# ON BASES IN THE DISC ALGEBRA

BY

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ABSTRACT. It is shown that the disc algebra has no Besselian basis. In fact, concrete minorations on certain Lebesgue functions are obtained. A consequence is the nonisomorphism of the disc algebra and the space of uniformly convergent Fourier series on the circle.

**0. Introduction.** The purpose of this paper is to prove the following result on the disc algebra $A$.

THEOREM. *Assume $m$ is a positive integer and $(\varphi_k)_{1 \leq k \leq n}$, $(x_k)_{1 \leq k \leq n}$ are biorthogonal sequences in $A$, $A^*$ resp., satisfying the properties:*
  1. $\|\varphi_k\|_\infty \leq M$ $(1 \leq k \leq n)$,
  2. $\|\sum a_k x_k\| \leq M(\sum |a_k|^2)^{1/2}$ *for all scalar sequences $(a_k)$.*
*Then, for some $z \in \mathbf{C}$, $|z| \leq 1$,*

$$\frac{1}{n} \sum_{1 \leq m \leq n} \left\| \sum_{1 \leq k \leq m} \varphi_k(z) x_k \right\| \geq \rho(M) \log n.$$

This theorem can be seen as an extension to the disc algebra of results of A. M. Olevskii [9] and S. V. Bochkarev [5]. The next consequence is immediate. Its analogue for the spaces $C$ and $L^1$ was proved by S. Szarek (see [11], also for definitions).

COROLLARY 1. *The space $A$ (resp. $L^1/H_0^1$) has no Besselian (resp. Hilbertian) basis.*

Recall that a sequence $(x_k)_{k=1,2,\ldots}$ is called Besselian (resp. Hilbertian) provided it admits a lower (resp. upper) estimation

$$\left\| \sum a_k x_k \right\| \geq c \left( \sum |a_k|^2 \right)^{1/2} \quad \left( \text{resp. } \left\| \sum a_k x_k \right\| \leq C \left( \sum |a_k|^2 \right)^{1/2} \right).$$

Notice that the space $A$ (resp. $L^1/H^1$) has a basis (see [6]).

The above theorem permits a local version of Corollary 1 in terms of finite-dimensional complemented subspaces of these spaces. Denote by $U$ the completion of the space of analytic polynomials on the circle (or on the disc) under the norm

$$\|p\|_U = \sup_m \|p * D_m\|_\infty$$

where $D_m(\theta) = \sum_{0 \leq k \leq m} e^{ik\theta}$ is the corresponding Dirichlet kernel.

Since obviously the characters form a Besselian basis for $U$, another consequence is (see [3]).

COROLLARY 2. *There is no linear isomorphism between the spaces $A$ and $U$.*

We use the notation $C$ for a numerical constant. In what follows, elements of $A$ and $H^\infty$ will be considered as functions on the circle $\Pi$.

The reader may consult [8] as standard reference for general Banach space theory and [10] for Banach spaces of analytic functions.

**1. Lifting and change of measure.** By an argument of local reflexivity (see [8, p. 33] for details), it is clear that we can assume $x_k \in L^1/H_0^1$. Consider the operator from $A$ into $l_n^2$ mapping $\varphi$ onto the sequence $(\langle \varphi, x_k \rangle)_{1 \le k \le n}$. The norm of this operator is bounded by $M$ and by the result of [1] (see §2) has 2-summing norm at most $CM$. Hence, using the Pietch-Grothendieck factorization theorem (cf. [11, p. 58]) and a change of density, we get

LEMMA 1. *There are functions $(f_k)_{1 \le k \le n}$ in $L^1(\Pi)$ and $\Delta \in L^1(\Pi)$, $\Delta > 0$ and $\int \Delta = 1$ satisfying:*
(i) $q(f_k) = x_k$ $(1 \le k \le n)$ *where $q$ denotes the quotient map from $L^1(\Pi)$ onto $L^1/H_0^1$,*
(ii) $(\int |f|^2 \Delta^{-1})^{1/2} \le CM (\sum |a_k|^2)^{1/2}$ *for each $f = \sum a_k f_k$.*

Note that indeed, since $x_k \in L^1/H_0^1$, we can assume that the corresponding Pietch measure is of the form $\Delta \, dm$ ($m$ is the Haar measure on $\Pi$).

