

INVOLUTIONS WITH ISOLATED FIXED POINTS ON ORIENTABLE 3-DIMENSIONAL FLAT SPACE FORMS

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ABSTRACT. In this paper we completely classify (up to conjugacy) all involutions $\iota: M \rightarrow M$, where M is an orientable connected flat 3-dimensional space form, such that ι has fixed points but only finitely many. If M_1, \dots, M_6 are the 6 space forms then only M_1, M_2, M_6 admit such involutions. Moreover, they are unique up to conjugacy. The main idea behind the proof is to find incompressible tori $T \subseteq M$ so that either $\iota(T) = T$ or $\iota(T) \cap T = \emptyset$ and then cut M into simpler pieces. These results lead to a complete classification of 3-manifolds containing $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ in their fundamental groups.

1. Introduction. In [1] it was shown that the only possible finitely generated abelian subgroups of the fundamental group of a 3-manifold are \mathbb{Z}_n , $\mathbb{Z} \oplus \mathbb{Z}_2$, \mathbb{Z} , $\mathbb{Z} \oplus \mathbb{Z}$, and $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Moreover, those manifolds M^3 such that $\pi_1(M)$ contains a finite subgroup or $\mathbb{Z} \oplus \mathbb{Z}_2$ as a subgroup were also characterized. In order to characterize those M^3 so that $\pi_1(M)$ contains $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ as a subgroup it is necessary to determine the involutions with isolated fixed points on 3-dimensional orientable flat space forms (see [7] for this characterization).

The purpose of this paper is to classify up to conjugacy all involutions $\iota: M \rightarrow M$, where M is an orientable flat 3-dimensional space form and ι has fixed points but only finitely many.

Throughout the paper we use the term space form as an abbreviation for a connected closed orientable 3-dimensional flat space form. According to Wolf (see [11, p. 117]) there are only 6 space forms up to affine equivalence. We use the notation M_1, \dots, M_6 to correspond to the order of these space forms on p. 117 of [11]. In particular M_1 is just the torus $S^1 \times S^1 \times S^1$ and M_6 is the so-called Hantzsche-Wendt manifold [2]. It turns out that M_1, \dots, M_5 are torus bundles over S^1 , whereas M_6 is the union along the boundaries of twisted orientable I -bundles over a Klein bottle [8]. For a complete topological description see [7].

In §5 we classify all involutions with isolated fixed points, on orientable torus bundles over S^1 . The result is that only M_1 and M_2 admit such involutions and they are unique up to conjugacy. Then in §6 we carry out the same program for unions along the boundaries of two orientable twisted I -bundles over Klein bottles. Among these is also the space form M_2 . We prove that besides M_2 only M_6 admits

Received by the editors November 17, 1982 and, in revised form, August 2, 1983.
1980 *Mathematics Subject Classification*. Primary 57S25.

involutions with isolated fixed points. Again the involutions are uniquely determined up to conjugacy. These involutions on M_1 , M_2 , M_6 have 8, 4, 2 fixed points, respectively. For further descriptions see [7].

If $\iota: M_i \rightarrow M_i$, $i = 1, 2$, or 6, is such an involution and x_1, \dots, x_k are the fixed points ($k = 8, 4, 2$) then there are disjoint 3-cells C_1, \dots, C_k so that $x_i \in \text{int } C_i$ and $\iota(C_i) = C_i$, $1 \leq i \leq k$. We let M_i^* denote $M_i - \text{int}(C_1 \cup \dots \cup C_k)/\iota$ the orbit manifold. In §7 we give explicit descriptions of M_1^* , M_2^* , and M_6^* .

All the space forms have incompressible tori and thus a certain hierarchy. The main idea behind the proofs is to find incompressible tori which either are disjoint from their image or which are invariant under the involution; thereby reducing the study of involutions on the various classes of manifolds to the study of involutions on simpler manifolds such as $T \times [0, 1]$, where T is the 2-dimensional torus, or W = the twisted orientable I -bundle over the Klein bottle.

In §3, we determine all involutions with only isolated fixed points on $T \times [0, 1]$ and then in §4 we do this for W .

For the torus $M = S^1 \times S^1 \times S^1$ a complete description of all possible involutions was given in [4]. Another relevant reference is [10].

We would like to thank the referee for his comments.

2. Notation and preliminaries. We will work throughout in the PL category. Our references are [3 or 10]. A PL homeomorphism we simply call an isomorphism.

All involutions are assumed to have only isolated fixed points. The following lemma will be applied several times.

LEMMA (2.1). *Suppose M is an n -manifold, $F \subseteq M$ is a proper bicollared $(n - 1)$ -submanifold, and $\iota: M \rightarrow M$ is an involution with $\iota(F) = F$. If U is any neighborhood of F in M , then there is a bicollar $F \times [-1, 1]$ of $F = F \times 0$ in M such that $F \times [-1, 1] \subset U$ and $\iota(F \times [-1, 1]) = F \times [-1, 1]$.*

Moreover, we may assume that $F \times [-1, 1] \cap \partial M = \partial F \times [-1, 1]$.

PROOF. There is a simplicial subdivision of M containing F as a subcomplex, with its simplicial neighborhood in U , and such that ι is simplicial. Let $F \times [-1, 1]$ be a regular neighborhood with respect to this subdivision. Q.E.D.

We have the following

PROPOSITION (2.2). *Suppose M is an n -manifold and $\iota: M \rightarrow M$ is an involution with only isolated fixed points. Then the fixed points of ι are in the interior of M .*

PROOF. There is a simplicial subdivision of M containing the fixed points as vertices such that ι is simplicial. If a fixed point is on the boundary of M , its link is an $(n - 1)$ -cell and it is invariant under ι . Therefore ι must have a fixed point on this link, and hence keep a 1-cell pointwise fixed; a contradiction. Q.E.D.

We will frequently apply the Lefschetz fixed point theorem as follows.

LEMMA (2.3). *Let M be a compact n -manifold and let $\iota: M \rightarrow M$ be an involution with fixed points x_1, \dots, x_k . If λ is the Lefschetz number of ι , then $\lambda = k$.*

PROOF. Choose a simplicial subdivision of M containing x_1, \dots, x_k as vertices and such that ι is simplicial. The star-neighborhoods of x_1, \dots, x_k in the first barycentric subdivision then define disjoint 3-cells $C_1, \dots, C_k \subset \text{int } M$ with $x_i \in \text{int } C_i$ and with $\iota(C_i) = C_i$, $1 \leq i \leq k$. Since $\iota|_{\partial C_i}$, $1 \leq i \leq k$, is a fixed point free involution of the $(n-1)$ -sphere ∂C_i , $\iota|_{\partial C_i}$ is orientation reversing if n is odd, and orientation preserving if n is even. The Lefschetz number of $\iota|_{\overline{M-C_1 \cup \dots \cup C_k}}$ can be computed by applying a Mayer-Vietoris sequence to $M = \overline{M-C_1 \cup \dots \cup C_k} \cup (C_1 \cup \dots \cup C_k)$. The result is $\lambda - k$. Therefore $\lambda = k$ as $\iota|_{\overline{M-C_1 \cup \dots \cup C_k}}$ has no fixed points. Q.E.D.

Note that if $k \geq 1$ and M is orientable then ι must be orientation reversing if n is odd and orientation preserving if n is even.

Two involutions $\iota, \iota': X \rightarrow X$ on a space X are said to be conjugate if there exists an isomorphism $h: X \rightarrow X$ with $\iota' = h^{-1}\iota h$.

A 3-manifold M is said to be irreducible if each 2-sphere in M bounds a 3-cell in M .

Let M be a 3-manifold and let $F \subset M$ be a compact 2-manifold which is either properly embedded in M or is contained in ∂M . We say that F is incompressible in M if no component of F is a 2-sphere and if for each 2-cell $B \subset M$ with $B \cap F = \partial B$ there is a 2-cell $B' \subset F$ with $\partial B' = \partial B$.

DEFINITION. A proper annulus A in a 3-manifold M is said to be trivial if A decomposes M into a solid torus $V = D^2 \times S^1$ and a submanifold M_0 such that:

(I) $M = M_0 \cup V$, $M_0 \cap V = \partial M_0 \cap \partial V = A$, and

(II) there exists a nonseparating proper 2-cell $B \subseteq V$ so that $B \cap A = \partial B \cap A$ is a nonseparating proper 1-cell in A . Otherwise we say that A is nontrivial.

Note that if the bounding 1-spheres of A are in different boundary components of M or if A does not separate M then A is nontrivial.

A nontrivial, incompressible, proper annulus in a 3-manifold we call an essential annulus. (Compare remark in [6].)

Note a nonseparating proper annulus in an irreducible 3-manifold with incompressible boundary is essential.

We now summarize for the convenience of the reader the results of [6] which are the basic tools of our proofs.

Suppose M is an irreducible 3-manifold and $\iota: M \rightarrow M$ is an involution. In Theorems (2.6) and (2.7) assume in addition that M is orientable.

THEOREM (2.4). *If there exists a proper nonseparating 2-cell $B_0 \subset M$, then there exists one with $B \cap \iota(B) = \emptyset$.*

Moreover, if $\partial B_0 \cap \iota(\partial B_0) = \emptyset$ we may assume that $\partial B = \partial B_0$.

THEOREM (2.5). *Suppose $A_0 \subset M$ is an essential annulus with $\partial A_0 \cap \iota(\partial A_0) = \emptyset$. Then either there is an essential annulus $A \subset M$ with*

$$A \cap \iota(A) = \emptyset \quad \text{and} \quad \partial A \cup \iota(\partial A) = \partial A_0 \cup \iota(\partial A_0).$$

or there are two disjoint essential annuli A_1, A_2 in M with $\iota(A_i) = A_i$, $i = 1, 2$, with $\partial(A_1 \cup A_2) = \partial A_0 \cup \iota(\partial A_0)$, and with no fixed points of ι on $A_1 \cup A_2$.

THEOREM (2.6). Suppose $A_0 \subset M$ is an essential annulus. Let $\partial A_0 = S_{01} \cup S_{02}$, and let $R_i \subset \partial M$ be the component with $S_{0i} \subset R_i$, $i = 1, 2$. If $R_1 = R_2$, assume that R_1 is incompressible. Then there is an essential annulus $A \subset M$ with either $A \cap \iota(A) = \emptyset$ or $\iota(A) = A$, and with $\partial A \cup \iota(\partial A) \subset R_1 \cup R_2 \cup \iota(R_1 \cup R_2)$. In the latter case there are no fixed points of ι on A .

Furthermore, if $R_1 \neq R_2$ and $\iota(R_1 \cup R_2) = R_1 \cup R_2$, we may assume that $\partial A = S_1 \cup S_2$ satisfies $S_i \subset R_i$, $i = 1, 2$.

THEOREM (2.7). Suppose $\iota: M \rightarrow M$ is an orientation reversing involution. If M contains an incompressible torus, then M contains an incompressible torus T with either $T \cap \iota(T) = \emptyset$ or $\iota(T) = T$. In the latter case there are no fixed points of ι on T .

3. Involutions on $T \times [0, 1]$. In order to describe the involutions on $T \times [0, 1]$, where T is the 2-dimensional torus $S^1 \times S^1$, we introduce the following involutions.

Let $\alpha: S^1 \rightarrow S^1$ be the antipodal map, $\alpha(x) = -x$; let $\kappa: S^1 \rightarrow S^1$ denote conjugation, $\kappa(x) = \bar{x}$; and let $\tau: [0, 1] \rightarrow [0, 1]$ be the map $\tau(t) = 1 - t$. Note that α is fixed point free, κ has 2 fixed points $\{1\}$ and $\{-1\}$, and τ has one fixed point $\{\frac{1}{2}\}$.

The main result of this section is in essence that any involution $\iota: T \times [0, 1] \rightarrow T \times [0, 1]$ is conjugate to exactly one of the following 5 involutions of $T \times [0, 1]$:

$$\alpha \times \text{id} \times \text{id}, \quad \alpha \times \kappa \times \text{id}, \quad \alpha \times \kappa \times \tau, \quad \alpha \times \text{id} \times \tau, \quad \kappa \times \kappa \times \tau.$$

The first 4 involutions are fixed point free and the last one has exactly 4 fixed points. There is also an all important relative version. If ι has no fixed points this result also follows from [3, Theorem 10.3, p. 91].

We now give more detailed descriptions of the above 5 standard involutions on $T \times [0, 1]$. In $T \times [0, 1]$ we select annuli A_{\pm} and solid tori V_{\pm} as follows:

$$\begin{aligned} A_{\pm} &= \{(\pm i, y, t) \in T \times [0, 1]\}, \\ V_+ &= \{(x, y, t) \in T \times [0, 1]; \operatorname{Re} x \geq 0\}, \\ V_- &= \{(x, y, t) \in T \times [0, 1]; \operatorname{Re} x \leq 0\}. \end{aligned}$$

Then $T \times [0, 1] = V_+ \cup V_-$ and $V_+ \cap V_- = A_+ \cup A_-$ (see Figure 1).

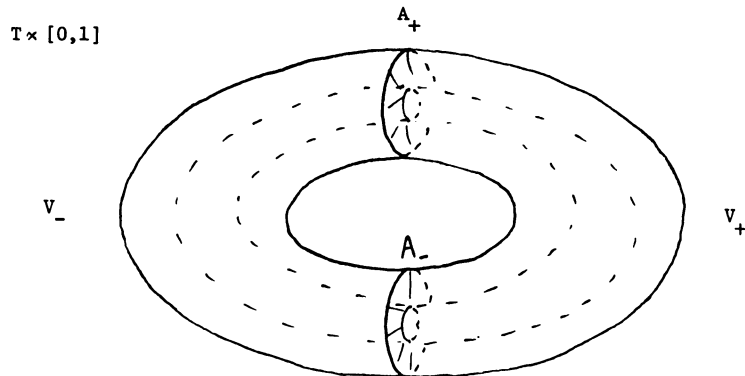


FIGURE 1

Each of the 5 standard involutions maps A_+ onto A_- . We have the following table:

involution	orientation type	fixed points	image of $T \times 0$	image of V_+
$\alpha \times \text{id} \times \text{id}$	preserving	0	$T \times 0$	V_-
$\alpha \times \kappa \times \text{id}$	reversing	0	$T \times 0$	V_-
$\alpha \times \kappa \times \tau$	preserving	0	$T \times 1$	V_-
$\alpha \times \text{id} \times \tau$	reversing	0	$T \times 1$	V_-
$\kappa \times \kappa \times \tau$	reversing	4	$T \times 1$	V_+

The first 3 columns of the table (i.e. the orientation type, the cardinality of the fixed point set, and the image of $T \times 0$) are conjugacy class invariants. Conversely, by our main result, if $\iota: T \times [0, 1] \rightarrow T \times [0, 1]$ is an involution, these 3 invariants determine uniquely its conjugacy class.

To classify the involutions of $T \times [0, 1]$ we proceed as follows. First we classify the involutions on a solid torus. If $\iota: T \times [0, 1] \rightarrow T \times [0, 1]$ is an involution we show that there exists an incompressible annulus $A \subset T \times [0, 1]$ that meets both boundary components of $T \times [0, 1]$ and so that $A \cap \iota(A) = \emptyset$. Then $A \cup \iota(A)$ decomposes $T \times [0, 1]$ into handlebodies V_{\pm} as in the situation of the 5 standard involutions of $T \times [0, 1]$. Applying the classification of the involutions on a solid torus will then lead to a classification of the involutions on $T \times [0, 1]$.

In order to classify the involutions on a solid torus we begin with the following lemma, which is an immediate consequence of Theorem 1 in [5].

LEMMA (3.1). *Suppose $\iota, \iota': S^2 \times [0, 1] \rightarrow S^2 \times [0, 1]$ are fixed point free involutions leaving the ends invariant. Then there exists an isomorphism $h: S^2 \times [0, 1] \rightarrow S^2 \times [0, 1]$ so that*

$$\iota' = h\iota h^{-1}.$$

Furthermore, if we are given an isomorphism $h_0: S^2 \times 0 \rightarrow S^2 \times 0$ satisfying $\iota'|_{S^2 \times 0} = h_0\iota h_0^{-1}$, then we may assume that $h|_{S^2 \times 0} = h_0$.

(Thus such an involution is conjugate to the involution $(x, t) \rightarrow (-x, t)$.)

