

CONTRIBUTIONS FROM CONJUGACY CLASSES OF REGULAR ELLIPTIC ELEMENTS IN $\mathrm{Sp}(n, \mathbf{Z})$ TO THE DIMENSION FORMULA

BY

MINKING EIE¹

ABSTRACT. The dimension of the space of cusp forms on the degree n Siegel upper half-space can be obtained from the Selberg trace formula; in this paper we compute the contribution from the conjugacy classes of regular elliptic elements in $\mathrm{Sp}(n, \mathbf{Z})$ using Weyl's character formula for representations of $\mathrm{GL}(n, \mathbf{C})$.

Introduction. Let H_n be the Siegel upper half-space; specifically,

$$H_n = \{ Z \in M_n(\mathbf{C}) \mid Z = {}^t\bar{Z}, \operatorname{Im} Z > 0 \}.$$

The real symplectic group of degree $2n$, $\mathrm{Sp}(n, \mathbf{R})$, acts transitively on H_n as a group of holomorphic automorphisms by the action,

$$M(Z) = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ in } \mathrm{Sp}(n, \mathbf{R}).$$

Let $\mathrm{Sp}(n, \mathbf{Z}) = \mathrm{Sp}(n, \mathbf{R}) \cap M_{2n}(\mathbf{Z})$ be the full modular subgroup of $\mathrm{Sp}(n, \mathbf{R})$.

Denote by $S(k; \mathrm{Sp}(n, \mathbf{Z}))$ the vector space of holomorphic cusp forms of weight k and degree n with respect to $\mathrm{Sp}(n, \mathbf{Z})$. (See [6] for the detail of definition.) If $k \geq 2n + 3$, k is even when n is odd and $n \geq 1$, the dimension of $S(k; \mathrm{Sp}(n, \mathbf{Z}))$ over \mathbf{C} is given by Selberg's trace formulas as follows [3]:

$$\dim_{\mathbf{C}} S(k; \mathrm{Sp}(n, \mathbf{Z})) = C(k, n) \int_F \sum_M K_M(Z, \bar{Z})^{-k} (\det Y)^{k-n-1} dX dY,$$

where

(1)

$$C(k, n) = 2^{-n} (2\pi)^{-n(n+1)/2} \prod_{i=0}^{n-1} \Gamma\left(k - \frac{n-i-1}{2}\right) \left[\prod_{i=0}^{n-1} \Gamma\left(k - n + \frac{i}{2}\right) \right]^{-1},$$

(2) F is a fundamental domain on H_n for $\mathrm{Sp}(n, \mathbf{Z})$,

(3) in the summation, M ranges over all matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\mathrm{Sp}(n, \mathbf{Z})/\{\pm 1\}$ and

$$K_M(Z, \bar{Z}) = \det[(1/2i)(Z - \overline{M(Z)})] \det(\overline{CZ + D}).$$

An element M of $\mathrm{Sp}(n, \mathbf{Z})$ is regular elliptic if M has an isolated fixed point on H_n . Let M be a regular elliptic element of $\mathrm{Sp}(n, \mathbf{Z})$. Then the fixed point of M can be

Received by the editors May 18, 1983 and, in revised form, October 11, 1983.

1980 *Mathematics Subject Classification*. Primary 10D20, 10D05; Secondary 10D30.

¹ Research supported by Academia Sinica and NSC of Republic of China.

©1984 American Mathematical Society
0002-9947/84 \$1.00 + \$.25 per page

transformed into the point iE of H_n by an action of $\mathrm{Sp}(n, \mathbf{R})$. Here E is the identity of the matrix ring $M_n(\mathbf{C})$. Note that the isotropy group of iE in $\mathrm{Sp}(n, \mathbf{R})$ is $U(n)$, the $n \times n$ unitary group, via the identification

$$\begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} \mapsto P + Qi.$$

Hence M is conjugate in $\mathrm{Sp}(n, \mathbf{R})$ to an element $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ with $A + Bi$ in $U(n)$. Furthermore, we can assume $A + Bi$ is a diagonal matrix since every unitary matrix can be diagonalized by a conjugation of a unitary matrix. Since $\mathrm{Sp}(n, \mathbf{Z})$ is discrete and $U(n)$ is compact, it follows that M is an element of finite order. Thus the eigenvalues of $A + Bi$ are roots of unity. Also, the centralizer $C_{M, \mathbf{Z}}$ of M in $\mathrm{Sp}(n, \mathbf{Z})/\{\pm 1\}$ is a group of finite order.

