

STABLE RANK 2 REFLEXIVE SHEAVES ON \mathbf{P}^3 WITH SMALL c_2 AND APPLICATIONS

BY

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ABSTRACT. We investigate the moduli spaces of stable rank two reflexive sheaves on \mathbf{P}^3 with small Chern classes. As an application to curves of low degree in \mathbf{P}^3 , we prove the curve has maximal rank and that the corresponding Hilbert scheme is irreducible and unirational.

Introduction. In the past few years, the subject of vector bundles on projective spaces and, in particular, the case of rank 2 on \mathbf{P}^3 , has received much attention. Several basic theorems have been proved, e.g., the existence of the moduli space of stable vector bundles [M], though so far very few of these moduli spaces have been studied in detail.

The interest of vector bundles partly stems from their connection with curves. However, the class of curves obtained in this way is rather restricted. Recently, Hartshorne [SRS] has focused attention on reflexive sheaves of rank 2 on \mathbf{P}^3 . On the one hand, most results proved for vector bundles also turn out to be true for reflexive sheaves. On the other hand, reflexive sheaves have two significant advantages. First, they correspond to quite general curves; second, and most importantly, one has the “reduction step” introduced by Hartshorne, which is an effective tool in studying vector bundles, by relating them to simpler reflexive sheaves.

In this paper, we investigate the moduli spaces of stable rank 2 reflexive sheaves on \mathbf{P}^3 with $c_2 \leq 3$. For $c_2 \leq 2$, we prove the moduli spaces are irreducible, nonsingular and rational; we also classify the related unstable planes. For $c_2 = 3$, in most cases we show that the moduli space is irreducible and unirational. As one of the applications to curves of low degree in \mathbf{P}^3 (cf. [SRS, 4.1]), we prove that the curve has maximal rank and that the corresponding Hilbert scheme is irreducible and unirational. Hence, as a corollary, we conclude that the variety of moduli \mathfrak{M}_g of curves of genus g is unirational, for $g = 5, 6, 7, 8$.

In §1, we give facts on multiple lines and deduce a criterion for unstable planes. §2 contains our classification of semistable sheaves with $c_2 \leq 2$. §3 is about sheaves with $c_2 = 3$. §4 gives the applications to curves.

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Berkeley a number of visiting scholars from abroad, and together they created a very active and stimulating atmosphere from which I benefited greatly.

CONVENTIONS. By $E(c_1, c_2, c_3)$ we will denote a rank 2 reflexive sheaf on \mathbf{P}^3 with Chern classes (c_1, c_2, c_3) . According to Maruyama [M], a moduli space $M(c_1, c_2, c_3)$ for stable rank 2 reflexive sheaves on \mathbf{P}_k^3 , with Chern classes (c_1, c_2, c_3) , exists as a scheme of finite type over k . The underlying variety of $M(c_1, c_2, c_3)$ will be denoted $M_{\text{red}}(c_1, c_2, c_3)$. One fact which will be used repeatedly is that any irreducible component of $M(c_1, c_2, c_3)$ has dimension $\geq l = 8c_2 - 3$ (respectively $8c_2 - 5$), if $c_1 = 0$ (resp. if $c_1 = -1$).

Let P be a property of a reflexive sheaf. We will say that P is *special* if every irreducible component of M contains a nonempty open set corresponding to sheaves which do not have property P . To show P is special, it suffices to show that the set of E having property P has dimension $< l$. The remark will be used extensively. A property which is not special is *general*.

Unless otherwise stated, a curve will be a generically locally complete intersection, Cohen-Macaulay closed scheme of pure dimension one. We shall work over an algebraically closed ground field k of arbitrary characteristic, except in §4, and 3.6.

As a general reference on reflexive sheaves, see [SRS].

1. Remarks on multiple lines. In this section we give one condition for a stable rank 2 reflexive sheaf E to have an unstable plane, when E corresponds to a multiple line.

DEFINITION. A curve Y is a *multiplicity n line* if Y is supported on a line and length $\mathcal{O}_{Y, \eta} = n$, η being the generic point.

LEMMA 1.0. *If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and $0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0$ are two exact sequences, then the sequence $A \otimes B' + B \otimes A' \rightarrow B \otimes B' \rightarrow C \otimes C' \rightarrow 0$ is exact.*

PROOF. Diagram chasing of the following:

$$\begin{array}{ccccccc}
 A \otimes A' & \rightarrow & A \otimes B' & \rightarrow & A \otimes C' & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 B \otimes A' & \rightarrow & B \otimes B' & \rightarrow & B \otimes C' & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 C \otimes A' & \rightarrow & C \otimes B' & \rightarrow & C \otimes C' & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & &
 \end{array}$$

PROPOSITION 1.1. *Let Y_n be a multiplicity n line in \mathbf{P}^3 . If $\chi(\mathcal{O}_{Y_n}) \leq n$, then either there exists a plane containing a subscheme of Y_n of degree ≥ 2 , or $H^1(\mathcal{O}_{Y_n}(-1)) = 0$. The latter occurs only when $\chi(\mathcal{O}_{Y_n}) = n$.*

PROOF. Suppose Y_n is supported on a line L defined by $x = y = 0$. We may assume $I_{Y_n} \not\subset I_L^2$; otherwise any plane $ax + by = 0$ will do. Define Y_i by $I_{Y_i} := I_{Y_n} + I_L^i$, then we have the filtration

$$\mathcal{O} \supset I_L \supset I_{Y_2} \supset \cdots \supset I_{Y_{n-1}} \supset I_{Y_n}$$

and hence

$$\chi(\mathcal{O}_{Y_n}) = \chi(\mathcal{O}_L) + \chi(I_L/I_{Y_2}) + \chi(I_{Y_2}/I_{Y_3}) + \cdots + \chi(I_{Y_{n-1}}/I_{Y_n}).$$

So we want to study $I_{Y_i}/I_{Y_{i+1}}$. By definition, $I_{Y_i} = I_{Y_{i+1}} + I_L^i$ and $I_L^i \cap I_{Y_{i+1}} \supset I_L^{i+1}$, whence an isomorphism $I_{Y_i}/I_{Y_{i+1}} \simeq I_L^i/(I_L^i \cap I_{Y_{i+1}})$ and an exact sequence

$$(1) \quad 0 \rightarrow (I_L^i \cap I_{Y_{i+1}})/I_L^{i+1} \rightarrow I_L^i/I_L^{i+1} \rightarrow I_L^i/(I_L^i \cap I_{Y_{i+1}}) \rightarrow 0.$$

For $i = 1$, using the fact that $I_{Y_2} \cap I_L = I_{Y_2}$, the above sequence becomes

$$(2) \quad 0 \rightarrow I_{Y_2}/I_L^2 \rightarrow I_L/I_L^2 \rightarrow I_L/I_{Y_2} \rightarrow 0.$$

I_L/I_{Y_2} is a rank 1 \mathcal{O}_L -module and $I_L/I_L^2 \simeq 2\mathcal{O}_L(-1)$ maps onto it, so we may assume $I_L/I_{Y_2} \simeq \mathcal{O}_L(r-1) + \text{torsion}$, where $r \geq 0$.

Claim. $I_{Y_{i+1}}/I_{Y_{i+2}} \simeq \mathcal{O}_L(r-1)^{\otimes i+1} + \text{torsion}$.

PROOF. The proof is by induction. Suppose $I_{Y_i}/I_{Y_{i+1}} \simeq \mathcal{O}_L(r-1)^{\otimes i} + \text{torsion}$. Applying Lemma 1.0 to the exact sequences (1) and (2), we get the top row in the diagram below. The bottom row is obtained from sequence (1) by substituting $i+1$ for i .

$$\begin{array}{ccccccc} K & \rightarrow & (I_L/I_L^2) \otimes (I_L^i/I_L^{i+1}) & \rightarrow & (I_L/I_{Y_2}) \otimes I_L^i/(I_L^i \cap I_{Y_{i+1}}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & I_L^{i+1}/I_L^{i+2} & \rightarrow & I_L^{i+1}/(I_L^{i+1} \cap I_{Y_{i+2}}) & \rightarrow & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

where $K := (I_{Y_2}/I_L^2) \otimes (I_L^i/I_L^{i+1}) + (I_L/I_L^2) \otimes (I_L^i \cap I_{Y_{i+1}})/I_L^{i+1}$. If we can show that the induced dotted map is well defined, then this gives a surjective map from $(I_L/I_{Y_2}) \otimes I_L^i/(I_L^i \cap I_{Y_{i+1}}) \simeq \mathcal{O}_L(r-1)^{\otimes i+1} + \text{torsion}$ to $I_L^{i+1}/(I_L^{i+1} \cap I_{Y_{i+2}}) \simeq I_{Y_{i+1}}/I_{Y_{i+2}}$, so $I_{Y_{i+1}}/I_{Y_{i+2}} \simeq \mathcal{O}_L(r-1)^{\otimes i+1} + \text{torsion}$, and we are done.

It suffices to check that any element in K is mapped to zero in $I_L^{i+1}/(I_L^{i+1} \cap I_{Y_{i+2}})$, i.e., if $a \in I_{Y_2} \subset I_L$, $b \in I_L^i$, or $b \in I_L^i \cap I_{Y_{i+1}}$, then $ab \in I_L^{i+1} \cap I_{Y_{i+2}}$. But this is clear from $I_L^k \cdot I_{Y_h} \subset I_{Y_{k+h}}$. Thus the claim is proved.

It follows from the claim that

$$\begin{aligned} \chi(\mathcal{O}_{Y_n}) &= \chi(\mathcal{O}_L) + \chi(\mathcal{O}_L(r-1)) + \chi(\mathcal{O}_L(r-1)^{\otimes 2}) \\ &\quad + \cdots + \chi(\mathcal{O}_L(r-1)^{\otimes n-1}) + t \end{aligned}$$

for $r \geq 0$ and $t \geq 0$. If $r \geq 1$, then $\chi(\mathcal{O}_L(r-1)^{\otimes i}) \geq 1$, and $\chi(\mathcal{O}_{Y_n}) \geq n + t$. So our hypothesis $\chi(\mathcal{O}_{Y_n}) \leq n$ implies either $r = 0$, or $r = 1$ and $t = 0$. If $r = 0$, then exact sequence (2) is

$$0 \rightarrow I_{Y_2}/I_L^2 \rightarrow I_L/I_L^2 \xrightarrow{\text{is}} \mathcal{O}_L(-1) + \text{torsion} \rightarrow 0$$

$$2\mathcal{O}_L(-1)$$

so $I_{Y_n} \subset I_{Y_2} \subset (ax + by, I_L^2)$, where a, b are scalars. The plane $ax + by = 0$ contains the double line defined by the ideal $(ax + by, I_L^2)$ which is a degree 2 subscheme of Y_n .

If $r = 1$ and $t = 0$, then $I_Y/I_{Y_{i+1}} \simeq \mathcal{O}_L$ and we have $0 \rightarrow \mathcal{O}_L \rightarrow \mathcal{O}_{Y_{i+1}} \rightarrow \mathcal{O}_{Y_i} \rightarrow 0$ for all i . Twisting by -1 and taking cohomology, $h^1(\mathcal{O}_L(-1)) = 0$ yields

$$h^1(\mathcal{O}_{Y_2}(-1)) = h^1(\mathcal{O}_{Y_3}(-1)) = \cdots = h^1(\mathcal{O}_{Y_n}(-1)) = 0.$$

REMARK 1.1.1. Let Y_n be a multiplicity n line in \mathbf{P}^3 . If $\chi(\mathcal{O}_{Y_n}) = n$, and Y_n is contained in a degree 2 surface, then either there exists a plane containing a subscheme of Y_n of degree ≥ 2 , or Y_n is type $(0, n)$ on a nonsingular quadric.

PROOF. As in the proof of the proposition above, we have $I_Y \subset I_{Y_2} = (ux + vy, I_L^2)$, where u, v are the other coordinates of \mathbf{P}^3 . Hence the nonsingular quadric $ux + vy + ax^2 + bxy + cy^2$ contains Y , where a, b, c are scalars. From arithmetic genus, it is easy to see that Y_n is type $(0, n)$.

REMARK 1.2. If E has $c_1 = 0$ or -1 , a section of $E(1)$ corresponds to a curve Y , let H be a plane containing a subscheme of Y of degree r . Then H is an unstable plane of order $r - c_1 - 1$. Moreover, the corresponding section gives a nonzero section of the kernel in the reduction sequence.

PROOF. Restricting on H the exact sequence $0 \rightarrow \mathcal{O} \rightarrow E(1) \rightarrow I_Y(2 + c_1) \rightarrow 0$, we get the surjective map $E_H \rightarrow I_{Z,H}(1 + c_1 - r) \rightarrow 0$. Combining with the map $E \rightarrow E_H \rightarrow 0$, we get the surjective map for an unstable plane.

REMARK 1.3. The family of all double lines Y_2 with $\chi(\mathcal{O}_{Y_2}) = r + 1$ is irreducible, nonsingular, rational and of dimension $2r + 5$. Also, $h^0(\mathcal{O}_{Y_2}(k)) = 2k - r - 1$, if $k \geq 1$.

