$ for all $g \in S_\Lambda$, and since $\mathfrak{A}_\Lambda^G$ is generated by linear combinations of the group elements g, we have written $\omega|_{\mathfrak{A}_\Lambda^G}$ as a convex sum of distinct states $\omega_\Lambda^+|_{\mathfrak{A}_\Lambda^G}$ and $\omega_\Lambda^-|_{\mathfrak{A}_\Lambda^G}$.

By our construction, the states $\omega_\Lambda^{\mp}$ are seen to be consistent: i.e., given two finite subsets of Λ, Λ' of $\mathbf{Z}^+$, both containing $\gamma_1 \cup \gamma_2$, if $g \in S_\Lambda \cap S_{\Lambda'}$, then it is straightforward to show that $\omega_\Lambda^{\mp}(g) = \omega_{\Lambda'}^{\mp}(g)$. Hence we are able to define positive linear functionals (of norm 1) ω^+, ω^- on $\mathfrak{A}_0^G$ given by $\omega^{\mp}(g) = \omega_\Lambda^{\mp}(g)$, for any Λ satisfying $g \in S_\Lambda$. Extending $\omega^{\mp}$ by continuity to states (also denoted $\omega^{\mp}$) on $\mathfrak{A}^G$, we have $\lambda\omega^+ + (1-\lambda)\omega^- = \omega^G$, and $\omega^+ \neq \omega^-$, hence ω^G cannot be pure. This completes the proof of the proposition.

PROPOSITION 3.7. *Preserve the notation of Theorem* 3.2. *Then* ω^G *is pure if* $0 < c_i < \infty$ *for at most one* $i \in \{1, 2, \ldots, n\}$.

PROOF. Suppose ω_1, ω_2 are states on $\mathfrak{A}^G$ such that $\lambda\omega_1 + (1-\lambda)\omega_2 = \omega^G$, for some $\lambda \in (0,1)$. Suppose further that $g \in S(\infty)$ satisfies $g\colon \gamma_i \to \gamma_i$, for $1 \leqslant i \leqslant n$. Then $\omega(g) = 1$, and since g is unitary (see §II), hence of norm 1, we may conclude that $\omega_1(g) = \omega_2(g) = 1$ as well. In fact, if $(\pi_{\omega_t}, H_{\omega_t}, \Omega_{\omega_t})$, $t = 1,2$, are the GNS constructions for ω_t, $t = 1,2$, respectively, then the relations $(\pi_{\omega_t}(g)\Omega_{\omega_t}, \Omega_{\omega_t}) = \omega_t(g) = 1$, $\|\pi_{\omega_t}(g)\| = 1$, together imply that $\pi_{\omega_t}(g)\Omega_{\omega_t} = \Omega_{\omega_t}$, for $t = 1,2$.

Now suppose that $g\colon \gamma_i \nrightarrow \gamma_i$, for some i. Then there exist positive integers p, q such that $p \in \gamma_i$, $q \in \gamma_j$, for some $j \neq i$, and $g\colon p \to q$. By assumption, either $c_i = \infty$ or $c_j = \infty$: assume, for the moment, that $c_j = \infty$. Then there exists a sequence $\{q_k\}$ of distinct positive integers, each lying in the set γ_j and satisfying the additional property that $q_k \notin s(g)$, $k = 1, 2, \ldots$. Define new operators g_k for $k = 1, 2, \ldots$ by $g_k = (qq_k)g(qq_k)$. We have $(qq_k)\colon \gamma_s \to \gamma_s$, for $s = 1, 2, \ldots, n$, hence $\pi_{\omega_1}(qq_k)\Omega_{\omega_1} = \Omega_{\omega_1}$, by the results of the preceding paragraph. In particular, since the transposition (qq_k) is selfadjoint, we have

$$\begin{aligned} \omega_1(g_k) &= \left(\left[\pi_{\omega_1}(qq_k)\pi_{\omega_1}(g)\pi_{\omega_1}(qq_k)\right]\Omega_{\omega_1}, \Omega_{\omega_1}\right) \\ &= \left(\left[\pi_{\omega_1}(g)\pi_{\omega_1}(qq_k)\right]\Omega_{\omega_1}, \pi_{\omega_1}(qq_k)\Omega_{\omega_1}\right) \\ &= \left(\pi_{\omega_1}(g)\Omega_{\omega_1}, \Omega_{\omega_1}\right) = \omega_1(g). \end{aligned}$$

Furthermore, for $k \neq k'$, we have $g_{k'}^* g_k = (g_{k'})^{-1} g_k$, and it is straigthforward to show from the definitions of the g_k that $(g_{k'})^{-1} g_k \colon p \to q_k$, where $p \in \gamma_i$, $q_k \in \gamma_j$, hence $g_{k'}^* g_k \colon \gamma_i \nrightarrow \gamma_i$, which implies that $\omega(g_{k'}^* g_k) = 0$. Define $x_m = \sum_{k=1}^m g_k$: then by applying the Schwarz Inequality we have

$$\lambda m^2 |\omega(g)|^2 = \lambda\big(|\omega_1(x_m)|\big)^2 \leqslant \lambda\omega_1(x_m^* x_m) \leqslant \omega(x_m^* x_m)$$
$$= \sum_{k' \neq k = 1}^{m} \omega(g_{k'}^* g_k) + \sum_{k=1}^{m} \omega(g_k^* g_k)$$

or

$$\lambda m^2 \big(|\omega_1(g)|\big)^2 \leqslant \sum_{k=1}^{m} \omega(g_k^* g_k) = \sum_{k=1}^{m} 1 = m.$$

Since $\lambda > 0$ by assumption, and since the above inequality holds for all m, we conclude that $\omega_1(g) = 0 = \omega(g)$. (A similar argument holds for the case when $c_i = \infty$.) Hence, if $\lambda\omega_1 \leqslant \omega^G$ for $\lambda > 0$ we have shown $\omega_1 = \omega^G$, and therefore ω^G must be pure, completing the proof of the proposition. This result, along with our previous proposition, completes the proof of Theorem 3.2.

IV. A factor condition. In this section we again consider pure product states on the lattice algebra: the following theorem demonstrates a sufficient condition for their restrictions to $\mathfrak{A}^G$ not to be a factor state.

THEOREM 4.1. *Suppose e is a rank-one projection of $N = B(H)$, H an n-dimensional Hilbert space, and $e = e_j$ are the corresponding projections of the algebras N_j. Suppose $\omega = \omega_{\phi_1} \otimes \omega_{\phi_2} \otimes \cdots$ is a pure product state on $\mathfrak{A}$ satisfying*

(i) $\sum_{j=1}^{\infty} \omega(e_j) = \infty$ *and*

(ii) $0 < \sum_{j=1}^{\infty} \omega(e_j)[1 - \omega(e_j)] < \infty$.

Then $\omega^G = \omega|_{\mathfrak{A}^G}$ is not a factor state.

This theorem was proved in the $n = 2$ case (i.e., where N is a 2×2 matrix algebra) by Saito in [**6**]. Here we shall extend his method of proof for the general case $n \geqslant 2$. Before proceeding with the proof we replace conditions (i) and (ii) of the theorem with an equivalent set of conditions which are somewhat easier to deal with.

PROPOSITION 4.2. *Retain the notation of the theorem. Then the following sets of conditions are equivalent.*

I. (i) $\sum_{j=1}^{\infty} \omega(e_j) = \infty$,

(ii) $0 < \sum_{j=1}^{\infty} \omega(e_j)[1 - \omega(e_j)] < \infty$.

II. *There exists a unit vector $f \in H$, and a sequence of unit vectors f_k: $k = 1, 2, \ldots$ such that*

(a) *$f_k = f$ for infinitely many k,*

(b) *either $(f_k, f) = 1$ or $(f_k, f) = 0$,*

(c) $\sum_{k=1}^{\infty} [1 - |(f_k, \phi_k)|^2] < \infty$,

(d) *for at least one k, $0 < |(f, \phi_k)| < 1$.*

PROOF. II $\Rightarrow$ I. Let $e \in N$ be the rank-one projection satisfying $ef = f$, and let $e_i \in N_i$ be the corresponding projection of N_i. Let $\Lambda = \{k: f_k = f\}$. Then Λ is an infinite set, by (a). Note that for $k \in \Lambda$, $\omega(e_k) = |(f, \phi_k)|^2$, hence

$$\infty > \sum_{k\in\Lambda} \left[1 - |(f_k, \phi_k)|^2\right] = \sum_{k\in\Lambda} \left[1 - |(f, \phi_k)|^2\right] = \sum_{k\in\Lambda} [1 - \omega(e_k)],$$

hence $\sum_{k=1}^{\infty} \omega(e_k) = \infty$, which gives (i).

By (d), $0 < \omega(e_k) < 1$ for at least one k, giving the first inequality of (ii). For $k \in \Lambda$, write $\phi_k = \alpha_k f_k + \beta_k \phi_k'$, where ϕ_k' is a unit vector orthogonal to f_k. Then $|\alpha_k|^2 + |\beta_k|^2 = 1$. Summing over $k \in \Lambda$, we have

$$\sum_{k\in\Lambda} \omega(e_k)[1 - \omega(e_k)] \leqslant \sum_{k\in\Lambda} [1 - \omega(e_k)] = \sum_{k\in\Lambda} \left[1 - |(f, \phi_k)|^2\right]$$
$$= \sum_{k\in\Lambda} \left[1 - |(f_k, \phi_k)|^2\right] < \infty,$$

by condition (c). For $k \in \mathbf{Z}^+ \setminus \Lambda$, note that

$$\omega(e_k) = (e_k\phi_k, \phi_k) = (e_k\beta_k\phi_k', \beta_k\phi_k') \leqslant |\beta_k|^2 = 1 - |(f_k, \phi_k)|^2,$$

hence

$$\sum_{k\in\mathbf{Z}^+\setminus\Lambda} \omega(e_k)[1 - \omega(e_k)] \leqslant \sum_{k\in\mathbf{Z}^+\setminus\Lambda} \omega(e_k) \leqslant \sum_{k\in\mathbf{Z}^+\setminus\Lambda} \left[1 - |(f_k, \phi_k)|^2\right] < \infty.$$

Combining these last two results yields the second inequality of condition (ii).

I $\Rightarrow$ II. Write $\phi_k = c_k f + d_k \phi_k''$, where f is a unit vector such that $ef = f$ as above, and ϕ_k'' is a unit vector orthogonal to f; then $|c_k|^2 + |d_k|^2 = 1$. Define $\Gamma = \{k \in \mathbf{Z}^+ : |c_k| > \frac{1}{2}\}$. Consider the sequence $\{f_k\}$ of unit vectors with $f_k = f$, for $k \in \Gamma$, and $f_k = \phi_k''$ otherwise. For $k \in \mathbf{Z}^+ \setminus \Gamma$, $\omega(e_k) = (e_k\phi_k, \phi_k) = (e_k c_k f, c_k f) \leqslant \frac{1}{4}$; hence

$$\infty > \sum_{k\in\mathbf{Z}^+\setminus\Gamma} \omega(e_k)[1 - \omega(e_k)] \geqslant \frac{3}{4} \sum_{k\in\mathbf{Z}^+\setminus\Gamma} \omega(e_k).$$

Combining this inequality with condition (i) shows that Γ is infinite, giving (a).

Using the second inequality of (ii) again, we have

$$\infty > \sum_{k\in\Gamma} \omega(e_k)[1 - \omega(e_k)] + \sum_{k\notin\Gamma} \omega(e_k)[1 - \omega(e_k)]$$
$$\geqslant \frac{1}{4} \cdot \sum_{k\in\Gamma} [1 - \omega(e_k)] + \frac{3}{4} \cdot \sum_{k\notin\Gamma} \omega(e_k)$$
$$= \frac{1}{4} \cdot \sum_{k\in\Gamma} \left[1 - |(f, \phi_k)|^2\right] + \frac{3}{4} \cdot \sum_{k\notin\Gamma} |c_k|^2$$
$$\geqslant \frac{1}{4} \cdot \sum_{k\in\Gamma} \left[1 - |(f, \phi_k)|^2\right] + \frac{3}{4} \cdot \sum_{k\notin\Gamma} \left[1 - |(f_k, \phi_k)|^2\right],$$

giving (c). Finally, (d) follows immediately from the left-hand inequality of condition (ii).