THEOREM (3.2). *Let C be a 3-cell and let $\iota, \iota': C \rightarrow C$ be involutions. Then there exists an isomorphism $h: C \rightarrow C$ so that $\iota' = h\iota h^{-1}$.*

Moreover, if $h_0: \partial C \rightarrow \partial C$ is an isomorphism satisfying $\iota'|_{\partial C} = h_0\iota h_0^{-1}$, we may assume that $h|_{\partial C} = h_0$.

PROOF. Note that since $\partial C/\iota|$ and $\partial C/\iota'|$ are projective planes, there always exists an isomorphism $h_0: \partial C \rightarrow \partial C$ with $\iota'|_{\partial C} = h_0\iota h_0^{-1}$. Thus we may assume that h_0 is given.

By Lemma (2.3), ι, ι' have exactly 1 fixed point each (and they must be orientation reversing).

Let K, K' be simplicial subdivisions of C so that $\iota: K \rightarrow K, \iota': K' \rightarrow K'$ are simplicial. Then the fixed points x_0, x'_0 of ι, ι' are vertices of K, K' , respectively. Let $K_0 = \text{St}(x_0, K), K'_0 = \text{St}(x'_0, K')$ be the star neighborhoods of x_0, x'_0 in K, K' ,

respectively. Then $\iota(K_0) = K_0$, $\iota'(K'_0) = K'_0$ and $\iota|_{K_0}$, $\iota'|_{K'_0}$ are conings of $\iota|_{\partial K}$, $\iota'|_{\partial K'_0}$, respectively. We may assume that $K_0 \cap \partial C = \emptyset = K'_0 \cap \partial C$. It then follows that $W = \overline{C} - |K_0|$, $W' = \overline{C} - |K'_0|$ are copies of $S^2 \times [0, 1]$.

By Lemma (3.1) we can extend h_0 to an isomorphism $\hat{h}_0: W \rightarrow W'$ so that $\iota'|_{W'} = \hat{h}_0 \iota \hat{h}_0^{-1}$. By coning $\hat{h}_0: |\partial K_0| \rightarrow |\partial K'_0|$ to the vertices x_0, x'_0 , we extend \hat{h}_0 to an isomorphism $h: C \rightarrow C$. Then $\iota' = h \iota h^{-1}$. Q.E.D.

Thus there is a unique conjugacy class of involutions on the 3-cell C . For example, let $C = c(S^2)$ (= the cone on S^2) and let $\alpha: S^2 \rightarrow S^2$ be the antipodal map. Then $\hat{\alpha}: C \rightarrow C$, $\hat{\alpha}(x, t) = (\alpha(x), t)$, is an involution with one fixed point. If $C = D^2 \times [0, 1]$, then a canonical involution is $(x, t) \rightarrow (-x, 1 - t)$.

Now we are ready to classify the involutions on the solid torus $V = D^2 \times S^1$.

THEOREM (3.3). *Suppose $\iota, \iota': V \rightarrow V$ are involutions and one of the following holds:*

- (I) ι, ι' are orientation preserving,
- (II) ι, ι' are orientation reversing and fixed point free,
- (III) ι, ι' are orientation reversing and have fixed points.

Then there is an isomorphism $h: V \rightarrow V$ so that $\iota' = h \iota h^{-1}$.

Assume $h_0: \partial V \rightarrow \partial V$ is an isomorphism with $\iota'|_{\partial V} = h_0 \iota h_0^{-1}$. In addition, for case (I) only, assume that h_0 extends to an isomorphism of V onto V . Then $h: V \rightarrow V$ can be chosen to satisfy $h|_{\partial V} = h_0$.

PROOF. Since both $\partial V/\iota|$ and $\partial V/\iota'|$ are either tori or Klein bottles there always exists an isomorphism $h_0: \partial V \rightarrow \partial V$ so that $\iota'|_{\partial V} = h_0 \iota h_0^{-1}$ and if ι, ι' are orientation preserving, extending to an isomorphism of V onto V . Thus we assume that $h_0: \partial V \rightarrow \partial V$ is given.

(1) First we show that if ι, ι' are orientation reversing, h_0 extends to an isomorphism of V onto V .

We have $H_1(\partial V) = \mathbb{Z}[\partial D] \oplus \mathbb{Z}[S]$.

Lemma (2.3) implies immediately that ι has a fixed point if and only if $\iota_*: H_1(V) \rightarrow H_1(V)$ is $\iota_* = -\text{id}$ (in which case it has exactly two fixed points). Therefore $(\iota)_*, (\iota')_*: H_1(\partial V) \rightarrow H_1(\partial V)$ have the following descriptions:

If ι, ι' are fixed point free, then

$$\begin{aligned} (\iota)_*[\partial D] &= -[\partial D], & (\iota')_*[\partial D] &= -[\partial D], \\ (\iota)_*[S] &= \alpha[\partial D] + [S], & (\iota')_*[S] &= \alpha'[\partial D] + [S] \end{aligned}$$

and if ι, ι' have fixed points, then

$$\begin{aligned} (\iota)_*[\partial D] &= [\partial D], & (\iota')_*[\partial D] &= [\partial D], \\ (\iota)_*[S] &= \beta[\partial D] - [S], & (\iota')_*[S] &= \beta'[\partial D] - [S]. \end{aligned}$$

Let $h_{0*}[\partial D] = a[\partial D] + b[S]$, $h_{0*}[S] = c[\partial D] + d[S]$ with $a \cdot d - b \cdot c = \pm 1$. Then we must have

$$\begin{bmatrix} -1 & 0 \\ \alpha' & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 0 \\ \alpha & 1 \end{bmatrix}$$

in the first case, and

$$\begin{bmatrix} 1 & 0 \\ \beta' & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \beta & -1 \end{bmatrix}$$

in the second case.

In both cases we obtain $-b = b$ and hence $b = 0$. Therefore $h_{0*}[\partial D] = \pm[\partial D]$. Consequently h_0 extends.

(2) By Theorem (2.4) there is a nonseparating proper 2-cell $B \subset V$ with

$$B \cap \iota(B) = \emptyset.$$

Then $B \cup \iota(B)$ decomposes V into 3-cells C_+, C_- : $V = C_+ \cup C_-$, $C_+ \cap C_- = \partial C_+ \cap \partial C_- = B \cup \iota(B)$. (See Figure 2).

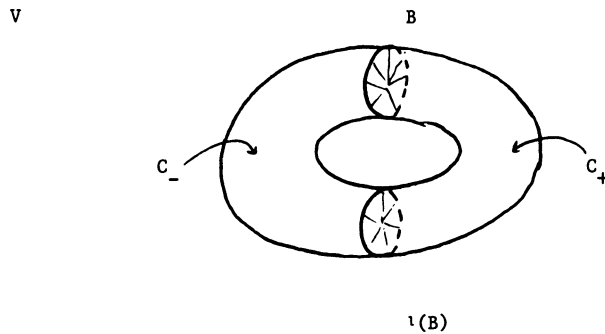


FIGURE 2

We have $\iota' h_0(\partial B) = h_0 \iota(\partial B)$ and hence

$$h_0(\partial B) \cap \iota'(h_0(\partial B)) = \emptyset.$$

By (1), and in case (I) by the hypothesis on h_0 , there is a proper 2-cell $B' \subset V$ with $\partial B' = h_0(\partial B)$. By Theorem (2.4) we may assume that $B' \cap \iota'(B') = \emptyset$.

Then $B' \cup \iota'(B')$ decomposes V again into 3-cells C'_+, C'_- : $V = C'_+ \cup C'_-$, $C'_+ \cap C'_- = \partial C'_+ \cap \partial C'_- = B' \cup \iota'(B')$.

Suppose that $h_0(C_+ \cap \partial V) = C'_+ \cap \partial V$. Let $h_{01}: B \rightarrow B'$ be an extension of $h_0|: \partial B \rightarrow \partial B'$ and define $h_{02}: \iota(B) \rightarrow \iota'(B')$ by equivariance $h_{02} = \iota' h_0 \iota|_{\iota(B)}$. Define $\hat{h}_0 = h_{01} \cup h_{02}: B \cup \iota(B) \rightarrow B' \cup \iota'(B')$. By construction, $\iota'|_{B' \cup \iota'(B')} = \hat{h}_0 \iota \hat{h}_0^{-1}$.

Case 1. Both ι and ι' have no fixed points.

As observed in (1), then $\iota_*, \iota'_*: H_1(V) \rightarrow H_1(V)$ are $\iota_* = \iota'_* = \text{id}$. Consequently, $\iota(C_+) = C_-$ and $\iota'(C'_+) = C'_-$.

Let $h_+: C_+ \rightarrow C'_+$ be an extension of $h_0|_{C_+ \cap \partial V} \cup \hat{h}_0: \partial C_+ \rightarrow \partial C'_+$ and define $h_-: C_- \rightarrow C'_-$ by $h_- = \iota' h_+ \iota|_{C_-}$. Then h_+ and h_- agree on $C_+ \cap C_- = B \cup \iota(B)$ and therefore piece together to give the conjugating isomorphism $h = h_+ \cup h_-: V \rightarrow V$ with $\iota' = h \iota h^{-1}$.

Case 2. Both ι and ι' have fixed points.

Then necessarily $\iota(C_i) = C_i$ and $\iota'(C'_i) = C'_i$, $i = +, -$. By Theorem (3.2) the isomorphism $h_0|_{C_i \cap \partial V} \cup \hat{h}_0: \partial C_i \rightarrow \partial C'_i$ extends to an isomorphism $h_i: C_i \rightarrow C'_i$ with

$\iota'|_{C_i} = h_i \iota h_i^{-1}$, $i = +, -$. Again h_+ and h_- agree on $C_+ \cap C_- = B \cup \iota(B)$ and hence $h = h_+ \cup h_-: V \rightarrow V$ is an isomorphism with $\iota' = h \iota h^{-1}$. Q.E.D.

Here are explicit representatives for the 3 conjugacy classes of involutions on V . Recall that $\alpha: S^1 \rightarrow S^1$ is the antipodal map and $\kappa: S^1 \rightarrow S^1$ is conjugation. Let $\hat{\alpha}, \hat{\kappa}: D^2 \rightarrow D^2$ denote the radial extensions. These are again involutions. The involution $\hat{\alpha}$ has precisely one fixed point. Then we have the following 3 involutions of $V = D^2 \times S^1$:

$$\begin{aligned}\omega &= \hat{\alpha} \times \alpha, & \omega(x, y) &= (-x, -y), \\ \iota_0 &= \hat{\kappa} \times \alpha, & \iota_0(x, y) &= (\bar{x}, -y), \\ \iota_2 &= \hat{\alpha} \times \kappa, & \iota_2(x, y) &= (-x, \bar{y}).\end{aligned}$$

We select proper 2-cells B_{\pm} and 3-cells C_{\pm} as follows:

$$\begin{aligned}B_{\pm} &= \{(x, \pm i) \in D \times S\}, \\ C_+ &= \{(x, y) \in D \times S; \operatorname{Re} y \geq 0\}, \\ C_- &= \{(x, y) \in D \times S; \operatorname{Re} y \leq 0\}.\end{aligned}$$

Then $V = C_+ \cup C_-$ and $C_+ \cap C_- = B_+ \cup B_-$. Compare Figure 2. Each of the 3 standard involutions maps B_+ onto B_- . We have the following table:

involution	orientation type	fixed points	image of C_+
$\omega = \hat{\alpha} \times \alpha$	preserving	0	C_-
$\iota_0 = \hat{\kappa} \times \alpha$	reversing	0	C_-
$\iota_2 = \hat{\alpha} \times \kappa$	reversing	2	C_+

The first 2 columns of the table (i.e. the orientation type, the cardinality of the fixed point set) are conjugacy class invariants. Conversely, by Theorem (3.3), if $\iota: V \rightarrow V$ is an involution, these 2 invariants determine uniquely its conjugacy class.

COROLLARY (3.4). *An involution $\iota: V \rightarrow V$ is conjugate to:*

- (I) ω , if ι is orientation preserving,
- (II) ι_0 , if ι is orientation reversing and fixed point free, and
- (III) ι_2 , if ι is orientation reversing with fixed points.

Also there is a relative version as in (3.3).

REMARK. If $\iota: V \rightarrow V$ is a fixed point free involution then V/ι is a solid torus again if ι is orientation preserving and a solid Klein bottle if ι is orientation reversing.

(Of course this result also follows immediately from Dehn's Lemma. It can also be used to prove cases (I) and (II) of Theorem (3.3).)

Next we continue with some technical lemmas. Recall that T denotes the 2-dimensional torus $S^1 \times S^1$.

LEMMA (3.5). *Let $A \subset T \times [0, 1]$ be a proper annulus. Then the following three statements are equivalent.*

(I) A is essential.

(II) A is incompressible and meets both boundary components of $T \times [0, 1]$.

(III) A is nonseparating.

LEMMA (3.6). (1) Suppose $A \subset T \times [0, 1]$ is an essential annulus. Then $(T \times [0, 1], A)$ is isomorphic to $(S^1 \times S^1 \times [0, 1], x_0 \times S^1 \times [0, 1])$.

(2) Suppose $A_1, A_2 \subset T \times [0, 1]$ are disjoint essential annuli. Then $(T \times [0, 1], A_1, A_2)$ is isomorphic to $(S^1 \times S^1 \times [0, 1], x_1 \times S^1 \times [0, 1], x_2 \times S^1 \times [0, 1])$.

PROOF. (1) We may assume that $x_0 \times S^1 \times 0$ is a boundary component of A and that $S^1 \times y_0 \times [0, 1]$ and A intersect transversally in a single proper 1-cell in A and in $S^1 \times y_0 \times [0, 1]$. With this it is not difficult to construct an isomorphism

$$(T \times [0, 1], A) \rightarrow (S^1 \times S^1 \times [0, 1], x_0 \times S^1 \times [0, 1]).$$

(2) By (1) we may assume that

$$(T \times [0, 1], A) = (S^1 \times S^1 \times [0, 1], x_1 \times S^1 \times [0, 1]).$$

We may assume that $S^1 \times y_0 \times [0, 1]$ and A_2 intersect transversally in a single proper 1-cell in A_2 and in $S^1 \times y_0 \times [0, 1]$. Then it is not difficult to construct an isomorphism

$$(T \times [0, 1], A_1, A_2) \rightarrow (S^1 \times S^1 \times [0, 1], x_1 \times S^1 \times [0, 1], x_2 \times S^1 \times [0, 1]).$$

Q.E.D.

MAIN LEMMA (3.7). Suppose $\iota: T \times [0, 1] \rightarrow T \times [0, 1]$ is an involution. Then there exists an essential annulus $A \subset T \times [0, 1]$ with $A \cap \iota(A) = \emptyset$.

PROOF. By Theorem (2.6) there is an essential annulus $A_0 \subset T \times [0, 1]$ with either $A_0 \cap \iota(A_0) = \emptyset$ or $\iota(A_0) = A_0$. In the latter case there are no fixed points of ι on A_0 . If $A_0 \cap \iota(A_0) = \emptyset$, let $A = A_0$. Hence suppose $\iota(A_0) = A_0$.

By Lemma (2.1) there is a bicollar $A_0 \times [-\epsilon, \epsilon]$ of $A_0 = A_0 \times 0$ in $T \times [0, 1]$ with $(A_0 \times [-\epsilon, \epsilon]) \cap \partial(T \times [0, 1]) = \partial A_0 \times [-\epsilon, \epsilon]$ and $\iota(A_0 \times [-\epsilon, \epsilon]) = A_0 \times [-\epsilon, \epsilon]$. If $\iota(A_0 \times \epsilon) = A_0 \times -\epsilon$, let $A = A_0 \times \epsilon$. Hence suppose $\iota(A_0 \times \epsilon) = A_0 \times \epsilon$ and thus $\iota(A_0 \times -\epsilon) = A_0 \times -\epsilon$. According to Lemma (3.6) we may assume that

$$(T \times [0, 1], A_0) = (S^1 \times S^1 \times [0, 1], x_0 \times S^1 \times [0, 1]).$$

Let $I \subset A_0$ be a nonseparating proper 1-cell with $I \cap \iota(I) = \emptyset$. (Consider A_0/ι . It is either an annulus or a Moebius strip. A nonseparating proper 1-cell in A_0/ι lifts to a nonseparating proper 1-cell I as required.) By choosing ϵ sufficiently small we can insure that ι has no fixed points on $A_0 \times [-\epsilon, \epsilon]$ and

$$(I \times [-\epsilon, \epsilon]) \cap \iota(I \times [-\epsilon, \epsilon]) = \emptyset.$$

Now consider the solid torus $V = \overline{T \times [0, 1] - A_0 \times [-\epsilon, \epsilon]}$. We have $\iota(V) = V$. Suppose $B \subset V$ is a proper 2-cell with

$$B \cap (A_0 \times [-\epsilon, \epsilon]) = I \times \epsilon \cup I \times -\epsilon \quad \text{and} \quad B \cap \iota(B) = \emptyset.$$

Then $A = I \times [-\epsilon, \epsilon] \cup B$ is an essential annulus with $A \cap \iota(A) = \emptyset$.