To $\Delta$ we apply Proposition 1.2 of [2], which we restate now.

LEMMA 2. *There are positive scalars $(c_i)$ and $H^\infty$-functions $(\theta_i)$, $(\tau_i)$ such that defining $\Delta_1 = \sum c_i |\tau_i|$ :*
(i) $\|\theta_i\|_\infty \le 1$,
(ii) $\| \sum |\tau_i| \|_\infty \le C$,
(iii) $\sum \tau_i^2 \theta_i = 1$ *a.e.,*
(iv) $\Delta_1 \ge \max(1, \Delta)$ *and* $\int \Delta_1 \le C$,
(v) $|\tau_i| \Delta_1 \le C c_i$ *a.e.*

It will be convenient to replace the Haar measure $m$ on $\Pi$ by the measure $d\mu = \Delta_1 dm$.

The following fact will be exploited in the sequel.

LEMMA 3. *Assume $\alpha \in L^2(\mu)$, $\alpha \ge 0$. Then there exists $\psi \in H^\infty$ so that:*
(i) $\|\psi\|_\infty \le C$,
(ii) $|\psi| \cdot \alpha \le 1$ *a.e.,*
(iii) $\|1 - \psi\|_{L^2(\mu)} \le C \|\alpha\|_{L^2(\mu)}$.

PROOF. It is clearly sufficient to estimate in (ii) by some constant $c$. Using the notations of Lemma 2, consider for each $i$ the function $\gamma_i = (1 + |\tau_i| \alpha)^{-1}$ and the outer functors $\psi_i$ with boundary value $\gamma_i \exp[i \mathcal{H}(\log \gamma_i)]$ ($\mathcal{H}$=Hilbert-transform). Then

$$\|1 - \psi_i\|_2 \le C \| |\tau_i| \cdot \alpha \|_2 \quad \text{and} \quad \alpha |\tau_i| \cdot |\psi_i| \le 1 \quad \text{a.e.}$$

Define $\psi = \sum \theta_i \tau_i^2 \psi_i$. Then, by (ii) of Lemma 2, $\|\psi\|_\infty \leq 1$ and $\alpha \cdot |\psi| \leq c$ a.e. By (iii) and (v) of Lemma 2

$$\int |1 - \psi|^2 \Delta_1 \leq C \int \sum |\tau_i|^2 |1 - \psi_i|^2 \Delta_1 \leq C \sum c_i \|1 - \psi_i\|_2^2$$

$$\leq C \int \alpha^2 (\sum c_i |\tau_i|^2) \leq C \int \alpha^2 \Delta_1,$$

which completes the proof.

Let us replace $f_k$ by $\overline{f}_k = f_k \Delta_1^{-1}$ for each $k = 1, \ldots, n$. Define further for $f \in L^1(\mu)$

$$\|f\|_* = \sup_{\substack{\varphi \in H^\infty \\ \|\varphi\|_\infty \leq 1}} \left| \int f \cdot \varphi \, d\mu \right|.$$

Hence $\|f\|_{L^2(\mu)} \leq CM(\sum |a_k|^2)^{1/2}$ for $f = \sum a_k \overline{f}_k$ and the inequality stated in the theorem becomes

$$(*) \qquad \frac{1}{n} \sum_{1 \leq m \leq n} \left\| \sum_{1 \leq k \leq m} \varphi_k(z) \overline{f}_k \right\|_* \geq \rho(M) \log n.$$