On the other hand, let $M \in \mathrm{Sp}(n, \mathbf{Z})$ be conjugate in $\mathrm{Sp}(n, \mathbf{R})$ to $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ with $U = A + Bi$ in $U(n)$, and U have eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$; $\lambda_i \lambda_j \neq 1$ for $1 \leq i \leq j \leq n$. Then the set of fixed points of M is isomorphic to

$$\Omega = \{W \in D_n \mid UW^tU = W\}$$

under the Cayley transform which maps H_n biholomorphically onto the generalized disc D_n defined by $D_n: W = {}^tW \in M_n(\mathbf{C}), E - W^t\bar{W} > 0$. Note that Ω is isomorphic to

$$\Omega' = \{W = [w_{ij}] \in D_n \mid (1 - \lambda_i \lambda_j)w_{ij} = 0 \text{ for all } i, j\}$$

which is a single point 0 because $1 - \lambda_i \lambda_j \neq 0$ for all i, j . Consequently, the set of fixed points of M is a single point and hence M is a regular elliptic element of $\mathrm{Sp}(n, \mathbf{Z})$. This proves that the following statements are equivalent:

- (1) M is a regular elliptic element of $\mathrm{Sp}(n, \mathbf{Z})$.
- (2) $M \in \mathrm{Sp}(n, \mathbf{Z})$ and has an isolated fixed point on H_n .
- (3) $M \in \mathrm{Sp}(n, \mathbf{Z})$ and is conjugate in $\mathrm{Sp}(n, \mathbf{R})$ to $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ with $A + Bi = \mathrm{diag}[\lambda_1, \lambda_2, \dots, \lambda_n] \in U(n)$; λ_i ($i = 1, 2, \dots, n$) are roots of unity and $\lambda_i \lambda_j \neq 1$ for all i, j .

In this paper, we shall prove the following Theorem with which we can compute the contribution from conjugacy classes of regular elliptic elements in $\mathrm{Sp}(n, \mathbf{Z})$ for general n .

THEOREM. Suppose $M \in \mathrm{Sp}(n, \mathbf{Z})$ is conjugate in $\mathrm{Sp}(n, \mathbf{R})$ to $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ with $U = A + Bi$ in $U(n)$ and U has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$; $\lambda_i \lambda_j \neq 1$ for $1 \leq i \leq j \leq n$. Then the contribution to $\dim_{\mathbf{C}} S(k; \mathrm{Sp}(n, \mathbf{Z}))$ ($k \geq 4$, k is even if n is odd) of elements in $\mathrm{Sp}(n, \mathbf{Z})$ which are conjugate in $\mathrm{Sp}(n, \mathbf{Z})/\{\pm 1\}$ to M is given by

$$N_{\{M\}} = |C_{M, \mathbf{Z}}|^{-1} \prod_{i=1}^n \bar{\lambda}_i^k \prod_{1 \leq i \leq j \leq n} (1 - \bar{\lambda}_i \bar{\lambda}_j)^{-1},$$

where $C_{M, \mathbf{Z}}$ is the centralizer of M in $\mathrm{Sp}(n, \mathbf{Z})/\{\pm 1\}$ and $|C_{M, \mathbf{Z}}|$ is its order.

1. Convergence of the series.

LEMMA 1 [4]. Suppose E is the $n \times n$ identity matrix and Z, W are $n \times n$ symmetric matrix variables satisfying $E - Z^t \bar{Z} \geq 0$ and $E - W^t \bar{W} \geq 0$. Then

$$\det(E - Z^t \bar{Z}) \det(E - W^t \bar{W}) \leq |\det(E - Z^t \bar{W})|^2.$$

The equality holds only when $Z = W$.