We now show the existence of such a 2-cell B .

Note that the inclusions $A_0 \times i\epsilon \rightarrow V$, $i = 1, -1$, induce isomorphisms

$$H_1(A_0 \times i\epsilon) \rightarrow H_1(V), \quad i = 1, -1.$$

It follows from Lemma (2.3) that $(\iota)_*: H_1(A_0 \times i\epsilon) \rightarrow H_1(A_0 \times i\epsilon)$ is $(\iota)_* = \text{id}$. By commutativity then $(\iota|_V)_*: H_1(V) \rightarrow H_1(V)$ is $(\iota|_V)_* = \text{id}$. By Corollary (3.5), $V/\iota|$ is a solid torus or a solid Klein bottle. Let $p: V \rightarrow V/\iota|$ be the covering projection, and let $\bar{B} \subset V/\iota|$ be a nonseparating proper 2-cell. We may assume that $\bar{B} \cap p(A_0 \times i\epsilon) = p(I \times i\epsilon)$, $i = 1, -1$. Let $B \subset p^{-1}(\bar{B})$ be the 2-cell with $B \cap A_0 \times \epsilon = I \times \epsilon$. If $B \cap A_0 \times -\epsilon = I \times -\epsilon$, B has the required property. If

$$B \cap A_0 \times -\epsilon = \iota(I \times -\epsilon),$$

we replace \bar{B} by a nonseparating proper 2-cell B' so that $B' \cap p(A_0 \times i\epsilon) = p(I \times i\epsilon)$, $i = 1, -1$, $B' \cap \bar{B} = p(I \times \epsilon) \cup p(I \times -\epsilon)$, and $H_1(B' \cup \bar{B}) \rightarrow H_1(V/\iota|)$ is an isomorphism. Again let $B \subset p^{-1}(B')$ be the 2-cell with $B \cap A_0 \times \epsilon = I \times \epsilon$. This time $B \cap A_0 \times -\epsilon = I \times -\epsilon$, and B has the required property. Q.E.D.

Now suppose $\iota: T \times [0, 1] \rightarrow T \times [0, 1]$ is an involution. By the preceding Main Lemma there exists an essential annulus $A \subset T \times [0, 1]$ satisfying $A \cap \iota(A) = \emptyset$.

By Lemma (3.6), (2), we may assume that

$$(T \times [0, 1], A, \iota(A)) = (S_1 \times S_2 \times [0, 1], x_+ \times S_2 \times [0, 1], x_- \times S_2 \times [0, 1]).$$

Let $S_1 = I_+ \cup I_-$, I_+ , I_- 1-cells with $I_+ \cap I_- = \partial I_+ \cap \partial I_- = \{x_+, x_-\}$, and let $V_i = I_i \times S_2 \times [0, 1]$, $i = +, -$. The solid tori V_+ , V_- decompose $T \times [0, 1]$:

$$T \times [0, 1] = V_+ \cup V_- \quad \text{and} \quad V_+ \cap V_- = \partial V_+ \cap \partial V_- = A \cup \iota(A).$$

Compare Figure 1.

We have $H_1(T \times [0, 1]) = \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$.

LEMMA (3.8). *If $\iota: T \times [0, 1] \rightarrow T \times [0, 1]$ is fixed point free, then we may assume that $\iota(V_+) = V_-$.*

PROOF. Suppose $\iota(V_+) = V_+$ and hence $\iota(V_-) = V_-$. Let $p: T \times [0, 1] \rightarrow T \times [0, 1]/\iota$ be the covering projection. Then $p(V_+)$, $p(V_-)$ are solid tori or solid Klein bottles. Choose a nonseparating proper 1-cell $I \subset A$. Let $\bar{B}_i \subset p(V_i)$ be a nonseparating proper 2-cell with $\bar{B}_i \cap p(A)$ consisting of exactly 2 proper 1-cells, one of them being $p(I)$, the other we denote by \bar{I} , $i = +, -$. Let $B_i \subset p^{-1}(\bar{B}_i)$ be the 2-cell with $B_i \cap A = I$, $i = +, -$. Then $B_i \cap \iota(A) \subset p^{-1}(\bar{I})$, $i = +, -$.

We may assume that $B_+ \cap \iota(A) = B_- \cap \iota(A)$. Otherwise we replace \bar{B}_- by a nonseparating proper 2-cell \bar{B}'_- satisfying $\bar{B}'_- \cap p(A) = \bar{B}_- \cap p(A) = \bar{B}_- \cap \bar{B}'_-$ and $H_1(\bar{B}_- \cup \bar{B}'_-) \rightarrow H_1(p(V_-))$ is an isomorphism.

The proper annulus $A' = B_+ \cup B_-$ is incompressible, meets both boundary components of $T \times [0, 1]$, and satisfies $A' \cap \iota(A') = \emptyset$ and $A' \cap A = I$.

Now $A' \cup \iota(A')$ decomposes $T \times [0, 1]$ again into solid tori V'_+ , V'_- :

$$T \times [0, 1] = V'_+ \cup V'_-, \quad V'_+ \cap V'_- = \partial V'_+ \cap \partial V'_- = A' \cup \iota(A').$$

We claim that $\iota(V'_+) = V'_-$. Namely, we have the presentation $T \times 0 = (T \times 0) \cap A \times (T \times 0) \cap A'$. If $\iota(V'_+) = V'_+$, then $\iota(V'_-) = V'_-$, and we conclude that

$$\iota_*: H_1(T \times [0, 1]) \rightarrow H_1(T \times [0, 1])$$

is $\iota_* = -\text{id}$. The Lefschetz number of ι is $\neq 0$ and hence ι has fixed points. Contradiction. Q.E.D.

LEMMA (3.9). *If $\iota: T \times [0, 1] \rightarrow T \times [0, 1]$ has fixed points, then ι has exactly 4 fixed points, $\iota_*: H_1(T \times [0, 1]) \rightarrow H_1(T \times [0, 1])$ is $\iota_* = -\text{id}$, $\iota(T \times 0) = T \times 1$.*

PROOF. Since ι has fixed points we must have $\iota(V_i) = V_i$, $i = +, -$.

Suppose a fixed point of ι is in V_+ . Then, by Lemma (2.3), $(\iota)_*: H_1(V_+) \rightarrow H_1(V_+)$ must be $(\iota)_* = -\text{id}$. We conclude that $\iota_*[S_2] = -[S_2]$ and by commutativity that $(\iota)_*: H_1(V_-) \rightarrow H_1(V_-)$ is also $(\iota)_* = -\text{id}$. Therefore, by Lemma (2.3), ι has exactly 4 fixed points and its Lefschetz number is 4. Hence $\iota_*[S_1] = -[S_1] + m[S_2]$. Since $\iota_*^2 = \text{id}$ we conclude that $m = 0$. Thus $\iota_* = -\text{id}$. Since ι is orientation reversing we must have $\iota(T \times 0) = T \times 1$. Q.E.D.

THEOREM (3.10). *Suppose $\iota, \iota': T \times [0, 1] \rightarrow T \times [0, 1]$ are involutions and one of the following holds:*

- (I) ι, ι' are orientation preserving and $\iota(T \times 0) = \iota'(T \times 0) = T \times 0$,
- (II) ι, ι' are orientation reversing and $\iota(T \times 0) = \iota'(T \times 0) = T \times 0$,
- (III) ι, ι' are orientation preserving and $\iota(T \times 0) = \iota'(T \times 0) = T \times 1$,
- (IV) ι, ι' are orientation reversing, $\iota(T \times 0) = \iota'(T \times 0) = T \times 1$, and ι, ι' are fixed point free, or
- (V) ι, ι' are orientation reversing, $\iota(T \times 0) = \iota'(T \times 0) = T \times 1$, and ι, ι' have fixed points.

Then there is an isomorphism $h: T \times [0, 1] \rightarrow T \times [0, 1]$ so that $\iota' = h\iota h^{-1}$.

Moreover, in case (V), if $h_0: \partial(T \times [0, 1]) \rightarrow \partial(T \times [0, 1])$ is an isomorphism with

$$\iota'|_{\partial(T \times [0, 1])} = h_0 \iota h_0^{-1}$$

we may assume that $h|_{\partial(T \times [0, 1])} = h_0$.

(Note that if $h': T \times 0 \rightarrow T \times 0$ is an arbitrary isomorphism, then $h_0 = h' \cup (\iota' h' \iota|_{T \times 1}): \partial(T \times [0, 1]) \rightarrow \partial(T \times [0, 1])$ is the unique isomorphism that extends h_0 and satisfies $\iota'|_{\partial(T \times [0, 1])} = h_0 \iota h_0^{-1}$. Thus we may assume that $h|_{T \times 0} = h'$.)

PROOF. (1) Suppose ι, ι' are fixed point free. Let $A, A' \subset T \times [0, 1]$ be essential annuli with $A \cap \iota(A) = A' \cap \iota'(A') = \emptyset$ and let V_{\pm}, V'_{\pm} be the corresponding solid tori. By Lemma (3.8) we may assume that $\iota(V_+) = V_-$, $\iota'(V'_+) = V'_-$.

Choose an arbitrary isomorphism $h_1: A \rightarrow A'$. Define $h_2: \iota(A) \rightarrow \iota'(A)$ by equivariance, $h_2 = \iota' h_1 \iota|_{\iota(A)}$. It follows from the hypothesis in cases (I)–(IV) that $h_1 \cup h_2: A \cup \iota(A) \rightarrow A' \cup \iota'(A')$ extends to an isomorphism $h_+: V_+ \rightarrow V'_+$. Define $h_-: V_- \rightarrow V'_-$ by equivariance, $h_- = \iota' h_+ \iota|_{V_-}$. Then h_+ and h_- piece together to define an isomorphism $h = h_+ \cup h_-: T \times [0, 1] \rightarrow T \times [0, 1]$. By construction, $\iota' = h\iota h^{-1}$.

(2) Suppose ι, ι' have fixed points. By Lemma (3.5) this is case (V).

By the remark in the statement of the theorem we may assume that h_0 is given.

Let $A \subset T \times [0, 1]$ be an essential annulus with $A \cap \iota(A) = \emptyset$. Let V_{\pm} be the corresponding solid tori. Since ι has fixed points we must have $\iota(V_i) = V_i$, $i = +, -$.

There is a proper annulus $A_0 \subset T \times [0, 1]$ with $\partial A_0 = h_0 \partial A$. We have

$$\partial A_0 \cap \iota'(\partial A_0) = h_0(\partial A) \cap h_0 \iota(\partial A) = h_0(\partial A \cap \iota(\partial A)) = \emptyset.$$

By Theorem (2.5), there is either an essential annulus $A' \subset T \times [0, 1]$ with $\partial A' \cup \partial \iota'(A') = \partial A_0 \cup \partial \iota'(A_0)$ and with $A' \cap \iota'(A') = \emptyset$, or there are two disjoint essential annuli $A_i \subset T \times [0, 1]$ with $\iota'(A_i) = A_i$, $i = 1, 2$; $\partial(A_1 \cup A_2) = \partial A_0 \cup \partial \iota'(A_0)$; and with no fixed points of ι' on $A_1 \cup A_2$.

The second possibility cannot occur. Namely, if $\iota'(A_i) = A_i$ the Lefschetz number of ι' : $A_i \rightarrow A_i$ would have to be 2 by Lemma (3.9), and hence there would have to be fixed points of ι' on A_i .

We may assume that $A' \cap T \times 0 = h_0(A \cap T \times 0)$. Then necessarily $A' \cap T \times 1 = h_0(A \cap T \times 1)$. Thus $\partial A' = h_0(\partial A)$.

Let V'_\pm be the solid tori determined by $A' \cup \iota'(A')$. We may assume that $h_0(V_i \cap \partial(T \times [0, 1])) = V'_i \cap \partial(T \times [0, 1])$, $i = +, -$. Since ι' has fixed points we again must have $\iota'(V'_i) = V'_i$, $i = +, -$.

The isomorphism $h_0|_{\partial A} : \partial A \rightarrow \partial A'$ can be extended to an isomorphism $h_1 : A \rightarrow A'$. The isomorphism $h_2 : \iota(A) \rightarrow \iota'(A)$ is defined by equivariance, $h_2 = \iota' h_1 \iota|_{\iota(A)}$. Piecing together we obtain the isomorphisms

$$h_i = h_0|_{V_i \cap \partial(T \times [0, 1])} \cup h_1 \cup h_2 : \partial V_i \rightarrow \partial V_i, \quad i = +, -,$$

which by construction satisfy $\iota'|_{\partial V'_i} = h_i \iota h_i^{-1}$, $i = +, -$.

By Theorem (3.3), h_i extends to an isomorphism $\hat{h}_i : V_i \rightarrow V'_i$ with $\iota'|_{V'_i} = \hat{h}_i \iota \hat{h}_i^{-1}$, $i = +, -$.

Then $h = \hat{h}_+ \cup \hat{h}_- : T \times [0, 1] \rightarrow T \times [0, 1]$ is the desired extension of h_0 with $\iota' = h \iota h^{-1}$. Q.E.D.

There are also relative versions of the theorem in cases (I)–(IV). They follow easily from Corollary (3.12). Since they are not needed in this work they are omitted.

COROLLARY (3.11). *Suppose $\iota : T \times [0, 1] \rightarrow T \times [0, 1]$ is an involution. Then ι is conjugate to:*

- (I) $\alpha \times \text{id} \times \text{id}$, if ι is orientation preserving and $\iota(T \times 0) = T \times 0$,
- (II) $\alpha \times \kappa \times \text{id}$, if ι is orientation reversing and $\iota(T \times 0) = T \times 0$,
- (III) $\alpha \times \kappa \times \tau$, if ι is orientation preserving and $\iota(T \times 0) = T \times 1$.
- (IV) $\alpha \times \text{id} \times \tau$, if ι is orientation reversing, $\iota(T \times 0) = T \times 1$, and ι has no fixed points, or

- (V) $\kappa \times \kappa \times \tau$, if ι is orientation reversing, $\iota(T \times 0) = T \times 1$, and ι has fixed points.

Moreover, in case (V), if $h_0 : \partial(T \times [0, 1]) \rightarrow \partial(T \times [0, 1])$ is an isomorphism with $\iota|_{\partial(T \times [0, 1])} = h_0(\kappa \times \kappa \times \tau)h_0^{-1}$, then h_0 extends to an isomorphism $h : T \times [0, 1] \rightarrow T \times [0, 1]$ with $\iota = h(\kappa \times \kappa \times \tau)h^{-1}$.

COROLLARY (3.12). *Suppose $\iota : T \times [0, 1] \rightarrow T \times [0, 1]$ is an involution without fixed points.*

- (I) *If ι is orientation preserving and $\iota(T \times 0) = T \times 0$, then $T \times [0, 1]/\iota = T \times [0, 1]$.*

- (II) *If ι is orientation reversing and $\iota(T \times 0) = T \times 0$, then $T \times [0, 1]/\iota = K \times [0, 1]$, K a Klein bottle.*

- (III) *If ι is orientation preserving and $\iota(T \times 0) = T \times 1$, then $T \times [0, 1]/\iota =$ orientable twisted I -bundle over a Klein bottle.*

- (IV) *If ι is orientation reversing and $\iota(T \times 0) = T \times 1$, then $T \times [0, 1]/\iota = MO \times S^1$, MO a Moebius strip.*

4. Involutions on the orientable twisted I -bundle over a Klein bottle. As a model for the oriented twisted I -bundle over a Klein bottle we take the following manifold W :

$$W = S^1 \times [0, 1] \times [-1, 1] / (x, 0, t) \sim (\bar{x}, 1, -t).$$

Let $q: S^1 \times [0, 1] \times [-1, 1] \rightarrow W$ be the natural map. Then $K = q(S^1 \times [0, 1] \times 0)$ is a Klein bottle and $\pi: W \rightarrow K$, $\pi(q(x, s, t)) = q(x, s, 0)$ defines an I -bundle structure over the Klein bottle K .