PROOF. Since Y_2 is Cohen-Macaulay, in the exact sequence $0 \rightarrow I_{L,Y_2} \rightarrow \mathcal{O}_{Y_2} \rightarrow \mathcal{O}_L \rightarrow 0$, I_{L,Y_2} is a torsion-free rank 1 \mathcal{O}_L -module; since $\chi(\mathcal{O}_{Y_2}) = r + 1$ implies $\chi(\mathcal{O}_{L,Y_2}) = r$, we have $I_{L,Y_2} = \mathcal{O}_L(r - 1)$. Substituting this in exact sequence (2) in 1.1, we get $I_{Y_2} = (fx + gy, I_L^2)$ where f and g are homogeneous forms of degree r in the remaining coordinates u, v of \mathbf{P}^3 . Y_2 is determined by L, f and g . The family of those Y_2 is isomorphic to an open subset of $\mathbf{P}_G(\pi_*(2\mathcal{O}_{\mathcal{L}}(r)))$ where \mathcal{L} is the universal \mathbf{P}^1 bundle considered as a subset of $G(1, 3) \times \mathbf{P}^3 \xrightarrow{\pi} G(1, 3) =: G$, $G(1, 3)$ being the Grassmann variety of lines in \mathbf{P}^3 . The choices of L, f and g depend on $4 + 2r + 1 = 5 + 2r$ parameters. Taking the dual of the sequence $0 \rightarrow \mathcal{O}_L(r - 1) \rightarrow \mathcal{O}_{Y_2} \rightarrow \mathcal{O}_L \rightarrow 0$, we get $0 \rightarrow \omega_L \rightarrow \omega_{Y_2} \rightarrow \omega_L(1 - r) \rightarrow 0$, so for $k \geq 1$, we have

$$h^0(\omega_{Y_2}(k)) = h^0(\omega_L(k)) + h^0(\omega_L(k + 1 - r)) = 2k - r - 1.$$

2. Semistable reflexive sheaves with $c_2 \leq 2$. In this section we shall give a complete classification of the semistable reflexive sheaves on \mathbf{P}^3 with $c_2 \leq 2$. For each set of Chern classes we shall give the spectrum and cohomology table; the latter is easy to compute from Riemann-Roch, Castelnuovo's theorem [Mu, p. 99], and spectrum, and so we shall omit the details.

LEMMA 2.0. *Let E be a properly semistable rank 2 reflexive sheaf on \mathbf{P}^3 .*

(a) *If $c_2 = 0$, then $\dim \text{End}(E) = 4$.*

(b) *If $c_2 > 0$, then $\dim \text{End}(E) = 2$.*

PROOF. (a) If $c_2 = 0$, then $E \simeq \mathcal{O} \oplus \mathcal{O}$ [SRS, 9.7] and $\dim \text{End}(E) = 4$.

(b) If $c_2 > 0$, then $H^0(E) \neq 0$ and $Y \neq \emptyset$ in the extension $0 \rightarrow \mathcal{O} \rightarrow E \rightarrow I_Y \rightarrow 0$ imply $h^0(E) = 1$. Applying $\text{Hom}(E, \cdot)$, we get

$$0 \rightarrow \text{Hom}(E, \mathcal{O}) \rightarrow \text{Hom}(E, E) \rightarrow \text{Hom}(E, I_Y).$$

So $\dim \text{Hom}(E, E) \leq \dim \text{Hom}(E, \mathcal{O}) + \dim \text{Hom}(E, I_Y) \leq 2 \dim \text{Hom}(E, \mathcal{O}) = 2h^0(E) = 2$; the second inequality holds because $I_Y \hookrightarrow \mathcal{O}$ implies $\text{Hom}(E, I_Y) \hookrightarrow \text{Hom}(E, \mathcal{O})$. On the other hand, the short exact sequence above gives the endomorphism $\sigma^0: E \rightarrow I_Y \hookrightarrow \mathcal{O} \rightarrow E$ which is not a scalar multiplication. So, $\dim \text{End}(E) \geq 2$, hence is 2.

LEMMA 2.1. *If $E(0, 1, c_3)$ is semistable, then $c_3 = 0$ or 2. If $c_3 = 0$, E is stable. If $c_3 = 2$, E is only properly semistable.*

(This is in [C2]; we give the proof for the sake of completeness.)

PROOF. Theorem 8.2 in [SRS] implies the first statement and $H^2(E) = 0$. Riemann-Roch gives $\chi(E) = \frac{1}{2}c_3$. Hence if $c_3 = 2$, then $h^0(E) \neq 0$ and E is not stable. Conversely, if E is properly semistable, then $H^0(E) \neq 0$, and we have the extension

$$(1) \quad 0 \rightarrow \mathcal{O} \rightarrow E \rightarrow I_L \rightarrow 0.$$

So, $c_3 = 2p_a - 2 + 4d = 2$.

REMARK 2.1.1. Stable vector bundles with Chern classes $(0, 1, 0)$ were studied in [B and W].

LEMMA 2.2. *There exists an irreducible, nonsingular, rational, 6-dimensional variety $V_{s,s}(0, 1, 2)$ whose closed points are in one-to-one correspondence with the isomorphism classes of properly semistable sheaves with Chern classes $(0, 1, 2)$.*

PROOF. E is one-to-one correspondent to the zero set L , and the image of ξ in $H^0(\omega_L(4))/k^* \simeq H^0(\mathcal{O}_L(3))/k^*$, so $V_{s,s}(0, 1, 2)$ is isomorphic to $\mathbf{P}_G(\pi_* \mathcal{O}_{\mathcal{L}}(2)^\vee)$, where \mathcal{L} is the universal \mathbf{P}^1 bundle considered as a subset of $G(1, 3) \times \mathbf{P}^3 \xrightarrow{\pi} G = G(1, 3)$, hence irreducible, nonsingular, and rational of dimension 6.

LEMMA 2.3. *A plane H is an unstable plane of order 1 of $E(0, 1, 2)$ in 2.2, if and only if H contains the unique line L .*

PROOF. If $H \cap L = P$ is a point, then restricting (1) on H gives $0 \rightarrow \mathcal{O}_H \rightarrow E_H \rightarrow I_P \rightarrow 0$, hence $H^2(E_H(-2)) = 0$, and H is not unstable for E . If $H \supset L$, then (1) on H gives

$$0 \rightarrow I_{W,H}(1) \rightarrow E_H \rightarrow \mathcal{O}_H(-1) \rightarrow 0$$

where W is the set of nonlocally-free points of E which has length 2.

$$\text{Ext}^1(\mathcal{O}_H(-1), I_{W,H}(1)) = H^1(I_{W,H}(2)) = 0,$$

so the sequence above splits, and

$$(2) \quad E_H = I_{W,H}(1) \oplus \mathcal{O}_H(-1).$$

TABLE 2.3.1. $h^i(E(l))$ for $E(0, 1, 2)$ semistable.

$i \backslash l$	-4	-3	-2	-1	0	1	2
3	1	0	0	0	0	0	0
2	2	2	1	0	0	0	0
1	0	0	0	0	0	0	0
0	0	0	0	0	1	6	17

Spectrum $\{0, -1\}$

For $E(-1, 2, c_3)$ stable, there are three cases. The cases $(-1, 2, 0)$ and $(-1, 2, 4)$ were studied in [H-S and SRS]. Here we do the case $c_3 = 2$.

LEMMA 2.4. *Let $E(-1, 2, 2)$ be stable. Then the zero set of $E(1)$ is a pair of skew lines (or its degeneration, a double line).*

PROOF. As in 2.1, Riemann-Roch and spectrum imply $H^0(E(1)) \neq 0$, and we have

$$(3) \quad 0 \rightarrow \mathcal{O} \rightarrow E(1) \rightarrow I_Y(1) \rightarrow 0$$

where Y has $\deg(Y) = 2$, and $\chi(\mathcal{O}_Y) = 2$. So the ones in the statement are the only possible Y .

REMARK 2.4.1. For Y as above, we always have $\omega_Y \simeq \mathcal{O}_Y(-2)$, and

$$(4) \quad 0 \rightarrow \mathcal{O}_L \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{L'} \rightarrow 0$$

where L' may be equal to L (see 1.3).

THEOREM 2.5. *The moduli of stable rank 2 reflexive sheaves with Chern classes $(-1, 2, 2)$ is irreducible, nonsingular, and rational of dimension 11.*

PROOF. In 2.4 we have seen that $h^0(E(1)) = 1$, so E is one-to-one correspondent to Y and ξ/k^* in $H^0(\omega_Y(3))/k^* = H^0(\mathcal{O}_Y(1))/k^*$, and the family is an open subset of $\mathbf{P}_{G \times G}(p_1^* \pi_* \mathcal{O}_{\mathbb{P}}(1) \oplus p_2^* \pi_* \mathcal{O}_{\mathbb{P}}(1))$, where π, G, \mathbb{P} are as in 2.2. The choice of Y and ξ/k^* is $8 + 3 = 11$. To conclude that this is the whole moduli, we claim $\text{Ext}^2(E, E) = 0$. Apply $\text{Ext}^i(\cdot, E)$ to (3), (4) in 2.4, 2.4.1, and the exact sequence for the ideal sheaf of Y . Using the fact that $\text{Ext}^i(\mathcal{O}(-k), E) = H^i(E(k)) = 0$ for $i = 1, 2, 3$ and $k \geq 0$, we get

$$\text{Ext}^2(E, E) = \text{Ext}^2(I_Y, E) = \text{Ext}^3(\mathcal{O}_Y, E)$$

which vanishes because $\text{Ext}^3(\mathcal{O}_L, E) = \text{Ext}^3(\mathcal{O}_{L'}, E) = 0$ by applying $\text{Ext}^i(\cdot, E)$ to the resolution $0 \rightarrow \mathcal{O}(-2) \rightarrow 2\mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_L \rightarrow 0$.

PROPOSITION 2.6. *Let $E(-1, 2, 2)$ be stable, as in 2.4. Then a plane H is unstable of order 1 if and only if either H contains one of the lines or H contains both nonlocally-free points of E .*

PROOF. H is unstable of order 1 if and only if the map given by the defining equation of H , $h: H^2(E(-3)) \rightarrow H^2(E(-2))$, is not surjective. Taking the long cohomology sequence of (3), dualizing it, and identifying $H^2(I_Y(-3))$ with

$$H^1(\mathcal{O}_Y(-3)) \perp H^0(\omega_Y(3)),$$

we have the diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & H^0(\omega_Y(2)) & \rightarrow & H^2(E(-2))^\perp & \rightarrow & 0 \\
 & & \downarrow \check{h} & & \downarrow \check{h} & & \\
 0 & \rightarrow & H^3(\mathcal{O}(-4))^\perp & \xrightarrow{\delta} & H^0(\omega_Y(3)) & \rightarrow & H^2(E(-3))^\perp \rightarrow 0
 \end{array}$$

where δ sends 1 to ξ which vanishes at the bad points of E . \check{h} is not injective on the right-hand column, if and only if either \check{h} is not injective on the left-hand column, or $\text{Im } \delta \subset \text{Im } \check{h}$. Namely, either $L \subset H$ or H contains both bad points.

TABLE 2.6.1. $h^i(E(l))$ for $E(-1, 2, 2)$ stable.

$i \backslash l$	-4	-3	-2	-1	0	1	2
3	1	0	0	0	0	0	0
2	2	3	2	0	0	0	0
1	0	0	0	0	1	0	0
0	0	0	0	0	0	1	8

Spectrum $\{-1, -1\}$

Stable vector bundles with Chern classes $(0, 2, 0)$ were done in [SVB]. For the rest of this section, we study the other two cases, $c_3 = 2$ and 4, when $c_1 = 0$, $c_2 = 2$.

LEMMA 2.7. *Let $E(0, 2, 2)$ be stable. Then the zero set of a general section of $E(1)$ is the disjoint union of a line and a conic lying on the unique unstable plane of order 1.*

PROOF. By Table 2.8.1, $h^2(E(-3)) = 3$ and $h^2(E(-2)) = 1$, E has an unstable plane of order 1. Making reduction [SRS, 9.1] on an unstable plane H , we get $E'(-1, 1, 1)$ stable, and

$$(5) \quad 0 \rightarrow E' \rightarrow E \rightarrow I_{P,H}(-1) \rightarrow 0.$$

We know that any line through P is a zero set of $E'(1)$, so the section corresponding to a line not in H induces a section of $E(1)$; by arithmetic genus computation and the formula $c_3 = 2p_a - 2 + 2d$, the corresponding zero set for $E(1)$ is a line disjoint union a plane curve of degree 2.

The unstable plane H is unique because, taking cohomology of the reduction sequence above, we have:

$$\begin{array}{ccccccc}
 0 & \rightarrow & H^2(E(-3)) & \rightarrow & H^2(I_{P,H}(-4)) & \rightarrow & 0 \\
 & & \downarrow h' & & \downarrow h' & & \\
 0 & \rightarrow & H^2(E(-2)) & \rightarrow & H^2(I_{P,H}(-3)) & \rightarrow & 0
 \end{array}$$

The map h' is never 0 unless $h' = h$, the defining function of H .

THEOREM 2.8. *The moduli of stable rank 2 reflexive sheaves with Chern classes $(0, 2, 2)$ is irreducible, nonsingular, and rational of dimension 13.*

PROOF. Dualizing (5), we have

$$\begin{array}{ccccccc}
 0 & \rightarrow & E & \rightarrow & E'(1) & \rightarrow & I_{Z,H}(2) \rightarrow 0 \\
 & & & & \downarrow & \nearrow f & \\
 & & & & E'_H(1) & &
 \end{array}$$

where E is determined by E' , H , Z and f . We note that H and P (hence E') are unique. H contains the nonlocally-free point P of E' , so $E'_H(1) \simeq \mathcal{O}_H \oplus I_{P,H}(1)$. Identify $M = M_s(-1, 1, 1)$ with \mathbf{P}^3 via the map sending E' to P . To give E is to give P , H and the surjective map $E'_H(1) \rightarrow I_{Z,H}(2)$. Since $E'_H(1) \simeq \mathcal{O}_H \oplus I_{P,H}(1)$, the set of surjective maps for various Z is an open subset of $\text{Hom}(E'_H(1), \mathcal{O}_H(2))$. This shows that our moduli is an open set of $\mathbf{P}(q_* \mathcal{H}om(p^* \mathcal{E}, \pi^* \mathcal{O}(1)))$ where \mathcal{E} is the universal sheaf on $\mathbf{P}^3 \times M$, $p: F = \{(H, x, m) \in \mathbf{P}^{3*} \times \mathbf{P}^3 \times M \mid x, m \in H\} \rightarrow \mathbf{P}^3 \times M$, $\pi: F \rightarrow \mathbf{P}^3$. The choice of E is the choice of the point P , the plane H through P , and the map $E'_H(1) \simeq \mathcal{O}_H \oplus I_{P,H}(1) \rightarrow \mathcal{O}_H(2)$; hence is $3 + 2 + (6 + 3 - 1) = 13$.