## 2. Expression of the mean using Haar and Schauder systems.

It is no restriction to take $n$ of the form $n = 2^R$, thus $R = \log n$. Denote by $(\chi_j(\omega))_j$ the $L^\infty$-normalized Haar system on $[0, 1]$ and $(\varsigma_j(s))_j$ the Schauder system on $[0, 1]$ (see [9, p. 36]). We reformulate the left member in $(*)$ as in [5]. Define the function $F$ on $\Pi \times \Pi \times [0, 1]$ by

$$F(\theta, \psi, s) = \varphi_k(\theta) \overline{f}_k(\psi) \quad \text{if } (k - 1)/n \leq s < k/n$$

and for $m = 1, 2, \ldots, n$

$$\Lambda_m(s) = \begin{cases} 1 & \text{if } s \in [0, m/n], \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for fixed $\theta$ and $\varsigma \in L^2[0, 1]$

$$\left\| \int F(\theta, \psi, s) \varsigma(s) \, ds \right\|_{L^2(\mu(d\psi))} \leq CM^2 n^{-1/2} \|\varsigma\|_2.$$

For $z = e^{i\theta}$, the left member of $(*)$ becomes

$$(**) \qquad \sum_{1 \leq m \leq n} \left\| \int F(\theta, \psi, s) \Lambda_m(s) \, ds \right\|_*.$$

For $\omega \in [(m - 1)n^{-1}, mn^{-1}]$, one has (see [7, pp. 120, 50])

$$\Lambda_m(s) = 1 - s - \frac{1}{2} \sum_{r=0}^{R-1} \sum_{j=2^r+1}^{2^{r+1}} \chi_j(\omega) \varsigma_j(s) + \lambda(\omega, s),$$

where $\lambda(\omega, s)$ is supported by $[(m-1)n^{-1}, mn^{-1}]$ and $|\lambda(\omega, s)| \leq 1$. Define the functions

$$c_0(\theta, \psi) = \int F(\theta, \psi, s)(1-s)\, ds,$$

$$c_1(\theta, \psi, \omega) = \int F(\theta, \psi, s)\lambda(\omega, s)\, ds,$$

$$c_j(\theta, \psi) = \int F(\theta, \psi, s)\varsigma_j(s)\, ds \qquad (1 \leq j \leq 2^R).$$

Substitution in $(**)$ and integration in $\theta$ with respect to $\mu$ leads to the expression

$$(*) \quad n \int_0^1 \int_\Pi \left\| c_0(\theta, \psi) + c_1(\theta, \psi, \omega) - \frac{1}{2} \sum_{r=0}^{R-1} \sum_{j=2^r+1}^{2^{r+1}} \chi_j(\omega)c_j(\theta, \psi) \right\|_{*(\psi)} d\omega \mu(d\theta).$$

The problem is reduced to minorating $(*)$. Notice that the contribution of the $c_1$-term in $(*)$ is bounded by $M^2$ and will therefore be negligible, provided the remainder has a minoration of order $\log n$.

## 3. Construction of an $H^\infty$-valued martingale.

The idea of the proof is the fact that the terms appearing after the summation $\sum_{r=0}^{R-1}$ will sum up in the $l^1$-sense. To formalize this, we will exhibit a martingale in the variable $\omega$ such that pointwise a $c_0$-sequence in $H^\infty$ is obtained.

Define the function $W$ on $\Pi \times [0, 1]$ by

$$W(\theta, s) = |\bar{f}_k(\theta)| \quad \text{for } (k-1)/n \leq s < k/n.$$

Let $0 < \varepsilon, \delta < 1$ and $2 > \rho > 1$ be constants to be specified later. Define

$$A_j(\theta, \psi) = \delta \left[ \int \varsigma_j + \varepsilon \int W(\theta, s)\varsigma_j(s)\, ds \right]^{-1} \int F(\psi, \theta, s)\varsigma_j(s)\, ds$$

which is an $\varepsilon^{-1}\delta M$-bounded $H^\infty$-function in $\psi$. Define for $r = 0, 1, \ldots, R-1$

$$\Phi_r(\omega, \theta, \psi) = \sum_{j=2^r+1}^{2^{r+1}} \chi_j(\omega)A_j(\theta, \psi)$$

and apply Lemma 3 to the function $\alpha(\psi) = \sum_{t=0}^{r-1} \rho^{r-t}|\phi_t(\omega, \theta, \psi)|$ for fixed $\omega, \theta$. This gives a function $\Psi_r(\omega, \theta, \psi)$ only dependent on the first $(r-1)$-levels of the diadic filtration. Therefore the sum