∂W is a torus and $\pi|_{\partial W}: \partial W \rightarrow K$ is the orientable double covering.

We introduce the annulus $A_* = q(S^1 \times 0 \times [-1, 1]) = q(S^1 \times 1 \times [-1, 1])$. Then A_* is nonseparating. By Corollary (4.3), this property determines A_* uniquely up to an ambient isotopy.

The main result of this section is in essence that any involution $\iota: W \rightarrow W$ is conjugate to exactly one of the 3 involutions $\nu, \rho_0, \rho_2: W \rightarrow W$:

$$\nu(q(x, s, t)) = q(-x, s, t),$$

$$\rho_0(q(x, s, t)) = q(-x, s, -t),$$

$$\rho_2(q(x, s, t)) = \begin{cases} q(-\bar{x}, \frac{1}{2} - s, -t), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ q(-x, \frac{3}{2} - s, t), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

There is also a relative version which is crucial for our application.

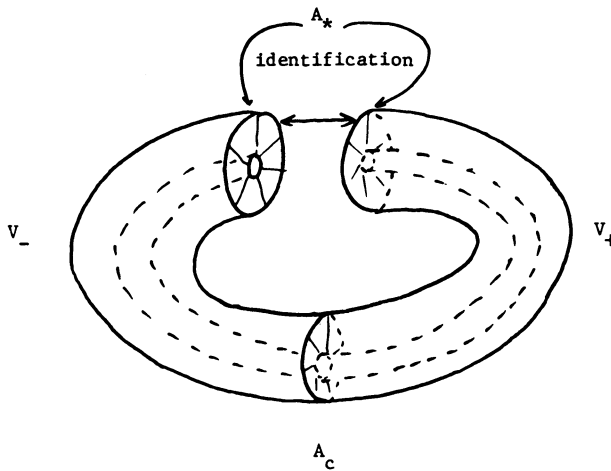
We now give more detailed descriptions of the above 3 standard involutions.

To describe ρ_2 , we introduce the nonseparating, nontrivial, and incompressible annulus $A_c = q(S^1 \times \frac{1}{2} \times [-1, 1])$. Then $A_* \cup A_c$ decomposes W into 2 solid tori V_{\pm} :

$$W = V_+ \cup V_- \quad \text{and} \quad V_+ \cap V_- = A_* \cup A_c.$$

(See Figure 3.)

Each of the 3 standard involutions maps V_i onto itself, $i = +, -$.



$$W = S^1 \times [0, 1] \times [-1, 1] / (x, 0, t) \sim (\bar{x}, 1, -t)$$

FIGURE 3

We have the following table:

involution	orientation type	fixed points	image of A_*
ν	preserving	0	A_*
ρ_0	reversing	0	A_*
ρ_2	reversing	2	A_*

The first 2 columns of the table (i.e. the orientation type and the cardinality of the fixed point set) are conjugation class invariants. Conversely, by the announced result, if $\iota: W \rightarrow W$ is an involution, these 2 invariants determine uniquely its conjugacy class.

If $\iota: W \rightarrow W$ is an involution, we will first find an annulus that has the same properties as A_* does with regard to the 3 standard involutions. This is the essential step in the classification of the involutions of W .

LEMMA (4.1). *Let K be a Klein bottle and let $S_1, S_2 \subset K$ be 2-sided nonseparating 1-spheres. Then there is an ambient isotopy on K that maps S_1 onto S_2 , the isotopy being a composition of ambient isotopies each the identity outside a 2-cell.*

PROOF. Let $S'_i \subset K$ be a 1-sphere that intersects S_i transversally in exactly 1 point. Then S'_i cannot be 2-sided. It follows then from [3, Lemma 2.13, p. 23], that there are embeddings $f_i: S^1 \rightarrow K$ with $f_i(S^1) = S_i$ and with f_1 and f_2 homotopic.

Let $p: T \rightarrow K$ be the orientable double covering and let $\tilde{f}_i: S^1 \rightarrow T$ be a lift of f_i , $i = 1, 2$, with \tilde{f}_1 and \tilde{f}_2 homotopic. We may assume that $\tilde{f}_1(S^1)$ and $\tilde{f}_2(S^1)$ intersect transversally with intersection number 0. We can find a 2-cell B with $\partial B \subset \tilde{f}_1(S^1) \cup \tilde{f}_2(S^1)$ and with exactly 2 intersection points of $\tilde{f}_1(S^1)$ and $\tilde{f}_2(S^1)$ on ∂B . Then $p(B) \subset K$ is a 2-cell with $\partial p(B) \subset S_1 \cup S_2$ and with exactly 2 intersection points of S_1 and S_2 on $\partial p(B)$. The 2-cell $p(B)$ can be used to eliminate these 2 intersection points.

Therefore we may assume that $S_1 \cap S_2 = \emptyset$. Then there is an annulus $A \subset K$ with $\partial A = S_1 \cup S_2$ and the proof can be completed. Q.E.D.

PROPOSITION (4.2). *Suppose $A \subset W$ is an essential annulus. Then there is an ambient isotopy on W mapping A to A_1 so that A_1 and K intersect transversally in a single 2-sided 1-sphere S^1 with $\pi^{-1}(S^1) = A_1$.*

If A does not separate W , then we may assume that $A_1 = A_$.*

PROOF. We may assume that A and A_* intersect transversally. Then $A \cap A_*$ consists of disjoint proper 1-cells and 1-spheres.

If a 1-sphere of $A \cap A_*$ bounds a 2-cell in A_* it must also bound a 2-cell in A and vice versa. Since W is irreducible these 1-spheres can be removed by applying an ambient isotopy to A .

If there is a proper 1-cell of $A \cap A_*$ in A_* that meets only one component of ∂A_* , then there is a 2-cell $D \subset A_*$ with $I = D \cap A = \partial D \cap A$ a proper 1-cell in A and with $D \cap \partial W = \overline{\partial D - I}$. Since A is nontrivial, it follows that there is a 2-cell $D' \subset A$

with $D' \cap D = \partial D' \cap \partial D = I$ and with $D' \cap \partial W = \overline{\partial D' - I}$. There is a 2-cell $D'' \subset \partial W$ with $\partial D'' = \partial(D \cup D')$. Using the 3-cell bounded by the 2-sphere $D \cup D' \cup D''$ an ambient isotopy on W can be constructed which applied to A removes I from $A \cap A_*$.

Thus we may assume that $A \cap A_*$ consists entirely of disjoint 1-spheres in A , A_* that do not bound 2-cells in A , A_* , or disjoint proper 1-cells in A , A_* that meet both boundary components of A , A_* , or $A \cap A_* = \emptyset$.

Suppose that $A \cap A_* \neq \emptyset$ consists of disjoint 1-spheres that do not bound 2-cells in A , A_* . Then there is an annulus $A' \subset A$ with $\partial A' = S_1 \cup S_2$, $S_2 \subset \partial A$, and $A' \cap A_* = S_1$; and there is an annulus $A'' \subset A_*$ with one component of $\partial A''$ being S_1 , the other component being in ∂W , and such that annuli A' , A'' determine a solid torus V with $V \cap \partial W = \overline{\partial V - (A' \cup A')}$. This solid torus can be used to construct an ambient isotopy on W which when applied to A removes S_1 from $A \cap A_*$.

Now suppose that $A \cap A_* \neq \emptyset$ consists of disjoint proper 1-cells in A , A_* that meet both boundary components of A , A_* . We have $A_* = q(S^1 \times 0 \times [-1, 1]) = q(S^1 \times 1 \times [-1, 1])$. Applying an ambient isotopy on A_* to $A \cap A_*$, extending it to an ambient isotopy on W , and applying it to A , we may assume that

$$A \cap A_* = q(x_1 \times 0 \times [-1, 1] \cup \cdots \cup x_m \times 0 \times [-1, 1]).$$

The disjoint 1-cells of $A \cap A_*$ in A decompose A into 2-cells B_1, \dots, B_m with

$$B_i \cap A_* = \partial B_i \cap A_* = q(x_i \times 0 \times [-1, 1]) \cup q(x_{i+1} \times 0 \times [-1, 1]),$$

$$i = 1, \dots, m \text{ (where } x_{m+1} = x_1).$$

Considering the proper 2-cells $\overline{q^{-1}(\text{int } B_i)} \subset S^1 \times [0, 1] \times [-1, 1]$, we can find an ambient isotopy on W keeping A_* pointwise fixed and mapping A to A_1 such that $A_1 \cap K = S^1$ is a single 1-sphere and such that $A_1 = \pi^{-1}(S^1)$. Necessarily, S^1 is 2-sided.

If $A \cap A_* = \emptyset$, then there is an ambient isotopy on W that maps A to A_* .

Lastly, suppose that A does not separate W . Then S^1 does not separate K . By Lemma (4.1), there is an ambient isotopy on K that maps S^1 to $A_* \cap K$; the isotopy being a composition of ambient isotopies each the identity outside a 2-cell. Therefore this isotopy extends to an ambient isotopy of W that maps K onto itself, preserves the fibers of π , and hence maps A_1 to A_* . Q.E.D.

COROLLARY (4.3). *If $A \subset W$ is a nonseparating proper annulus, then there is an ambient isotopy on W that maps A onto A_* .*

PROOF. A nonseparating proper annulus is necessarily nontrivial and incompressible.

MAIN LEMMA (4.4) *Suppose $\iota: W \rightarrow W$ is an involution. Then there is a nonseparating proper annulus $A \subset W$ with either $A \cap \iota(A) = \emptyset$ or $\iota(A) = A$. In the latter case there are no fixed points of ι on A . If $A \cap \iota(A) = \emptyset$, then ι has exactly 2 fixed points. Conversely, if ι has a fixed point there is a nonseparating proper annulus $A \subset W$ with $A \cap \iota(A) = \emptyset$.*

PROOF. By Theorem (2.6) there is an essential annulus $A_0 \subset W$ with either $A_0 \cap \iota(A_0) = \emptyset$ or $\iota(A_0) = A_0$ and in the latter case with no fixed points of ι on A_0 .

(1) Suppose A_0 separates W and $\iota(A_0) = A_0$. We show that there is a nonseparating proper annulus $A \subset W$ with $\iota(A) = A$ and with no fixed points of ι on A .

By Proposition (4.2) we may assume that A_0 and K intersect transversally in a single 2-sided 1-sphere S^1 with $\pi^{-1}(S^1) = A_0$. Suppose $W = V_+ \cup V_-$ and $V_+ \cap V_- = \partial V_+ \cap \partial V_- = A_0$. Then $V_i \cap K$ is a Moebius strip, $i = +, -$. We have $V_i = \pi^{-1}(V_i \cap K)$. Therefore V_i is a solid torus.

Case 1. $\iota(V_+) = V_+$. Since there are no fixed points of ι on A_0 , we must have $(\iota)_*: H_1(A_0) \rightarrow H_1(A_0)$ is the identity.

Let $j: A_0 \rightarrow V_+$ be the inclusion. Then $j_* H_1(A_0) = 2\mathbb{Z} \subset H_1(V_+) = \mathbb{Z}$. Therefore, by commutativity, $(\iota)_*: H_1(V_+) \rightarrow H_1(V_+)$ is also the identity. By Lemma (2.3), $\iota|: V_+ \rightarrow V_+$ has no fixed points. By Corollary (3.5) V_+/ι is a solid torus or a solid Klein bottle.

Consider the inclusion $\tilde{j}: A_0/\iota \rightarrow V_+/\iota$ and $\tilde{j}_*: H_1(A_0/\iota) \rightarrow H_1(V_+/\iota)$. By commutativity $\tilde{j}_* H_1(A_0/\iota) = 2\mathbb{Z} \subset H_1(V_+/\iota) = \mathbb{Z}$. Therefore the inclusion $\tilde{j}: A_0/\iota \rightarrow V_+/\iota$ lifts to two embeddings of A_0/ι into ∂V_+ . Hence $A_0 \cap \iota A_0 = \emptyset$. A contradiction. Thus $\iota(V_+) = V_+$ is not possible.

Case 2. $\iota(V_+) = V_-$. Consider the annulus or Moebius strip A_0/ι (depending on whether $\iota|: A_0 \rightarrow A_0$ is orientation preserving or orientation reversing, respectively). A nonseparating proper 1-cell in A_0/ι determines a nonseparating proper 1-cell $I \subset A_0$ with $I \cap \iota(I) = \emptyset$. There is a nonseparating proper 2-cell $B \subset V_+$ so that $B \cap A_0 = I \cup \iota(I)$.

Define $A = B \cup \iota(B)$. Then A is a nonseparating proper annulus with $\iota(A) = A$ and with no fixed points of ι on A .

(2) Suppose A_0 separates W and $A_0 \cap \iota(A_0) = \emptyset$.

We show again that there is a nonseparating proper annulus $A \subset W$ with $\iota(A) = A$ and with no fixed points of ι on A .

Again by Proposition (4.2), we may assume that A_0 and K intersect transversally in a single 2-sided 1-sphere S^1 in K with $\pi^{-1}(S^1) = A_0$.

Suppose that $W = V \cup V'$ with $V \cap V' = \partial V \cap \partial V' = A_0$ and with $\iota(A_0) \subset V'$. Since $V \cap K$ and $V' \cap K$ are Moebius strips it follows that V and V' are solid tori. Now $\iota(A_0)$ decomposes V' into two components, one being $\iota(V)$. Thus we have the following decomposition of W into 3 solid tori $V_0, V, \iota(V)$:

$$W = V \cup V_0 \cup \iota(V), \quad V \cap V_0 = A_0, \quad \iota(V) \cap V_0 = \iota(A_0), \quad V \cap \iota(V) = \emptyset.$$

Therefore $\iota(V_0) = V_0$.

There is a nonseparating proper 2-cell $B_0 \subset V_0$ such that $B_0 \cap A_0$ is a nonseparating proper 1-cell in A_0 , $B_0 \cap \iota(A_0)$ is a nonseparating proper 1-cell in $\iota(A_0)$, and $B_0 \cap \iota(B_0) = \emptyset$.

PROOF. First suppose that $\iota|: V_0 \rightarrow V_0$ has no fixed points. By Corollary (3.5) V_0/ι is a solid torus or a solid Klein bottle. Let $\bar{B}_0 \subset V_0/\iota$ be a nonseparating proper 2-cell with $\bar{B}_0 \cap A_0/\iota$ consisting of 2 disjoint nonseparating proper 1-cells. Let B_0 be

a lift of \bar{B}_0 . Next suppose that $\iota|_V: V_0 \rightarrow V_0$ has fixed points. Define the involution $\iota_2: S^1 \times [0, 1] \times [-1, 1] \rightarrow S^1 \times [0, 1] \times [-1, 1]$ by $\iota_2(x, s, t) = (\bar{x}, 1 - s, -t)$. There is an isomorphism

$$\begin{aligned} h_0: (\partial(S^1 \times [0, 1] \times [-1, 1]), S^1 \times 0 \times [-1, 1], S^1 \times 1 \times [-1, 1]) \\ \rightarrow (\partial V_0, A_0, \iota(A_0)) \end{aligned}$$

with $\iota|_{\partial V_0} = h_0 \iota_2 h_0^{-1}$. By Theorem (3.3), h_0 extends to an isomorphism $h: S^1 \times [0, 1] \times [-1, 1] \rightarrow V_0$ with $\iota|_{V_0} = h \iota_2 h^{-1}$. Choose $s_0 \in S^1$ with $\bar{s}_0 \neq s_0$. Define

$$B_0 = h(s_0 \times [0, 1] \times [-1, 1]).$$

There is a nonseparating proper 2-cell $B \subset V$ with $B \cap A_0 = (B_0 \cup \iota(B_0)) \cap A_0$.

Then $A = B \cup B_0 \cup \iota(B \cup B_0)$ is a nonseparating proper annulus with $\iota(A) = A$ and with no fixed points of ι on A .

(3) Suppose that $A \subset W$ is a nonseparating proper annulus with $A \cap \iota(A) = \emptyset$. We show that ι has exactly 2 fixed points. For the following arguments we may assume that $A = A_*$.