To show that $\text{Ext}^2(E, E) = 0$, we use the resolution $0 \rightarrow \mathcal{O}(-1) \rightarrow E \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}_Y(1) \rightarrow 0$ where $Y = L \amalg C$, a line disjoint union a degree 2 plane curve. This is the same reasoning as in 2.5, except that we use $H^i(E(l)) = 0$ for $i = 2, 3$ and $l \geq -1$, or $i = 1$ and $l \geq 1$. Note that for the conic C we have

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1) \oplus \mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}_C(1) \rightarrow 0.$$

For the rest of the proof we ditto the last paragraph of 3.1 in [H-S].

TABLE 2.8.1. $h^i(E(l))$ for $E(0, 2, 2)$ stable.

$i \backslash l$	-4	-3	-2	-1	0	1	2	3
3	0	0	0	0	0	0	0	0
2	3	3	1	0	0	0	0	0
1	0	0	0	1	1	0	0	0
0	0	0	0	0	0	3	13	31

Spectrum $\{0, -1\}$

LEMMA 2.9. *Let $E(0, 2, 4)$ be stable; then we have*

$$(6) \quad 0 \rightarrow 2\mathcal{O}(-1) \rightarrow 4\mathcal{O} \rightarrow E(1) \rightarrow 0.$$

PROOF. By 2.12.2 and Castelnuovo's theorem, $E(1)$ is generated by four global sections; in fact, we have the surjective map $4\mathcal{O} \rightarrow E(1)$. The kernel is obtained by Chern classes computation and [SRS, 9.7].

LEMMA 2.10. *Let $E(0, 2, 4)$ be as in 2.9; then $\text{Ext}^2(E, E) = 0$.*

PROOF. Applying $\text{Ext}^i(\cdot, E)$ to (6), we get

$$\text{Ext}^1(2\mathcal{O}(-2), E) \rightarrow \text{Ext}^2(E, E) \rightarrow \text{Ext}^2(4\mathcal{O}(-1), E).$$

The right and left terms are $4H^2(E(1))$ and $2H^1(E(2))$ which are zero by 2.12.2. Q.E.D.

For any set K , define

$$\mathfrak{M}_{m \times n}(K) := \{m \times n \text{ matrix over } K\}.$$

LEMMA 2.11. *Define $\mathfrak{N} := \mathfrak{M}_{2 \times 4}(k^{4*})/\text{GL}(2) \times \text{GL}(4)$ through the equivalence relation $A \sim A'$ if there exist B in $\text{GL}(2)$, and C in $\text{GL}(4)$ such that $A = BA'C$. Then \mathfrak{N} is birational to \mathbf{P}^{13} .*

PROOF. An element in \mathfrak{N} is a matrix whose entries are linear forms. Such a matrix can be written as

$$A = A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3$$

where the x_i 's are the coordinates, and $A_i \in \mathfrak{N}_{2 \times 4}(k)$. For A general,

$$D := \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} \in \mathfrak{N}_{4 \times 4}(k)$$

is invertible, i.e., $\exists D^{-1} \in \text{GL}(4)$, such that $A_0D^{-1} = (I_{2 \times 2}, 0_{2 \times 2})$ and $A_1D^{-1} = (0_{2 \times 2}, I_{2 \times 2})$. If $B \in \text{GL}(2)$ and

$$C := \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \in \text{GL}(4)$$

such that $B(A_iD^{-1})C = A_iD^{-1}$ for $i = 0, 1$, then

$$B(I, 0) \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = (BC_1, BC_2) = (I, 0)$$

implies $C_1 = B^{-1}$ and $C_2 = 0$. Similarly, we get $C_4 = B^{-1}$ and $C_3 = 0$, i.e.,

$$C = \begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}.$$

Write A_2D^{-1} as (A'_2, A''_2) , where $A'_2, A''_2 \in \mathfrak{N}_{2 \times 2}(k)$, then

$$A_2D^{-1} \sim B(A'_2, A''_2) \begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = (BA'_2B^{-1}, BA''_2B^{-1}).$$

For A_2 general, A'_2 can be diagonalized uniquely, and the new A''_2 can be written as $\begin{pmatrix} * & a \\ * & * \end{pmatrix}$, where $a \neq 0$. Let $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ which makes the diagonal matrix invariant and the new A''_2 equivalent to

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} * & a \\ * & * \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} * & 1 \\ * & * \end{pmatrix}.$$

Therefore, for A general, we can find a unique $B \in \text{GL}(2)$ such that

$$\begin{aligned} A &\sim BA \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}^{-1} \begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \\ &= (I, 0)x_0 + (0, I)x_1 + \begin{pmatrix} * & 0 & * & 1 \\ 0 & * & * & * \end{pmatrix} x_2 + \bar{A}_3 x_3. \end{aligned}$$

So, an open dense set of \mathfrak{N} is isomorphic to \mathbf{A}^{13} .

THEOREM 2.12. *The moduli of stable rank 2 reflexive sheaves on \mathbf{P}^3 with Chern classes $(0, 2, 4)$ is irreducible, nonsingular, and rational of dimension 13. There is a universal family over an open set.*

PROOF. Use Lemmas 2.9, 2.10 and 2.11.

REMARK 2.12.1. Okonek [O] shows that for stable rank 2 reflexive sheaves on \mathbf{P}^4 the sequence (6) is still valid. One sees easily by the same argument that the

statement in Theorem 2.12 holds for these sheaves except that the dimension is 21 rather than 13.

TABLE 2.12.2. $h^i(E(I))$ for $E(0, 2, 4)$ stable.

$i \backslash I$	-4	-3	-2	-1	0	1	2
3	0	0	0	0	0	0	0
2	4	4	2	0	0	0	0
1	0	0	0	0	0	0	0
0	0	0	0	0	0	4	14

Spectrum $\{-1, -1\}$

LEMMA 2.13. *Let E be a general member of $M(0, 2, 4)$. Then the zero set of a general section of $E(1)$ is a nonsingular twisted cubic curve through four distinct bad points which determine four unstable, order-1 planes containing three points at a time.*

PROOF. It is clear that such a curve is a zero set of $E(1)$, and if we use these curves, we can construct an open set in $M(0, 2, 4)$ through the extension $0 \rightarrow \mathcal{O} \rightarrow E(1) \rightarrow I_Y(2) \rightarrow 0$. Reasoning as in 2.6, we have

$$\begin{array}{ccccccc}
 & & & & H^0(\mathcal{O}_{\mathbf{P}^1}(1)) & & \\
 & & & & \wr & & \\
 0 & \rightarrow & H^0(\omega_Y(1)) & \rightarrow & H^2(E(-2))^\perp & \rightarrow & 0 \\
 & & \downarrow \check{h} & & \downarrow \check{h} & & \\
 0 & \rightarrow & H^3(\mathcal{O}(-4)) & \xrightarrow{\delta} & H^0(\omega_Y(2)) & \rightarrow & H^2(E(-3))^\perp \rightarrow 0 \\
 & & \wr & & \wr & & \\
 & & k & & H^0(\mathcal{O}_{\mathbf{P}^1}(4)) & &
 \end{array}$$

where δ sends 1 to ξ which vanishes at the four bad points of E . We also identify Y with \mathbf{P}^1 by $(t^3, t^2s, ts^2, s^3) \longleftrightarrow (t, s)$. \check{h} is not injective, if and only if $\text{Im } \delta \subset \text{Im } \check{h}$, if and only if H contains three out of the four bad points of E .

REMARK 2.13.1. For any stable $E(0, 2, 4)$, and any unstable plane H , we have $E_H = \mathcal{O}_H(-1) \oplus I_{W,H}(1)$, where W has length 3.

LEMMA 2.14. *Let E be as in 2.13, and the four bad points are $P_0 = (1, 0, 0, 0)$, $P_1 = (0, 1, 0, 0)$, $P_2 = (0, 0, 1, 0)$, and $P_3 = (0, 0, 0, 1)$. Then E can be represented by*

$$\begin{pmatrix} x_0 & 0 & x_2 & x_3 \\ 0 & x_1 & x_2 & \lambda x_3 \end{pmatrix} \text{ in } \mathfrak{M}$$

(cf. 2.11), where λ is a scalar.

PROOF. Again, we write $A = \sum_{i=0}^3 A_i x_i$, where $A_i \in \mathfrak{M}_{2 \times 4}(k)$. E is not locally free at P_i 's means A has rank 1 at P_i 's (see exact sequence (6)), namely, A_i 's are rank 1 matrices, so

$$A_i = \begin{pmatrix} k_i \mathcal{Q}_i \\ k'_i \mathcal{Q}_i \end{pmatrix}$$

where k_i, k'_i are scalars, and $\mathcal{Q}_i \in \mathfrak{M}_{1 \times 4}(k)$. If A is general, \mathcal{Q}_i 's are linear independent, and

$$D := \begin{pmatrix} \mathcal{Q}_0 \\ \mathcal{Q}_1 \\ \mathcal{Q}_2 \\ \mathcal{Q}_3 \end{pmatrix} \in \mathrm{GL}(4),$$

hence

$$\begin{aligned} A \sim AD^{-1} &= \begin{pmatrix} k_0 & 0 & 0 & 0 \\ k'_0 & 0 & 0 & 0 \end{pmatrix} x_0 + \begin{pmatrix} 0 & k_1 & 0 & 0 \\ 0 & k'_1 & 0 & 0 \end{pmatrix} x_1 \\ &+ \begin{pmatrix} 0 & 0 & k_2 & 0 \\ 0 & 0 & k'_2 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 & 0 & k_3 \\ 0 & 0 & 0 & k'_3 \end{pmatrix} x_3. \end{aligned}$$

Multiplying by

$$\begin{pmatrix} k_0 & k_1 \\ k'_0 & k'_1 \end{pmatrix}^{-1}$$

on the left, we get

$$\begin{aligned} A &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_0 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x_1 \\ &+ \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & \alpha' & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & \beta' \end{pmatrix} x_3. \end{aligned}$$

Multiplying by

$$\begin{pmatrix} I & 0 \\ 0 & \begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/\beta \end{pmatrix} \end{pmatrix},$$

we get

$$\begin{aligned} A &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_0 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x_1 \\ &+ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha'/\alpha & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 & 0 & 1/\beta \\ 0 & 0 & 0 & \beta'/\beta \end{pmatrix} x_3 \end{aligned}$$

and last, applying

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha/\alpha' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha'/\alpha \end{pmatrix} & 0 \\ 0 & I \end{pmatrix}$$

together,

$$\begin{aligned} A &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_0 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x_1 \\ &\quad + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} x_3 \\ &= \begin{pmatrix} x_0 & 0 & x_2 & x_3 \\ 0 & x_1 & x_2 & \lambda x_3 \end{pmatrix} \end{aligned}$$

where $\lambda = \beta'\alpha/\beta\alpha$. Q.E.D.

Note that to an unordered set of four distinct points, P_1, \dots, P_4 , on a nonsingular rational curve Y , one can assign a well-defined j -invariant $[\mathbf{AG}]$, namely, let $\phi: Y \rightarrow \mathbf{P}^1$ be an isomorphism such that $\phi(P_1) = 0$, $\phi(P_2) = 1$, $\phi(P_3) = \infty$, $\phi(P_4) = \lambda$; and define $j = 2^8(\lambda^2 - \lambda + 1)^3/\lambda^2(\lambda - 1)^2$. Given two sets, there exists an automorphism of Y taking one set to the other set if and only if the j -invariant is the same.

PROPOSITION 2.15. *Let $M_0 \subset M = M(0, 2, 4)$ be the subset of sheaves described in 2.14, and $G_0 \subset G = \mathrm{PGL}(3)$ be the subgroup which fixes all the P_i 's. Then:*

- (a) G_0 acts trivially on M_0 .
- (b) Two twisted cubics through four given fixed points have the same j -invariant, if and only if they correspond to the same E .
- (c) G does not act transitively on M .

PROOF. An element in G_0 can be represented as

$$\sigma = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix},$$

namely, $\sigma: (x_0, x_1, x_2, x_3) \rightarrow (\alpha x_0, \beta x_1, \gamma x_2, \delta x_3)$. As we have seen in 2.14, any A in M_0 can be represented as

$$\begin{pmatrix} x_0 & 0 & x_2 & x_3 \\ 0 & x_1 & x_2 & \lambda x_3 \end{pmatrix}.$$

Thus

$$\sigma(A) = \begin{pmatrix} \alpha x_0 & 0 & \gamma x_2 & \delta x_3 \\ 0 & \beta x_1 & \gamma x_2 & \lambda \delta x_3 \end{pmatrix}.$$

Multiplying by

$$\begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & \gamma^{-1} & 0 \\ 0 & 0 & 0 & \delta^{-1} \end{pmatrix},$$

we see that $\sigma(A) \sim A$, and G_0 acts trivially on M_0 . This proves (a).