$$\Gamma = \sum_{r=0}^{R-1} \Phi_r \cdot \Psi_r \qquad (\Psi_0 = 1)$$

is a diadic martingale difference sequence with respect to the Haar system in $\Omega$. Moreover, $\Gamma$ is $H^\infty$ in $\psi$ and since by construction

$$|\Psi_r| \cdot |\Phi_t| \leq \rho^{-(r-t)} \quad \text{for } r > t$$

it follows from the lemma stated at the end of this section that

$$\|\Gamma\|_\infty \leq C(\varepsilon^{-1}\delta M)(\rho - 1)^{-1} \quad \text{for } \delta \leq \varepsilon M^{-1}.$$

Also (by (iii) of Lemma 3), freezing $\omega, \theta$ and considering the $L^2(\mu)$-norm in $\psi$, we have

$$\|1 - \Psi_r\|_{L^2(\mu)} \leq \sum_{t < r} \rho^{r-t} \|\Phi_t\|_{L^2(\mu)}.$$

Using $\Gamma_{\omega, \theta}$ as a test function, the following minoration of $(*)$ is obtained:

$$nC^{-1}\varepsilon^{1/2}\delta^{-1/2}M^{-1/2} \int_0^1 \int_\Pi \int_\Pi \left\{ \sum_{r=0}^{R-1} \sum_{j=2^r+1}^{2^{r+1}} \chi_j(\omega)^2 c_j(\theta, \psi) \right.$$

$$\left. A_j(\theta, \psi)\Psi_r(\omega, \theta, \psi) \right\} \omega(d\theta)\mu(d\psi)\, d\omega.$$

The integral is minorated as

$(**)$

$$\left| \sum_{0 \leq r < R} 2^{-r} \sum_j \int_\Pi \int_\Pi c_j(\theta, \psi) A_j(\theta, \psi)\mu(d\theta)\mu(d\psi) \right|$$

$$- \varepsilon^{-1}\delta M \sum_{r=0}^{R-1} \int_0^1 \int_\Pi \int_\Pi \left| \sum_j \chi_j(\omega)^2 c_j(\theta, \psi) \right| |1 - \psi_r(\omega, \theta, \psi)|\, \mu(d\theta)\mu(d\psi)\, d\omega.$$

LEMMA 4. *Assume* $0 \leq a_k \leq A$, $0 \leq b_k \leq B$ *and* $a_k b_{k+j} < \rho^{-j}$ *with* $\rho > 1$. *Then*

$$\sum a_k b_k \leq 2\rho AB/\rho - 1.$$

We leave the proof as an exercise.

**4. Contribution of main terms.** It will be convenient to consider the function $d$ on $[0, 1]^2$ defined by

$$d(s, s') = \begin{cases} 1 & \text{if } s, s' \text{ belong to the same interval } [\frac{k-1}{n}, \frac{k}{n}[, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the biorthogonality hypothesis that

$$\int F(\theta, \psi, s)F(\psi, \theta, s')\mu(d\psi) = d(s, s')F(\theta, \theta, s).$$

Hence

$$\int_\Pi \int_\Pi c_j(\theta, \psi) A_j(\theta, \psi) = \delta \int_\Pi \int_0^1 \int_0^1 d(s, s')\varsigma_j(s)\varsigma_j(s')F(\theta, \theta, s)$$

$$\times \left[ \int \varsigma_j + \varepsilon \int W(\theta, s)\varsigma_j(s)\, ds \right]^{-1} ds\, ds'\mu(d\theta),$$

where

$$\left| [\cdots]^{-1} - \left( \int \varsigma_j \right)^{-1} \right| \leq \varepsilon \left( \int \varsigma_j \right)^{-2} \int W(\theta, s)\varsigma_j(s)\, ds.$$

This gives the contribution $C^{-1}\delta/n$ up to a perturbation of at most

$$\frac{1}{n}\varepsilon\delta \left( \int \varsigma_j \right)^{-2} \left\{ \int \|W(\theta, s)\|_{L^2(\mu)}\varsigma_j(s)\, ds \right\} \left\{ \int \|F(\theta, \theta, s)\|_{L^2(\mu)}\varsigma_j(s)\, ds \right\}$$

and thus $(C/n)\varepsilon\delta M^3$. Consequently, for $\varepsilon \sim M^{-3}$, the sum of the first integral in
(**) has the contribution $\delta R/Cn$.