$A \cup \iota(A)$ decomposes W into 2 solid tori V_{\pm} :

$$W = V_+ \cup V_- \quad \text{and} \quad V_+ \cap V_- = \partial V_+ \cap \partial V_- = A \cup \iota(A).$$

It cannot happen that $\iota(V_+) = V_-$. Namely if so, let $I \subset A$ be a nonseparating proper 1-cell and let $B \subset V_+$ be a nonseparating proper 2-cell with $B \cap (A \cup \iota(A)) = I \cup \iota(I)$. Then $B \cup \iota(B)$ must be a Moebius strip. But $\iota|_B: B \cup \iota(B) \rightarrow B \cup \iota(B)$ is a fixed point free involution, a contradiction: A Moebius strip does not admit a fixed point free involution. (Otherwise, the quotient space would have to be a Moebius strip again. A 2-fold covering of a Moebius strip, however, is an annulus.)

Thus we must have $\iota(V_+) = V_+$ and $\iota(V_-) = V_-$. We may assume that $V_+ = q(S^1 \times [0, \frac{1}{2}] \times [-1, 1])$, $V_- = q(S^1 \times [\frac{1}{2}, 1] \times [-1, 1])$, $\iota(A) = q(S^1 \times \frac{1}{2} \times [-1, 1])$ (compare Figure 3). Consider the isomorphisms

$$\iota_+ = (q|_V)^{-1} \iota(q|_V): S^1 \times [0, \frac{1}{2}] \times [-1, 1] \rightarrow S^1 \times [0, \frac{1}{2}] \times [-1, 1],$$

$$\iota_- = (q|_V)^{-1} \iota(q|_V): S^1 \times [\frac{1}{2}, 1] \times [-1, 1] \rightarrow S^1 \times [\frac{1}{2}, 1] \times [-1, 1].$$

We must have $(\iota_-|_{S^1 \times \frac{1}{2} \times [-1, 1]})_* = -(\iota_+|_{S^1 \times \frac{1}{2} \times [-1, 1]})_*$ and we conclude that exactly one of $(\iota_+)_*$, $(\iota_-)_*$ is $-\text{id}$ on the first homology group. Hence, exactly one of $(\iota|_{V_i})_*: H_1(V_i) \rightarrow H_1(V_i)$, $i = +, -$, must be $-\text{id}$. By Lemma (2.3), exactly one of $\iota|_{V_i}: V_i \rightarrow V_i$, $i = +, -$, will have 2 fixed points, and the other will be fixed point free.

(4) Suppose ι has a fixed point. We show that there is a nonseparating proper annulus $A \subset W$ with $A \cap \iota(A) = \emptyset$.

By (1) and (2), let $A_0 \subset W$ be a nonseparating proper annulus with either $A_0 \cap \iota(A_0) = \emptyset$ or $\iota(A_0) = A_0$ and with no fixed points of ι on A_0 in the latter case.

If $A_0 \cap \iota(A_0) = \emptyset$, let $A = A_0$.

Now assume $\iota(A_0) = A_0$. By Lemma (2.1), let $A_0 \times [-\epsilon, \epsilon]$ be a bicollar of $A_0 = A_0 \times 0$ in W with $(A_0 \times [-\epsilon, \epsilon]) \cap \partial W = \partial A_0 \times [-\epsilon, \epsilon]$, $\iota(A_0 \times [-\epsilon, \epsilon]) = A_0 \times [-\epsilon, \epsilon]$, and with no fixed points of ι on $A_0 \times [-\epsilon, \epsilon]$.

If $\iota(A_0 \times \varepsilon) = A_0 \times -\varepsilon$, let $A = A_0 \times \varepsilon$.

Lastly suppose that $\iota(A_0 \times \varepsilon) = A_0 \times \varepsilon$. For the following argument we may assume that $A_0 \times \varepsilon = A_*$. Let $V = \overline{W - A_0 \times [-\varepsilon, \varepsilon]}$. Then V is a solid torus and the inclusion $j: A \times \varepsilon \rightarrow V$ induces an isomorphism $j_*: H_1(A \times \varepsilon) \rightarrow H_1(V)$. The involution $\iota|_V: V \rightarrow V$ has a fixed point. By Lemma (2.3), then, $(\iota|_V)_*: H_1(V) \rightarrow H_1(V)$ is $(\iota|_V)_* = -\text{id}$. On the other hand, since $\iota|_V: A \times \varepsilon \rightarrow A \times \varepsilon$ is fixed point free, we must have $(\iota|_V)_*: H_1(A \times \varepsilon) \rightarrow H_1(A \times \varepsilon)$ is $(\iota|_V)_* = \text{id}$. By commutativity with respect to j_* it follows that $(\iota|_V)_* = \text{id}$. This contradiction proves that $\iota(A \times \varepsilon) = A \times \varepsilon$ is not possible. Q.E.D.

We now introduce canonical generators of $H_1(\partial W)$ and $H_1(W)$.

Define $S_0 = q(S^1 \times 0 \times 1) = q(S^1 \times 1 \times -1) \subset \partial W$, $S_1 = q(1 \times [0, 1] \times 0) \subset W$, and $S_2 = q(1 \times [0, 1] \times 1) \cup q(1 \times [0, 1] \times -1) \subset \partial W$. Then

$$H_1(\partial W) = \mathbf{Z}[S_0] \oplus \mathbf{Z}[S_2] \quad \text{and} \quad H_1(W) = \mathbf{Z}_2[S_0] \oplus \mathbf{Z}[S_1].$$

Let $j: \partial W \rightarrow W$ be the inclusion. Then $j_*[S_0] = [S_0]$ and $j_*[S_2] = 2[S_1]$.

Note that $H_i(W) = H_i(K) = 0$, $i \geq 2$.

Now let $\iota: W \rightarrow W$ be an involution and consider $(\iota)_*: H_1(\partial W) \rightarrow H_1(\partial W)$.

PROPOSITION (4.5). (I) If ι is orientation preserving, then $(\iota)_* = \text{id}$.

(II) If ι is orientation reversing without fixed points, then

$$(\iota)_*[S_0] = -[S_0], \quad (\iota)_*[S_2] = [S_2] + k[S_0].$$

(III) If ι is orientation reversing with fixed points, then

$$(\iota)_*[S_0] = [S_0], \quad (\iota)_*[S_2] = -[S_2] + l[S_0].$$

PROOF. By Main Lemma (4.4), there is a nonseparating proper annulus $A \subset W$ with either $A \cap \iota(A) = \emptyset$ or with $\iota(A) = A$. By Corollary (4.3) we may assume that $A = A_*$. We conclude that

$$(\iota)_*[S_0] = \pm [S_0].$$

Consider $\iota_*: H_1(W) \rightarrow H_1(W)$. If ι is fixed point free, then its Lefschetz number must be 0, and therefore $\iota_* = \text{id}$. If ι has fixed points, then by Lemma (2.3) its Lefschetz number must be 2, and hence $\iota_* = -\text{id}$. It follows from commutativity that $(\iota)_*[S_2] = [S_2] + k[S_0]$ if ι is fixed point free, and that $(\iota)_*[S_2] = -[S_2] + l[S_0]$ if ι has fixed points.

If ι is orientation preserving so is $\iota|_{\partial W}$ and we must have $(\iota)_*[S_0] = [S_0]$. Since $(\iota|_*)^2 = \text{id}$, we conclude that $(\iota)_*[S_2] = [S_2]$, i.e. $(\iota)_* = \text{id}$.

If ι is orientation reversing so is $\iota|_{\partial W}$. Therefore $(\iota)_*[S_0] = -[S_0]$ if ι is fixed point free, and $(\iota)_*[S_0] = [S_0]$ if ι has fixed points. Q.E.D.

LEMMA (4.6). Let $A \subset W$ be a nontrivial proper annulus with one component S' of ∂A satisfying $[S'] = [S_0]$ in $H_1(\partial W)$. Then A is nonseparating.

PROOF. We may assume that $S' = S_0$.

Consider the proper Moebius strip $MO = q(1 \times [0, 1] \times [-1, 1]) \subset W$. Then ∂MO and S_0 intersect transversally in exactly one point $q(1 \times 1 \times 1)$. Applying an ambient isotopy to A keeping S_0 pointwise fixed, we may assume that A and MO

intersect transversally in a single 1-cell I , which is proper in both A and MO . It must meet both components of ∂A . Since A is nontrivial, it cannot separate MO . Hence A does not separate W . Q.E.D.

THEOREM (4.7). *Suppose $\iota, \iota' : W \rightarrow W$ are involutions and one of the following holds:*

(I) ι, ι' are orientation preserving,

(II) ι, ι' are orientation reversing without fixed points, or

(III) ι, ι' are orientation reversing with fixed points.

Then there is an isomorphism $h : W \rightarrow W$ so that $\iota' = h\iota h^{-1}$.

Moreover, in case (III), if $h_0 : \partial W \rightarrow \partial W$ is an isomorphism with $\iota'|_{\partial W} = h_0 \iota h_0^{-1}$, we may assume that $h|_{\partial W} = h_0$.

PROOF. (1) Suppose ι, ι' have no fixed points. By Main Lemma (4.4), there is a nonseparating proper annulus $A \subset W$ with $\iota(A) = A$. By Lemma (2.1), let $A \times [-\epsilon, \epsilon] \subset W$ be a bicollar of $A = A \times 0$ in W with $A \times [-\epsilon, \epsilon] \cap \partial W = \partial A \times [-\epsilon, \epsilon]$ and $\iota(A \times [-\epsilon, \epsilon]) = A \times [-\epsilon, \epsilon]$. By Main Lemma (4.4), $\iota(A \times i\epsilon) = A \times i\epsilon$, $i = \pm$.

For the following compare the proof of lemma (3.7).

Let $I \subset A$ be a nonseparating proper 1-cell with $I \cap \iota(I) = \emptyset$. We may assume that ϵ is sufficiently small so that $(I \times [-\epsilon, \epsilon]) \cap \iota(I \times [-\epsilon, \epsilon]) = \emptyset$.

By Corollary (4.3) we may assume $A = A_*$. Hence $V = \overline{W - A \times [-\epsilon, \epsilon]}$ is a solid torus. By the argument in Lemma (3.7), there is a nonseparating proper 2-cell $B \subset V$ with $B \cap A \times i\epsilon = I \times i\epsilon$, $i = \pm$, and with $B \cap \iota(B) = \emptyset$.

$I \times [-\epsilon, \epsilon] \cup \iota(I \times [-\epsilon, \epsilon])$ decomposes $A \times [-\epsilon, \epsilon]$ into the 3-cells C , $\iota(C)$:

$$A \times [-\epsilon, \epsilon] = C \cup \iota(C) \quad \text{and} \quad C \cap \iota(C) = I \times [-\epsilon, \epsilon] \cup \iota(I \times [-\epsilon, \epsilon]).$$

$B \cup \iota(B)$ decomposes V into the 3-cells E , $\iota(E)$:

$$V = E \cup \iota(E) \quad \text{and} \quad E \cap \iota(E) = B \cup \iota(B).$$

We may assume that $E \cap A \times \epsilon = C \cap A \times \epsilon$.

Similarly, A' , $A' \times [-\epsilon', \epsilon']$, I' , V' , B' , C' , and E' are defined with respect to ι' .

Let $\partial I = \{x_0, x_1\}$, $\partial I' = \{x'_0, x'_1\}$. Choose an arbitrary isomorphism

$$\begin{aligned} h_0 : (I \times [-\epsilon, \epsilon], x_0 \times [-\epsilon, \epsilon], x_1 \times [-\epsilon, \epsilon], I \times -\epsilon, I \times \epsilon) \\ \rightarrow (I' \times [-\epsilon', \epsilon'], x'_0 \times [-\epsilon', \epsilon'], x'_1 \times [-\epsilon', \epsilon'], I' \times -\epsilon', I' \times \epsilon'). \end{aligned}$$

Define $h_1 : \iota(I \times [-\epsilon, \epsilon]) \rightarrow \iota'(I' \times [-\epsilon', \epsilon'])$ by equivariance, $h_1 = \iota' h_0 \iota|_{\iota(I \times [-\epsilon, \epsilon])}$. Since ι, ι' have the same orientation type, the isomorphism

$$h_0 \cup h_1 : I \times [-\epsilon, \epsilon] \cup \iota(I \times [-\epsilon, \epsilon]) \rightarrow I' \times [-\epsilon', \epsilon'] \cup \iota'(I' \times [-\epsilon', \epsilon'])$$

extends to an isomorphism

$$\begin{aligned} h_2 : (C, I \times [-\epsilon, \epsilon], \iota(I \times [-\epsilon, \epsilon]), C \cap A \times \epsilon, C \cap A \times -\epsilon) \\ \rightarrow (C', I' \times [-\epsilon', \epsilon'], \iota'(I' \times [-\epsilon', \epsilon']), C' \cap A' \times \epsilon', C' \cap A' \times -\epsilon'). \end{aligned}$$

The isomorphism $h_0| : (I \times \epsilon, I \times -\epsilon) \rightarrow (I' \times \epsilon', I' \times -\epsilon')$ extends to an isomorphism $h_3 : B \rightarrow B'$, and the isomorphism

$$\begin{aligned} h_3 \cup h_2 : B \cup (E \cap A \times \epsilon) \cup (E \cap A \times -\epsilon) \\ \rightarrow B' \cup (E' \cap A' \times \epsilon') \cup (E' \cap A' \times -\epsilon') \end{aligned}$$

extends to an isomorphism

$$h_4: (E, B, \iota(B)) \rightarrow (E', B', \iota'(B')).$$

The isomorphisms h_2 and h_4 piece together.

Finally, by equivariance define

$$h = h_2 \cup h_4 \cup \iota'(h_2 \cup h_4)|_{\iota(C) \cup \iota(E)}: W \rightarrow W.$$

Then h is a well-defined isomorphism satisfying $\iota' = h\iota h^{-1}$.

(2) Suppose ι, ι' have fixed points. Since $\partial W/\iota|, \partial W/\iota'|$ are Klein bottles there exists an isomorphism $h_0: \partial W \rightarrow \partial W$ so that $\iota'|_{\partial W} = h_0\iota h_0^{-1}$. Thus we may assume that h_0 is given.

By Main Lemma (4.4) there is a nonseparating proper annulus $A \subset W$ with $A \cap \iota(A) = \emptyset$. By Corollary (4.3) we may assume that $A = A_*$ and $\iota(A) = A_c$. We make the following observations:

There are no annuli $A_1, A_2 \subset \partial W$ with $\partial A_1 = \partial A$, $\partial A_2 = \partial \iota(A)$, and with $A_1 \cap A_2 = \emptyset$.

Now $A \cup \iota(A)$ decomposes W into 2 solid tori V_{\pm} :

$$W = V_+ \cup V_- \quad \text{and} \quad V_+ \cap V_- = \partial V_+ \cap \partial V_- = A \cup \iota(A).$$

By Proposition (4.5), $(\iota|)_*[S_0] = [S_0]$, $(\iota|)_*[S_2] = -[S_2] + l[S_0]$ and $(\iota'|)_*[S_0] = [S_0]$, $(\iota'|)_*[S_2] = -[S_2] + l'[S_0]$. The same computation as in part (1) of the proof of Theorem (3.3) gives

$$(h_0|_{\partial W})_*[S_0] = \pm[S_0].$$

Therefore there is an ambient isotopy on W that maps $h_0(S_0)$ to S_0 .

Consider next $h_0(\partial A)$ and $h_0(\iota \partial A) = \iota' h_0 \partial A$ in W . We have $h_0(\partial A) \cap \iota' h_0(\partial A) = h_0(\partial A \cap \iota \partial A) = \emptyset$.

Let $A_0 \subset W$ be a nontrivial proper annulus with $A_0 \cap \partial W = \partial A_0 = h_0(\partial A)$. By Theorem (2.5) there is a nontrivial proper annulus $A' \subset W$ with either

$$A' \cap \iota'(A') = \emptyset \quad \text{and} \quad \partial A' \cup \iota'(\partial A') = \partial A_0 \cup \iota'(\partial A_0),$$

or there are 2 disjoint nontrivial proper annuli \tilde{A}_1, \tilde{A}_2 in W with $\iota'(\tilde{A}_i) = \tilde{A}_i$, $i = 1, 2$; $\partial(\tilde{A}_1 \cup \tilde{A}_2) = \partial A_0 \cup \iota'(\partial A_0)$; and with no fixed points of ι' on $\tilde{A}_1 \cup \tilde{A}_2$.