REMARK. We choose matrix units e_{ij}^k for the $n \times n$ algebras N_k so that, under the identification of the Hilbert spaces H_k with H, $e_{11}^k = e_k$ is the one-dimensional projection on the vector f.

Using the set II of conditions equivalent to those of the theorem, the proof of the theorem will be carried out by demonstrating the existence of a nontrivial element $V_0(t)$ in the center of $\pi_{\omega^G}(\mathfrak{A})''$, where π_{ω^G} is the GNS representation associated with ω^G. The construction of this element rests on certain strong convergence properties of a set of number operators in $\mathfrak{A}$ (operators with integer spectrum) which we define as follows.

DEFINITION 4.1. Given any finite subset Λ of $\mathbf{Z}^+$, define $K_\Lambda \in \mathfrak{A}_\Lambda$ to be the positive operator $\Sigma_{k \in \Lambda} e_k$, and let $\hat{K}_\Lambda = K_\Lambda - \omega(K_\Lambda)I$, where I is the identity operator.

DEFINITION 4.2. Let $(\pi_\omega, H_\omega, \Omega_\omega)$ be the GNS construction of the state ω on $\mathfrak{A}$: then for finite $\Lambda \subset \mathbf{Z}^+$, and $t \in \mathbf{R}$, the real numbers, define $V_\Lambda(t)$ to be the (unitary) operator $V_\Lambda(t) = \pi_\omega(\exp[it\hat{K}_\Lambda]) \in \pi_\omega(\mathfrak{A})$.

One can show that K_Λ has spectrum $\sigma(K_\Lambda) = \{0, 1, 2, \ldots, N(\Lambda)\}$, where $N(\Lambda)$ is the order of the set Λ, hence $\|N_\Lambda\| = N(\Lambda)$. Moreover, since K_Λ is a sum of commuting projections e_k, for $k \in \Lambda$, the unitary operators $\exp[itK_\Lambda]$ have the form, for $t \in \mathbf{R}$, $\exp[itK_\Lambda] = \exp[\Sigma_{k \in \Lambda} ite_k] = \Pi_{k \in \Lambda} \exp[ite_k]$, where $\exp[ite_k] \in N_k$, hence $\exp[itK_\Lambda]$ is a product of commuting unitary operators. In particular, since ω is a product state on $\mathfrak{A}$ we have $\omega(\exp[itK_\Lambda]) = \Pi_{k \in \Lambda} \omega(\exp[ite_k])$.

Although the K_Λ enjoy the properties mentioned above, they do not lie in the invariant algebra $\mathfrak{A}$, hence neither do their exponentials $\exp[itK_\Lambda]$, $t \in \mathbf{R}$. We shall show, however, that there exits a sequence $\Gamma_1 \subset \Gamma_2 \subset \cdots$ of finite subsets with union $\mathbf{Z}^+$ such that $V(t) = \text{st-lim}_{m \to \infty} V_{\Gamma_m}(t)$ exists, and furthermore, the $V(t)$ restrict to unitary operators $V_0(t)$ on the Hilbert subspace $H_0 = [\pi_\omega(\mathfrak{A}^G)\Omega_\omega]^-$ of H_ω. The cyclic representation π_0: $\mathfrak{A}^G \to B(H_0)$ formed by restricting $\pi_\omega|_{\mathfrak{A}^G}$ to H_0 is unitarily equivalent to the GNS representation π_{ω^G} of ω^G (see Lemma 4.11); hence, identifying π_0 with π_{ω^G}, the $V_0(t)$ are identified as elements of $B(H_{\omega^G})$. A straightforward argument (see paragraph after Lemma 4.12) establishes that for some t, $V_0(t) \in B(H_{\omega^G})$ is nontrivial.

In order to prove that the $V_0(t)$ actually lie in the center of $\pi_{\omega^G}(\mathfrak{A})''$ we define a sequence of selfadjoint operators $\hat{J}_m$, central in $\mathfrak{A}_{\Gamma_m}^G$, so that

(i) $U(t) = \text{st-lim}_{m \to \infty} U_m(t)$ exists, where $U_m(t) = \pi_\omega(\exp[it\hat{J}_m])$,

(ii) $U(t)$ maps H_0 to H_0; this restriction defines a unitary operator $U_0(t) \in \pi_{\omega^G}(\mathfrak{A})'' \cap \pi_{\omega^G}(\mathfrak{A})'$, and

(iii) $U_0(t) = V_0(t)$.

The $\hat{J}_m$ will be defined, up to multiplication by a scalar, as the sum of all transpositions $(ij) \in S_{\Gamma_m}$ (see Definition 4.6). As we shall see, the strong convergence of the $U_m(t)$ depends on making a suitable choice for the sets $\Gamma_1, \Gamma_2, \ldots$; i.e., each Γ_m must contain "many" more indices k with $f_k = f$ than with f_k orthogonal to f. We make this more precise in the following definition.

DEFINITION 4.3. Define inductively a sequence $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \mathbf{Z}^+$ of finite sets with union equal to the positive integers, where Γ_m has order m, and Γ_m is the union of disjoint sets γ_m and γ'_m, where

(i) $\gamma_m = \{k \in \Gamma_m: f_k = f\}$; $\gamma'_m = \{k \in \Gamma_m: (f_k, f) = 0\}$,

(ii) γ_m has order r_m satisfying $(m - r_m) < m^{1/3}$, $m = 1, 2, \ldots$.

Furthermore, let $\gamma = \bigcup_{m=1}\gamma_m$; $\gamma' = \bigcup_{m=1}\gamma'_m$. Then $\mathbf{Z}^+$ is the disjoint union of γ and γ'.

We remark that the operators J_m are introduced into the proof in order to show that the unitary elements $V_0(t) = \text{st-lim}_{m\to\infty}U_m(t)$ are central in $\pi_{\omega^G}(\mathfrak{A}^G)''$. On the other hand, we identify $V_0(t)$ with $(\text{st-lim}_{m\to\infty}V_{\Gamma_m}(t))|_{\pi_0(\mathfrak{A}^G)''}$ as a means of showing that for some t, $V_0(t)$ is a nontrivial element of the center of $\pi_{\omega^G}(\mathfrak{A}^G)''$.

The following two results, found in **[6]**, will be useful in treating the numerous calculations of Lemmas 4.7 and 4.8.

LEMMA 4.3. *Let $\{a_k\}$ be a sequence of nonnegative numbers satisfying $\sum_{k=1}^{\infty}(a_k)^2 < \infty$. Then $\lim_{m\to\infty}(1/m)(\sum_{k=1}^{m}a_k)^2 = 0$.*

PROOF. For given $\varepsilon > 0$, choose R sufficiently large so that $\sum_{k=R}^{\infty}(a_k^2) < \varepsilon$, and let $M = \sum_{k=1}^{R-1}a_k$. Then for $m > R$ we note that by the Hölder Inequality,

$$\sum_{k=1}^{m} a_k = M + \sum_{k=R}^{m} a_k \leqslant M + (m-R)^{1/2}\left(\sum_{k=R}^{m} a_k^2\right)^{1/2}.$$

Squaring the left-hand and right-hand sides of the above inequality, and taking the limit as $m \to \infty$, we have

$$\lim_{m\to\infty}\frac{1}{m}\cdot\left(\sum_{k=1}^{m} a_k\right)^2$$

$$\leqslant \lim_{m\to\infty}\left\{\frac{1}{m}\cdot\left[M^2 + 2M(m-R)^{1/2}\left(\sum_{k=R}^{m} a_k^2\right)^{1/2}\right] + \frac{m-R}{m}\cdot\left(\sum_{k=R}^{m} a_k^2\right)\right\}$$

$$= \sum_{k=R}^{m}(a_k)^2 < \varepsilon.$$

Since ε is arbitrary, our assertion holds.

LEMMA 4.4. *Let B be a bounded selfadjoint operator on a Hilbert space H. Then for all $h \in H$ and $t \in \mathbf{R}$, $\|(e^{itB} - I)h\| \leqslant |t|\cdot\|Bh\|$.*

PROOF. First observe that the inequality $|e^{it} - 1| \leqslant |t|$ holds for all real t. Now, letting $\{E_\lambda: \lambda \in \mathbf{R}\}$ denote the spectral measure for the operator B, we have

$$\begin{aligned}\|(e^{itB} - I)h\|^2 &= (\{e^{itB} - I\}^*\{e^{itB} - I\}h, h)\\ &= \int_{-\infty}^{\infty}\{e^{-it\lambda} - 1\}\{e^{it\lambda} - 1\}d(E_\lambda h, h)\\ &\leqslant \int_{-\infty}^{\infty} t^2\lambda^2 d(E_\lambda h, h) = t^2(B^2h, h) = t^2\cdot\|Bh\|^2.\end{aligned}$$

Taking square roots of this inequality yields the desired result.

Retaining the notation of condition set II of Proposition 4.2, we write $\phi_k = \alpha_k f_k + \beta_k \phi'_k$, $k = 1,2,\ldots$, where ϕ'_k is a unit vector orthogonal to f_k. Note that $|\alpha_k|^2 + |\beta_k|^2 = 1$. We adopt the notation $\omega(e^k_{ij}) = t^k_{ij}$, $1 \leqslant i, j \leqslant n$, and $\omega(e_k) = 1 - t_k$ for $k \in \gamma$. Observing that $\omega(e^k_{ij}) = (e_{ij}\phi_k, \phi_k)$, the following lemma is an immediate consequence of condition set II.

LEMMA 4.5. *Let ω be a product state on $\mathfrak{A}$ satisfying the equivalent conditions of Proposition* 4.2, *and retain the notation of the previous paragraph. Then*:

(a) *For any $k \in \mathbf{Z}^+$, if either i or j equals* 1, *but not both, then* $|t^k_{ij}| \leqslant |\beta_k|$.

(b) *For* $k \in \gamma'$, $|t^k_{11}| = t^k_{11} = \omega(e_k) \leqslant |\beta_k|^2$.

(c) *For* $k \in \gamma$, *and* $i \neq 1, j \neq 1$, $|t^k_{ij}| \leqslant |\beta_k|^2$.

(d) *For* $k \in \gamma$, $t_k = |\beta_k|^2$.

DEFINITION 4.4. For $m = 1,2,\ldots$, define $K_m = K_{\Gamma_m}$, and $\hat{K}_m = \hat{K}_{\Gamma_m}$; let $V_m(t)$ be the unitary operators in $\pi_\omega(\mathfrak{A})''$ defined by $V_m(t) = \exp[\pi_\omega(it\hat{K}_m)]$, $t \in \mathbf{R}$.

DEFINITION 4.5. Letting $\mathfrak{A}_0 = \bigcup_{m=1}^\infty \mathfrak{A}_m$, define ϑ to be the dense linear subspace of H_ω given by $\vartheta = \{\pi_\omega(A)\Omega_\omega : A \in \mathfrak{A}_0\}$.

LEMMA 4.6. For $m = 1,2,\ldots$, let $\{K_m\}$, $\{\hat{K}_m\}$ be defined as above. Then the sequence $\{\pi_\omega(\hat{K}_m)\}$ converges strongly on all vectors $h \in \vartheta$.