**5. Estimation of error terms.** Again by Cauchy-Schwarz, the $r$th triple
integral appearing in (**) is dominated by

$$\int_0^1 \left\| \sum_{j=2^r+1}^{2^{r+1}} \chi_j(\omega)^2 c_j(\theta,\psi) \right\|_{L^2(\mu(d\theta)\otimes\mu(d\psi))} \|1-\Psi_r(\omega,\theta,\psi)\|_{L^2} \, d\omega.$$

For each $\omega \in [0,1]$, the first factor is dominated by

$$\sup_{2^r \le j \le 2^{r+1}} \|c_j(\theta,\psi)\|_{L^2} \le CM^2 n^{-1/2}\sqrt{2^r}.$$

Since for fixed $\omega,\theta$

$$\|1-\psi_r\|_{L^2}(\mu(d\psi)) \le \sum_{t<r} \rho^{r-t}\|\phi_t\|_{L^2(\mu(d\psi))}$$

we find

$$\int_0^1 \|1-\Psi_r(\omega,\theta,\psi)\|_{L^2} \, d\omega \le \sum_{t<r} \rho^{r-t} \sup_{2^t < j \le 2^{t+1}} \|A_j\|_{L^2}$$

where

$$\|A_j\|_{L^2} \le \delta \left(\int \varsigma_j\right)^{-1} \left\| \int F(\psi,\theta,s)\varsigma_j(s)\,ds \right\|_{L^2} \le C\delta M^2 \left(\int \varsigma_j\right)^{-1} \|\varsigma_j\|_2 n^{-1/2}.$$

Thus

$$\int_0^1 \|1-\Psi_r\|_{L^2} \, d\omega \le C \left(\sum_{t<r} \rho^{r-t}\sqrt{2^t}\right) \delta M^2 n^{-1/2} \le C\delta M^2 n^{-1/2}\sqrt{2^r}$$

choosing $2\rho = 1 + \sqrt{2}$. Therefore, the second term in (**) is at most

$$\frac{C}{n}\varepsilon^{-1}\delta^2 M^5 R = \frac{C}{n}M^8\delta^2 R.$$

**6. End of proof.** The left member in the inequality stated in the Theorem
dominates (*) and hence, collecting previous estimates,

$$C^{-1}M^{-4}\delta^{-1}n\left\{\frac{\delta R}{Cn} - \frac{C}{n}M^8\delta^2 R\right\} = C^{-1}M^{-4}R(1-C\delta M^8).$$

Taking $\delta \sim M^{-8}$, the Theorem follows with $\rho(M) \sim M^{-4}$.

Denote by $L_n^\infty$ (resp. $U_n$) the linear space $[1, e^{i\theta}, \ldots, e^{in\theta}]$ equipped with $L^\infty$-
norm (resp. $\| \ \|_U$). As a consequence of the proof of the Theorem and a result of
[4] we have

COROLLARY 3. *There exists $\kappa > 0$ such that $d(L_n^\infty, U_n) \ge (\log n)^\kappa$, where $d$ is
the Banach-Mazur distance of normed spaces.*

One can give a more qualitative version of the Theorem (in particular, not in-
volving a change of measure) and extend Szarek's results [12] to the spaces $A$ and
$L^1/H_0^1$. Thus

PROPOSITION 4. *Any normalized basis of $L^1/H_0^1$ has a subsequence which is
equivalent to the usual $l^1$-unit vector basis. If $(g_n)$ is a normalized basis of $A$, then,*

*for some increasing sequence $(n_k)$ of positive integers, the map $\sum c_n g_n \mapsto (c_{n_k})_{k=1}^{\infty}$ takes $A$ onto $c_0$.*

This answers affirmatively Proposition 4.10 of [12].

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