Now take E_0 in M_0 . Then G_0 acts on $H^0(E_0(1))/k^*$ (since scalar multiplication is the only automorphism of E_0) which is one-to-one correspondent to $T := \{Y \mid Y \text{ is a}$

Any E with four general nonlocally-free points can be sent to a member E_0 in M_0 by $\mathrm{PGL}(3)$, hence we combine the above and conclude (b) and (c).

PROOF. E is properly semistable, so we have

where Y is a curve of degree 2 and $\chi(\mathcal{O}_Y) = 4 - \frac{1}{2}c_3$, for the case Y is a double line. Since all such E is one-to-one correspondent to Y and $\xi/k^* \in H^0(\omega_Y(4))/k^*$, by 1.3 there is a universal subscheme \mathcal{Q} of $H \times \mathbf{P}^3$, where H is the parameter space of all such Y . So E is parametrized by an open subset of $\mathbf{P}_H(\pi_*\omega_{\mathcal{Q}}(4))$. Using 1.3 for $r = 3 - \frac{1}{2}c_3$ and $k = 4$, we get that the dimension of such E is $14 - \frac{1}{2}c_3$.

TABLE 2.16.1. $h^i(E(l))$ for properly semistable $E(0, 2, c_3)$.

$E_{s,s}(0, 2, 0)$

$i \backslash l$	-4	-3	-2	-1	0	1	2	3
3	1	0	0	0	0	0	0	0
2	3	2	1	0	0	0	0	0
1	0	0	1	2	3	2	1	0
0	0	0	0	0	1	4	13	30

Spectrum $\{-1, 1\}$

$E_{s,s}(0, 2, 2)$

$i \backslash l$	-4	-3	-2	-1	0	1	2	3
3	1	0	0	0	0	0	0	0
2	4	3	1	0	0	0	0	0
1	0	0	0	1	2	1	0	0
0	0	0	0	0	1	4	13	31

Spectrum $\{-1, 0\}$

$E_{s,s}(0, 2, 4)$

$i \backslash l$	-4	-3	-2	-1	0	1	2	3
3	1	0	0	0	0	0	0	0
2	5	4	2	0	0	0	0	0
1	0	0	0	0	1	0	0	0
0	0	0	0	0	1	4	14	32

Spectrum $\{-1, -1\}$

$E_{s,s}(0, 2, 6)$

$i \backslash l$	-4	-3	-2	-1	0	1	2	3
3	1	0	0	0	0	0	0	0
2	6	5	3	1	0	0	0	0
1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	5	15	33

Spectrum $\{-1, -2\}$

Maruyama [M] constructed $\bar{M} = \bar{M}(c_1, c_2, c_3)$ which contains an open set $M = M(c_1, c_2, c_3)$, the coarse moduli of stable sheaves. The points of $\bar{M} - M$ are in one-to-one correspondence with S -equivalence classes of properly semistable sheaves. If E is properly semistable, then it follows from exact sequence (7) that the S -equivalence class of E is determined by $\mathcal{O} \oplus I_Y$, hence by Y alone.

PROPOSITION 2.17. *We have the following table for stable and semistable reflexive sheaves with $c_1 = 0$, $c_2 = 2$:*

c_3	0	2	4	6
$\dim V_{s,s}$	14	13	13	14
$\dim M$	13	13	13	$-1 (M = \emptyset)$
$\dim(\bar{M} - M)$	11	9	8	8
M dense in \bar{M} ?	$no (M \cap (\bar{M} - M) = \emptyset)$	yes	yes	$no (M = \emptyset)$

PROOF. The proof follows from 2.8, 2.12, 2.16 [SVB]. We see that M is dense in \bar{M} in the cases $c_3 = 0, 2$ and 4. For $c_3 = 0$, a stable sheaf and a properly semistable sheaf have different α -invariant, hence $\bar{M} - M$ and M are disjoint. For $c_3 = 4$, a general member E in $M(0, 2, 4)$ (resp. $V_{s,s}(0, 2, 4)$) can be constructed by a nonsingular curve of degree 6, genus 3, not lying in any quadric surface (resp. type (2,4) on a nonsingular quadric) (for detailed reason, see §4). By 2.17.1, M is dense in \bar{M} .

For $c_3 = 2$, we note that if $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are nonsingular plane cubic curves, meeting at one point, 2.17.2 implies that Y is the limit of degree 6, genus 2, nonsingular curves which are never in any quadric surface.

REMARK 2.17.1. It is easy to see that for $d \geq 2g - 2$, $H_{d,g}$, the Hilbert scheme of nonsingular curves of degree d , genus g in \mathbf{P}^3 , is irreducible.

REMARK 2.17.2. A result of Hirschowitz says that if Y_1 and Y_2 are two nonsingular curves meeting at one point, and $H^1(N_{Y_1}) = H^1(N_{Y_2}) = 0$, then $Y = Y_1 \cup Y_2$ can be smoothed.

REMARK 2.17.3. For $c_3 = 0$, \bar{M} has two irreducible disjoint components $\bar{M} - M$ and M , of dimension 13 and 11. The latter shows that the expected lower bound $8c_2 - 3$ on dimension of moduli does not hold for properly semistable sheaves.

3. Stable reflexive sheaves with $c_2 = 3$. Our goal for this section is to prove the irreducibility of some moduli spaces $M(c_1, c_2, c_3)$. According to [SRS, 3.4.1], it is sufficient to show that the set of *general* E in $M(c_1, c_2, c_3)$ forms an irreducible family. Our method is to discuss what kind of curves or what “dereduction” E may have, and count the maximal possible dimension of E constructed in this way. (We do not care if different reductions give the same E .)

We do the case $c_1 = 0$ first.

REMARK. Stable rank 2 vector bundles with Chern classes $(0, 3, 0)$ were studied by Ellingsrud and Strømme [E-S].

LEMMA 3.1. *Let $E(0, 3, c_3)$ be stable, where $c_3 = 2, 4, 6, 8$, and assume that $H^0(E(1)) \neq 0$. Then either:*

(a) $E(1)$ has a section whose zero set is the disjoint union of a line and a nonsingular twisted cubic (in which case $c_3 = 4$), or

- (b) $E(1)$ has a section whose zero set is a nonsingular rational quartic or a nonsingular twisted cubic with a line attached at one point (in which case $c_3 = 6$), or
- (c) $E(1)$ has a section whose zero set is a nonsingular elliptic quartic or a nonsingular twisted cubic with a line intersecting at two points (in which case $c_3 = 8$), or
- (d) E has an unstable plane of order at least 1.

PROOF. Since $H^0(E(1)) \neq 0$, we have the extension

$$(1) \quad 0 \rightarrow \mathcal{O} \rightarrow E(1) \rightarrow I_Y(2) \rightarrow 0$$

where Y is a curve of degree 4 and $\chi(\mathcal{O}_Y) = 4 - \frac{1}{2}c_3 \leq 3$. By 1.2, it suffices to show that there exists a plane containing a subscheme of degree ≥ 2 of Y , except in cases (a)–(c) above. This is very clear from the degree of the support of Y except in the following cases:

- (i) $\deg(\text{Supp } Y) = 1$. Then $Y = Y^4$, a multiplicity 4 line with $\chi(\mathcal{O}_{Y^4}) < 4$.
- (ii) $\deg(\text{Supp } Y) = 2$, and $\text{Supp } Y$ is a pair of skew lines. Then $Y = L \amalg Y^3$, disjoint union of a line L and a multiplicity 3 line Y^3 with $\chi(\mathcal{O}_{Y^3}) < 3$; or $Y = Y^1 \amalg Y^2$, disjoint union of two double lines with $\chi(\mathcal{O}_{Y^2}) \leq \chi(\mathcal{O}_{Y^1})$. (So, $\chi(\mathcal{O}_{Y^2}) < 2$, because $\chi(\mathcal{O}_Y) \leq 3$.)

- (iii) $\deg(\text{Supp } Y) = 3$, and $\text{Supp } Y$ is three disjoint lines. Then $Y = Y^2 \amalg L \amalg L^1$, disjoint union of a pair of skew lines L, L^1 , and a double line Y^2 with $\chi(\mathcal{O}_{Y^2}) < 2$.

1.1 implies Y^m has a degree ≥ 2 part contained in a plane, where $m = 4, 3, 2$.

REMARK. It is easy to see that for $c_3 = 0$, E has an unstable plane of order 1 if and only if E has α -invariant $\alpha = 1$. (It follows from spectra that $h^2(E(-3)) = 3$ and $h^2(E(-2)) = \alpha$.)

LEMMA 3.2. *The property of having an unstable plane of order 1 is a special property for stable sheaves with Chern classes $(0, 3, c_3)$; $c_3 = 2, 4, 6$.*

PROOF. Reduction step on the plane H gives $E'(-1, 2, c'_3)$ stable, sequence (2), and its dual sequence (2) $^\vee$:

$$(2) \quad \begin{array}{ccccccc} 0 & \rightarrow & E' & \rightarrow & E & \rightarrow & I_{Z,H}(-1) \rightarrow 0, \\ & & & & & & \\ & 0 & \rightarrow & E & \rightarrow & E'(1) & \rightarrow I_{Z',H}(2) \rightarrow 0 \\ & & & & & \downarrow & \nearrow f \\ & & & & & E'_H(1) & \end{array}$$

Let s be the length of Z . Then $c'_3 = c_3 + 2s - 4 \geq s$ (this follows from the proof of [SRS, 9.1]) and Z' has length $s' = 4 - s$. In fact, we have the table of s :

c'_3	c_3	2	4	6
4	4	3	2	1
2	2	2	1	0
0	0	--	0	--

The dimension of each irreducible component of the moduli space is $\geq 8c_2 - 3 = 21$ [SRS, 3.4.1]; so in order to prove the lemma it suffices to show that the dimension of any such possible family is < 21 by counting the dimension of choices of Z' ,

H , f/k^* , and E' in (2), or the dimension of choices of Y , ξ in the extension (1). (Note that $H^0(E'(1)) \neq 0$ implies $H^0(E(1)) \neq 0$ and such an extension.)

Case 1. $c'_3 = 4$. Recall that $E'(-1, 2, 4)$ has a unique unstable plane H' containing conic Y' which is a general zero set of $E'(1)$, and note that Z' has length $s' \leq 3$; therefore impose s' conditions on quadric or cubic forms on H .

Case 1.1. $H = H'$. $E'_H(1) = \mathcal{O}_H(-1) \oplus I_{W,H}(2)$, where W is of length 4 [SRS, 9.12], so f and Z' are given by a scalar and a cubic form with s' conditions. The choices of E' , f/k^* , and Z' are $11 + 10 - s' = 21 - s' < 21$.

Case 1.2. $H \supset$ a degree 1 component of Y' . Restricting the extension $0 \rightarrow \mathcal{O} \rightarrow E'(1) \rightarrow I_{Y'}(1) \rightarrow 0$ on H , and computing the Chern classes of the kernel of $E'_H(1) \rightarrow \mathcal{O}_H \rightarrow 0$, we have

$$0 \rightarrow I_{W',H}(1) \rightarrow E'_H(1) \rightarrow \mathcal{O}_H \rightarrow 0,$$

where W' has length 2. Easy calculation gives $\text{Ext}^1(\mathcal{O}_H, I_{W',H}(1)) = H^1(I_{W',H}(1)) = 0$, therefore $E'_H(1) = \mathcal{O}_H \oplus I_{W',H}(1)$.

Claim. The choice of f/k^* and Z' has dimension ≤ 7 . If $Z' \subset W'$, then f is given by a linear form and a quadric form through Z' , so the dimension is $3 + 6 - s' - 1 \leq 7$.

If $Z' \not\subset W'$, then the choice of each additional point is ≤ 2 dimensions, but this point imposes one condition on each of the forms, hence does not increase the dimension of the choices of f/k^* and Z' as in the previous case.

Since Y' is reducible, E' constructed from Y' has dimension ≤ 10 . The possible choices of H have dimension ≤ 1 . Adding together, the dimension of choices of E' , f/k^* , H , Z' is $\leq 18 < 21$.

Case 1.3. $H \cap Y'$ has dimension 0. In sequence (2), $s \geq 1$ implies $H^0(E'(1)) = H^0(E(1))$, and a general zero set Y is a conic Y' union another conic on H . From arithmetic genus of Y , we know there are two situations.

(a) $c_3 = 4$, and Y is disjoint union of 2 conics. So the dimension of the family through the correspondence $(E(1), s) \leftrightarrow (Y, \xi)$ is the dimension of the choices of $Y + h^0(\omega_Y(2)) - h^0(E(1)) = 16 + 6 - 2 = 20 < 21$.

(b) $c_3 = 6$, and Y is the union of 2 conics intersecting at one point. As above, the dimension of the family is $15 + 7 - 2 = 20 < 21$.

Case 2. $c'_3 = 2$. Recall that $E'(-1, 2, 2)$ has a pair of skew lines (or its degeneration) as the unique zero set Y' of $E'(1)$.