PROOF. Given $h \in \vartheta$, we may write $h = \pi_\omega(A)\Omega_\omega$, where $A \in \mathfrak{A}_p$, for some p, and let us assume that $m \geqslant n$ are sufficiently large so that $\{1,2,\ldots,p\}$ is contained in both Γ_m and Γ_n. Letting $\pi = \pi_\omega$, we have

$$\left\|\left[\pi(\hat{K}_m) - \pi(\hat{K}_n)\right]\pi(A)\Omega_\omega\right\|^2 = \omega\left(A^*[\hat{K}_m - \hat{K}_n]^2 A\right).$$

But $\hat{K}_m - \hat{K}_n$ is an operator lying in $\mathfrak{A}_{\Gamma_0}$, where $\Gamma_0 = \Gamma_m \setminus \Gamma_n$, and Γ_0 is disjoint from $\{1,2,\ldots,p\}$. Hence, since ω is a product state,

$$\begin{aligned}\omega\left(A^*[\hat{K}_m - \hat{K}_n]^2 A\right) &= \omega\left(A^*A[\hat{K}_m - \hat{K}_n]^2\right) = \omega(A^*A)\omega\left([\hat{K}_m - \hat{K}_n]^2\right)\\ &= \omega(A^*A)\left\{\omega\left([K_m - K_n]^2\right) - [\omega(K_m - K_n)]^2\right\}.\end{aligned}$$

Noting that $K_m - K_n = \sum_{k\in\gamma_m\setminus\gamma_n} e_k + \sum_{k\in\gamma'_m\setminus\gamma'_n} e_k$, and using the product state property for ω again (in particular, $\omega(e_i e_j) = \omega(e_i)\omega(e_j)$ for $i \neq j$), we have

$$\begin{aligned}\omega\left([K_m - K_n]^2\right) - [\omega(K_m - K_n)]^2 &= \left[\sum_{k\in\gamma_m\setminus\gamma_n} \omega(e_k) + \sum_{k\in\gamma'_m\setminus\gamma'_n} \omega(e_k)\right]\\ &\quad - \left[\sum_{k\in\gamma_m\setminus\gamma_n} \{\omega(e_k)\}^2 + \sum_{k\in\gamma'_m\setminus\gamma'_n} \{\omega(e_k)\}^2\right]\\ &= \sum_{k\in\gamma_m\setminus\gamma_n} \left[(1 - t_k) - (1 - t_k)^2\right] + \sum_{k\in\gamma'_m\setminus\gamma'_n} \left[t^k_{11} - (t^k_{11})^2\right]\\ &= \sum_{k\in\gamma_m\setminus\gamma_n} \left[t_k - (t_k)^2\right] + \sum_{k\in\gamma'_m\setminus\gamma'_n} \left[t^k_{11} - (t^k_{11})^2\right].\end{aligned}$$

Recalling that $\sum_{k=1}^\infty |\beta_k|^2 < \infty$, an application of Lemma 4.5 shows that the equation above tends to 0 as n, m tends to infinity, and therefore we may combine the above results to conclude that $0 = \lim_{m,n\to\infty} \|[\pi(\hat{K}_m) - \pi(\hat{K}_n)]\pi(A)\Omega_\omega\|^2$. Thus the sequence $\{\pi(\hat{K}_n)\}$ converges strongly on ϑ, as asserted.

LEMMA 4.7. *The sequence of unitary groups* $\{V_m(t)\colon t \in \mathbf{R}\}$ *converges strongly to a strongly continuous unitary group of operators* $\{V(t)\colon t \in \mathbf{R}\}$ *in* $\pi_\omega(\mathfrak{A})''$.

PROOF. We first establish that for any vector $\phi \in H_\omega$, and for fixed $t \neq 0$ in $\mathbf{R}$, the sequence $\{V_m(t)\phi\}$ of vectors is Cauchy. For given $\varepsilon > 0$ we may find an operator $A \in \mathfrak{A}_0$ such that if $h = \pi(A)\Omega_\omega$, then $\|h - \phi\| < \varepsilon/4$, and by the preceding lemma we may choose integers $m \geq n$ sufficiently large so that $\|[\pi(\hat{K}_m) - \pi(\hat{K}_n)]h\| < \varepsilon/2|t|$. Then we have

$$\begin{aligned}\|[V_m(t) - V_n(t)]\phi\| &\leq \|[V_m(t) - V_n(t)](\phi - h)\| + \|[V_m(t) - V_n(t)]h\| \\ &\leq 2\|\phi - h\| + \|[\exp(\pi(it\hat{K}_m)) - \exp(\pi(it\hat{K}_n))]h\| \\ &< \varepsilon/2 + \|[\exp(it\pi[\hat{K}_m - \hat{K}_n]) - I]h\| \\ &\leq \varepsilon/2 + |t| \cdot \|\pi(\hat{K}_m - \hat{K}_n)h\| \\ &< \varepsilon/2 + |t|\varepsilon/2|t| = \varepsilon,\end{aligned}$$

where we have invoked the result from Lemma 4.4. Hence $\{V_m(t)\}$ is Cauchy, and if we define $V(t)$ to be the limit of the $V_m(t)$, it is straightforward to show that $V(t)$ is unitary, and that the set $\{V(t)\colon t \in \mathbf{R}\}$ satisfies $V(t)V(s) = V(t+s)$, $s, t \in \mathbf{R}$.

We now wish to show that the group $\{V(t)\colon t \in \mathbf{R}\}$ is strongly continuous. Let ϕ and h be as above. Since $\{\pi_\omega(\hat{K}_m)h\colon m = 1, 2, \ldots\}$ is convergent, there exists a number M dominating the norms $\|\pi_\omega(K_m)h\|$. Choose $\varepsilon > 0$, and let $s, t \in \mathbf{R}$ satisfy $|t - s| < \varepsilon/6M$. Applying Lemma 4.4 once again, and choosing m sufficiently large so that $\|[V(t) - V_m(t)]h\| < \varepsilon/6$, $\|[V(s) - V_m(s)]h\| < \varepsilon/6$, we have

$$\begin{aligned}\|[V(t) - V(s)]\phi\| &\leq \|[V(t) - V(s)](\phi - h)\| + \|[V(t) - V(s)]h\| \\ &< \varepsilon/2 + \|[V(t) - V_m(t)]h\| + \|[V_m(s) - V(s)]h\| + \|[V_m(t) - V_m(s)]h\| \\ &< \varepsilon/2 + \varepsilon/6 + \varepsilon/6 + \|[\exp(it\pi(\hat{K}_m)) - \exp(is\pi(\hat{K}_m))]h\| \\ &= 5\varepsilon/6 + \|[\exp\{i(t-s)\pi(\hat{K}_m)\} - I]h\| \\ &< 5\varepsilon/6 + |t - s| \cdot \|\pi(\hat{K}_m)h\| < \varepsilon,\end{aligned}$$

and thus the unitary group $\{V(t)\}$ is strongly continuous.

DEFINITION 4.6. For $m = 1, 2, \ldots$, let $J_m \in \mathfrak{A}^G_{\Gamma_m}$ be the selfadjoint operator $[\Sigma_{i\neq j\in\Gamma_m}(ij)]/2m$; and define $\hat{J}_m = J_m - \omega(J_m)I$. Define $U_m(t)$ in $\pi_\omega(\mathfrak{A})''$ by $U_m(t) = \pi_\omega(\exp[it\hat{J}_m])$, for $t \in \mathbf{R}$.

REMARK. Observe that $gJ_m = J_m g$ for any $g \in S_{\Gamma_m}$, hence J_m (and thus $\hat{J}_m$) lies in the center of $\mathfrak{A}^G_{\Gamma_m}$.

What we now wish to show, in the following two lemmas, is that if we form the operators $\pi(\hat{J}_m) = \pi_\omega(\hat{J}_m)$ in $\pi_\omega(\mathfrak{A})$, then $\text{st-lim}_{m\to\infty} \pi(\hat{J}_m)h$ exists for all $h \in \vartheta$, and that this limit coincides with the strong limit of the vectors $\pi(\hat{K}_m)h$.

LEMMA 4.8. *Let J_m, $\hat{J}_m$ be defined as above, and let Ω_ω be the cyclic generating vector in the GNS construction for ω. Then* st-$\lim_{m\to\infty}\pi(\hat{J}_n)\Omega_\omega$ *exists* (*where* $\pi = \pi_\omega$) *and agrees with* st-$\lim_{m\to\infty}\pi(\hat{K}_m)\Omega_\omega$.

REMARK. A number of occasions will arise over the course of the proof where we shall apply Lemma 4.3 to conclude that limits of certain expressions tends to 0, e.g., $\lim_{m\to\infty}(1/m)\cdot\sum_{i\neq j\in\gamma_m}t_it_j = 0$. When we wish to indicate that a certain summation tends to 0 as $m\to\infty$, we shall enclose the term in double braces $\{\{\ \}\}$, and then replace it in the next line of the calculation by $o(1)$: e.g., we shall write

$$\frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}(1-t_i)(1-t_j) = \frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}\left(1-t_i-t_j+\{\{t_it_j\}\}\right)$$
$$= \frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}(1-t_i-t_j)+o(1).$$

In other instances we shall use the fact that $r = r_m$ has been chosen so that $(m-r) < m^{(1/3)}$ to conclude that certain summations tend to 0 as $m\to\infty$: we shall enclose these terms in double brackets [[ ]]. For example, we shall write

$$\frac{1}{m}\cdot\sum_{i\neq j\in\gamma'_m}\omega(ij) = \left[\left[\frac{1}{m}\cdot\sum_{i\neq j\in\gamma'_m}\omega(ij)\right]\right].$$

PROOF OF LEMMA 4.8. As before, fix a positive integer m and let $r = r_m$. We shall establish the result by verifying that $\lim_{m\to\infty}\|[\pi(\hat{K}_m)-\pi(\hat{J}_m)]\Omega_\omega\|^2 = 0$. Noting that the operators $\hat{J}_m$ and $\hat{K}_m$ commute, we have

$$\left\|\left[\pi(\hat{K}_m)-\pi(\hat{J}_m)\right]\Omega_m\right\|^2 = \omega\left(\left[\hat{K}_m-\hat{J}_m\right]^2\right) = \omega\left([(K_m-J_m)-\omega(K_m-J_m)]^2\right)$$
$$= \omega\left([K_m-J_m]^2\right)-[\omega(K_m-J_m)]^2 = \left\{\omega(K_m^2)-[\omega(K_m)]^2\right\}$$
$$-2\{\omega(K_mJ_m)-\omega(K_m)\omega(J_m)\}+\left\{\omega(J_m^2)-[\omega(J_m)]^2\right\}.$$

We shall examine each of the terms in the brackets separately, and we begin with the least cumbersome calculation. Using the product property of ω we have

$$\omega(K_m^2)-\omega(K_m)^2 = \sum_{i,j\in\Gamma_m}\omega(e_ie_j)-\sum_{i,j\in\Gamma_m}\omega(e_i)\omega(e_j) = \sum_{i\in\Gamma_m}\left[\omega(e_i)-\omega(e_i)^2\right]$$
$$= \sum_{i\in\gamma_m}\left[\omega(e_i)-\omega(e_i)^2\right]+\sum_{i\in\gamma'_m}\left[\omega(e_i)-\omega(e_i)^2\right]$$
$$= \sum_{i\in\gamma_m}\left[(1-t_i)-(1-t_i)^2\right]+\sum_{i\in\gamma'_m}\left[t^i_{11}-(t^i_{11})^2\right]$$
$$= \sum_{i\in\gamma_m}\left[t_i-(t_i)^2\right]+\sum_{i\in\gamma'_m}\left[t^i_{11}-(t^i_{11})^2\right].$$