With this lemma, we are able to prove

PROPOSITION. For $k > n$ and an element M of finite order in $\mathrm{Sp}(n, \mathbf{Z})$ such that M has an isolated fixed point on H_n , we have:

- (1) $\int_{H_n} |K_M(Z, \bar{Z})|^{-k} (\det Y)^{k-n-1} dZ$ is convergent,
- (2) the contribution of elements in $\mathrm{Sp}(n, \mathbf{Z})$ which are conjugate in $\mathrm{Sp}(n, \mathbf{Z})/\{\pm 1\}$ to M is given by

$$N_{\{M\}} = C(k, n) |C_{M, \mathbf{Z}}|^{-1} \int_{H_n} K_M(Z, \bar{Z})^{-k} (\det Y)^{k-n-1} dZ.$$

PROOF. Since M is conjugate in $\mathrm{Sp}(n, \mathbf{R})$ to an element $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ with $U = A + iB = \mathrm{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$, $\lambda_i \lambda_j \neq 1$, we may assume $M = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ in the computation of $K_M(Z, \bar{Z})$. Then under the Cayley transform $Z = i(E + W)(E - W)^{-1}$, the integral in (1) is transformed into

$$2^{n(n+1)} \int_{D_n} \frac{\det(E - W \bar{W})^{k-n-1} dW}{|\det(E - {}^t \bar{U} W \bar{U} W)|^k},$$

where D_n is the generalized disc defined by

$$D_n: W = {}^t W \in M_n(\mathbf{C}), \quad E - W^t \bar{W} > 0.$$

For $k > n$, we have

$$\int_{D_n} \det(E - W \bar{W})^{k-n-1} dW < \infty.$$

To prove the integral is convergent, it suffices to prove that $\det(E - {}^t \bar{U} W \bar{U} W) \neq 0$ if $E - W^t \bar{W} \geq 0$. Suppose on the contrary that $\det(E - {}^t \bar{U} W \bar{U} W) = 0$; then Lemma 1 with $Z = {}^t \bar{U} W \bar{U}$ implies $\det(E - W \bar{W}) = 0$ and ${}^t \bar{U} W \bar{U} = W$. Let $W = [w_{ij}]$; then ${}^t \bar{U} W \bar{U} = W$ implies $(1 - \bar{\lambda}_i \lambda_j) w_{ij} = 0$. Since $1 - \bar{\lambda}_i \lambda_j \neq 0$ for all i, j ; this forces $W = 0$ which contradicts $\det(E - W \bar{W}) = 0$. Hence $\det(E - {}^t \bar{U} W \bar{U} W) \neq 0$ for all $E - W \bar{W} \geq 0$ and (1) follows.

To prove (2), we use the notation

$$K_\gamma(Z, \bar{Z}) = \det \left[\frac{1}{2i} (Z - \overline{\gamma(Z)}) \right] \det \overline{(C_1 Z + D_1)} \quad \text{if } \gamma = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}.$$

According to the Selberg trace formula, the contribution is given by

$$\begin{aligned} N_{\{M\}} &= C(k, n) \int_{F_{M_1 \in \bar{\Gamma}_n / C_{M, \mathbf{Z}}}} \sum K_{M_1 M M_1^{-1}}(Z, \bar{Z})^{-k} (\det Y)^{k-n-1} dZ \\ &= C(k, n) \cdot |C_{M, \mathbf{Z}}|^{-1} \int_F \sum_{M_1 \in \bar{\Gamma}_n} K_{M_1 M M_1^{-1}}(Z, \bar{Z})^{-k} (\det Y)^{k-n-1} dZ. \end{aligned}$$

Here $\bar{\Gamma}_n = \mathrm{Sp}(n, \mathbf{Z}) / \{\pm 1\}$ is the projective full modular subgroup of $\mathrm{Sp}(n, \mathbf{R})$. Note that

$$\begin{aligned} &\int_F \sum_{M_1 \in \bar{\Gamma}_n} |K_{M_1 M M_1^{-1}}(Z, \bar{Z})|^{-k} (\det Y)^{k-n-1} dZ \\ &= \int_{H_n} |K_M(Z, \bar{Z})|^{-k} (\det Y)^{k-n-1} dZ < +\infty. \end{aligned}$$

Thus, we can exchange the order of integration and summation, so our assertion in (2) follows.