Case 2.1. $H \cap Y'$ contains a line. Restricting on H the extension $0 \rightarrow \mathcal{O} \rightarrow E'(1) \rightarrow I_{Y'}(1) \rightarrow 0$ gives $0 \rightarrow I_{P',H}(1) \rightarrow E'_H(1) \rightarrow I_{P,H} \rightarrow 0$. Applying $\text{Hom}(\cdot, I_{Z',H}(2))$,

$$0 \rightarrow \text{Hom}(I_{P,H}, I_{Z',H}(2)) \rightarrow \text{Hom}(E'_H(1), I_{Z',H}(2)) \rightarrow \text{Hom}(I_{P',H}(1), I_{Z',H}(2)).$$

Claim. The choices of Z' and $f/k^* \in \text{Hom}(E'_H(1), I_{Z',H}(2))/k^*$ have dimension ≤ 8 . Because

$$\begin{aligned} \dim \text{Hom}(E'_H(1), I_{Z',H}(2)) &\leq \dim \text{Hom}(I_{P,H}, I_{Z',H}(2)) \\ &\quad + \dim \text{Hom}(I_{P',H}(1), I_{Z',H}(2)), \quad \text{if } z' \subset \{P, P'\}, \end{aligned}$$

then the choices have dimension $\leq 6 + 3 - 1 = 8$; if $Z' \not\subset \{P, P'\}$, then each additional point needs ≤ 2 parameters and imposes one condition of each form in

$\text{Hom}(I_{P,H}, I_{Z',H}(2))$ or $\text{Hom}(I_{P',H}(1), I_{Z',H}(2))$, so the dimension is still ≤ 8 . The parameters of E' , H , f/k^* , and Z' are $\leq 11 + 1 + 8 = 20 < 21$.

Case 2.2. $H \cap Y'$ has dimension 0. As in Case 1.3 above, we have the following situations.

(a) $c_3 = 2$, and Y is disjoint union of Y' and a conic. So in the $(E(1), s) \leftrightarrow (Y, \xi)$ correspondence, the dimension of the family of E is $16 + h^0(\omega_Y(2)) - h^0(E(1)) = 16 + 5 - 1 = 20 < 21$.

(b) $c_3 = 4$, and Y is the union of Y' and a conic intersecting Y' at length 1. As above, we get $15 + 6 - 1 = 20 < 21$.

(c) $c_3 = 6$, and Y is the union of Y' and a conic intersecting Y' at length 2. The dimension we are counting is $\leq 14 + 7 - 1 = 20 < 21$.

Case 3. $c'_3 = 0$. Completing $(2)^\vee$ on H , we get the exact sequence

$$0 \rightarrow \mathcal{O}_H(-1) \rightarrow E'_H(1) \xrightarrow{f} I_{Z',H}(2) \rightarrow 0$$

where f and Z' are determined by a section of $E'_H(2)$. So the parameter space of E' , H , f/k^* and Z' has dimension $11 + 3 + h^0(E'_H(2)/k^*) = 11 + 3 + 7 - 1 = 20 < 21$.

LEMMA 3.3. *Let $E(0, 3, c_3)$ be stable, where $c_3 = 2, 4, 6, 8$. If E has an unstable plane of order 2, then E has either $c_3 = 8$, or $c_3 = 6$ and spectrum $\{0, -1, -2\}$. This is a special property.*

PROOF. If H is an unstable plane of order 2, then $H^2(E_H(-1)) \neq 0$, hence $H^2(E(-1)) \neq 0$. So the cases above are the only possibilities.

Case 1. $c_3 = 6$. Reduction step gives $E'(-1, 1, 1)$ stable, and

$$0 \rightarrow E' \rightarrow E \rightarrow I_{P,H}(-2) \rightarrow 0.$$

A general section of $E(1)$ has zero set $Y = L \amalg C$, where L , a general zero set of $E'(1)$, is a line through P ; C is a plane “cubic.” (This is an easy exercise in degree and arithmetic genus calculation.) $h^0(E(1)) = h^0(E'(1)) = 3$, and the dimension we are counting is

$$\dim\{Y\} + h^0(\omega_Y(2)) - h^0(E(1)) = (4 + 12) + (1 + 6) - 3 = 20 < 21.$$

Case 2. $c_3 = 8$. By the same reasoning as in Case 1, except that L and C intersect at one point, and the reduction sequence is $0 \rightarrow E' \rightarrow E \rightarrow \mathcal{O}_H(-2) \rightarrow 0$ we conclude that the dimension of the family is < 21 .

REMARK 3.3.1. In Case 2, we have $\text{Ext}^2(E, E) = 0$.

PROOF. Apply $\text{Ext}^2(\cdot, E)$ to the sequence in Case 2, $0 \rightarrow \mathcal{O}(-2) \rightarrow 3\mathcal{O}(-1) \rightarrow E' \rightarrow 0$ [SRS, 9.3], and the resolution $0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}_H(-2) \rightarrow 0$, and use the facts that $\text{Ext}^1(\mathcal{O}(-l), E) = H^1(E(l)) = 0$ for $l = 2, 3$, and $\text{Ext}^2(\mathcal{O}(-l), E) = H^2(E(l)) = 0$ for $l = 1, 2$ (3.9.1).

THEOREM 3.4. *The moduli of stable rank 2 reflexive sheaves on \mathbf{P}^3 with Chern classes $(0, 3, 4)$ is irreducible, and generically reduced of dimension 21. Its underlying variety $M_{\text{red}}(0, 3, 4)$ is rational.*

PROOF. Riemann-Roch gives $\chi(E(1)) = 1$, and spectrum gives $h^2(E(1)) = 0$, hence $H^0(E(1)) \neq 0$. Lemmas 3.1–3.3 imply that for E general, $E(1)$ has zero set $Y = L \amalg Y_1$, where L is a line and Y_1 is a nonsingular twisted cubic. So $h^0(E(1)) = 1$, and E is one-to-one correspondent to Y and $\xi/k^* \in H^0(\omega_Y(2))/k^*$, and needs $(4 + 12) + (1 + 5) - 1 = 21$ parameters. $M_{\text{red}}(0, 3, 4)$ is isomorphic to an open subset of $\mathbf{P}_{G \times H}(p_1^* \pi_* \mathcal{O}_{\mathcal{L}}^\vee \oplus p_2^* \pi'_* \omega_{\mathcal{Y}_1}(2)^\vee)$ where $G = G(1, 3)$ is the Grassmann variety of lines in \mathbf{P}^3 , $H = H_{3,0}$ is the Hilbert scheme of nonsingular twisted cubic curves, \mathcal{L} and \mathcal{Y}_1 are the universal curves on $G \times \mathbf{P}^3$ and $H \times \mathbf{P}^3$, and π, π' are the appropriate maps from $G \times \mathbf{P}^3$ and $H \times \mathbf{P}^3$ to G and H , p_1, p_2 are the maps from $G \times H$ to G and H , so $M_{\text{red}}(0, 3, 4)$ is irreducible and rational. To conclude the moduli has the properties stated, we claim $\text{Ext}^2(E, E) = 0$ for E general: Extension (1) gives the first of the following resolutions:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}(-1) \rightarrow E \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}_L(1) \oplus \mathcal{O}_{Y_1}(1) \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}(-1) \rightarrow 2\mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}_L(1) \rightarrow 0 \quad (\text{Koszul}), \\ 0 &\rightarrow 2\mathcal{O}(-2) \rightarrow 3\mathcal{O}(-1) \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}_{Y_1}(1) \rightarrow 0 \quad (\text{well-known or cf. 2.9}). \end{aligned}$$

With the same reasoning as in 2.5, and using the facts that $h^i(E(l)) = 0$ for $i = 2, 3$, and $l \geq -1$, or $i = 1$ and $l \geq 1$, from 3.4.1, we have $\text{Ext}^2(E, E) = 0$.

TABLE 3.4.1. $h^i(E(l))$ for $E(0, 3, 4)$ stable.

General E							
$i \backslash l$	-4	-3	-2	-1	0	1	2
3	0	0	0	0	0	0	0
2	6	5	2	0	0	0	0
1	0	0	0	1	2	0	0
0	0	0	0	0	0	1	10

Special E			
$i \backslash l$	0	1	2
1	2	1	0
0	0	2	10

Spectrum $\{-1, -1, 0\}$

THEOREM 3.5. $M_{\text{red}}(0, 3, 6)$ is irreducible, and unirational of dimension 21.

PROOF. The proof is the same as that of Theorem 3.4, a general $E(1)$ has nonsingular rational quartic as zero set. The dimension counting gives $16 + 7 - 2 = 21$.

TABLE 3.5.1. $h^i(E(l))$ for $E(0, 3, 6)$ stable.General E

$i \backslash l$	-4	-3	-2	-1	0	1	2
3	0	0	0	0	0	0	0
2	7	6	3	0	0	0	0
1	0	0	0	0	1	0	0
0	0	0	0	0	0	2	11

Spectrum $\{-1, -1, -1\}$ Special E

$i \backslash l$	-4	-3	-2	-1	0	1	2
3	0	0	0	0	0	0	0
2	7	6	3	1	0	0	0
1	0	0	0	1	1	1	0
0	0	0	0	0	0	3	11

Spectrum $\{0, -1, -2\}$

REMARK 3.5.2. Note that the irreducible moduli space $M(0, 3, 6)$ has two different spectra.

LEMMA 3.6 (Char 0). *Let $E(0, 3, 2)$ be stable and general. Then a general section of $E(2)$ has irreducible and nonsingular curve of degree 7 and genus 2 as zero set.*

PROOF. Lemmas 3.1 and 3.2 imply that a general E has $h^0(E(1)) = 0$. By Riemann-Roch, $\chi(E(1)) = 0$, so $h^0(E(1)) = 0$ and $E(2)$ is generated by global sections (3.7.1 and Castelnuovo's theorem [Mu, p. 99]). A general zero set Y of $E(2)$ is nonsingular except possibly two points where $\xi \in H^0(\omega_Y)$ fails to generate $0 \rightarrow \mathcal{O} \rightarrow E(2) \rightarrow I_Y(4) \rightarrow 0$ where $d(Y) = 7$ and $\chi(\mathcal{O}_Y) = -1$.

Claim. Y is irreducible. If Y has three components, meeting at no more than two points, then from $\sum_{i=1}^3 \chi(\mathcal{O}_{Y_i}) \leq \chi(\mathcal{O}_Y) + 2 = 1$, they are two elliptic curves and one rational curve, so Y is contained in a cubic surface and $h^0(I_Y(3)) \neq 0$ implies $h^0(E(1)) \neq 0$. This is a contradiction. If Y has two components meeting at exactly two points P and Q , then the same reason as above implies that Y is the union of a rational curve Y_0 and an elliptic curve Y_e . Restricting ω_Y at Y_e we get $\omega_Y|_{Y_e} = \mathcal{O}_{Y_e}(P + Q)$, namely, ξ vanishes at P, Q on Y_e , but on the other hand, $\omega_Y|_{Y_0} = \mathcal{O}_{Y_0}$, ξ generates $P, Q \in Y_0$, again a contradiction.

If Y has two components meeting at one point, then Y is the union of an elliptic cubic (which is contained in a plane) and an elliptic quartic (which is in quadric surfaces), so Y is contained in a cubic surface. Thus as before, E is not general. This proves the claim.

If Y is a nonsingular curve of degree 7, genus 2, then the dimension of the family of sheaves is 21. In fact, the choice of an abstract curve Y of genus 2 is 3 parameters.

The choice of an invertible sheaf \mathcal{L} of degree 7 is 2 parameters. The choice of a 4-dimensional linear subspace of $H^0(Y, \mathcal{L})$, which is 6-dimensional, is 8 parameters. Then add automorphisms of \mathbf{P}^3 (15 parameters). This gives 28 parameters of Y . Add $h^0(\omega_Y) = 2$, and subtract $h^0(E(2)) = 9$; we have 21, the expected dimension.

If Y is not nonsingular everywhere, then we claim that the choice of Y is < 28 parameters (hence the dimension of the family of sheaves is < 21 , by the same calculation as above). Using the notation in [AG, IV, Exercise 1.8], Y has abstract model either an elliptic curve ($\delta_P = 1$ means P is a node), or a rational curve (with two nodes, or one singular point Q with $\delta_Q = 2$). Modifying the parameter counting above, we have that the choice of such Y is < 28 parameters, e.g., the abstract model of an elliptic curve with a node needs two parameters (one for the nonsingular elliptic curve, one for identifying two specified points as a node), the choice of degree 7 line bundle is still 2, and the Riemann-Roch formula works for such a curve.

As a corollary to Lemma 3.6, we have the following

THEOREM 3.7. *The family $M_{\text{red}}(0, 3, 2)$ is irreducible of dimension 21.*

TABLE 3.7.1. $h^i(E(l))$ for $E(0, 3, 2)$ stable.

General E							
$i \backslash l$	-4	-3	-2	-1	0	1	2
3	0	0	0	0	0	0	0
2	5	4	1	0	0	0	0
1	0	0	0	2	3	0	0
0	0	0	0	0	0	0	9

Special E

$i \backslash l$	1	2
1	2	0
0	2	9

$i \backslash l$	1	2
1	1	0
0	1	9

Spectrum $\{-1, 0, 0\}$

LEMMA 3.8. *Let $E(0, 3, 8)$ be stable and without any unstable planes of order ≥ 1 . Then $E(1)$ is generated by global sections.*

PROOF. The spectrum and Riemann-Roch give $h^2(E(1)) = 0$ and $\chi(E(1)) = 3$, respectively, so in the extension

$$0 \rightarrow \mathcal{O} \rightarrow E(1) \rightarrow I_Y(2) \rightarrow 0,$$

$h^0(E(1)) \geq 3$ implies $h^0(I_Y(2)) \geq 2$, hence is 2, and Y is a complete intersection of two degree 2 surfaces without common component. (A reducible quadric surface containing Y would give an unstable plane, by 1.2.) Restricting the sequence on one of the quadric surfaces Q , we obtain $E_Q(1) \rightarrow \mathcal{O}_Q \rightarrow 0$, and combined with the surjection $E(1) \rightarrow E_Q(1)$, the kernel of the surjection $E(1) \rightarrow \mathcal{O}_Q$ is properly semistable, having Chern classes $(0, 0, 0)$, so it is $2\mathcal{O}$ [SRS, 9.7].

$$0 \rightarrow 2\mathcal{O} \rightarrow E(1) \rightarrow \mathcal{O}_Q \rightarrow 0,$$

$H^1(2\mathcal{O}(l)) = 0$ for all l , and both $2\mathcal{O}$ and \mathcal{O}_Q are generated by global sections, so $E(1)$ is generated by global sections.