Noting that $\omega(e_k(ij)) = \omega(e_k)\omega(ij)$ for distinct indices i, j, k of $\mathbf{Z}^+$, we undertake the calculation of

$$\begin{aligned}
\omega(K_m J_m) - \omega(K_m)\omega(J_m) &= \omega\left(\left[\sum_{k\in\Gamma_m} e_k\right]\left[\frac{1}{2m}\cdot\sum_{i\neq j\in\Gamma_m}(ij)\right]\right) \\
&\quad -\left[\sum_{k\in\Gamma_m}\omega(e_k)\right]\cdot\left[\frac{1}{2m}\cdot\sum_{i\neq j\in\Gamma_m}\omega(ij)\right] \\
&= \left[\frac{1}{2m}\cdot\underbrace{\sum_{k\in\Gamma_m}\sum_{i\neq j\in\Gamma_m}}_{\text{n.d.}}\omega(e_k(ij))\right] - \left[\frac{1}{2m}\cdot\underbrace{\sum_{k\in\Gamma_m}\sum_{i\neq j\in\Gamma_m}}_{\text{n.d.}}\omega(e_k)\omega(ij)\right] \\
&= \mathrm{I} - \mathrm{II},
\end{aligned}$$

where n.d. means that the indices i, j, k are not distinct, i.e. either $k = i$ or $k = j$. Thus we have

$$\begin{aligned}
\mathrm{I} &= \frac{1}{2m}\cdot\underbrace{\sum_{k\in\Gamma_m}\sum_{i\neq j\in\Gamma_m}}_{\text{n.d.}}\omega(e_k(ij)) = \frac{2}{2m}\cdot\sum_{i\neq j\in\Gamma_m}\omega(e_i(ij)) \\
&= \frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}\omega(e_i(ij)) + \left[\left[\frac{1}{m}\cdot\sum_{i\neq j\in\gamma'_m}\omega(e_i(ij))\right]\right] \\
&\quad + \frac{1}{m}\cdot\sum_{i\in\gamma_m}\sum_{j\in\gamma'_m}\omega(e_i(ij)) + \frac{1}{m}\cdot\sum_{i\in\gamma'_m}\sum_{j\in\gamma_m}\omega(e_i(ij)) \\
&= \mathrm{I}_1 + o(1) + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

Note that

$$e_i(ij) = \sum_{k,l=1}^{n} e^i_{11}e^i_{kl}e^j_{lk} = \sum_{l=1}^{n} e^i_{1l}e^j_{l1} = e^i_{11}e^j_{11} + \sum_{l=2}^{n} e^i_{1l}e^j_{l1},$$

hence

$$\begin{aligned}
\mathrm{I}_1 &= \frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}\omega(e_i(ij)) = \frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}(1-t_i)(1-t_j) + \left\{\left\{\frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}\sum_{l=2}^{n} t^i_{1l}t^j_{l1}\right\}\right\} \\
&= \frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}(1-t_i-t_j) + \left\{\left\{\frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}(t_it_j)\right\}\right\} + o(1) \\
&= \frac{r(r-1)}{m} - \left[\frac{2(r-1)}{m}\cdot\sum_{i\in\gamma_m} t_i\right] + o(1).
\end{aligned}$$

Now

$$
\begin{aligned}
\mathrm{I}_2 &= \frac{1}{m}\cdot\sum_{i\in\gamma_m}\sum_{j\in\gamma'_m}\omega(e_i(ij))\\
&= \frac{1}{m}\cdot\sum_{i\in\gamma_m}\sum_{j\in\gamma'_m}(1-t_i)t^j_{11} + \left\{\left\{\frac{1}{m}\cdot\sum_{i\in\gamma_m}\sum_{j\in\gamma'_m}\sum_{l=2}^{n}\left(t^i_{1l}t^j_{l1}\right)\right\}\right\}\\
&= \frac{1}{m}\cdot\sum_{i\in\gamma_m}\sum_{j\in\gamma'_m}\left(t^j_{11}\right) - \left\{\left\{\frac{1}{m}\cdot\sum_{i\in\gamma_m}\sum_{j\in\gamma'_m}t_it^j_{11}\right\}\right\} + o(1)\\
&= \frac{r}{m}\cdot\sum_{j\in\gamma'_m}t^j_{11} + o(1).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathrm{I}_3 &= \frac{1}{m}\cdot\sum_{i\in\gamma'_m}\sum_{j\in\gamma_m}\omega(e_i(ij))\\
&= \frac{1}{m}\cdot\sum_{i\in\gamma'_m}\sum_{j\in\gamma_m}t^i_{11}(1-t_j) + \left\{\left\{\frac{1}{m}\cdot\sum_{i\in\gamma'_m}\sum_{j\in\gamma_m}\sum_{l=2}^{n}t^i_{1l}t^j_{l1}\right\}\right\}\\
&= \frac{1}{m}\cdot\sum_{i\in\gamma'_m}\sum_{j\in\gamma_m}t^i_{11} - \left\{\left\{\frac{1}{m}\cdot\sum_{i\in\gamma'_m}\sum_{j\in\gamma_m}t^i_{11}t_j\right\}\right\} + o(1)\\
&= \frac{r}{m}\cdot\sum_{i\in\gamma'_m}t^i_{11} + o(1).
\end{aligned}
$$

Assembling our calculations, we have

$$
\mathrm{I} = \frac{r(r-1)}{m} - \frac{2(r-1)}{m}\cdot\sum_{i\in\gamma_m}t_i + \frac{2r}{m}\cdot\sum_{i\in\gamma'_m}t^i_{11} + o(1).
$$

Now for II,

$$
\begin{aligned}
\mathrm{II} = \frac{1}{2m}\cdot\underbrace{\sum_{k\in\Gamma_m}\sum_{i\neq j\in\Gamma_m}}_{\text{n.d.}}\omega(e_k)\omega(ij) &= \frac{1}{m}\cdot\sum_{i\neq j\in\Gamma_m}\omega(e_i)\omega(ij)\\
= \frac{1}{m}\cdot\sum_{i\neq j\in\gamma_m}\omega(e_i)\omega(ij) &+ \left[\left[\frac{1}{m}\cdot\sum_{i\neq j\in\gamma'_m}\omega(e_i)\omega(ij)\right]\right]\\
+\frac{1}{m}\cdot\sum_{i\in\gamma_m}\sum_{j\in\gamma'_m}\omega(e_i)\omega(ij) &+ \frac{1}{m}\cdot\sum_{i\in\gamma'_m}\sum_{j\in\gamma_m}\omega(e_i)\omega(ij)\\
= \mathrm{II}_1 + o(1) + \mathrm{II}_2 + \mathrm{II}_3.
\end{aligned}
$$

Evaluating these terms separately, we have

$$\mathrm{II}_1 = \frac{1}{m} \cdot \sum_{i \neq j \in \gamma_m} \sum_{k,l=1}^{n} \omega(e_i)\omega(e^i_{kl})\omega(e^j_{lk})$$

$$= \frac{1}{m} \cdot \sum_{i \neq j \in \gamma_m} (1 - t_i)(1 - t_i)(1 - t_j) + \left\{\left\{ \frac{1}{m} \cdot \underbrace{\sum_{i \neq j \in \gamma_m} \sum_{k,l=1}^{n}}_{\text{n.b.1}} (1 - t_i) t^i_{kl} t^j_{lk} \right\}\right\}$$

(where n.b.1 means k and l cannot both be 1)

$$= \frac{1}{m} \cdot \sum_{i \neq j \in \gamma_m} \left(1 - 2t_i - t_j + t_i^2\right) + \left\{\left\{ \frac{1}{m} \cdot \sum_{i \neq j \in \gamma_m} \left[2 t_i t_j - (t_i)^2 t_j\right] \right\}\right\} + o(1)$$

$$= \frac{r(r-1)}{m} - \frac{3(r-1)}{m} \cdot \sum_{i \in \gamma_m} (t_i) + \frac{(r-1)}{m} \cdot \sum_{i \in \gamma_m} (t_i)^2 + o(1).$$

We have

$$\mathrm{II}_2 = \frac{1}{m} \cdot \sum_{i \in \gamma_m} \sum_{j \in \gamma'_m} \omega(e_i)\omega(ij)$$

$$= \frac{1}{m} \cdot \sum_{i \in \gamma_m} \sum_{j \in \gamma'_m} \sum_{k,l=1}^{n} (1 - t_i)\omega(e^i_{kl})\omega(e^j_{lk})$$

$$= \frac{1}{m} \cdot \sum_{i \in \gamma_m} \sum_{j \in \gamma'_m} (1 - t_i)(1 - t_i) t^j_{11}$$

$$+ \left\{\left\{ \frac{1}{m} \cdot \sum_{i \in \gamma_m} \sum_{j \in \gamma'_m} \sum_{l=2}^{n} (1 - t_i)\left[t^i_{1l} t^j_{l1} + t^i_{l1} t^j_{1l}\right] \right\}\right\}$$

$$+ \left\{\left\{ \frac{1}{m} \cdot \sum_{i \in \gamma_m} \sum_{j \in \gamma'_m} \sum_{k,l=1}^{n} (1 - t_i) t^i_{kl} t^j_{lk} \right\}\right\}.$$

(Note that the first term in double braces is dominated in absolute value by $(1/m) \cdot \Sigma_{i \in \gamma_m} \Sigma_{j \in \gamma'_m} 2(n-1)(1 - t_i)|\beta_i||\beta_j|$, and the second term by

$$\left[(n-1)^2/m\right] \cdot \sum_{i \in \gamma_m} \sum_{j \in \gamma'_m} (1 - t_i)|\beta_i|^2,$$

both of which tend to 0 as $m \to \infty$.) Hence

$$\mathrm{II}_2 = \frac{1}{m} \cdot \sum_{i \in \gamma_m} \sum_{j \in \gamma'_m} (1 - t_i)^2 t^j_{11} + o(1)$$

$$= \frac{1}{m} \cdot \sum_{i \in \gamma_m} \sum_{j \in \gamma'_m} t^j_{11} + \left\{\left\{ \frac{1}{m} \cdot \sum_{i \in \gamma_m} \sum_{j \in \gamma'_m} \left[-2 t_i t^j_{11} + (t_i)^2 t^j_{11}\right] \right\}\right\} + o(1)$$

$$= \frac{r}{m} \cdot \sum_{j \in \gamma'_m} t^j_{11} + o(1).$$

Similarly,

$$\begin{aligned}
\mathrm{II}_3 &= \frac{1}{m} \cdot \sum_{i\in\gamma'_m} \sum_{j\in\gamma_m} \omega(e_i)\omega(ij) \\
&= \frac{1}{m} \sum_{i\in\gamma'_m} \sum_{j\in\gamma_m} \sum_{k,l=1}^{n} \left(t^i_{11}\right)\omega\left(e^i_{kl}\right)\omega\left(e^j_{lk}\right) \\
&= \frac{1}{m} \cdot \sum_{i\in\gamma'_m} \sum_{j\in\gamma_m} t^i_{11}\left(t^i_{11}[1-t_j]\right) + \left\{\left\{\frac{1}{m} \cdot \sum_{i\in\gamma'_m} \sum_{j\in\gamma_m} \sum_{l=2}^{n} t^i_{11}\left(t^i_{1l}t^j_{l1} + t^i_{l1}t^j_{1l}\right)\right\}\right\} \\
&\quad + \left\{\left\{\frac{1}{m} \cdot \sum_{i\in\gamma'_m} \sum_{j\in\gamma_m} \sum_{k,l=2}^{n} t^i_{11}t^i_{kl}t^j_{lk}\right\}\right\} \\
&= \frac{1}{m} \cdot \sum_{i\in\gamma'_m} \sum_{j\in\gamma_m} \left(t^i_{11}\right)^2 - \left\{\left\{\frac{1}{m} \cdot \sum_{i\in\gamma'_m} \sum_{j\in\gamma_m} \left(t^i_{11}\right)^2 t_j\right\}\right\} + o(1) \\
&= \frac{r}{m} \cdot \sum_{i\in\gamma'_m} \left(t^i_{11}\right)^2 + o(1).
\end{aligned}$$