2. Proof of the Theorem. To prove the Theorem, we need the following two lemmas.

LEMMA 2 (THEOREM 3-3 IN [7]). Suppose that $\{\psi_i(W)\}$, $i = 1, 2, \dots, n, \dots$, is an orthonormal base of the Hilbert space $H^2(k, D_n)$ with $k \geq 4$. Then

$$2^{n(n+1)} C(k, n) \det(E - Z\bar{W})^{-k} = \sum_{i=1}^{\infty} \psi_i(Z) \overline{\psi_i(W)}.$$

Here D_n : $W = {}^t W \in M_n(\mathbf{C})$, $E - W\bar{W} > 0$, and $H^2(k, D_n)$ is the space of holomorphic functions f defined on D_n satisfying

$$\int_{D_n} \det(E - W\bar{W})^{k-n-1} |f(W)|^2 dW < \infty.$$

LEMMA 3 (THEOREM 5 IN III OF [4]). Let $f = [f_1, f_2, \dots, f_n]$ be n -tuple integers such that $f_1 \geq f_2 \geq \dots \geq f_n \geq 0$ and $\xi = [\xi_1, \xi_2, \dots, \xi_n]$ with $|\xi_i| \leq 1$. Suppose that $M_{2f}(\xi) = \det[\xi_i^{2f_j + n-j}]$ and $0 < t < 1$. Then

$$\sum_f M_{2f}(\xi) t^{f_1 + f_2 + \dots + f_n} = \prod_{i < j} (\xi_i - \xi_j) \prod_{i \leq j} (1 - t \xi_i \xi_j)^{-1}.$$

PROOF OF THE THEOREM. We divide the proof into the following steps.

Step I. Since M has only isolated fixed points and $C_{M, \mathbf{Z}}$ is a discrete subgroup of the stabilizer of the fixed point, $C_{M, \mathbf{Z}}$ is a group of finite order. The contribution of the conjugacy class represented by M to the dimension formula is

$$N_{\{M\}} = C(k, n) |C_{M, \mathbf{Z}}|^{-1} \int_{H_n} K_M(Z, \bar{Z})^{-k} (\det Y)^{k-n-1} dX dY,$$

with

$$K_M(Z, \bar{Z}) = \det \left[\frac{1}{2i} \left(Z - \overline{(AZ + B)(-BZ + A)^{-1}} \right) \right] \det \overline{(-BZ + A)}.$$

Step II. With the Cayley transform $Z = i(E + W)(E - W)^{-1}$, the half-space H_n is transformed into the generalized disc D_n as defined in Lemma 2. Also we have

$$\begin{aligned} K_M(Z, \bar{Z}) &= \det \left[\frac{-1}{2i} (ZB\bar{Z} - ZA + A\bar{Z} + B) \right] \\ &= \det \left[(E - W)^{-1} (U - W\bar{U}\bar{W}) (E - \bar{W})^{-1} \right] \end{aligned}$$

and

$$(\det Y)^{-n-1} dX dY = 2^{n(n+1)} \det \left[(E - W)^{-1} (E - W\bar{W}) (E - \bar{W})^{-1} \right]^{-n-1} dW.$$

Hence we can express $N_{\{M\}}$ as

$$\begin{aligned} N_{\{M\}} &= C(k, n) 2^{n(n+1)} |C_{M, Z}|^{-1} \int_{D_n} \frac{\det(E - W\bar{W})^{k-n-1} dW}{\det(U - W\bar{U}\bar{W})^k} \\ &= \frac{C(k, n) 2^{n(n+1)}}{|C_{M, Z}| (\det U)^k} \int_{D_n} \frac{\det(E - W\bar{W})^{k-n-1} dW}{\det(E - {}^t\bar{U}W\bar{U}\bar{W})^k}. \end{aligned}$$

There exists a unitary matrix V such that ${}^t\bar{V}UV = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$. With the transformation $W \rightarrow VW{}^tV$, we get

$$N_{\{M\}} = \frac{C(k, n) 2^{n(n+1)}}{|C_{M, Z}| (\det U)^k} \int_{D_n} \frac{\det(E - W\bar{W})^{k-n-1} dW}{\det(E - {}^t\bar{U}'W\bar{U}'W)^k}, \quad U' = {}^t\bar{V}UV.$$

Hence we can assume that U is a diagonal matrix in the beginning. Note that the operation of U on D_n is $W \rightarrow UW{}^tU = [\lambda_i \lambda_j w_{ij}]$.