THEOREM 3.9. *The moduli of stable rank 2 reflexive sheaves on \mathbf{P}^3 with Chern classes $(0, 3, 8)$ is irreducible, nonsingular, and rational of dimension 21.*

PROOF. The property of having an unstable plane of order $d \geq 1$ is *special* for E : if $d = 2$, this is 3.3; if $d = 1$, the proof in 3.2 applies. In either case, we have $\text{Ext}^2(E, E) = 0$ (cf. 3.3.1). For a general E , $E(1)$ is generated by three global sections, and we have $0 \rightarrow \mathcal{O}(-2) \rightarrow 3\mathcal{O} \rightarrow E(1) \rightarrow 0$. (The kernel is obtained by Chern classes computation and the fact that it is rank 1, reflexive, and hence locally free.) Dualizing the exact sequence, we get

$$0 \rightarrow E(-1) \rightarrow 3\mathcal{O} \rightarrow I_{Z,Q}(2) \rightarrow 0$$

where Z is the set of nonlocally-free points of E , a complete intersection of three quadrics which determine E . Thus the family is parametrized by an open subset of Grassmann variety of 2-planes in $\mathbf{P}(H^0(\mathcal{O}(2)))$, namely $G(2, 9)$, so it is irreducible, nonsingular, and rational of dimension 21.

Applying $\text{Hom}(\cdot, E)$ to $0 \rightarrow \mathcal{O}(-3) \rightarrow 3\mathcal{O}(-1) \rightarrow E \rightarrow 0$, we get

$$\text{Ext}^1(\mathcal{O}(-3), E) \rightarrow \text{Ext}^2(E, E) \rightarrow \text{Ext}^2(3\mathcal{O}(-1), E).$$

The first and last terms are $H^1(E(3))$ and $H^2(3E(1))$, respectively, which are both 0, so $\text{Ext}^2(E, E) = 0$, and the theorem is proved.

TABLE 3.9.1. $h^i(E(l))$ for $E(0, 3, 8)$ stable.

$i \backslash l$	-4	-3	-2	-1	0	1	2
3	0	0	0	0	0	0	0
2	8	7	4	1	0	0	0
1	0	0	0	0	0	0	0
0	0	0	0	0	0	3	12

Spectrum $\{-1, -1, 2\}$

Now we will treat the cases where $c_1 = -1$.

LEMMA 3.10. *The property of $H^0(E(1)) \neq 0$ is a special property for stable sheaves with Chern classes $(-1, 3, c_3)$; $c_3 = 1, 3, 5$.*

PROOF. A section of $E(1)$ gives $0 \rightarrow \mathcal{O} \rightarrow E(1) \rightarrow I_Y(1) \rightarrow 0$ where Y is a degree 3 curve with $\chi(\mathcal{O}_Y) = 4\frac{1}{2} - \frac{1}{2}c_3 \geq 2$. As in Lemma 3.2, we want to show that the dimension of any of the families is < 19 .

Case 1. $Y = L \amalg L' \amalg L''$. Then the dimension of the family constructed from Y is obtained by computing the parameters of Y , $h^0(\omega_Y(3))$ and $h^0(E(1))$, which are 12, 6 and 1, respectively, and hence the dimension is $17 < 19$.

Case 2. $Y = L \amalg Y'$, where Y' is a connected degree 2 curve with $\chi(\mathcal{O}_{Y'}) = 3\frac{1}{2} - \frac{1}{2}c_3$. If Y' is a double line, then using 1.3 for $r = 2\frac{1}{2} - \frac{1}{2}c_3$ and $k = 3$, we get the parameters of Y , $h^0(\omega_Y(3))$ and $h^0(E(1))$ equal to $(2r + 5) + 4$, $(2k - r - 1) + 2$, and 1. Therefore, the dimension of the family is $r + 2k + 9 = 17\frac{1}{2} - \frac{1}{2}c_3 < 19$. If Y' is a conic, then the choice of Y is 12 parameters, and $h^0(\omega_Y(2)) = 7$. So we have the dimension $18 < 19$.

Case 3. Y has only one connected component. Then Y is either a line union a double line intersecting it (by 1.2, E has an unstable plane H of order 2) or a multiplicity 3 line. It follows from 1.1 that the latter has either a degree 2 part contained in a plane H , or $\chi(\mathcal{O}_Y) = 3$ or 4. If $\chi(\mathcal{O}_Y) = 3$, then Y is type $(3, 0)$ on a nonsingular quadric surface (see Remark 1.1.1), hence this is a special case of Case 1 above. If $\chi(\mathcal{O}_Y) = 4$, then a plane containing the support of Y is unstable of order ≥ 1 (in which case $c_3 = 1$).

Case 3.1. H is of order 2. A necessary condition for unstable planes of such existing order is $H^2(E(-1)) \neq 0$, hence (from spectrum) the only possibilities are $c_3 = 5, 7$, and $c_3 = 3$ with spectrum $\{0, -1, -2\}$. Reduction step gives $E'(0, 1, c'_3)$ properly semistable (therefore $c'_3 = 2$ by 2.1), and the sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & E'(-1) & \rightarrow & E & \rightarrow & I_{Z,H}(-2) \rightarrow 0, \\ & & & & & & \\ 0 & \rightarrow & E(1) & \rightarrow & E'(1) & \rightarrow & I_{Z',H}(3) \rightarrow 0 \\ & & & & \downarrow & \nearrow f & \\ & & & & E'_H(1) & & \end{array}$$

where Z has length s , and Z' has length $s' = 5 - s$:

$$\begin{array}{c} c^3 \\ s \end{array} \begin{array}{|c|c|c|c|} \hline 3 & 5 & 7 & \\ \hline 2 & 1 & 0 & \\ \hline \end{array}$$

The section in $H^0(E') = H^0(E(1))$ gives the unique zero set L of E' , and Y of $E(1)$. Since H contains the support of Y , and L and Y are isomorphic outside of H , we have $L \subset H$, and $E'_H(1) = I_{W,H}(2) \oplus \mathcal{O}_H$, where W has length 2 (cf. 2.3). Now we compute the dimension of the family. The choices of E' and H are 6 and 1, respectively. f and Z' are determined by a plane cubic form imposed by $s' - 3$ conditions, and a plane linear form. But an automorphism of E' acts on f , and gives the same kernel E . So we need to subtract $\dim \text{End}(E')$ from the choice of the 1-form and 3-form. Therefore, the family has dimension $\leq 6 + 1 + 13 - (s' - 3) - 2 = 21 - s' = 16 + s \leq 18 < 19$.

Case 3.2. H is of order 1 and $c_3 = 1$. Reduction step gives $E'(0, 2, c'_3)$ properly semistable, and the sequences:

$$(3) \quad \begin{array}{ccccccc} 0 & \rightarrow & E'(-1) & \rightarrow & E & \rightarrow & I_{Z,H}(-1) \rightarrow 0, \\ & & & & & & \\ & & 0 & \rightarrow & E & \rightarrow & E' \rightarrow I_{Z',H}(1) \rightarrow 0 \\ & & & & & & \downarrow \nearrow f \\ & & & & & & E'_H \end{array}$$

Let s be the length of Z . Then $c'_3 = 2s - 2 \geq s$ (since $Z \subset$ nonlocally-free set of E') and Z' has length $s' = 3 - s$.

TABLE OF s

	c_3	1
c'_3	2	2
	4	3

Claim. The dimension of the family is < 19 . In 2.16 we have the extension $0 \rightarrow \mathcal{O} \rightarrow E' \rightarrow I_{Y'} \rightarrow 0$. By the same reasoning as in the previous case, the plane H contains the support of Y' , hence $\text{Supp } Y'$ has degree 1 (note that $\text{Supp } Y$ has degree 1), and Y' is a double line. Therefore the choices of E' and H are $14 - \frac{1}{2}c'_3$ parameters and 1 parameter, respectively. Restricting the extension on H , we have

$$(3.a) \quad 0 \rightarrow I_{W_2,H}(1) \rightarrow E'_H \rightarrow I_{W_1,H}(-1) \rightarrow 0$$

where $\text{length}(W_1) = 3 - \frac{1}{2}c'_3$ and $\text{length}(W_2) = \frac{1}{2}c'_3$. Applying $\text{Hom}(\cdot, I_{Z',H}(1))$,

$$\begin{aligned} 0 &\rightarrow \text{Hom}(I_{W_1,H}(-1), I_{Z',H}(1)) \rightarrow \text{Hom}(E'_H, I_{Z',H}(1)) \\ &\rightarrow \text{Hom}(I_{W_2,H}(1), I_{Z',H}(1)). \end{aligned}$$

If the last term is 0, then the choice of each point which cuts 1 dimension on the first term is 1 parameter (because Z' is contained in the nonlocally-free set of E , hence lies on Y which has the same support as Y'). So the choice of f and Z' is $\leq \dim \text{Hom}(I_{W_1,H}(-1), I_{Z',H}(1)) \leq 6$. If the last term is not 0, then Z' is determined, and $f \in \text{Hom}(E'_H, I_{Z',H}(1))$ has dimension $\leq \dim \text{Hom}(I_{W_1,H}(-1), I_{Z',H}(1)) + \dim \text{Hom}(I_{W_2,H}(1), I_{Z',H}(1)) \leq 7$. Adding all the parameters together, and subtracting the dimension of $\text{End}(E')$, we have that the dimension of the family is $\leq (14 - \frac{1}{2}c'_3) + 1 + 7 - 2 = 20 - \frac{1}{2}c'_3$, which is < 19 , if $c'_3 > 2$. We claim that if $c'_3 = 2$, the dimension is actually $\leq 19 - \frac{1}{2}c'_3$ by showing that if E' is general in the family, and H is general in the pencil, then $\text{Hom}(I_{W_2,H}(1), I_{Z',H}(1)) = 0$. E' has an extension (cf. 2.17),

$$0 \rightarrow \mathcal{O} \rightarrow E'(2) \rightarrow I_{Y_1}(4) \rightarrow 0$$

where Y_1 is the union of two nonsingular plane cubics. If H is general, Y_1 intersects H at six points. Twisting sequence (3) by $\mathcal{O}(3)$, we see that Y_1 gives a zero set Y_0 of $E(3)$. From arithmetic genus computation, $Y_0 = Y_1 \cup C$, where C is a plane cubic on

H , intersecting Y_1 at four points. The extension

$$0 \rightarrow \mathcal{O} \rightarrow E(3) \rightarrow I_{Y_0}(5) \rightarrow 0$$

when restricted on H , yields (3), hence $Z \cap C = \emptyset$. The nonlocally-free point Z' of E lies on C , hence Z' is not contained in Z which contains W_2 (cf. sequence (3a)). Therefore $\text{Hom}(I_{W_2, H}(1), I_{Z', H}(1)) = 0$ and the family has dimension < 19 .

LEMMA 3.11. *Let $E(-1, 3, c_3)$ be stable, where $c_3 = 1$ or $c_3 = 3$ and E has spectrum $\{-1, -1, -1\}$. Then E has an unstable plane H of order 1, and the reduction sequences:*

$$(4) \quad \begin{array}{ccccccc} 0 & \rightarrow & E'(-1) & \rightarrow & E & \rightarrow & I_{Z, H}(-1) \rightarrow 0, \\ & & & & & & \\ & & 0 & \rightarrow & E(1) & \rightarrow & E'(1) \rightarrow I_{Z', H}(2) \rightarrow 0 \\ & & & & & & \downarrow \nearrow f \\ & & & & & & E'_H(1) \end{array}$$

If $H^0(E(1)) = 0$, then $E'(0, 2, c'_3)$ is stable, with $c'_3 = 0, 2, 4$ if $c_3 = 3$, and $c'_3 = 2, 4$ if $c_3 = 1$. The property that $E(2)$ has no reduced zero set is a special property.

PROOF. The existence of an unstable plane is proved by the same reasoning as in 3.7. Note that the values of s are given by the accompanying table, and that the length of $Z' := s' = 3 - s$. The only nontrivial part is to show that such a property is special (cf. 3.2). (The family has dimension < 19 .)

TABLE OF s

$c'_3 \backslash c_3$	1	3
0	x	0
2	2	1
4	3	2

Case 1. $c'_3 = 2$. Recall that $E'(0, 2, 2)$ has the extension

$$(5) \quad 0 \rightarrow \mathcal{O} \rightarrow E'(1) \rightarrow I_{Y'}(2) \rightarrow 0$$

where $Y' = L \amalg C$, C is a conic lying on the unique unstable order 1 plane H' (cf. 3.7).

Case 1.1. $H = H'$. Restricting (5) on H , we have

$$0 \rightarrow I_{W, H}(2) \rightarrow E'_H(1) \rightarrow I_{P, H} \rightarrow 0$$

where W has length 2 and is the nonlocally-free set of E' , hence lying on C , and $P = H \cap L$. Therefore $P \cap W = \emptyset$. Applying $\text{Hom}(\cdot, I_{Z', H}(2))$, we get

$$0 \rightarrow \text{Hom}(I_{P, H}, I_{Z', H}(2)) \rightarrow \text{Hom}(E'_H(1), I_{Z', H}(2)) \rightarrow \text{Hom}(I_{W, H}(2), I_{Z', H}(2)),$$

as in Case 2.1 of Lemma 3.2. (Note that Z' lies on Y' . This follows from $Y' \subset Y$, which is the induced zero set of $E(2)$, and H can contain at most a degree 2 subscheme of Y .) We conclude that the choice of f/k^* and Z' is

$$\begin{aligned} \dim \text{Hom}(E'_H(1), I_{Z', H}(2)) - 1 &\leq \dim \text{Hom}(I_{P, H}, I_{Z', H}(2)) \\ &\quad + \dim \text{Hom}(I_{W, H}(2), I_{Z', H}(2)) - 1 \\ &\leq 5 \text{ parameters.} \end{aligned}$$

$\dim M(0, 2, 2) = 13$. Therefore, the dimension of the family is $\leq 5 + 13 = 18 < 19$.