Thus we have, putting everything together,

$$\begin{aligned}
\mathrm{II} &= \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 + o(1) \\
&= \frac{r(r-1)}{m} - \frac{3(r-1)}{m} \cdot \sum_{i\in\gamma_m} t_i + \frac{r-1}{m} \cdot \sum_{i\in\gamma_m} (t_i)^2 + o(1) \\
&\quad + \frac{r}{m} \cdot \sum_{j\in\gamma'_m} t^j_{11} + o(1) + \frac{r}{m} \cdot \sum_{i\in\gamma'_m} \left(t^i_{11}\right)^2 + o(1) + o(1).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\omega(K_m J_m) - \omega(K_m)\omega(J_m) &= \mathrm{I} - \mathrm{II} \\
&= \frac{(r-1)}{m} \cdot \sum_{i\in\gamma_m} \left[t_i - (t_i)^2\right] + \frac{r}{m} \cdot \sum_{j\in\gamma'_m} \left[t^j_{11} - \left(t^j_{11}\right)^2\right] + o(1).
\end{aligned}$$

Now we must consider $\omega(J_m^2) - \omega(J_m)^2$. A routine calculation yields

$$\begin{aligned}
\omega\left(J_m^2\right) - \omega(J_m)^2 &= \left\{\frac{1}{4m^2} \cdot \underbrace{\sum_{i\neq j\in\Gamma_m} \sum_{k\neq l\in\Gamma_m}}_{\text{(n.d.)}} \omega((ij)(kl))\right\} \\
&\quad - \left\{\frac{1}{4m^2} \cdot \underbrace{\sum_{i\neq j\in\Gamma_m} \sum_{k\neq l\in\Gamma_m}}_{\text{(n.d.)}} \omega((ij)\omega(kl))\right\} \\
&= \mathrm{III} - \mathrm{IV},
\end{aligned}$$

where n.d. means that the 4 indices $i \neq j$, $k \neq l$ are not distinct.

Now

$$\mathrm{III} = \frac{1}{4m^2}\cdot\left[\underbrace{\sum_{i\neq j\in\Gamma_m}\sum_{k\neq l\in\Gamma_m}}_{\text{(n.d.)}}\omega((ij)(kl))\right]$$

$$= \frac{4}{4m^2}\cdot\omega\left(\left[\underbrace{\sum_{i,j,l\in\Gamma_m}}_{\text{(d.)}}(ij)(jl)\right]\right) + \frac{2}{4m^2}\cdot\omega\left(\left[\sum_{i\neq j\in\Gamma_m}(ij)(ij)\right]\right)$$

(where d. signifies that i, j, l are distinct)

$$= \frac{1}{m^2}\cdot\omega\left(\left[\underbrace{\sum_{i,j,l\in\Gamma_m}}_{\text{(d.)}}(ijl)\right]\right) + \frac{m^2-m}{2m^2}$$

$$= \frac{1}{m^2}\cdot\omega\left(\left[\underbrace{\sum_{i,j,l\in\Gamma_m}}_{\text{(d.)}}(ijl)\right]\right) + \frac{1}{2} + o(1)$$

$$= \frac{1}{m^2}\left[\underbrace{\sum_{i,j,l\in\gamma_m}}_{\text{(d.)}}(ijl)\right] + \left[\left[\frac{1}{m^2}\cdot\underbrace{\sum_{i,j,l\in\gamma_m'}}_{\text{(d.)}}\omega(ijl)\right]\right]$$

$$+ \frac{3}{m^2}\left[\sum_{i\neq j\in\gamma_m}\sum_{l\in\gamma_m'}\omega(ijl)\right] + \left[\left[\frac{3}{m^2}\cdot\sum_{i\in\gamma_m}\sum_{j\neq l\in\gamma_m'}\omega(ijl)\right]\right] + \frac{1}{2} + o(1)$$

$$= \mathrm{III}_1 + o(1) + \mathrm{III}_2 = o(1) + \frac{1}{2} + o(1).$$

Recalling that $(ijl) = \sum_{p,q,s=1}^{n} e^i_{pq}e^j_{qs}e^l_{sp}$, we have

$$\mathrm{III}_1 = \frac{1}{m^2}\cdot\underbrace{\sum_{i,j,l\in\gamma_m}}_{\text{(d.)}}\omega(ijl)$$

$$= \frac{1}{m^2}\cdot\underbrace{\sum_{i,j,l\in\gamma_m}}_{\text{(d.)}}\sum_{p,q,s=1}^{n}\left[\omega\left(e^i_{pq}\right)\omega\left(e^j_{qs}\right)\omega\left(e^l_{sp}\right)\right]$$

$$= \frac{1}{m^2}\cdot\underbrace{\sum_{i,j,l\in\gamma_m}}_{\text{(d.)}}\left[(1-t_i)(1-t_j)(1-t_l)\right]$$

$$+ \left\{\left\{\frac{1}{m^2}\cdot\underbrace{\sum_{i,j,l\in\gamma_m}}_{\text{(d.)}}\underbrace{\sum_{p,q,s=1}^{n}}_{\substack{\text{not}\\ \text{all 1}}}\left[t^i_{pq}t^j_{qs}t^l_{sp}\right]\right\}\right\}$$

$$= \frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma_m}}_{\text{(d.)}} \left[1 - t_i - t_j - t_l\right]$$

$$+ \left\{\left\{\frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma_m}}_{\text{(d.)}} \left[t_i t_l + t_i t_j + t_j t_l - t_i t_j t_l\right]\right\}\right\} + o(1)$$

$$= \frac{r(r-1)(r-2)}{m^2} - \frac{3(r-1)(r-2)}{m^2} \cdot \sum_{i \in \gamma_m} (t_i) + o(1).$$

For III_2,

$$\mathrm{III}_2 = \frac{3}{m^2} \cdot \sum_{i \neq j \in \gamma_m} \sum_{l \in \gamma_m'} \omega(ijl)$$

$$= \frac{3}{m^2} \cdot \left[\sum_{i \neq j \in \gamma_m} \sum_{l \in \gamma_m'} \sum_{p,q,s=1}^{n} \left\{\omega\left(e_{pq}^i\right)\omega\left(e_{qs}^j\right)\omega\left(e_{sp}^l\right)\right\}\right]$$

$$= \frac{3}{m^2} \cdot \sum_{i \neq j \in \gamma_m} \sum_{l \in \gamma_m'} (1 - t_i)(1 - t_j) t_{11}^l$$

$$+ \left\{\left\{\frac{3}{m^2} \cdot \sum_{i \neq j \in \gamma_m} \sum_{l \in \gamma_m'} \underbrace{\sum_{p,q,s=1}^{n}}_{\substack{\text{not} \\ \text{all } 1}} \omega\left(e_{pq}^i\right)\omega\left(e_{qs}^j\right)\omega\left(e_{sp}^l\right)\right\}\right\}$$

$$= \frac{3}{m^2}\left[\sum_{i \neq j \in \gamma_m} \sum_{l \in \gamma_m'} t_{11}^l\right] + \left\{\left\{\frac{3}{m^2}\left[\sum_{i \neq j \in \gamma_m} \sum_{l \in \gamma_m'} \left(-t_{11}^l t_i - t_{11}^l t_j + t_i t_j t_{11}^l\right)\right]\right\}\right\} + o(1)$$

$$= \left[\frac{3r(r-1)}{m^2}\right] \cdot \sum_{l \in \gamma_m'} t_{11}^l + o(1).$$

Hence

$$\mathrm{III} = \mathrm{III}_1 + \mathrm{III}_2 + \frac{1}{2} + o(1)$$

$$= \frac{r(r-1)(r-2)}{m^2} - \frac{3(r-1)(r-2)}{m^2} \cdot \sum_{i \in \gamma_m} t_i$$

$$+ \frac{3r(r-1)}{m^2} \cdot \sum_{l \in \gamma_m'} t_{11}^l + \frac{1}{2} + o(1).$$

Finally, we examine term IV,

$$\mathrm{IV} = \frac{1}{4m^2} \cdot \underbrace{\sum_{i \neq j \in \Gamma_m} \sum_{k \neq l \in \Gamma_m}}_{\text{(n.d.)}} \omega(ij)\omega(kl)$$

$$= \frac{4}{4m^2} \cdot \underbrace{\sum_{i,j,l \in \Gamma_m}}_{(\text{d.})} \omega(ij)\omega(jl) + \frac{2}{4m^2} \cdot \sum_{i \neq j \in \Gamma_m} [\omega(ij)]^2$$

$$= \mathrm{IV}_1 + \mathrm{IV}_2.$$

For IV_1,

$$\begin{aligned}
\mathrm{IV}_1 = {} & \frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma_m}}_{(\text{d.})} \omega(ij)\omega(jl) + \left[\left[\frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma'_m}}_{(\text{d.})} \omega(ij)\omega(jl)\right]\right] \\
& + \frac{2}{m^2} \cdot \sum_{i \neq j \in \gamma_m} \sum_{l \in \gamma'_m} \omega(ij)\omega(jl) + \left[\left[\frac{2}{m^2} \cdot \sum_{i \in \gamma_m} \sum_{j \neq l \in \gamma'_m} \omega(ij)\omega(jl)\right]\right] \\
& + \left[\left[\frac{1}{m^2} \cdot \sum_{j \in \gamma_m} \sum_{i \neq l \in \gamma'_m} \omega(ij)\omega(jl)\right]\right] + \frac{1}{m^2} \cdot \sum_{i \neq l \in \gamma_m} \sum_{j \in \gamma'_m} \omega(ij)\omega(jl) \\
= {} & \mathrm{IV}_{1\mathrm{a}} + o(1) + \mathrm{IV}_{1\mathrm{b}} + o(1) + o(1) + \mathrm{IV}_{1\mathrm{c}}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{IV}_{1\mathrm{a}} = {} & \frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma_m}}_{(\text{d.})} \omega(ij)\omega(jl) \\
= {} & \frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma_m}}_{(\text{d.})} \sum_{p,q=1}^{n} \sum_{u,v=1}^{n} \omega\left(e^j_{pq}\right)\omega\left(e^i_{qp}\right)\omega\left(e^j_{uv}\right)\omega\left(e^l_{vu}\right) \\
= {} & \frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma_m}}_{(\text{d.})} (1 - t_i)(1 - t_j)^2(1 - t_l) \\
& + \left\{\left\{\frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma_m}}_{(\text{d.})} \underbrace{\sum_{p,q,u,v=1}^{n}}_{\substack{\text{not} \\ \text{all } 1}} \left(t^i_{pq} t^j_{qp} t^j_{uv} t^l_{vu}\right)\right\}\right\} \\
= {} & \frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma_m}}_{(\text{d.})} \left[1 - t_i - t_l - 2t_j + (t_j)^2\right] \\
& + \left\{\left\{\frac{1}{m^2} \cdot \underbrace{\sum_{i,j,l \in \gamma_m}}_{(\text{d.})} \left[t_i t_l + 2t_j t_i + 2t_j t_l - 2t_i t_j t_l - t_i t_j^2 - t_l t_j^2 + t_i t_l t_j\right]\right\}\right\} \\
& + o(1) \\
= {} & \frac{r(r-1)(r-2)}{m^2} - \frac{4(r-1)(r-2)}{m^2} \cdot \sum_{i \in \gamma_m} t_i \\
& + \frac{(r-1)(r-2)}{m^2} \cdot \sum_{j \in \gamma_m} t_j^2 + o(1).
\end{aligned}$$