By Lemma (4.6), $A', \tilde{A}_1, \tilde{A}_2$ are all nonseparating.

The second case cannot occur. (Namely otherwise by Corollary (4.3), we may assume that $\tilde{A}_1 = A_*$, $\tilde{A}_2 = A$. We have the decomposition $W = \tilde{V}_+ \cup \tilde{V}_-$, \tilde{V}_{\pm} solid tori with $\tilde{V}_+ \cap \tilde{V}_- = \partial \tilde{V}_+ \cap \partial \tilde{V}_- = \tilde{A}_1 \cup \tilde{A}_2$. Necessarily $\iota'(\tilde{V}_i) = \tilde{V}_i$, $i = \pm$. The inclusion $j_i: \tilde{A}_i \rightarrow \tilde{V}_i$ induces an isomorphism $j_{i*}: H_1(\tilde{A}_i) \rightarrow H_1(\tilde{V}_i)$. Since $\iota': \tilde{A}_i \rightarrow \tilde{A}_i$ is fixed point free, we must have $(\iota')_* = \text{id}$. By commutativity $(\iota'|_{\tilde{V}_i})_* = \text{id}$ and the Lefschetz number of $\iota'|_{\tilde{V}_i}$ is 0. By Lemma (2.3), $\iota'|_{\tilde{V}_i}$ has no fixed points, $i = \pm$, and hence ι' is fixed point free. Contradiction.)

Therefore there is a nonseparating proper annulus $A' \subset W$ with $A' \cap \iota'(A') = \emptyset$ and $\partial A' \cup \iota'(\partial A') = \partial A_0 \cup \iota'(\partial A_0)$.

It is not possible that one component of $\partial A'$ is in ∂A_0 and the other is in $\iota'(\partial A_0)$. Namely, otherwise there are annuli $A'_1, A'_2 \subset \partial W$ with $\partial A'_1 = \partial A_0$, $\partial A'_2 = \iota' \partial A_0$, and with $A'_1 \cap A'_2 = \emptyset$. Then $A_i = h_0^{-1}(A'_i)$, $i = 1, 2$, contradicting our observation.

Thus we have $\partial A' = \partial A_0 = h_0(\partial A)$ and hence $\iota'(\partial A') = h_0 \iota \partial A$.

Again $A' \cup \iota'(A')$ decomposes W into 2 solid tori V'_\pm :

$$W = V'_+ \cup V'_- \quad \text{and} \quad V'_+ \cap V'_- = \partial V'_+ \cap \partial V'_- = A' \cup \iota'(A').$$

The isomorphism $h_0|_{\partial A} : \partial A \rightarrow \partial A'$ can be extended to an isomorphism $\hat{h}_0 : A \rightarrow A'$. Define the isomorphism

$$h_i = \hat{h}_0 \cup h_0|_{V_i \cap \partial W} \cup \iota' \hat{h}_0 \iota|_{\iota(A)} : \partial V_i \rightarrow \partial V'_i, \quad i = \pm.$$

By construction $\iota'|_{\partial V'_i} = h_i \iota h_i^{-1}$, $i = \pm$. By Theorem (3.3), h_i extends to an isomorphism $\hat{h}_i : V_i \rightarrow V'_i$ with $\iota'|_{V'_i} = \hat{h}_i \iota \hat{h}_i^{-1}$, $i = \pm$. Finally, define $h = \hat{h}_+ \cup \hat{h}_- : W \rightarrow W$. Then h is a well-defined isomorphism extending h_0 and satisfying $\iota' = h \iota h^{-1}$. Q.E.D.

COROLLARY (4.8). *Suppose $\iota : W \rightarrow W$ is an involution. Then ι is conjugate to:*

- (I) ν , if ι is orientation preserving,
- (II) ρ_0 , if ι is orientation reversing and fixed point free, or
- (III) ρ_2 , if ι is orientation reversing with fixed points. Moreover, in case (III), if $h_0 : \partial W \rightarrow \partial W$ is an isomorphism with $\iota|_{\partial W} = h_0 \rho_2 h_0^{-1}$, then h_0 extends to an isomorphism $h : W \rightarrow W$ with $\iota = h \rho_2 h^{-1}$.

Let MO be a Moebius strip and let $\psi : MO \rightarrow MO$ be an involution with fixed set an isolated point and a proper 1-cell. Then $MO \times [0, 1]/(x, 0) \sim (\psi(x), 1)$ is the non-orientable twisted I-bundle over a Klein bottle.

COROLLARY (4.9). *Suppose $\iota : W \rightarrow W$ is a fixed point free involution.*

- (I) *If ι is orientation preserving, then $W/\iota = W$.*
- (II) *If ι is orientation reversing, then $W/\iota =$ nonorientable twisted I-bundle over a Klein bottle.*

PROOF. By Corollary (4.8), we may assume that $W/\iota = W/\nu$ in case (I) and that $W/\iota = W/\rho_0$ in case (II). The corollary is easily verified for W/ν and for W/ρ_0 . Q.E.D.

5. Involution with fixed points on orientable torus bundles over the 1-sphere. Let S_1, S_2 be copies of the 1-sphere and let $T = S_1 \times S_2$ be the 2-dimensional torus. If $g : T \rightarrow T$ is an isomorphism then the associated torus bundle M_g over the 1-sphere is defined as follows:

$$M_g = T \times [0, 1]/(x, 0) \sim (g(x), 1).$$

Let $q : T \times [0, 1] \rightarrow M_g$ be the identification map and abbreviate $q(x, y, t) = [x, y, t]$. Let $S^1 = [0, 1]/0 \sim 1$. Define $\pi : M_g \rightarrow S^1$ by $\pi[x, y, t] = t$. Note that each fiber $\pi^{-1}(t) = q(T \times t)$ is a nonseparating, 2-sided, incompressible torus. Clearly M_g is irreducible.

Every torus bundle over S^1 arises this way.

The isomorphism $g : T \rightarrow T$ is up to isotopy uniquely determined by the isomorphism $g_* : H_1(T) \rightarrow H_1(T)$. With respect to the basis $[S_1], [S_2]$ of $H_1(T) = \mathbf{Z} \oplus \mathbf{Z}$, g_* is given by a matrix $A \in \text{GL}_2(\mathbf{Z})$. The torus bundle M_g is orientable if, and only if, the isomorphism $g : T \rightarrow T$ is orientation preserving. This is equivalent to $\det A = 1$.

It is also possible to have separating incompressible tori in M_g . As an example of this consider the space form M_2 (see [7, 8]). It is the torus bundle corresponding to $g(x, y) = (\bar{x}, \bar{y})$. The annuli $A_{\pm} = \{\pm i\} \times S_2 \times [0, 1]$ piece together to give the separating incompressible torus $q(A_+ \cup A_-)$. Cutting M_2 along this torus then gives 2 copies of W = the oriented twisted I -bundle over the Klein bottle.

The main result of this section is the complete classification of involutions with fixed points on orientable torus bundles over S^1 . It turns out that only $M_1 = S^1 \times S^1 \times S^1$ and M_2 admit such involutions, and moreover they are unique up to conjugacy.

The unique involution on M_1 is $\iota_8: M_1 \rightarrow M_1$, $\iota_8(x_1, x_2, x_3) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$. It has 8 fixed points $(\pm 1, \pm 1, \pm 1)$.

The unique involution on M_2 is given by

$$\iota_4[x, y, t] = \begin{cases} [-x, -y, \frac{1}{2} - t] & \text{if } 0 \leq t \leq \frac{1}{2}, \\ [-\bar{x}, -\bar{y}, \frac{3}{2} - t] & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

There are 4 fixed points $[\pm i, \pm i, \frac{3}{4}]$.

LEMMA (5.1). *Suppose $T_0 \subset \text{int } T \times [0, 1]$ is an incompressible torus. Then there is an ambient isotopy of $T \times [0, 1]$ taking T_0 onto $T \times \frac{1}{2}$ and leaving $\partial(T \times [0, 1])$ pointwise fixed.*

PROOF. Applying an ambient isotopy to T_0 keeping the boundary pointwise fixed, we may assume that $T_0 \cap (S_1 \times y_0 \times [0, 1]) = S_1 \times y_0 \times \frac{1}{2}$ and then that

$$T_0 \cap (x_0 \times S_2 \times [0, 1]) = x_0 \times S_2 \times \frac{1}{2}.$$

Now there is an ambient isotopy that keeps the boundary and $S_1 \times y_0 \times [0, 1] \cup x_0 \times S_2 \times [0, 1]$ pointwise fixed, and maps T_0 to $T \times \frac{1}{2}$. Q.E.D.

LEMMA (5.2). *Let W be the orientable twisted I -bundle over a Klein bottle. If $T \subset \text{int } W$ is an incompressible torus, then there is a collar $\partial W \times [0, 1]$ of $\partial W = \partial W \times 0$ in W so that $\partial W \times 1 = T$.*

PROOF. Let $p: \tilde{W} \rightarrow W$ be the 2-fold covering with $p_*\pi_1(\tilde{W}) = j_*\pi_1(\partial W)$, where $j: \partial \tilde{W} \rightarrow \tilde{W}$ is the inclusion. Then $\tilde{W} = \partial \tilde{W} \times [0, 3]$, $p|: \partial \tilde{W} \times i \rightarrow \partial W$ is an isomorphism, $i = 0, 3$, and $p^{-1}(T)$ must consist of 2 components T_1 and T_2 , each an incompressible torus. By Lemma (5.1) we may assume that $T_1 = \partial \tilde{W} \times 1$ and that $T_2 = \partial \tilde{W} \times 2$. Let $\partial W \times [0, 1] = p(\partial \tilde{W} \times [0, 1])$. Q.E.D.

LEMMA (5.3). *Let M be an orientable torus bundle over S^1 and let $T \subset M$ be an incompressible torus.*

(1) *Suppose T is nonseparating. Then $\overline{M - T \times [-1, 1]}$, where $T \times [-1, 1]$ is a bicollar of $T = T \times 0$ in M , is another copy of $T \times [-1, 1]$.*

(2) *Suppose T is separating. Then T separates M into orientable twisted I -bundles W_1, W_2 over a Klein bottle. Moreover, there are nonseparating proper annuli $A_{i*} \subset W_i$, $i = 1, 2$, so that $A_{1*} \cap A_{2*}$ is a boundary component of both A_{1*} and A_{2*} .*

PROOF. First note that T is 2-sided and so has a bicollar $T \times [-1, 1]$. Also $\overline{M - T \times [0, 1]}$ will have 1 or 2 components.

Let $T_* = q(T \times 0)$ be a distinguished fiber. Applying an ambient isotopy to T we may assume that T and T_* intersect transversally in disjoint 1-spheres S_1, \dots, S_n . Since M is irreducible, by standard arguments we may assume that none of the 1-spheres S_1, \dots, S_n bound 2-cells in T or T_* . If $T \cap T_* = \emptyset$, then Lemma (5.1) implies that case (1) holds.

Now assume that $n = 1$. Then $q^{-1}(T) \subset T \times [0, 1]$ is an incompressible proper annulus meeting both boundary components. By Lemma (3.6) $(T \times [0, 1], q^{-1}(T))$ is isomorphic to $(S^1 \times S^1 \times [0, 1], x_0 \times S^1 \times [0, 1])$. Let $T \times [-1, 1] \subset M$ be a bicollar of $T = T \times 0$ in M with $(T \times [-1, 1]) \cap T_* = S_*^1 \times [-1, 1]$. Then we may assume that $q^{-1}(T \times [-1, 1]) = I \times S^1 \times [0, 1]$ where $I \subset S^1$ is a 1-cell with $x_0 \in \text{int } I$ and with $q(I \times S^1 \times 0) = q(I \times S^1 \times 1)$. Since $\overline{M - T \times [0, 1]}$ has two boundary components and is connected we conclude that

$$\overline{M - T \times [0, 1]} = q(\overline{S^1 - I \times S^1 \times [0, 1]}) = T \times [-1, 1].$$

Finally suppose that $n > 1$. Now the 1-spheres S_1, \dots, S_n decompose T into annuli A_1, \dots, A_n . We arrange the ordering so that $\partial A_i = S_i \cup S_{i+1}$, $i = 1, \dots, n$, where $S_{n+1} = S_1$.

Consider the proper annuli $B_i = \overline{q^{-1}(\text{int } A_i)} \subset T \times [0, 1]$, $i = 1, \dots, n$.

If an annulus B_i has both boundary components in the same boundary component of $T \times [0, 1]$ then there is a solid torus $V \subset T \times [0, 1]$ with $B_i \subset \partial V$, $V \cap \partial(T \times [0, 1]) = \partial V - B_i$, and such that there is a nonseparating proper 2-cell $B \subset V$ with $B \cap B_i$ a 1-cell. Using $q(V) \subset M$, we can construct an ambient isotopy on M which applied to T will remove S_i and S_{i+1} from $T \cap T_*$. Thus we may assume that each annulus $B_i \subset T \times [0, 1]$ meets both boundary components of $T \times [0, 1]$.

Again by Lemma (3.6) $(T \times [0, 1], B_1, \dots, B_n)$ is isomorphic to $(S^1 \times S^1 \times [0, 1], x_1 \times S^1 \times [0, 1], \dots, x_n \times S^1 \times [0, 1])$. Let $T \times [-1, 1]$ be a bicollar of $T = T \times 0$ in M with $T \times [-1, 1] \cap T_* = S_i \times [-1, 1]$, $i = 1, \dots, n$. Then $q^{-1}(\text{int } A_i \times [-1, 1]) = I_i \times S^1 \times [0, 1]$, where $I_1, \dots, I_n \subset S^1$ are disjoint 1-cells with $x_i \in \text{int } I_i$ and $q(I_i \times S^1 \times 1) = q(I_{i+1} \times S^1 \times 0)$, and where $I_{n+1} = I_1$. It follows that

$$\overline{M - T \times [-1, 1]} = \bigcup_{i=1}^n q(J_i \times S^1 \times [0, 1]),$$

where the $J_i \subset S^1$ are the complementary intervals determined by $I_1 \cup \dots \cup I_n$ in S^1 .

The components of $\bigcup_{i=1}^n q(J_i \times S^1 \times [0, 1])$ are either copies of $T \times [-1, 1]$ or W . If T is nonseparating then it automatically follows that the only component of $\overline{M - T \times [-1, 1]}$ is a copy of $T \times [-1, 1]$. On the other hand if T separates M , then the 2 components of $\overline{M - T \times [-1, 1]}$ must necessarily be copies of W .

Finally, let A_{1*}, A_{2*} be adjacent annuli of $W_1 \cap T_*, W_2 \cap T_*$, respectively. Q.E.D.

MAIN LEMMA (5.4). *Let M be an orientable torus bundle over the 1-sphere and let $\iota: M \rightarrow M$ be an involution with fixed points. Then there is a nonseparating incompressible torus $T \subset M$ with $T \cap \iota(T) = \emptyset$.*

PROOF. By Theorem (2.7) there exists an incompressible torus $T_0 \subset M$ so that either $T_0 \cap \iota(T_0) = \emptyset$ or $\iota(T_0) = T_0$. In the latter case there are no fixed points of ι on T_0 .

(1) Suppose $\iota(T_0) = T_0$ and T_0 is nonseparating. By Lemma (2.1), let $T_0 \times [-1, 1]$ be a bicollar so that $\iota(T_0 \times [-1, 1]) = T_0 \times [-1, 1]$ and ι has no fixed points on $T_0 \times [-1, 1]$. By Lemma (5.3) $M - T_0 \times [-1, 1] = T_1 \times [-1, 1]$ and ι has a fixed point on $T_1 \times [-1, 1]$. Finally by Lemma (3.9) we must have $\iota(T_1 \times 1) = T_1 \times -1$. Consequently $T = T_0 \times 1$ will satisfy the conclusion of the Main Lemma.

(2) Now suppose $T_0 \cap \iota(T_0) = \emptyset$ and T_0 separates M .

By Lemma (5.3) we have $M = W_1 \cup W'_2$, where W_1, W'_2 are copies of W and $W_1 \cap W'_2 = \partial W_1 = \partial W'_2 = T_0$. Suppose $\iota(T_0) \subset W'_2$. Then by Lemma (5.2) we have the following decomposition of M : $M = W_1 \cup T' \times [0, 1] \cup W_2$, W_1, W_2 disjoint copies of W so that $W_1 \cap T' \times [0, 1] = \partial W_1 = T' \times 0$, and $W_2 \cap T' \times [0, 1] = \partial W_2 = T' \times 1$, where $T' \times 1 = \iota(T_0)$. Necessarily $\iota(W_1) = W_2$ and $\iota(T' \times [0, 1]) = T' \times [0, 1]$.