Step III. For each n -tuple $f = [f_1, f_2, \dots, f_n]$ of integers $f_1 \geq f_2 \geq \dots \geq f_n \geq 0$, we let A_{2f} denote the irreducible unitary representation of $\text{GL}_n(\mathbb{C})$ on the Hilbert space H corresponding to the highest weight

$$\chi_{2f}(\text{diag}[a_1, a_2, \dots, a_n]) = a_1^{2f_1} a_2^{2f_2} \dots a_n^{2f_n}.$$

Then H is a finite dimension vector space of dimension

$$N[2f] = \prod_{i < j} (2f_i - 2f_j + j - i).$$

Furthermore, H is spanned by homogeneous polynomial functions in $n(n+1)/2$ variables of degree $f_1 + f_2 + \dots + f_n$. Let $\psi_{2f}^{(i)}(W)$, $i = 1, 2, \dots, N[2f]$, be an orthonormal base of H , which consist of homogeneous polynomial functions. By Hua's argument (p. 234 of [4]), $\{\psi_{2f}^{(i)}\}$ is an orthonormal base of $H^2(k, D_n)$ when $2f$ ranges over all n -tuples of nonnegative even integers $2f_1 \geq 2f_2 \geq \dots \geq 2f_n \geq 0$. Consequently, we have the decomposition of the kernel

$$C(k, n) 2^{n(n+1)} \det(E - {}^t\bar{U}W\bar{U}\bar{W})^{-k} = \sum_f \sum_{i=1}^{N[2f]} \psi_{2f}^{(i)}({}^t\bar{U}W\bar{U}) \overline{\psi_{2f}^{(i)}(W)},$$

by Lemma 2. When W is transformed into ${}^t\bar{U}W\bar{U}$, we have

$$\psi_{2f}^{(i)}({}^t\bar{U}W\bar{U}) = A_{2f}(\bar{U})(\psi_{2f}^{(i)}(W)) \quad \text{for } i = 1, 2, \dots, N.$$

Let $N_{\{M\}}(tU)$ be the integral with U replaced by tU ($0 < t < 1$). Then we have

$$N_{\{M\}} = \lim_{t \rightarrow 1} N_{\{M\}}(tU)$$

by continuity and $N_{\{M\}}(tU)$ can be expressed as the series

$$\begin{aligned} N_{\{M\}}(tU) &= |C_{M,\mathbf{Z}}|^{-1} \prod_{i=1}^n (t\lambda_i)^{-k} \int_{D_n} \det(E - W\bar{W})^{k-n-1} \\ &\quad \times \sum_f \sum_{i=1}^{N[2f]} A_{2f}(t\bar{U}) (\psi_{2f}^{(i)}(W)) \cdot \overline{\psi_{2f}^{(i)}(W)} dW \\ &= |C_{M,\mathbf{Z}}|^{-1} \prod_{i=1}^n (t\lambda_i)^{-k} \sum_f \text{Trace } A_{2f}(t\bar{U}) \end{aligned}$$

which is convergent when $0 < t < 1$ we shall see soon. Here we use the identity

$$\int_{D_n} \det(E - W\bar{W})^{k-n-1} \sum_{i=1}^{N[2f]} A_{2f}(t\bar{U}) (\psi_{2f}^{(i)}(W)) \overline{\psi_{2f}^{(i)}(W)} dW = \text{Trace } A_{2f}(t\bar{U})$$

which follows from the orthonormality of $\{\psi_{2f}^{(i)}(W)\}$, $i = 1, 2, \dots, N[2f]$, in $H^2(k; D_n)$ for $k \geq 4$.