$$\begin{aligned}
\mathrm{IV}_{1b} &= \frac{2}{m^2}\cdot \sum_{i\neq j\in\gamma_m}\ \sum_{l\in\gamma'_m} \omega(ij)\omega(jl)\\
&= \frac{2}{m^2}\cdot \sum_{i\neq j\in\gamma_m}\ \sum_{l\in\gamma'_m} (1-t_i)(1-t_j)^2 t^l_{11}\\
&\quad + \left\{\left\{\frac{2}{m^2}\cdot \sum_{i\neq j\in\gamma_m}\ \sum_{l\in\gamma'_m}\ \underbrace{\sum_{p,q,u,v=1}^{n}}_{\substack{\text{not}\\ \text{all }1}} \left(t^i_{pq}t^j_{qp}t^j_{uv}t^l_{vu}\right)\right\}\right\}\\
&= \frac{2}{m^2}\cdot \sum_{i\neq j\in\gamma_m}\ \sum_{l\in\gamma'_m} t^l_{11}\\
&\quad + \left\{\left\{\frac{2}{m^2}\cdot \sum_{i\neq j\in\gamma_m}\ \sum_{l\in\gamma'_m} \left(t_it^l_{11} - 2t_jt^l_{11} + 2t_it_jt^l_{11} + t_j^2t^l_{11} - t_it_j^2t^l_{11}\right)\right\}\right\} + o(1)\\
&= \frac{2r(r-1)}{m^2}\cdot \sum_{l\in\gamma'_m} t^l_{11} + o(1).
\end{aligned}$$

$$\begin{aligned}
\mathrm{IV}_{1c} &= \frac{1}{m^2}\cdot \sum_{i\neq l\in\gamma_m}\ \sum_{j\in\gamma'_m} \omega(ij)\omega(jl)\\
&= \frac{1}{m^2}\cdot \sum_{i\neq l\in\gamma_m}\ \sum_{j\in\gamma'_m} (1-t_i)\left(t^j_{11}\right)^2(1-t_l)\\
&\quad + \left\{\left\{\frac{1}{m^2}\cdot \sum_{i\neq l\in\gamma_m}\ \sum_{j\in\gamma'_m}\ \underbrace{\sum_{p,q,u,v=1}^{n}}_{\substack{\text{not}\\ \text{all }1}} \left(t^i_{pq}t^j_{qp}t^j_{uv}t^l_{vu}\right)\right\}\right\}\\
&= \frac{1}{m^2}\cdot \sum_{i\neq l\in\gamma_m}\ \sum_{j\in\gamma'_m} \left(t^j_{11}\right)^2\\
&\quad + \left\{\left\{\frac{1}{m^2}\cdot \sum_{i\neq l\in\gamma_m}\ \sum_{j\in\gamma'_m} \left[-t_i\left(t^j_{11}\right)^2 - t_l\left(t^j_{11}\right)^2 + t_it_l\left(t^j_{11}\right)^2\right]\right\}\right\} + o(1)\\
&= \frac{r(r-1)}{m^2}\cdot \sum_{j\in\gamma'_m} \left(t^j_{11}\right)^2 + o(1).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{IV}_1 &= \mathrm{IV}_{1a} + \mathrm{IV}_{1b} + \mathrm{IV}_{1c} + o(1)\\
&= \frac{r(r-1)(r-2)}{m^2} - \frac{4(r-1)(r-2)}{m^2}\cdot \sum_{i\in\gamma_m} t_i\\
&\quad + \frac{(r-1)(r-2)}{m^2}\cdot \sum_{j\in\gamma_m} t_j^2 + \frac{2r(r-1)}{m^2}\cdot \sum_{l\in\gamma'_m} t^l_{11}\\
&\quad + \frac{r(r-1)}{m^2}\cdot \sum_{j\in\gamma'_m} \left(t^j_{11}\right)^2 + o(1).
\end{aligned}$$

Now for IV_2,

$$\begin{aligned}
\mathrm{IV}_2 &= \frac{1}{2m^2}\cdot \sum_{i\neq j\in\gamma_m} [\omega(ij)]^2 \\
&= \frac{1}{2m^2}\cdot \sum_{i\neq j\in\gamma_m} [\omega(ij)]^2 + \left[\left[\frac{1}{2m^2}\cdot \sum_{i\neq j\in\gamma'_m} [\omega(ij)]^2\right]\right] \\
&\quad + \left[\left[\frac{2}{2m^2}\cdot \sum_{i\in\gamma_m}\sum_{j\in\gamma'_m} [\omega(ij)]^2\right]\right] \\
&= \frac{1}{2m^2}\cdot \sum_{i\neq j\in\gamma_m} [\omega(ij)]^2 + o(1) \\
&= \frac{1}{2m^2}\cdot \sum_{i\neq j\in\gamma_m} (1-t_i)^2(1-t_j)^2 + \left\{\left\{\frac{1}{2m^2}\cdot \sum_{i\neq j\in\gamma_m}\underbrace{\sum_{p,q=1}}_{\substack{\text{not}\\ \text{both } 1}}\left(t^i_{pq}t^j_{qp}\right)\right\}\right\} + o(1) \\
&= \frac{1}{2m^2}\cdot \sum_{i\neq j\in\gamma_m} 1 \\
&\quad + \left\{\left\{\frac{1}{2m^2}\cdot \sum_{i\neq j\in\gamma_m}\left[-2t_i - 2t_j + t_i^2 + t_j^2 + 4t_it_j - 2t_it_j^2 - 2t_jt_i^2 + t_i^2t_j^2\right]\right\}\right\} \\
&\quad + o(1) \\
&= \frac{r(r-1)}{2m^2} + o(1).
\end{aligned}$$

Combining the above calculations, we have

$$\begin{aligned}
\omega(J_m^2) - \omega(J_m)^2 &= \mathrm{III} - \mathrm{IV} \\
&= \frac{(r-1)(r-2)}{m^2}\cdot \sum_{i\in\gamma_m}\left[t_i - (t_i)^2\right] + \frac{r(r-1)}{m^2}\cdot \sum_{l\in\gamma'_m}\left[t^l_{11} - (t^l_{11})^2\right] + o(1).
\end{aligned}$$

A routine calculation now shows that

$$\begin{aligned}
\lim_{m\to\infty} \left\{\omega(K_m^2) - \omega(K_m)^2\right\} - 2\{\omega(K_mJ_m) - \omega(K_m)\omega(J_m)\}& \\
+ \left\{\omega(J_m^2) - \omega(J_m)^2\right\} = 0&,
\end{aligned}$$

hence $\lim_{m\to\infty}\|[\pi(\hat{K}_m) - \pi(\hat{J}_m)]\Omega_\omega\|^2 = 0$, and the lemma holds.

LEMMA 4.9. *The operators* $\{\pi(\hat{J}_m)\colon m = 1,2,\ldots\}$ (*where* $\pi = \pi_\omega$) *converge strongly on all vectors* $h\in\vartheta$, *with* $\text{st-lim}_{m\to\infty}\pi(\hat{J}_m)h = \text{st-lim}_{m\to\infty}\pi(\hat{K}_m)h$.

PROOF. Given $h\in\vartheta$, h is of the form $\pi(A)\Omega_\omega$, where $A\in\mathfrak{A}_p\subset\mathfrak{A}_0$, for some positive integer p. Letting $\mathcal{I}$ denote the set of p-tuples I of the form $I = (i_1,\ldots,i_p)$, where $i_1,\ldots,i_p \in \{1,2,\ldots,n\}$, A may be written as

$$A = \sum_{I,J\in\mathfrak{I}} \alpha(I:J)e^1_{i_1j_1}\cdots e^p_{i_pj_p}, \qquad \alpha(I:J)\in\mathbf{C}.$$

For fixed I, $J\in\mathfrak{I}$, define a positive linear functional $\rho_{I:J}$ on $\mathfrak{A}$ given by

$$\rho_{I:J}(x) = \left(\pi_\omega(x)\pi_\omega\left(e^1_{i_1j_1}\cdots e^p_{i_pj_p}\right)\Omega_\omega, \pi_\omega\left(e^1_{i_1j_1}\cdots e^p_{i_pj_p}\right)\Omega_\omega\right).$$

Defining $\psi_k = e^k_{i_kj_k}(\phi_k)$, for $k=1,2,\ldots,p$, it is straightforward to show that one of the following two conditions must hold.

(i) $e^k_{i_kj_k}(\phi_k)=0$ for some k, $1\leqslant k\leqslant p$, in which case $\rho_{I:J}$ is the 0 functional, or

(ii) there exist positive constants $a_1, a_2,\ldots,a_p$ such that $\xi_k = a_k\psi_k = a_ke^k_{i_kj_k}(\phi_k)$ is a unit vector, for $k=1,2,\ldots,p$, and taking $a=(a_1)^{-1}\cdots(a_p)^{-1}$, we have $\rho_{I:J}=a\cdot\omega_{I:J}$, where $\omega_{I:J}$ is the product state on $\mathfrak{A}$ defined by $\omega_{I:J}=\omega_{\xi_1}\otimes\cdots\otimes\omega_{\xi_p}\otimes\omega_{\phi_{p+1}}\otimes\omega_{\phi_{p+2}}\otimes\cdots$.

Now we have (again denoting π_ω by π)

$$\left\|\left[\pi(\hat{K}_m)-\pi(\hat{J}_m)\right]\pi(A)\Omega_\omega\right\| = \left\|\pi(\hat{K}_m-\hat{J}_m)\left[\sum_{I,J\in\mathfrak{I}}\alpha(I:J)\pi\left(e^1_{i_1j_1}\cdots e^p_{i_pj_p}\right)\right]\Omega_\omega\right\|$$
$$\leqslant \sum_{I,J}|\alpha(I:J)|\cdot\left\|\pi(\hat{K}_m-\hat{J}_m)\left[\pi\left(e^1_{i_1j_1}\cdots e^p_{i_pj_p}\right)\right]\Omega_\omega\right\|,$$

and for a fixed pair of indices I, J, we have

$$\left\|\pi(\hat{K}_m-\hat{J}_m)\pi\left(e^1_{i_1j_1}\cdots e^p_{i_pj_p}\right)\Omega_\omega\right\|^2$$
$$=\left(\left[\pi(\hat{K}_m-\hat{J}_m)\right]^2\pi\left(e^1_{i_1j_1}\cdots e^p_{i_pj_p}\right)\Omega_\omega, \pi\left(e^1_{i_1j_1}\cdots e^p_{i_pj_p}\right)\Omega_\omega\right)$$
$$=\rho_{I:J}\left([\hat{K}_m-\hat{J}_m]^2\right).$$

If condition (i) holds we have $\rho_{I:J}([\hat{K}_m-\hat{J}_m]^2)=0$, for all m, and if condition (ii) holds, then $\rho_{I:J}([\hat{K}_m-\hat{J}_m]^2)=a\omega_{I:J}([\hat{K}_m-\hat{J}_m]^2)$. But $\omega_{I:J}$ is a product state on $\mathfrak{A}$ which satisfies conditions a, b, c of condition set II of Proposition 4.2, and therefore we may apply the proof of the preceding lemma to the state $\omega_{I:J}$ to conclude that $\lim_{m\to\infty}\omega_{I:J}([\hat{K}_m-\hat{J}_m]^2)=0$. Retracing our steps, we have verified that $\lim_{m\to\infty}\|\pi(\hat{K}_m-\hat{J}_m)\pi(A)\Omega_\omega\|=0$, and the lemma is proved.

Lemma 4.10. *For $t\in\mathbf{R}$, the unitary operators $U_m(t)$ converge strongly to $V(t)$, as $m\to\infty$.*

Proof. Since the $U_m(t)$ are unitary we need only verify that $\lim_{m\to\infty}U_m(t)h = V(t)h$, for all h in the dense subset ϑ of H. We proceed by showing that the sequence $\{V_m(t)h - U_m(t)h\}$ converges to 0, for $h\in\vartheta$. Recalling that the operators $\hat{J}_m$, $\hat{K}_m$ commute, and applying Lemma 4.4, we have

$$\|V_m(t)h-U_m(t)h\| = \left\|\left[\exp\{\pi(it\hat{K}_m)\}-\exp\{\pi(it\hat{J}_m)\}\right]h\right\|$$
$$=\left\|\left[\exp\{\pi(it[\hat{K}_m-\hat{J}_m])\}-I\right]h\right\|\leqslant|t|\cdot\left\|\left[\pi(\hat{K}_m)-(\hat{J}_m)\right]h\right\|,$$

and since the last expression tends to 0 as $m\to\infty$, by Lemma 4.8, we have $\lim_{m\to\infty}\|[V_m(t)-U_m(t)]h\|=0$.