Case 1.2. $H \supset$ a degree 1 subscheme of Y' . Restricting (5) on H , we get

$$0 \rightarrow I_{W_2, H}(1) \rightarrow E'_H(1) \rightarrow I_{W_1, H}(1) \rightarrow 0.$$

Applying $\text{Hom}(\cdot, I_{Z', H}(2))$ and using an argument identical to that in Case 2.1 of 3.2, we conclude that the choice of f/k^* and Z' is ≤ 5 parameters. If the conic C is nonsingular, then H contains L and Z' (note that $Z' \subset C$), and hence there are only a finite number of H . If C is reducible, then E' is special (hence only needs ≤ 12 parameters), and the choice of H is 1 parameter. Putting together, we have the dimension of the family is $\leq 18 < 19$.

Case 1.3. $H \cap Y'$ has dimension 0. Twisting (4) by $\mathcal{O}(2)$, we see that Y' induces a zero set Y of $E(2)$. Compute arithmetic genus for the case $c_3 = 1$, and we have Y as the union of $Y' (= L \amalg C)$ and a conic C' intersecting L at one point. To compute the dimension of the family of E constructed from Y , we modulo only 3-dimensional sections which are from $E'(1)$, since we do not know if a general section corresponds to such Y . Therefore, the dimension is $\leq (4 + 8 + 8 - 1) + (1 + 1 + 1) - 3 = 19$. Equality holds, only if both conics are nonsingular, so Y is reduced. The same reasons work for the case $c_3 = 3$.

Case 2. $c'_3 = 4$. Recall that on an unstable plane H' for E' , $E'_{H'} = \mathcal{O}_{H'}(-1) \oplus I_{W, H'}(1)$, and a general E' has a nonsingular twisted cubic as zero set.

Case 2.1. $H = H'$. Then in (3) $^\vee$, Z' and f/k^* are determined by a degree 2 form with s' conditions imposed. So the dimension of the family is $13 + (6 - s') = 19 - s'$, which is < 19 if $c_3 = 3$, and $= 19$ if $c_3 = 1$. For the latter case, a general $E(2)$ has nonsingular zero sets, i.e., a nonsingular twisted cubic disjoint union a conic.

Cases 2.2 and 2.3. Ditto Cases 1.2 and 1.3.

Case 3. $c'_3 = 0$. In this case c_3 can only be 3. For the same reason as in 3.2, f/k^* and Z' are determined by a section in $H^0(E'_H(1))/k^*$. The dimension is $13 + 3 + (3 - 1) = 19$. If E' is general, then a general zero set of $E'(1)$ is 3 skew lines; this induces a zero set as a conic with three lines attached to it. Counting dimension as in Case 1.3, for $E(2)$ general, the zero set is reduced.

THEOREM 3.12. *The moduli of stable reflexive sheaves on \mathbf{P}^3 with Chern classes $(-1, 3, 1)$ is generically reduced and irreducible of dimension 19. The underlying reduced scheme is a unirational variety.*

PROOF. By the previous lemma, if E is general, then $H^0(E(1)) = 0$ and $E(2)$ has a reduced zero set. Reducedness is an open property, so a general zero set Y of $E(2)$ is reduced. *Claim.* Y is the disjoint union of a twisted cubic and a conic. We know Y has $d = 5$ and $p_a = -1$. Suppose Y is the union of two connected (may not be irreducible) components intersecting at k points, then $2 = 1 - p_a = (1 - p_{a_1}) + (1 - p_{a_2}) - k$ gives $p_{a_1} + p_{a_2} + k = 0$, namely, Y is the disjoint union of two rational curves, neither of them can be a line (because $H^0(\omega_Y(1)) \neq 0$). Counting dimension, we see that when both the twisted cubic and the conic are nonsingular, the family has dimension 19. The way to parametrize is irreducible and unirational (see the proof of Theorem 3.4).

TABLE 3.12.1. $h^i(E(l))$ for $E(-1, 3, 1)$ stable.

General E							
$i \backslash l$	-3	-2	-1	0	1	2	3
3	0	0	0	0	0	0	0
2	4	2	0	0	0	0	0
1	0	0	1	3	2	0	0
0	0	0	0	0	0	4	17

Special E			
$i \backslash l$	1	2	3
1	3	0	0
0	1	4	17

Spectrum $\{-1, -1, 0\}$ THEOREM 3.13. $M_{\text{red}}(-1, 3, 3)$ is irreducible and rational of dimension 19.

PROOF. For the same reason as in the previous theorem, a general $E(2)$ has a nonsingular rational quintic as zero set. Thus $M_{\text{red}}(-1, 3, 3)$ is irreducible of dimension 19. On the other hand, take $E'(0, 2, 0)$ (there is a universal family, by [H-S]), a general plane H , a section of $E'_H(2)$, we construct a nonsingular, irreducible and rational of dimension 19, open subset of $M_{\text{red}}(-1, 3, 3)$. Therefore $M_{\text{red}}(-1, 3, 3)$ is rational.

TABLE 3.13.1. $h^i(E(l))$ for $E(-1, 3, 3)$ stable.

General E							
$i \backslash l$	-3	-2	-1	0	1	2	3
3	0	0	0	0	0	0	0
2	5	3	0	0	0	0	0
1	0	0	0	2	1	0	0
0	0	0	0	0	0	5	18

Special E			
$i \backslash l$	1	2	3
1	2	0	0
0	1	5	18

Spectrum $\{-1, -1, -1\}$

$i \backslash l$	-3	-2	-1	0	1	2	3
3	0	0	0	0	0	0	0
2	5	3	1	0	0	0	0
1	0	0	1	2	2	1	1
0	0	0	0	0	1	6	19

Spectrum $\{0, -1, -2\}$

REMARK 3.13.2. Note that the irreducible moduli space $M(-1, 3, 3)$ has two different spectra.

THEOREM 3.14. $M_{\text{red}}(-1, 3, 5)$ is irreducible and unirational of dimension 19.

PROOF. By 3.10, if E is general, then $h^0(E(1)) = 0$. Riemann-Roch gives $\chi(E(1)) = 0$, hence $h^1(E(1)) = 0$ and $E(2)$ is generated by six global sections (Castelnuovo and 3.14.1). Since E is reflexive the kernel E' of

$$(6) \quad 0 \rightarrow E' \rightarrow 6\mathcal{O} \rightarrow E(2) \rightarrow 0$$

is a rank 4 vector bundle. E' has Chern classes $(-3, -4, -2)$ and $h^1(E') = h^2(E'(-1)) = h^3(E'(-2)) = 0$, so $E'(1)$ is generated by global section. Take a section of $E'(1)$; we get that the quotient E'' in the sequence

$$(7) \quad 0 \rightarrow \mathcal{O} \rightarrow E'(1) \rightarrow E'' \rightarrow 0$$

is a vector bundle. E'' has Chern classes $(1, 1, 1)$ and is generated by four sections. Computing Chern classes of the kernel, we have the sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow 4\mathcal{O} \rightarrow E'' \rightarrow 0.$$

Hence $E'' \simeq T_{\mathbb{P}^3}(-1)$ and $\text{Ext}^1(E'', \mathcal{O}) = \text{Ext}^1(T(-1), \mathcal{O}) = H^1(\Omega(1)) = 0$. So sequence (7) splits, and $E'(1) = \mathcal{O} \oplus T(-1)$. By (6), $E(2)$ is the quotient of $\mathcal{O}(-1) \oplus T(-2) \rightarrow 6\mathcal{O}$. Thus, the family is irreducible and unirational of

dimension = $\dim \text{Hom}(\mathcal{O}(-1) \oplus T(-2), 6\mathcal{O}) - \dim \text{End}(6\mathcal{O})$

$$- \dim \text{End}(\mathcal{O}(-1) \oplus T(-2)) + 1 = 36 + 24 - 36 - 6 + 1 = 19.$$

TABLE 3.14.1 $h^i(E(l))$ for $E(-1, 3, 5)$ stable.

General E

$i \backslash l$	-3	-2	-1	0	1	2	3
3	0	0	0	0	0	0	0
2	6	4	1	0	0	0	0
1	0	0	0	1	0	0	0
0	0	0	0	0	0	6	19

$i \backslash l$	1	2	3
1	1	0	0
0	1	6	19

Spectrum $\{-1, -1, -2\}$

THEOREM 3.15. $M_{\text{red}}(-1, 3, 7)$ is irreducible and rational of dimension 19. $M(-1, 3, 7)$ is generically reduced.

PROOF. This follows from the facts that $h^0(E(1)) = 1$ and a general $E(1)$ has a nonsingular twisted cubic as zero set (cf. 3.13).

TABLE 3.15.1. $h^i(E(l))$ for $E(-1, 3, 7)$ stable.

$i \backslash l$	-3	-2	-1	0	1	2
3	0	0	0	0	0	0
2	7	5	2	0	0	0
1	0	0	0	0	0	0
0	0	0	0	0	1	7

Spectrum $\{-2, -2, -1\}$

4. Some applications to curves in \mathbf{P}^3 . In this section we will construct some nonsingular curves of high degree, most of which are of maximal rank, and some even projectively normal. For some d and g , we use §§2 and 3 to conclude that the Hilbert scheme $H_{d,g}$ of nonsingular curves with degree d , genus g , is irreducible and unirational. Throughout the whole section we shall work over a ground field with $\text{char} = 0$. Curves are always irreducible and nonsingular.

The following proposition, due to Hartshorne and Hirschowitz, will be useful to us.

PROPOSITION H. *Let E be a rank 2 reflexive sheaf on \mathbf{P}^3 . If E is generated by global sections, and through each nonlocally-free point P there is a section whose zero set is nonsingular at P , then E corresponds to a nonsingular curve.*

Fact 4.1. Let Y be a curve in \mathbf{P}^3 with degree d , genus g . If $g \geq d - 2$ (resp. $g > 0$), then $\omega_Y(-1)$ (resp. ω_Y) has a section ξ generating it except at finitely many points. Hence we can construct a reflexive sheaf $E(3)$ (resp. $E(2)$), where E has $c_1 = -1$ (resp. $c_1 = 0$). If Y is not contained in any quadric surface, then E is stable.

PROOF. This is proved by [SRS, 4.1, 4.2] and Riemann-Roch.

Hence we have the following:

TABLE 4.1.1. $(Y, \xi) \leftrightarrow (E(3), s)$, where E is stable and $\xi \in H^0(\omega_Y(-1))$ is the extension $0 \rightarrow \mathcal{O} \rightarrow E(3) \rightarrow I_Y(5) \rightarrow 0$.

(c_1, c_2, c_3)	$(-1, 3, 1)$	$(-1, 3, 3)$	$(-1, 3, 5)$	$(-1, 3, 7)$	$(-1, 2, 2)$	$(-1, 2, 4)$
(d, g)	$(9, 6)^1$	$(9, 7)$	$(9, 8)$	$(9, 9)$	$(8, 6)$	$(8, 7)$

$(Y, \xi) \leftrightarrow (E(2), s)$, where E is stable and $\xi \in H^0(\omega_Y)$ is the extension $0 \rightarrow \mathcal{O} \rightarrow E(2) \rightarrow I_Y(4) \rightarrow 0$.

(c_1, c_2, c_3)	$(0, 3, 2)$	$(0, 3, 4)$	$(0, 3, 6)$	$(0, 3, 8)$
(d, g)	$(7, 2)$	$(7, 3)$	$(7, 4)^1$	$(7, 5)$

¹For those Y not in a quadric surface. (All the other curves are not in a quadric surface, by [AG, IV, 6.4.1].)

PROPOSITION 4.2. *For all $l \geq c_2 \geq 4$, there are curves with degree $d = l^2 + c_2$, and genus $g = (l^2 + c_2)(l - 2) + \frac{1}{2}(c_2^2 - c_2) + 1$.*

PROOF. We can construct a sheaf $E(0, c_2, c_2^2 - c_2)$ through the extension $0 \rightarrow \mathcal{O} \rightarrow E(1) \rightarrow I_{Y_0}(2) \rightarrow 0$, where Y_0 is a line disjoint union a plane curve of degree c_2 [C3, 10.1]. $E(l)$ is generated by global sections for $l \geq c_2$. Therefore Proposition H implies that a general zero set Y of $E(l)$ is nonsingular. $H^1(I_Y) = H^1(E(-l)) = 0$ [SRS, 7.6], hence Y is connected, with d, g as above [SRS, 4.1].