Let $A_0 \subset W_1$ be a nonseparating proper annulus and let $\partial A_0 = S' \cup S''$. Then there is a proper annulus $A' \subset T' \times [0, 1]$ so that $\partial A' = S' \cup \iota(S'')$. Thus $\partial A' \cap \iota(\partial A') = \emptyset$. We apply Theorem (2.5): Either there is an incompressible proper annulus $A \subset T' \times [0, 1]$ meeting both boundary components with $A \cap \iota(A) = \emptyset$ and with $\partial A \cup \iota(\partial A) = \partial A' \cup \iota(\partial A')$, or there are 2 disjoint incompressible annuli $A_1, A_2 \subset T' \times [0, 1]$ meeting both boundary components with $\iota(A_i) = A_i$, $i = 1, 2$; $\partial(A_1 \cup A_2) = \partial A' \cup \iota(\partial A')$; and with no fixed points of ι on $A_1 \cup A_2$.

The second alternative is not possible. Namely, by Lemma (3.9)

$$(\iota)_*: H_1(T' \times [0, 1]) \rightarrow H_1(T' \times [0, 1])$$

is $(\iota)_* = -\text{id}$. By Lemma (3.6) we may assume that A_1, A_2 are in standard position. We conclude that $(\iota)_*: H_1(A_i) \rightarrow H_1(A_i)$ is also $(\iota)_* = -\text{id}$. Therefore the Lefschetz number of $\iota|_{A_i}: A_i \rightarrow A_i$ is 2 and $\iota|_{A_i}$ must have fixed points on A_i . A contradiction.

Thus we may assume that $A \cap \iota(A) = \emptyset$. Then $T_1 = A_0 \cup A \cup \iota(A_0) \cup \iota(A)$ is a nonseparating incompressible torus with $\iota(T_1) = T_1$ and with no fixed points of ι on T_1 . We have arrived at case (1).

(3) Finally suppose that $\iota(T_0) = T_0$ and T_0 separates. By Lemma (5.3), we have the decomposition $M = W_1 \cup W_2$, $W_1, W_2 \subset M$ copies of W with $W_1 \cap W_2 = \partial W_1 = \partial W_2 = T_0$. Necessarily $\iota(W_i) = W_i$, $i = 1, 2$. Suppose that ι has fixed points in W_1 . By Main Lemma 4.4 there is a nonseparating proper annulus $A_1 \subset W_1$ with $A_1 \cap \iota(A_1) = \emptyset$. It follows from Lemma (5.3) and from Corollary (4.3) that there is a nonseparating proper annulus $A'_2 \subset W_2$ with $\partial A'_2 = \partial A_1$. We apply Theorem (2.5) to $\iota|_{W_2}: W_2 \rightarrow W_2$ with respect to A'_2 . By the same argument as in part (2) of the proof of Theorem (4.7) we obtain a nonseparating proper annulus $A_2 \subset W_2$ with $\partial A_2 = \partial A_1$ and with $A_2 \cap \iota(A_2) = \emptyset$. Define $T = A_1 \cup A_2$. Then T is 2-sided. Therefore T is a nonseparating incompressible torus with $T \cap \iota(T) = \emptyset$. Q.E.D.

THEOREM (5.5). *Let M be an orientable torus bundle over S^1 and let $\iota: M \rightarrow M$ be an involution with fixed points. Then ι has exactly 4 or 8 fixed points.*

PROOF. Let $T \subset M$ be the torus of Main Lemma (5.4). Then Lemmas (5.1) and (5.3) imply that M decomposes as $M = (T_1 \times [0, 1]) \cup (T_2 \times [0, 1])$ with $(T_1 \times [0, 1]) \cap (T_2 \times [0, 1]) = \partial(T_1 \times [0, 1]) = \partial(T_2 \times [0, 1]) = T \cup \iota(T)$. We must have

$$\iota(T_i \times [0, 1]) = T_i \times [0, 1], \quad i = 1, 2.$$

By Corollary (3.11), ι has either 0 or 4 fixed points in $T_i \times [0, 1]$, $i = 1, 2$. Q.E.D.

THEOREM (5.6). *Let M, M' be orientable torus bundles over S^1 and let $\iota: M \rightarrow M, \iota': M' \rightarrow M'$ be involutions with fixed points. If ι, ι' have the same number of fixed points then there exists an isomorphism $h: M \rightarrow M'$ satisfying $\iota' = h\iota h^{-1}$.*

PROOF. As in the proof of Theorem (5.5) we have decompositions $M = (T_1 \times [0, 1]) \cup (T_2 \times [0, 1])$, $M' = (T'_1 \times [0, 1]) \cup (T'_2 \times [0, 1])$ with respect to ι, ι' , respectively. We may assume that $\iota|_{T_2 \times [0, 1]}: T_2 \times [0, 1] \rightarrow T_2 \times [0, 1]$ and $\iota'|_{T'_2 \times [0, 1]}: T'_2 \times [0, 1] \rightarrow T'_2 \times [0, 1]$ have 4 fixed points each. Then $\iota|_{T_1 \times [0, 1]}: T_1 \times [0, 1] \rightarrow T_1 \times [0, 1]$ and $\iota'|_{T'_1 \times [0, 1]}: T'_1 \times [0, 1] \rightarrow T'_1 \times [0, 1]$ have the same number of fixed points (0 or 4).

By Theorem (3.10) there is an isomorphism $h_1: T_1 \times [0, 1] \rightarrow T'_1 \times [0, 1]$ with $\iota'|_{T'_1 \times [0, 1]} = h_1 \iota|_{T_1 \times [0, 1]} h_1^{-1}$. Then

$$\iota'|_{\partial(T'_2 \times [0, 1])} = h_1|_{\partial(T_2 \times [0, 1])} \iota|_{\partial(T_2 \times [0, 1])}.$$

By Theorem (3.10), there is an isomorphism $h_2: T_2 \times [0, 1] \rightarrow T'_2 \times [0, 1]$ with $\iota'|_{T'_2 \times [0, 1]} = h_2 \iota|_{T_2 \times [0, 1]} h_2^{-1}$ and with $h_2|_{\partial(T_2 \times [0, 1])} = h_1|_{\partial(T_1 \times [0, 1])}$. Then the isomorphism $h = h_1 \cup h_2: M \rightarrow M'$ satisfies $\iota' = h\iota h^{-1}$. Q.E.D.

COROLLARY 5.7. *The only orientable torus bundles over S^1 admitting involutions with fixed points are M_1, M_2 . Moreover, all such involutions on M_1, M_2 are conjugate to ι_8, ι_4 , respectively.*

6. Involutions with fixed points on unions along the boundaries of 2 orientable twisted I -bundles over Klein bottles. Let W_1, W_2 be copies of the orientable twisted I -bundle W over a Klein bottle. If $g: \partial W_1 \rightarrow \partial W_2$ is an isomorphism then the associated union M_g of W_1 and W_2 is defined as follows:

$$M_g = W_1 \cup W_2/x \sim g(x) \quad \text{for } x \in \partial W_1.$$

Orientations on W_1 and W_2 can be chosen so that M_g is orientable.

The manifold M_g admits a 2-fold covering by an orientable torus bundle over S^1 . We observe that M_g is irreducible.

We have $M_{\text{id}} = D(W)$, the double of W . It has a 2-fold covering by $S^1 \times S^1 \times S^1$. From this it can be deduced that $M_{\text{id}} = M_2$. Thus $D(\rho_2): D(W) \rightarrow D(W)$, the double of the involution $\rho_2: W \rightarrow W$ with 2 fixed points, is an involution with 4 fixed points on M_2 .

The main result of this section is the complete classification of involutions with fixed points on unions along the boundaries of 2 orientable twisted I -bundles over Klein bottles. We prove that besides M_2 —which we have dealt with in §5—only the

space form M_6 admits such involutions. Furthermore, they are unique up to conjugacy.

The space form M_6 , which is frequently called the Hantzsche-Wendt manifold in reference to [2], has the following description:

Let $[S_{0i}], [S_{2i}] \in H_1(\partial W_i)$, $i = 1, 2$, be the canonical generators which were introduced in §4. Let $g_0: \partial W_1 \rightarrow \partial W_2$ be an isomorphism so that $g_{0*}[S_{01}] = \pm[S_{22}]$, $g_{0*}[S_{21}] = \pm[S_{02}]$. Then $M_6 = M_{g_0}$.

Let $\rho_2: W_1 \rightarrow W_1$ be the involution with 2 fixed points and let $\rho_0: W_2 \rightarrow W_2$ be the fixed point free orientation reversing involution. We may choose g_0 so that $\rho_0 g_0 = g_0 \rho_2|_{\partial W_1}$. Then $\iota_2 = \rho_2 \cup \rho_0: M_6 \rightarrow M_6$ is a well-defined involution. It has 2 fixed points, namely those of ρ_2 .

Now suppose M is the union along the boundaries of 2 orientable I -bundles over Klein bottles.

PROPOSITION (6.1). *If M contains a nonseparating incompressible torus, then M is a torus bundle over S^1 .*

PROOF. Let $p: \tilde{M} \rightarrow M$ be a 2-fold covering of M by a torus bundle \tilde{M} over a 1-sphere. Let $T \subset M$ be a nonseparating incompressible torus. Then each component of $p^{-1}(T)$ is an incompressible torus in \tilde{M} .

(1) Suppose $p^{-1}(T)$ consists of 2 tori T_1, T_2 . Then T_1, T_2 must be nonseparating. Otherwise, if one of T_1, T_2 is separating, by Lemmas (5.3) and (5.2) we obtain a decomposition of \tilde{M} into 3 submanifolds. Since p is a 2-fold covering, we conclude that T must be separating, a contradiction.

Thus by Lemmas (5.3) and (5.1) we have the decomposition $\tilde{M} = (T_1 \times [0, 1]) \cup (T_2 \times [0, 1])$ with $(T_1 \times [0, 1]) \cap (T_2 \times [0, 1]) = \partial(T_1 \times [0, 1]) = \partial(T_2 \times [0, 1]) = T_1 \cup T_2$. Then M can be obtained from $T_i \times [0, 1]$ by identifying $T_i \times 0$ with $T_i \times 1$, i.e. M is a torus bundle over S^1 .

(2) Suppose $p^{-1}(T) = \tilde{T}$ is a single torus. Let $T \times [-1, 1] \subset M$ be a bicollar of $T = T \times 0$ in M . Then $p^{-1}(T \times [0, 1]) = \tilde{T} \times [-1, 1]$ is a bicollar of $\tilde{T} = \tilde{T} \times 0$ in \tilde{M} . The deck transformation $\iota: \tilde{M} \rightarrow \tilde{M}$ is orientation preserving and we have $\iota(\tilde{T} \times [-1, 1]) = \tilde{T} \times [-1, 1]$. As $\iota: \tilde{T} \rightarrow \tilde{T}$ is orientation preserving, we must have $\iota(\tilde{T} \times i) = \tilde{T} \times i$, $i = 1, -1$. Consequently \tilde{T} is nonseparating. By Lemma (5.3), $\overline{\tilde{M} - \tilde{T} \times [-1, 1]} = \tilde{T}' \times [-1, 1]$. By Corollary (3.12), $p(\tilde{M} - \tilde{T} \times [-1, 1]) = T' \times [-1, 1]$. Hence M is a torus bundle over S^1 . Q.E.D.

LEMMA (6.2). *A separating incompressible torus in M decomposes M into 2 orientable twisted I -bundles over Klein bottles.*

PROOF. We proceed as in the proof of Proposition (6.1). Let $T \subset M$ be a separating incompressible torus.

(1) Suppose $p^{-1}(T)$ consists of 2 tori T_1, T_2 . If one of the tori T_i is separating, both are separating, and we obtain the decomposition $\tilde{M} = W_1 \cup T \times [0, 1] \cup W_2$, where W_1, W_2 are copies of W with $\partial W_1 = T \times 0 = T_1$ and $\partial W_2 = T \times 1 = T_2$. Necessarily, $\iota(T \times 0) = T \times 1$ and hence $\iota(W_1) = W_2$, $\iota(T \times [0, 1]) = T \times [0, 1]$. Then

$M = p(W_1) \cup p(T \times [0, 1])$ with $p(W_1) \cap p(T \times [0, 1]) = \partial p(W_1) = \partial(T \times [0, 1]) = T$. Now $p(W_1)$ is a copy of W , and by Corollary (3.12) $p(T \times [0, 1])$ is a copy of W also.

Suppose both T_1 and T_2 are nonseparating. We obtain the decomposition $\tilde{M} = (T_1 \times [0, 1]) \cup (T_2 \times [0, 1])$. Then $M = p(T_1 \times [0, 1]) \cup p(T_2 \times [0, 1])$ and necessarily $p(T_1 \times [0, 1]) \cap p(T_2 \times [0, 1]) = T$. By Corollary (3.12), $p(T_i \times [0, 1])$ is a copy of W , $i = 1, 2$.

(2) Suppose $p^{-1}(T) = \tilde{T}$ is a single torus. By Lemma (5.3), \tilde{T} decomposes \tilde{M} into 2 copies of W : $\tilde{M} = \tilde{W}_1 \cup \tilde{W}_2$ with $\tilde{W}_1 \cap \tilde{W}_2 = \partial \tilde{W}_1 = \partial \tilde{W}_2 = \tilde{T}$. Then $M = p(\tilde{W}_1) \cup p(\tilde{W}_2)$ and necessarily $p(\tilde{W}_1) \cap p(\tilde{W}_2) = T$. By Corollary (4.9), $p(\tilde{W}_i)$ is a copy of W , $i = 1, 2$. Q.E.D.

MAIN LEMMA (6.3). *Suppose M is not a torus bundle over S^1 . If $\iota: M \rightarrow M$ is an involution with fixed points, then there is a separating incompressible torus $T \subset M$ with $\iota(T) = T$ and with no fixed points of ι on T .*

PROOF. Note by Proposition (6.1) each incompressible torus in M is separating. By Theorem (2.7) there is an incompressible torus $T \subset M$ with either $T \cap \iota(T) = \emptyset$ or with $\iota(T) = T$ and with no fixed points of ι on T .

Suppose $T \cap \iota(T) = \emptyset$. By Lemmas (6.2) and (5.2) we have the following decomposition of M : $M = W_1 \cup T \times [0, 1] \cup W_2$, where W_1, W_2 are disjoint copies of W with $W_1 \cap T \times [0, 1] = \partial W_1 = T \times 0 = T$ and $W_2 \cap T \times [0, 1] = \partial W_2 = T \times 1 = \iota(T)$.

Necessarily, $\iota(W_1) = W_2$ and $\iota(T \times [0, 1]) = T \times [0, 1]$. By the same arguments as in part (2) of the proof of Main Lemma (5.4) a nonseparating incompressible torus $T \subset M$ can be constructed. Thus $T \cap \iota(T) = \emptyset$ is not possible. Q.E.D.

THEOREM (6.4). *Suppose M is not a torus bundle over S^1 and $\iota: M \rightarrow M$ is an involution with fixed points. Then ι has exactly 2 fixed points.*

PROOF. Let $T \subset M$ be the torus of Main Lemma (6.3). By Lemma (6.2), T decomposes M into 2 copies W_1, W_2 of W :

$$M = W_1 \cup W_2 \quad \text{and} \quad W_1 \cap W_2 = \partial W_1 = \partial W_2 = T.$$

We must have $\iota(W_i) = W_i$, $i = 1, 2$. By Corollary (4.8), ι has either 0 or 2 fixed points in W_i , $i = 1, 2$. Hence ι has either 2 or 4 fixed points.

Suppose ι has 4 fixed points. Then $\iota|_{W_i}$ has 2 fixed points, $i = 1, 2$. Consider $D(W) = W \cup W'$ and the involution $D(\rho_2): D(W) \rightarrow D(W)$. By Theorem (4.7), there is an isomorphism $h_1: W \rightarrow W_1$ so that $\iota|_{W_1} = h_1 \rho_2 h_1^{-1}$. Then $\iota|_{\partial W_2} = h_1 \rho_2 (h_1|_{\partial W_2})^{-1}$. Again by Theorem (4.7) there is an isomorphism $h_2: W' \rightarrow W_2$ with $\iota|_{W_2} = h_2 \rho_2 h_2^{-1}$ extending $h_1|_{\partial W_2}$. Thus there is an isomorphism $h = h_1 \cup h_2: D(W) \rightarrow M$. Therefore $M = D(W) = M_2$ and M is a torus bundle over S^1 , a contradiction. Q.E.D.