DEFINITION 4.7. Given the GNS construction $\pi = (\pi_\omega, H_\omega, \Omega_\omega)$ of ω on $\mathfrak{A}$, and letting $\omega^G = \omega|_{\mathfrak{A}^G}$, define H_0 to be the Hilbert subspace $[\pi(\mathfrak{A}^G)\Omega_\omega]^-$ of H, and define π_0: $\mathfrak{A}^G \to B(H_0)$ to be the representation on $\mathfrak{A}^G$ given by $\pi_0(x)[\pi(A)\Omega_\omega] = \pi(xA)\Omega_\omega$, for $x \in \mathfrak{A}^G$, where $\pi(A)\Omega_\omega \in [\pi(\mathfrak{A}^G)\Omega_\omega]$, and extended to all of H_0 by continuity.

LEMMA 4.11. *The representation* π_0: $\mathfrak{A}^G \to B(H_0)$ *is unitarily equivalent to the representation* π_{ω^G} *of* $\mathfrak{A}^G$ *in the GNS construction* $(\pi_{\omega^G}, H_{\omega^G}, \Omega_{\omega^G})$ *of* ω^G.

PROOF. For $x \in \mathfrak{A}^G$,

$$(\pi_0(x)\Omega_\omega, \Omega_\omega) = (\pi_\omega(x)\Omega_\omega, \Omega_\omega) = \omega(x) = \omega^G(x) = (\pi_{\omega^G}(x)\Omega_{\omega^G}, \Omega_{\omega^G});$$

hence [**8**, Proposition 2.4.1], π_0 and π_{ω^G} induce unitarily equivalent representations.

REMARK. By an abuse of terminology, we shall refer to the triple $(\pi_0, H_0, \Omega_\omega)$ as the GNS construction for the state ω^G.

LEMMA 4.12. *The unitary group* $\{V(t): t \in \mathbf{R}\}$ *maps the subspace* H_0 *onto itself. Hence we may define a unitary group* $\{V_0(t): t \in \mathbf{R}\}$ *in* $B(H_0)$, *given by* $V_0(t)h = V(t)h$, *for* $h \in H_0$. *We have* $V_0(t) = \text{st-lim}_{m\to\infty} \exp[\pi_0(it\hat{J}_m)]$, *hence* $V_0(t) \in \pi_0(\mathfrak{A}^G)'' \cap \pi_0(\mathfrak{A}^G)'$.

PROOF. Define ϑ_0 to be the linear subspace of H_0 given by $\vartheta_0 = \{\pi_0(A)\Omega_\omega$ $(= \pi_\omega(A)\Omega_\omega)$: $A \in \mathfrak{A}_0^G\}$. Since $\mathfrak{A}_0^G$ is norm dense in $\mathfrak{A}^G$, ϑ_0 is dense in H_0. Let $h = \pi_\omega(A)\Omega_\omega \in \vartheta_0$, for some $A \in \mathfrak{A}_0^G$. Then $[\exp(it\hat{J}_m)]A \in \mathfrak{A}_0^G$; hence, $\pi_0([\exp(it\hat{J}_m)]A)\Omega_m = [\exp\{it\pi_\omega(\hat{J}_m)\}] \cdot \pi_\omega(A)\Omega_\omega$ lies in ϑ_0, and we have, applying Lemma 4.10,

$$\begin{aligned} V(t)h &= \lim_{m\to\infty} U_m(t)h = \lim_{m\to\infty} \left[\exp\left\{\pi_\omega\left(it\hat{J}_m\right)\right\}\right]\pi_\omega(A)\Omega_\omega \\ &= \lim_{m\to\infty} \left[\exp\left\{\pi_0\left(it\hat{J}_m\right)\right\}\right]\pi_0(A)\Omega_\omega; \end{aligned}$$

hence, $V(t)h$, a limit of vectors in ϑ_0, must itself lie in H_0. By continuity, we have $V(t)h \in H_0$ for all $h \in H_0$, and it is now straightforward to show that the $V(t)$ restrict to a unitary group $\{V_0(t): t \in \mathbf{R}\}$ on H_0. Furthermore, the above calculation shows that $V_0(t) = \text{st-lim}_{m\to\infty}\exp[\pi_0(it\hat{J}_m)]$. It remains to verify that $V_0(t)$ lies in the center of $\pi_0(\mathfrak{A}^G)''$. Now given $A \in \mathfrak{A}_0^G$, we have $A \in \mathfrak{A}_p^G$ for some $p > 0$. Furthermore, for sufficiently large m (say $m > M$) we have $\{1,\ldots,p\} \subset \Gamma_m$. Since $A \in \mathfrak{A}_p^G \subset \mathfrak{A}_{\Gamma_m}^G$, and $\hat{J}_m$ lies in the commutant of $\mathfrak{A}_{\Gamma_m}^G$ (see the remark after Definition 4.6), A and $\hat{J}_m$ are commuting operators for $m > M$; hence, A and $\exp(it\hat{J}_m)$ commute. Therefore,

$$\begin{aligned} V_0(t)\pi_0(A) &= \underset{m\to\infty}{\text{st-lim}} \left[\exp\left\{\pi_0\left(it\hat{J}_m\right)\right\}\right]\pi_0(A) = \underset{m\to\infty}{\text{st-lim}}\, \pi_0\left[\exp\left(it\hat{J}_m\right) \cdot A\right] \\ &= \underset{m\to\infty}{\text{st-lim}}\, \pi_0\left[A \cdot \exp\left(it\hat{J}_m\right)\right] = \underset{m\to\infty}{\text{st-lim}}\, \pi_0(A) \cdot \exp\left[it\pi_0\left(\hat{J}_m\right)\right] \\ &= \pi_0(A)V_0(t). \end{aligned}$$

We have shown that $V_0(t)$ commutes with all operators in the strongly dense set $\pi_0(\mathfrak{A}_0^G)$ of $\pi_0(\mathfrak{A}^G)''$, hence by continuity, the unitary operators $V_0(t)$ lie in $\pi_0(\mathfrak{A}^G)'' \cap \pi_0(\mathfrak{A}^G)'$, for $t \in \mathbf{R}$. This completes the proof of the lemma.

We are now in a position to prove our theorem.

PROOF OF THEOREM 4.1. We have shown that for $t \in \mathbf{R}$, $V_0(t)$ is a unitary operator lying in the center of $\pi_0(\mathfrak{A}^G)''$. We wish to show that for some $t \in \mathbf{R}$, $V_0(t)$ is not a scalar multiple of the identity, i.e., $V_0(t)$ is not of the form $V_0(t) = e^{i\alpha}I$, $\alpha \in \mathbf{R}$, for some $t \in \mathbf{R}$.

We consider the unitary operators $V_m(t) = \exp[it\pi_\omega(\hat{K}_m)]$. Recall that $\hat{K}_m = \sum_{j\in\Gamma_m}[e_j - \omega(e_j)I]$. Since the projections $e_j = e^j_{11}$, $j = 1, 2, \ldots$, commute with each other, we may write

$$\exp(it\hat{K}_m) = \exp\left[it\sum_{j\in\Gamma_m}\{e_j - \omega(e_j)I\}\right] = \prod_{j\in\Gamma_m}\left[\exp\{it(e_j - \omega(e_j)I)\}\right]$$
$$= c_m(t)\cdot\prod_{j\in\Gamma_m}\left[\exp(ite_j)\right],$$

where $c_m(t)$ is the complex number $\prod_{j\in\Gamma_m}[\exp\{-it\omega(e_j)\}]$ of modulus 1. Recalling the definition of the product state ω, and using our previous results, we have

$$|(V_0(t)\Omega_\omega, \Omega_\omega)|^2 = |(V(t)\Omega_\omega, \Omega_\omega)|^2 = \lim_{m\to\infty}|(V_m(t)\Omega_m, \Omega_\omega)|^2$$
$$= \lim_{m\to\infty}\left|\left(\exp\{\pi_\omega(it\hat{K}_m)\}\Omega_\omega, \Omega_\omega\right)\right|^2 = \lim_{m\to\infty}\left|\omega\left(\exp[it\hat{K}_m]\right)\right|^2$$
$$= \lim_{m\to\infty}\left|\omega\left(\prod_{j\in\Gamma_m}\left[\exp(ite_j)\right]\right)\right|^2 = \lim_{m\to\infty}\left|\prod_{j\in\Gamma_m}\left(\exp[ite_j]\phi_j, \phi_j\right)\right|^2.$$

Using the fact that $\exp(ite_j)$ is unitary, for $j = 1, 2, \ldots$, it follows that

$$\left|\left(\exp[ite_j]\phi_j, \phi_j\right)\right|^2 \leqslant 1,$$

hence, singling out the $k \in \mathbf{Z}^+$ guaranteed in condition (iv) of the statement of the theorem, we conclude from the limit above that

$$|(V_0(t)\Omega_\omega, \Omega_\omega)|^2 \leqslant |(\exp[ite_k]\phi_k, \phi_k)|^2.$$

Since $0 < |(f, \phi_k)| < 1$ by assumption, we may write $\phi_k = af + b\phi'$, where ϕ' is a unit vector orthogonal to f, and $|a|^2 + |b|^2 = 1$; and if we write $s = |a|^2$, then $0 < s < 1$. Then noting that $\exp(ite_k) = I + ite_k + [(it)^2/2!]e_k + \cdots = I + (e^{it} - 1)e_k$, and that $e_k\phi_k = af$, we have

$$|(\exp\{ite_k\}\phi_k, \phi_k)|^2 = \left|\left(\left[I + (e^{it} - 1)e_k\right]\phi_k, \phi_k\right)\right|^2 = \left|1 + \left[(e^{it} - 1)\cdot|a|^2\right]\right|^2$$
$$= |1 + (e^{it} - 1)\cdot s|^2 = 1 + 2(s - s^2)[\cos(t) - 1].$$

Since $0 < s < 1$, we have $s - s^2 > 0$, and thus we can find a t such that $1 + 2(s - s^2)[\cos(t) - 1]$ is less than 1. Retracing our inequalities, we conclude that $\|(V_0(t)\Omega_\omega, \Omega_\omega)|^2 < 1$ for this choice of t, hence $V_0(t)$ is not of the form $e^{i\alpha}\cdot I$. Since $V_0(t)$ is in the center of $\pi_0(\mathfrak{A}^G)''$, ω^G cannot be a factor state. This concludes the proof of the theorem.

V. Unitary equivalence of pure states on $\mathfrak{A}^G$. In this section we consider pure states on $\mathfrak{A}^G$ of the form described in Theorem 3.2, and determine necessary and sufficient conditions for two such states to be unitarily equivalent. By [**8**, Proposition 5.3.3] the GNS representations π_ω, π_ρ, associated with pure states ω, ρ, on a C^*-algebra A are unitarily equivalent if and only if they are quasi-equivalent. Hence Bratteli's result on the quasi-equivalence of factor states is applicable to our situation.