PROPOSITION 4.3. *The following table gives curves of maximal rank, with degree d , genus g .*

d	g	Projectively normal	Cf.	Remarks
$l^2 + 5$	$l^2 - 2l^2 + 5$ for $l \geq 1$	yes	[C3, 12.1]	
$l^2 + 1$	$(l^2 + 1)(l - 2) + 1$		2.1.1	
	$(l^2 + 1)(l - 2) + 2$	yes	2.3.1	
$l^2 + 2$ $l \geq 2$	$(l^2 + 2)(l - 2) + 1$ $(l^2 + 2)(l - 2) + 2$		[SVB] 2.8.1	Every nonsingular curve with $d = 6, g = 1, 2$, is of maximal rank.
	$(l^2 + 2)(l - 2) + 3$ $(l^2 + 2)(l - 2) + 4$	yes	2.12.2 2.16.1	Every curve with $d = 6, g = 3$ and not lying on a quadric is projectively normal.
$l^2 + 3$ $l \geq 2$	$(l^2 + 3)(l - 2) + 2$ $(l^2 + 3)(l - 2) + 3$ $(l^2 + 3)(l - 2) + 4$		3.7.1 3.4.1 3.5.1	
	$(l^2 + 3)(l - 2) + 5$	yes	3.9.1	Every curve with $d = 7, g = 5$ is projectively normal.
$l^2 - l + 1$ $l \geq 2$	$(l^2 - l + 1)(l - \frac{5}{2}) + \frac{3}{2}$ $(l^2 - l + 1)(l - \frac{5}{2}) + \frac{5}{2}$	yes	[SRS]	
$l^2 - l + 2$ $l \geq 3$	$(l^2 - l + 2)(l - \frac{5}{2}) + 2$		2.6.1	Every curve with $d = 8, g = 6$ has maximal rank.
	$(l^2 - l + 2)(l - \frac{5}{2}) + 3$	yes	[SRS]	Every curve with $d = 8, g = 7$ is projectively normal.
$l^2 - l + 3$	$(l^2 - l + 3)(l - \frac{5}{2}) + \frac{3}{2}$ $(l^2 - l + 3)(l - \frac{5}{2}) + \frac{5}{2}$ $(l^2 - l + 3)(l - \frac{5}{2}) + \frac{7}{2}$		3.12.1 3.13.1 3.14.1	
	$(l^2 - l + 3)(l - \frac{5}{2}) + \frac{9}{2}$ $(l^2 - l + 3)(l - \frac{5}{2}) + \frac{11}{2}$	yes	3.15.1 [SRS]	

PROOF. The proof is the same as that of Proposition 4.2. We take Y corresponding to $E(l)$.

REMARK 4.3.1. If a curve Y in \mathbf{P}^3 with $d = 7, g = 2$, is not of maximal rank, then Y has a 5-secant.

PROOF. By 4.1.1, Y corresponds to $E(0, 3, 2)$ stable. If Y is not of maximal rank, i.e., Y is in a cubic surface (possibly singular), then untwisting the sequence $0 \rightarrow \mathcal{O} \rightarrow E(2) \rightarrow I_Y(4) \rightarrow 0$, we see that $H^0(E(1)) \neq 0$. Lemma 3.1 implies that E has an unstable plane H . Twisting (2) in 3.2 by $\mathcal{O}(2)$ gives

$$0 \rightarrow E(2) \rightarrow E'(3) \rightarrow I_{Z', H}(4) \rightarrow 0,$$

where E' has Chern classes $(-1, 2, c'_3)$; $c'_3 = 2$ or 4 . Y induces a zero set of Y' of $E'(3)$, with $d = 8$ and $\chi(\mathcal{O}_{Y'}) = -5$ or -6 [SRS, 4.1]. Therefore, Y' is the union of Y and a line L intersecting it at r points, and $\chi(\mathcal{O}_{Y'}) = \chi(\mathcal{O}_Y) + \chi(\mathcal{O}_L) - r = -r$. Hence $r = 5$ or 6 , and L is either a 5-secant or a 6-secant of Y . Because $g = 2$, the degree-one divisor $\mathcal{O}_Y(1) \otimes \mathcal{O}(-\sum_{i=1}^6 P_i)$ has no nontrivial section; i.e., Y cannot have any 6-secant.

EXAMPLE. It is easy to see that a nonsingular curve of type $(7; 4 \ 3 \ 3 \ 2 \ 1 \ 1)$ on a nonsingular cubic surface has a 5-secant; namely, the line of type $(1; 0 \ 0 \ 0 \ 0 \ 1 \ 1)$ [AG, V].

THEOREM K. Let Y be a curve in \mathbf{P}^3 with minimal resolution

$$0 \rightarrow \oplus \mathcal{O}(-n_{3i}) \rightarrow \oplus \mathcal{O}(-n_{2i}) \rightarrow \oplus \mathcal{O}(-n_{1i}) \rightarrow I_Y \rightarrow 0.$$

Define $c = \max\{l \mid h^l(I_Y(l)) \neq 0\}$ and $e = \max\{l \mid h^l(\mathcal{O}_Y(l)) \neq 0\}$. If $c < \min\{n_{2i}\}$ and $h^l(I_Y(n_{1i} - 4)) = 0$ for all i , then

$$h^l(N_Y) = \sum h^l(\mathcal{O}_Y(n_{1i})) - \sum h^l(\mathcal{O}_Y(n_{2i})) + \sum h^l(\mathcal{O}_Y(n_{3i})).$$

PROOF. See [K].

PROPOSITION 4.4. The Hilbert scheme $H_{d,g}$ of nonsingular curves with degree d , genus g , is nonsingular for $d = 9, g = 8$, or $d = 8, g = 6$. $H_{9,7}$ is generically reduced.

PROOF. Case 1. $d = 9, g = 8$. By Table 4.1.1, Y yields stable $E(-1, 3, 5)$ through the extension $0 \rightarrow \mathcal{O} \rightarrow E(3) \rightarrow I_Y(5) \rightarrow 0$. Using notation as in Theorem K, $3 \leq \min\{n_{1i}\}$ = the smallest degree of a surface containing Y . $c = \max\{l \mid h^l(E(l - 2)) \neq 0\} = 2$ or 3 , so $\min\{n_{2i}\} > \min\{n_{1i}\} \geq 3 \geq c$. It is easy to see from the minimal resolution, in our case, that $\max\{n_{1i} - 4\} < \max\{n_{2i} - 4\} < \max\{n_{3i} - 4\} = c \leq 3$. Table 3.14.1 implies $h^l(I_Y(l)) = h^l(E(l - 2)) = 0$ for all $l \leq 1$, hence

$$h^l(I_Y(n_{1i} - 4)) = 0 \quad \text{for all } i.$$

$h^l(\mathcal{O}_Y(l)) = h^2(I_Y(l)) = h^2(E(l - 2)) = 0$ for all $l \geq 2$ (see 3.14.1), therefore $h^l(N_Y) = 0$ and the proposition follows from Theorem K.

Case 2. $d = 8, g = 6$ (resp., $d = 9, g = 7$), similarly, using 2.6.1 (resp. 3.13.1) instead of 3.14.1.

The following lemma allows us to compute the dimension of a family of curves corresponding to a family of reflexive sheaves.

LEMMA 4.5. Let Y be a curve in \mathbf{P}^3 . If Y is the zero set of two sections s and s' of a reflexive sheaf E , then there exists $\sigma \in \text{Aut } E$, such that $s = \sigma(s')$.

PROOF. Suppose we have the correspondences $(E, s) \leftrightarrow (Y, \xi)$ and $(E, s') \leftrightarrow (Y, \xi')$. Then since ξ and ξ' vanish at the same points on Y , we have $\xi = \lambda \xi'$, where

$\lambda \in k^*$. The second row of the following commutative diagram gives $(E, s/\lambda) \leftrightarrow (Y, \lambda\xi)$:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O} & \rightarrow & E & \xrightarrow{s} & I_Y(k) \rightarrow 0 \\ & & \downarrow \lambda & & \parallel & & \parallel \\ 0 & \rightarrow & \mathcal{O} & \rightarrow & E & \xrightarrow{s/\lambda} & I_Y(k) \rightarrow 0 \end{array}$$

Since $(E, s/\lambda)$ and (E, s') correspond to equivalent extensions, there is an automorphism sending s/λ to s' , hence there exists one sending s to s' .

REMARK 4.5.1. It is clear that s and $\sigma(s)$ have the same zero set.

COROLLARY 4.5.2. *Let E be stable. If s and s' have the same nonsingular curve Y as the zero set, then $s' = \lambda s$ for some $\lambda \in k^*$ (cf. [SVB, 1.3]).*

PROOF. The only automorphisms are scalar multiplications.

EXAMPLE 4.5.3. We can show that $\dim H_{9,9} = 36$. In fact, Fact 4.1 implies that any curve Y in $H_{9,9}$ yields $E(3)$, where $E(-1, 3, 7)$ is stable. $E(3)$ is generated by global sections, and a general $E(2)$ corresponds to a nonsingular curve (cf. 3.15), hence Proposition H implies that a general zero set of $E(3)$ is nonsingular. $h^0(\omega_Y(-1)) + \dim H_{9,9} = \dim M(-1, 3, 7) + h^0(E(3))$, hence $\dim H_{9,9} = 19 + 20 - 3 = 36$.

LEMMA 4.6. *Let E be properly semistable with $c_1 = 0$. If for $k > 0$, $s \in H^0(E(k))$ has zero set of codimension 2, then $\{\sigma(s) \mid \sigma \in \text{Aut } E\}$ has dimension 2.*

PROOF. By 2.0, σ can be written as $\lambda \text{id} + \mu \sigma^0$, where $\lambda, \mu \in k^*$, and $\sigma_0: E \rightarrow I_{Y_0} \hookrightarrow \mathcal{O} \rightarrow E$, where Y_0 is the zero set of E . Therefore $\sigma(s) = \lambda s + \mu s_0 f$, where $s_0 \in H^0(E)$, and f is a degree- k form. If $\lambda's + \mu's_0 f = \lambda s + \mu s_0 f$, then $\lambda = \lambda'$ and $\mu = \mu'$, since s does not vanish on any surface.

EXAMPLE 4.6.1. Let $E(0, 2, 4)$ be properly semistable (cf. 2.16.1) and general in the family $V_{s,s}(0, 2, 4)$. Then a general zero set of $E(2)$ is a curve Y of the type $(2, 4)$ on a nonsingular quadric surface.

PROOF. It is clear that such a curve Y corresponds to $E(2)$. We show that the E constructed from such a Y form an open set of $V_{s,s}(0, 2, 4)$ by counting dimension. Y is determined by the choices of a quadric Q and a section of $\mathcal{O}_Q(2, 4)$, so it needs $9 + h^0(\mathcal{O}_Q(2, 4)) - 1 = 23$ parameters. Therefore the dimension of the family constructed from Y is $23 + h^0(\omega_Y) - h^0(E(2)) + 1 = 23 + 3 - 14 + 1 = 13 = \dim V_{s,s}(0, 2, 4)$.

Fact 4.7. If $h^0(E)$ is constant for all E in an irreducible (resp. unirational) family M , and there is a universal sheaf \mathcal{E} over $M \times \mathbf{P}^3 \xrightarrow{p} M$, then $H = \{Y \mid Y \text{ corresponds to some } E \text{ in the family } M\}$ is irreducible (resp. unirational).

PROOF. Since $h^0(E)$ is constant, $p_* \mathcal{E}$ is locally free over M and commutes with base change. Hence $\{(E, s)\}$ can be identified with $\mathbf{P}(p_* \mathcal{E})$. Note the correspondence $H_{d,g} \supseteq H = \{Y\} \leftarrow \{(Y, \xi)\} \leftrightarrow \{(E, s)\} \rightarrow M$.

REMARK 4.7.1. The constancy of $h^0(E)$ is indeed necessary. For example, consider the irreducible family $M(0, 3, 4)$. A general $E(1)$ corresponds to the disjoint union of a line and a twisted cubic, but the zero set of the general section of a special $E(1)$ is two disjoint conics.

PROPOSITION 4.8. $H_{d,g}$ is irreducible and unirational of dimension $4d$, if $d = 7$, $g = 3, 4, 5$; or $d = 8$, $g = 6, 7$; or $d = 9$, $g = 6, 7, 8, 9$.

PROOF. Cf. 4.1.1, 4.5.3, 2.5, [SRS, 9.3], 3.4, 3.5, 3.9 and 3.12–3.15.

COROLLARY 4.8.1. The variety of moduli \mathfrak{N}_g of curves of genus g is unirational, for $g = 5, 6, 7, 8$.

PROOF. Eisenbud and Harris [E-H] have shown that if $d \geq \frac{3}{4}g + 3$, then a general curve of genus g can be embedded as a curve of degree d in \mathbf{P}^3 . In other words, the natural map $H_{d,g} \rightarrow \mathfrak{N}_g$ is dominant. Thus the result follows from 4.8.

REMARK. It is well known that \mathfrak{N}_g is unirational for $g \leq 10$.

As a consequence of the irreducibility of $M(c_1, c_2, c_3)$, one can prove that certain curves are limits of irreducible, nonsingular curves. We will write down the results obtained from one particular family of sheaves, namely, $M(0, 3, 4)$.

COROLLARY 4.8.2. Let C_1 (resp. C_2) be an irreducible, nonsingular curve of degree d_1 , genus g_1 (resp. d_2, g_2), such that C_1 and C_2 meet transversally at r points and where d_1, g_1, d_2, g_2 and r are given by the following table. Then the curve $Y = C_1 \cup C_2$ is the limit in the Hilbert scheme of irreducible, nonsingular curves:

r	(d_1, g_1)	(d_2, g_2)
2	(2, 0)	(5, 3)
2	(1, 0)	(6, 3)
3	(3, 1)	(4, 1)
3	(3, 0)	(4, 2)
3	(2, 0)	(5, 2)
3	(1, 0)	(6, 2)
4	(3, 0)	(4, 1)
4	(2, 0)	(5, 1)
4	(1, 0)	(6, 1)
5	(3, 0)	(4, 0)
5	(2, 0)	(5, 0)
5	(1, 0)	(6, 0)

PROOF. Consider, for instance, line 4. It is easy to see that ω_Y has a section generating it almost everywhere. Thus Y corresponds to $E(0, 3, 4)$. By 4.7 and 3.4.1, Y is the limit of a general zero set Y' of a general sheaf E' ; as in 4.2 and 4.5.3, we see that Y' is nonsingular and irreducible. Q.E.D.

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