THEOREM (6.5). *Let M, M' each be unions along the boundary of 2 orientable twisted I -bundles over Klein bottles and let $\iota: M \rightarrow M, \iota': M' \rightarrow M'$ be involutions with fixed points. If ι, ι' have the same number of fixed points, then there exists an isomorphism $h: M \rightarrow M'$ satisfying $\iota' = h\iota h^{-1}$.*

PROOF. By Theorem (6.4) and Corollary (5.7) we may assume that M, M' are not torus bundles over S^1 and that ι, ι' have 2 fixed points each. As in the proof of Theorem (6.4) we have decompositions of M, M' into copies of W with regard to ι, ι' , respectively: $M = W_1 \cup W_2$ and $M' = W'_1 \cup W'_2$. We may assume that $\iota|_{W_1}, \iota'|_{W'_1}$ are fixed point free and that $\iota|_{W_2}, \iota'|_{W'_2}$ have 2 fixed points each. By Theorem (4.7) there is an isomorphism $h_1: W_1 \rightarrow W'_1$ with $\iota'|_{W'_1} = h_1\iota h_1^{-1}$ and there is an isomorphism $h_2: W_2 \rightarrow W'_2$ with $\iota'|_{W'_2} = h_2\iota h_2^{-1}$ and extending $h_1|_{\partial W_2}$. Then $h = h_1 \cup h_2: M \rightarrow M'$ is an isomorphism with $\iota' = h\iota h^{-1}$. Q.E.D.

COROLLARY (6.6). *The only unions along the boundary of 2 orientable twisted I -bundles over Klein bottles that admit involutions with fixed points are M_2, M_6 . Moreover, all such involutions on M_2, M_6 are conjugate to ι_4, ι_2 , respectively.*

PROOF. $M_1 = S^1 \times S^1 \times S^1$ is not the union along the boundary of 2 orientable twisted I -bundles over Klein bottles. (There is no epimorphism $H_1(W) \oplus H_1(W) \rightarrow H_1(S^1 \times S^1 \times S^1)$.)

Corollary (5.7), Theorems (6.4) and (6.5) complete the proof. Q.E.D.

Finally, Corollaries (5.7) and (6.6) prove our main result.

THEOREM (6.7). *The only space forms from the orientable ones M_1, \dots, M_6 which admit involutions with fixed points are M_1, M_2, M_6 . Moreover these involutions are unique up to conjugacy and have 8, 4, 2 fixed points respectively.*

7. The orbit manifolds M_1^*, M_2^* , and M_6^* . In this section we exhibit canonical decompositions of the manifolds M_1^*, M_2^* , and M_6^* . P^2 denotes the projective plane.

PROPOSITION (7.1). *Let C be a 3-cell and let $\iota: C \rightarrow C$ be an involution with fixed point x_0 . Suppose $C_0 \subset \text{int } C$ is a 3-cell with $x_0 \in \text{int } C_0$ and $\iota(C_0) = C_0$. Then $\overline{C - C_0/\iota} = P^2 \times [0, 1]$.*

PROOF. We may assume that $\overline{C - C_0} = S^2 \times [0, 1]$ and by Lemma (3.1) that $\iota(x, t) = (-x, t)$. Q.E.D.

DEFINITION. A projective solid V^* is defined as follows. Let P_1, P_2 be projective planes, let $D_i \subset P_i \times 1$ be 2-cells, $i = 1, 2$, and let $h: D_1 \rightarrow D_2$ be an isomorphism. Define

$$V^* = P_1 \times [0, 1] \cup P_2 \times [0, 1] / (x, 1) \sim (h(x), 1) \quad \text{for } x \in D_1.$$

The usual notation is $V^* = (P_1 \times [0, 1]) \#_{\partial} (P_2 \times [0, 1])$. V^* does not depend on the choice of D_1, D_2 , or h . Note that ∂V^* consists of 2 projective planes, $P_1 \times 0, P_2 \times 0$, and a Klein bottle, $(P_1 \times 1) \# (P_2 \times 1)$.

PROPOSITION (7.2). *Let V be a solid torus and let $\iota: V \rightarrow V$ be an involution with fixed points x_1, x_2 . Suppose $C_1, C_2 \subset \text{int } V$ are disjoint 3-cells with $x_i \in \text{int } C_i$ and $\iota(C_i) = C_i, i = 1, 2$. Then $\overline{V - (C_1 \cup C_2)}/\iota = V^*$.*

PROOF. By Corollary (3.4) we may assume that $\iota = \iota_2$. For ι_2 the proposition is easily verified. Q.E.D.

DEFINITION. Let V_1^*, V_2^* be 2 projective solids. On the Klein bottles $K_i \subset \partial V_i^*$ we choose nonseparating annuli A_i , $i = 1, 2$. By Lemma (4.1), A_i is uniquely determined up to an ambient isotopy on K_i and hence on V_i^* , $i = 1, 2$. Let $h: A_1 \rightarrow A_2$ be an isomorphism. Define

$$T \times [0, 1]^* = V_1^* \cup V_2^* / x \sim h(x) \quad \text{for } x \in A_1.$$

$T \times [0, 1]^*$ does not depend on the choice of h .

Note that $\partial(T \times [0, 1]^*)$ consists of 4 projective planes and 1 torus.

PROPOSITION (7.3). Let $\iota: T \times [0, 1] \rightarrow T \times [0, 1]$ be an involution with fixed points x_1, x_2, x_3, x_4 . Suppose $C_1, C_2, C_3, C_4 \subset \text{int}(T \times [0, 1])$ are disjoint 3-cells with $x_i \in \text{int } C_i$ and with $\iota(C_i) = C_i$, $i = 1, 2, 3, 4$. Then

$$\overline{T \times [0, 1] - (C_1 \cup C_2 \cup C_3 \cup C_4)} / \iota = T \times [0, 1]^*.$$

PROOF. By Corollary (3.11) we may assume that $\iota = \kappa \times \kappa \times \tau$. The proposition is easily verified in this case. Q.E.D.

LEMMA (7.4). Let K be a Klein bottle and let $S_1, S_2 \subset K$ be nontrivial separating 1-spheres. Then there is an ambient isotopy on K that maps S_1 onto S_2 .

PROOF. This is well known. We sketch a brief proof. Let $K = S^1 \times [0, 1] / (x, 0) \sim (\bar{x}, 1)$ and let $q: S^1 \times [0, 1] \rightarrow K$ be the natural identification map. Then $\pi_1(K) = \langle a, b: bab^{-1} = a^{-1} \rangle$ where $a = [q(S^1 \times 0)]$ and $b = [q(1 \times [0, 1])]$. Now S_i is the boundary of a Moebius strip which is the regular neighborhood of a 1-sphere S'_i . We can find a 2-sided 1-sphere in K which intersects S'_i transversally in a single point. Thus we can apply, [3, Lemma 2.13, p. 23], and conclude that $[S'_i] = a^{r_i} b^{\pm 1}$, $i = 1, 2$. But $[S_i] = a^{r_i} b^{\pm 1} a^{r_i} b^{\pm 1}$, $i = 1, 2$. It follows from the identity on p. 23 of [3] that $[S_i] = b^{\pm 2}$, $i = 1, 2$. Hence there are embeddings $f_i: S^1 \rightarrow K$ with $f_i(S^1) = S_i$, $i = 1, 2$, and with f_1 and f_2 homotopic. Then there is an ambient isotopy that maps S_1 onto S_2 (see e.g. [3, Lemma 2.10, p. 21]). Q.E.D.

DEFINITION. Let V^* be a projective solid and let N be a solid Klein bottle. Let $K \subset \partial V^*$ be the Klein bottle. On K we choose a nonseparating annulus A_1 and on ∂N a separating annulus A_2 that does not bound a 2-cell on ∂N . By Lemmas (4.1) and (7.4), A_1, A_2 are uniquely determined up to ambient isotopy on $K, \partial N$ respectively. Let $h: A_1 \rightarrow A_2$ be an isomorphism. Define

$$W^* = V^* \cup N / x \sim h(x) \quad \text{for } x \in A_1.$$

W^* does not depend on the choice of h .

Note that ∂W^* consists of 2 projective planes and 1 Klein bottle.

PROPOSITION (7.5). Let $\iota: W \rightarrow W$ be an involution with fixed points x_1, x_2 . Suppose $C_1, C_2 \subset \text{int } W$ are disjoint 3-cells with $x_i \in \text{int } C_i$ and with $\iota(C_i) = C_i$, $i = 1, 2$. Then $\overline{W - C_1 \cup C_2} / \iota = W^*$.

PROOF. By Corollary (4.8) we may assume that $\iota = \rho_2$. It is not difficult to verify the proposition in this case. Q.E.D.

DEFINITION. Let $(T_1 \times [0, 1])^*$, $(T_2 \times [0, 1])^*$ be 2 copies of $(T \times [0, 1])^*$, let $T'_i \subset \partial(T_i \times [0, 1])^*$ be the torus, $i = 1, 2$, and let $h: T'_1 \rightarrow T'_2$ be an arbitrary isomorphism. Define

$$M_1^* = (T_1 \times [0, 1])^* \cup (T_2 \times [0, 1])^* / x \sim h(x) \quad \text{for } x \in T'_1.$$

Note that ∂M_1^* consists of 8 projective planes.

Let $\iota: T \times [0, 1] \rightarrow T \times [0, 1]$ be an orientation reversing fixed point free involution with $\iota(T \times 0) = T \times 1$. By Corollary (3.12), then $T \times [0, 1]/\iota = MO \times S^1$, MO a Moebius strip.

DEFINITION. Let $T' \subset \partial(T \times [0, 1])^*$ be the torus and let $h: T' \rightarrow \partial(MO \times S^1)$ be an arbitrary isomorphism. Define

$$M_2^* = (T \times [0, 1])^* \cup MO \times S^1 / x \sim h(x) \quad \text{for } x \in T'.$$

Note that ∂M_2^* consists of 4 projective planes.

Let $\iota: W \rightarrow W$ be an orientation reversing fixed point free involution. By Corollary (4.9), then $W/\iota = W/\rho_0$ is the nonorientable twisted I -bundle over a Klein bottle.

DEFINITION. Let $K \subset \partial W^*$ be the Klein bottle and let $h: K \rightarrow \partial(W/\rho_0)$ be an arbitrary isomorphism. Define

$$M_6^* = W^* \cup W/\rho_0 / x \sim h(x) \quad \text{for } x \in K.$$

Note that ∂M_6^* consists of 2 projective planes.

It will follow from Theorem (7.6) that M_1^* , M_2^* , M_6^* do not depend on the choice of h .

THEOREM (7.6). Let $\iota: M_i \rightarrow M_i$, $i = 1, 2, 6$ be an involution with isolated fixed points x_1, \dots, x_k , $k = 8, 4, 2$ respectively. Suppose $C_1, \dots, C_k \subset M_i$ are disjoint 3-cells with $x_j \in \text{int } C_j$ and $\iota(C_j) = C_j$, $j = 1, \dots, k$. Then

$$\overline{M_i - (C_1 \cup \dots \cup C_k)} / \iota = M_i^*, \quad i = 1, 2, 6.$$

Also $M_2^* = D(W^*)$, the double of W^* .

PROOF. Let $p: \tilde{M}_i \rightarrow M_i^*$ be the orientable double covering. Adding k 3-cells to $\partial \tilde{M}_i$, we obtain the closed manifold \hat{M}_i . The deck transformation on \tilde{M}_i extends to an involution on \hat{M}_i with k fixed points.

If $i = 1$, $p^{-1}(T_j \times [-1, 1])^*$, $j = 1, 2$, are 2 copies of $T \times [0, 1]$ with four 3-cells removed from each.

If $i = 2$, $p^{-1}(T \times [-1, 1])^*$ is a copy of $T \times [0, 1]$ with four 3-cells removed, and $p^{-1}(MO \times S^1)$ is another copy of $T \times [-1, 1]$.

If $i = 6$, $p^{-1}(W^*)$ is a copy of W with two 3-cells removed, and $p^{-1}(W/\rho_2)$ is another copy of W .

Therefore \hat{M}_1 , \hat{M}_2 are torus bundles over S^1 and \hat{M}_6 is the union along the boundary of 2 orientable twisted I -bundles over Klein bundles. By Corollaries (5.7) and (6.6), $\hat{M}_i = M_i$, $i = 1, 2, 6$, and by uniqueness of involutions we obtain an isomorphism $h: \hat{M}_i \rightarrow \overline{M_i - (C_1 \cup \dots \cup C_k)}$ which commutes with the deck transformation on \hat{M}_i and with $\iota|_{\overline{M_i - (C_1 \cup \dots \cup C_k)}}$. Therefore h defines an isomorphism $\bar{h}: M_i^* \rightarrow \overline{M - (C_1 \cup \dots \cup C_k)} / \iota$.

Considering the orientable double covering of $D(W^*)$ we obtain similarly \hat{M} , and we conclude by Corollary (6.6) that $\hat{M} = M_2$, and then that $M_2^* = D(W^*)$. Q.E.D.

THEOREM (7.7). *There are natural double coverings $M_1^* \xrightarrow{2} M_2^* \xrightarrow{2} M_6^*$.*

PROOF. (1) We may assume that $M_1^* = D(T \times [0, 1])^*$, the double of $(T \times [0, 1])^*$. We decompose M_1^* by a bicollar $T' \times [-1, 1] \subset M_1^*$ of the torus $T' \subset \partial(T \times [0, 1])^*$:

$$M_1^* = (T_- \times [0, 1])^* \cup T' \times [-1, 1] \cup (T_+ \times [0, 1])^*$$

with

$$(T_i \times [0, 1])^* \cap T' \times [-1, 1] = T' \times i1, \quad i = +, -.$$

Define the involution $\lambda: M_1^* \rightarrow M_1^*$ as follows. Let $\lambda|_{T' \times [-1, 1]} = \alpha \times \text{id} \times \tau: T' \times [-1, 1] \rightarrow T' \times [-1, 1]$ (see Corollary (3.11)) and extend the isomorphism $\lambda|_{T' \times -1}: T' \times -1 \rightarrow T' \times 1$ to an isomorphism

$$\lambda|_{(T_- \times [0, 1])^*}: (T_- \times [0, 1])^* \rightarrow (T_+ \times [0, 1])^*.$$

Let

$$\lambda = \lambda|_{(T_- \times [0, 1])^*} \cup \lambda|_{T' \times [-1, 1]} \cup \left(\lambda|_{(T_+ \times [0, 1])^*} \right)^{-1}.$$

Then λ is fixed point free and

$$M_1^*/\lambda = (T_- \times [0, 1])^* \cup T' \times [-1, 1]/\lambda = M_2^*.$$

(2) We may assume that $M_2^* = D(W^*)$, the double of W^* . We decompose M_2^* by a bicollar $K \times [-1, 1] \subset M_2^*$ of the Klein bottle $K \subset \partial W^*$:

$$M_2^* = W_+^* \cup K \times [-1, 1] \cup W_-^* \quad \text{with } W_i^* \cap K \times [-1, 1] = K \times i1, \quad i = +, -.$$

Define the involution $\mu: M_2^* \rightarrow M_2^*$ as follows. Let $\mu|_{K \times [-1, 1]}: K \times [-1, 1] \rightarrow K \times [-1, 1]$ be the involution with $K \times [-1, 1]/\mu| = \text{nonorientable twisted } I\text{-bundle over the Klein bottle}$. Extend the isomorphism $\mu|_{K \times -1}: K \times -1 \rightarrow K \times 1$ to an isomorphism $\mu|_{W_-^*}: W_-^* \rightarrow W_+^*$. Let

$$\mu = \mu|_{W_-^*} \cup \mu|_{K \times [-1, 1]} \cup \left(\mu|_{W_+^*} \right)^{-1}.$$

Then μ is fixed point free and

$$M_2^*/\mu = W_-^* \cup K \times [-1, 1]/\mu = M_6^*. \quad \text{Q.E.D.}$$

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