THEOREM 5.1 [**1**, THEOREM 4.5]. *Let A be an AF-algebra $A = \overline{\bigcup_{m=1}^{\infty} A_m}$, where $A_1 \subset A_2 \subset \cdots$ are finite-dimensional C^*-algebras, let ω, ρ be factor states on A, and π_ω, π_ρ the corresponding GNS representations of ω, ρ, respectively. Suppose* $\ker \pi_\omega = \ker \pi_\rho$. *Then ω, ρ are quasi-equivalent if and only if for each $\varepsilon > 0$, there exists a positive integer q such that $|\omega(x) - \rho(x)| \leq \varepsilon \|\pi_\omega(x)\|$, all $x \in A_q^c$.*

The notation of §III will be employed throughout the discussion. In particular, we construct pure states ω^G, ρ^G on $\mathfrak{A}^G$ as follows: let $\{f_1,\ldots,f_n\}$, $\{f_1',\ldots,f_n'\}$ be orthonormal bases of H, let r, r': $\mathbf{Z}^+ \to \{1,2,\ldots,n\}$ be maps, and define $\omega = \omega_{f_{r(1)}} \otimes \omega_{f_{r(2)}} \otimes \cdots$, $\rho = \omega_{f'_{r'(1)}} \otimes \omega_{f'_{r'(2)}} \otimes \cdots$, with $0 < c_j < \infty$ (resp. $0 < c_j' < \infty$) for at most one j, where c_j (resp. c_j') denotes the cardinality of the set $\gamma_j = \{k \in \mathbf{Z}^+ : r(k) = j\}$ (resp. $\gamma_j' = \{k \in \mathbf{Z}^+ : r'(k) = j\}$). Let ω^G, ρ^G be the restrictions to $\mathfrak{A}^G$ of ω, ρ, respectively. By Theorem 3.2, both ω^G and ρ^G are pure. As an application of Lemma 3.5, the following relations hold for a cycle $g \in S(\infty)$:

$$
(1)\qquad
\begin{aligned}
\omega^G(g) = \omega(g) &= \begin{cases} 1, & \text{if } s(g) \subset \gamma_j \text{ for some } j, \\ 0, & \text{otherwise,} \end{cases} \\
\rho^G(g) = \rho(g) &= \begin{cases} 1, & \text{if } s(g) \subset \gamma_j \text{ for some } j, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

LEMMA 5.2. *Let ω^G, ρ^G be as above. Then ω^G, ρ^G are quasi- (hence unitarily) equivalent only if there exists a positive integer M such that for i, $j > M$, $r(i) = r(j)$ if and only if $r'(i) = r'(j)$.*

PROOF. Assume ω^G, ρ^G are quasi-equivalent, then given $0 < \varepsilon < 1$ there exists, by Theorem 5.1, an M such that $|\omega^G(x) - \rho^G(x)| < \varepsilon \|\pi_{\omega^G}(x)\|$ for all $x \in (\mathfrak{A}_M^G)^c$. Assuming that the condition of the lemma fails to hold there exist, without loss of generality, indices i, $j > M$ such that $r(i) = r(j)$ but $r'(i) \neq r'(j)$. Let x be the transposition (ij); then clearly $x \in (\mathfrak{A}_M^G)^c$ and $\|\pi_{\omega^G}(ij)\| = \|(ij)\| = 1$. Applying (1), we have

$$1 = |1 - 0| = |\omega^G(ij) - \rho^G(ij)| \leq \varepsilon,$$

a contradiction, which yields the lemma.

REMARK. Applying the preceding lemma and the relations (1) we see that by reordering the basis vectors $f_1',\ldots,f_n'$ if necessary, we may assume $r(i) = r'(i)$ for all $i > M$, and that if $0 < c_j < \infty$, $0 < c_k' < \infty$ for some indices j, k, then $j = k$. Moreover, we may assume that after this reordering $f_i = f_i'$ for $i = 1,2,\ldots,n$ again by applying (1). Thus if ω^G and ρ^G are quasi-equivalent, we may assume, finally, that ρ has the form $\rho = \omega_{f_{r'(1)}} \otimes \omega_{f_{r'(2)}} \otimes \cdots$, where $r(i) = r'(i)$ for $i > M$.

LEMMA 5.3. *Let* $\omega^G = \omega|_{\mathfrak{A}^G}$ (*resp.* $\rho^G = \rho|_{\mathfrak{A}^G}$) *where* $\omega = \omega_{f_{r(1)}} \otimes \omega_{f_{r(2)}} \otimes \cdots$ (*resp.* $\rho = \omega_{f_{r'(1)}} \otimes \cdots$) *and* $r(i) = r'(i)$ *for all* $i > M$. *Let* $\Lambda = \{1, 2, \ldots, M\}$; *then* ω^G *and* ρ^G *are quasi-equivalent only if for all* l *with* $c_l = \infty = c'_l$, *the sets* $\gamma_l \cap \Lambda$ *and* $\gamma'_l \cap \Lambda$ *have the same order.*

PROOF. Suppose $c_l = c'_l = \infty$ and $\#(\gamma_l \cap \Lambda) - \#(\gamma'_l \cap \Lambda) = k$, where $\#$ denotes the order of a set. Then we may construct subsets $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \cdots \subset \mathbf{Z}^+$ of order $\#\,\Gamma_m = m$ and union $\cup_{m=1}^{\infty} \Gamma_m = \mathbf{Z}^+$ such that for sufficiently large m

$$\#(\gamma_l \cap \Gamma_m) = r_m, \tag{2}$$
$$\#(\gamma'_l \cap \Gamma_m) = r_m - k \quad \text{and} \quad m - r_m, m - (r_m - k) < (m)^{1/3}.$$

Furthermore, let $f = f_l$, let $e \in N$ be the one-dimensional projection satisfying $ef = f$, and let $e_j \in N_j$, $j = 1, 2, 3, \ldots$, be the corresponding one-dimensional projections. Then we have

$$\sum_{j=1}^{\infty} \omega(e_j) = \infty; \quad \sum_{j=1}^{\infty} \rho(e_j) = \infty, \tag{3}$$

and

$$0 = \sum_{j=1}^{\infty} \omega(e_j)\big[1 - \omega(e_j)\big]; \quad 0 = \sum_{j=1}^{\infty} \rho(e_j)\big[1 - \rho(e_j)\big]. \tag{4}$$

As in §IV, construct operators $K_m = \Sigma_{j \in \Gamma_m} e_j$, $\hat{K}_m = K_m - \omega(K_m)I$, $K'_m = K_m - \rho(K_m)I$, lying in $\mathfrak{A}$, and operators $J_m = [\Sigma_{i \neq j \in \Gamma_m}(ij)] \cdot (1/2m)$, $\hat{J}_m = J_m - \omega(J_m)I$, $J'_m = J_m - \rho(J_m)I$, lying in $\mathfrak{A}^G$. Then applying (3) and (4) we remark that the following relations are obtained as in the proof of Theorem 4.1:

$$V(t) = \operatorname*{st-lim}_{m \to \infty} \pi_{\omega^G}(\exp[it\hat{J}_m]) \in \pi_{\omega^G}(\mathfrak{A}^G)'' \cap \pi_{\omega^G}(\mathfrak{A}^G)'; \tag{5}$$
$$V'(t) = \operatorname*{st-lim}_{m \to \infty} \pi_{\omega^G}(\exp[itJ'_m]) \in \pi_{\omega^G}(\mathfrak{A}^G)'' \cap \pi_{\omega^G}(\mathfrak{A}^G)',$$

$$\lim_{m \to \infty} \omega\big(\exp[it\hat{J}_m] - \exp[it\hat{K}_m]\big) = 0; \quad \lim_{m \to \infty} \rho\big(\exp[itJ'_m] - \exp[itK'_m]\big) = 0. \tag{6}$$

Furthermore, a routine calculation gives $\omega(\exp[it\hat{K}_m]) = 1$; and similarly, $\rho(\exp[itK'_m]) = 1$, for all m.

Now suppose ω^G, ρ^G are unitarily equivalent. Then [**9**, Corollary 9] there exists a unitary operator $U \in \mathfrak{A}^G$ such that $\rho^G(x) = \omega^G(U^* x U)$ for all $x \in \mathfrak{A}^G$. Applying (6) we then have

$$\begin{aligned} 1 &= \lim_{m \to \infty} \rho\big(\exp[itK'_m]\big) = \lim_{m \to \infty} \rho^G\big(\exp[itJ'_m]\big) \\ &= \lim_{m \to \infty} \omega^G\big(U^* \exp[it\hat{J}_m] \cdot \exp(it[\omega(J_m) - \rho(J_m)])U\big) \\ &= \lim_{m \to \infty} \exp\big(it[\omega(J_m) - \rho(J_m)]\big) \cdot \omega^G\big(U^* \exp[it\hat{J}_m]U\big). \end{aligned}$$

A straightforward calculation (using (1) and (2)) shows that

$$\lim_{m\to\infty}\left[\omega(J_m)-\rho(J_m)\right]=k.$$

Furthermore, since st-$\lim_{m\to\infty}\pi_{\omega^G}(\exp[it\hat{J}_m])$ lies in $\pi_{\omega^G}(\mathfrak{A}^G)'$, we verify easily that $\lim_{m\to\infty}\omega^G(U^*\exp[it\hat{J}_m]U)=\lim_{m\to\infty}\omega^G(\exp[it\hat{J}_m])$, and thus by (6) we have $\lim_{m\to\infty}\omega^G(U^*\exp[it\hat{J}_m]U)=\lim_{m\to\infty}\omega(\exp[it\hat{K}_m])=1$. Combining our calculations gives $1=\lim_{m\to\infty}\exp(it[\omega(J_m)-\rho(J_m)])\cdot\omega^G(U^*\exp[it\hat{J}_m]U)=\exp(itk)$, for all t, and thus we conclude that $k=0$. Hence the lemma is proved.

THEOREM 5.4. *Let ω^G and ρ^G be pure states on $\mathfrak{A}^G$ which are restrictions of pure product states on $\mathfrak{A}$ of the form considered in Theorem* 3.2. *Then ω^G and ρ^G are unitarily equivalent if and only if there exists a basis $\{f_1,\ldots,f_n\}$ and mappings r, r': $\mathbf{Z}^+\to\{1,2,\ldots,n\}$ such that $\omega^G=\omega|_{\mathfrak{A}^G}$, $\rho^G=\rho|_{\mathfrak{A}^G}$, where*

(i) $\omega=\omega_{f_{r(1)}}\otimes\omega_{f_{r(2)}}\otimes\cdots$; $\rho=\omega_{f_{r'(1)}}\otimes_{f_{r'(2)}}\otimes\cdots$,

(ii) *there exists an M such that for $i>M$, $r(i)=r'(i)$,*

(iii) $\#\{k\in\Lambda\colon r(k)=j\}=\#\{k\in\Lambda\colon r'(k)=j\}$, $j=1,2,\ldots,n$, *where* $\Lambda=\{1,2,\ldots,M\}$.

PROOF. Combining the results of Lemmas 5.2 and 5.3 yields the "only if" part of the argument. For the other direction, there exists, by condition (iii), a permutation $t\in S(\infty)$ such that $r'(i)=r(t^{-1}(i))$, for all i. Then it is straightforward to show that $\rho^G(g)=\omega^G(t^{-1}gt)$ for all g in $S(\infty)$, and by linearity and continuity we then have $\rho^G(x)=\omega^G(t^{-1}xt)$, for all $x\in\mathfrak{A}^G$. Thus t is the unitary operator of $\mathfrak{A}^G$ which implements the unitary equivalence of ω^G and ρ^G.

NOTE ADDED IN PROOF. Baker and Powers have recently extended the results of this paper by considering restrictions of product states on $\mathfrak{A}$ to the group-invariant algebras $\mathfrak{A}^G$, where $G=\mathrm{SU}(n)$ or $G=U(1)$, [**12**, **13**]. In the cases $G=U(1)$ (where $\mathfrak{A}^G$ is the gauge-invariant CAR algebra) and $G=\mathrm{SU}(2)$, they have determined necessary and sufficient conditions for the restricted states to be factor states, and they have classified the restricted factor states according to type, [**13**].

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