Now choose positive numbers $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ so that $\prod_{i \neq j} (\varepsilon_i \bar{\lambda}_i - \varepsilon_j \bar{\lambda}_j) \neq 0$ and let $U_\varepsilon = \text{diag}[\varepsilon_1 \lambda_1, \varepsilon_2 \lambda_2, \dots, \varepsilon_n \lambda_n]$. The character formula of Hermann Weyl (Theorem 1.2, Chapter VI of [1]) tells us

$$\prod_{i < j} (t\varepsilon_i \bar{\lambda}_i - t\varepsilon_j \bar{\lambda}_j) \cdot \text{Trace } A_{2f}(t\bar{U}_\varepsilon) = M_{2f}(\bar{U}_\varepsilon) t^{f_1 + f_2 + \dots + f_n} t^{n(n-1)/2}.$$

By Lemma 3, we get

$$N_{\{M\}}(tU_\varepsilon) = |C_{M,\mathbf{Z}}|^{-1} \prod_{i=1}^n (t\varepsilon_i \lambda_i)^{-k} \prod_{i \leq j} (1 - t\varepsilon_i \bar{\lambda}_i \varepsilon_j \bar{\lambda}_j)^{-1}.$$

As t and ε_i ($i = 1, 2, \dots, n$) approach 1, we get

$$N_{\{M\}} = |C_{M,\mathbf{Z}}|^{-1} \prod_{i=1}^n \bar{\lambda}_i^k \prod_{1 \leq i \leq j \leq n} (1 - \bar{\lambda}_i \bar{\lambda}_j)^{-1}.$$

This proves our Theorem.

3. Application. There are three conjugacy classes of regular elliptic elements in $\text{PSL}_2(\mathbf{Z}) = \text{SL}_2(\mathbf{Z})/\{\pm 1\}$, represented by

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Their total contributions to the dimension formula with respect to $\text{SL}_2(\mathbf{Z})$ is

$$\frac{1}{4}(-i)^k + \frac{1}{9}(\rho^k + \bar{\rho}^k - \rho^{k-2} - \bar{\rho}^{k-2}), \quad \rho = e^{\pi i/3}.$$

Combining this with the contribution from identity and conjugacy classes represented by $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, n a nonzero integer, we get that the dimension for the vector space of cusp forms of degree 1 and weight k (k even, $k \geq 4$) with respect to $\text{SL}_2(\mathbf{Z})$ is

$$\dim_{\mathbf{C}} S(k; \text{SL}_2(\mathbf{Z})) = \frac{k-1}{12} - \frac{1}{2} + \frac{1}{4}(-i)^k + \frac{1}{9}(\rho^k + \bar{\rho}^k - \rho^{k-2} - \bar{\rho}^{k-2}).$$

It is more complicated for the case of degree two. By the computation in [2], there are 22 conjugacy classes of regular elliptic elements in $\mathrm{Sp}(2, \mathbf{Z})$ and the total contribution from these conjugacy classes is $N_1 + N_2$ with

$$N_1 = \begin{cases} 2^{-7}3^{-3} \times [1131, 229, -229, -1131, 427, -571, 123, \\ \quad -203, 203, -123, 571, -427] \\ \text{if } k \equiv [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] \pmod{12}; \end{cases}$$

$$N_2 = \begin{cases} 5^{-1} & \text{if } k \equiv 0 \pmod{5}, \\ -5^{-1} & \text{if } k \equiv 3 \pmod{5}, \\ 0 & \text{otherwise.} \end{cases}$$

Here N_1 is the total contribution from regular elliptic conjugacy classes except conjugacy classes of order 5 and N_2 is the total contributions from conjugacy classes of order 5.

Let N_3 and N_4 be the contribution from conjugacy classes in $\mathrm{Sp}(2, \mathbf{Z})$, represented by elements of the form

$$\begin{bmatrix} a & 0 & b & * \\ * & 1 & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (c \neq 0) \quad \text{and} \quad \begin{bmatrix} U & S'U^{-1} \\ 0 & 'U^{-1} \end{bmatrix},$$

respectively. Then by the computation in [2], we get

$$N_3 = \begin{cases} 2^{-5}3^{-3} \times [17k - 294, -25k + 325, -25k + 254, \\ \quad 17k - 261, 17k - 86, -k + 53, -k - 42, \\ \quad -7k + 91, -7k + 2, -k - 27, -k + 166, 17k - 181], \\ \text{if } k \equiv [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] \pmod{12} \end{cases}$$

and

$$N_4 = \begin{cases} 2^{-7}3^{-3}5^{-1}(2k^3 + 96k^2 - 52k - 3231) & \text{if } k \text{ is even,} \\ 2^{-7}3^{-3}5^{-1}(2k^3 - 114k^2 + 2018k - 9051) & \text{if } k \text{ is odd.} \end{cases}$$

Consequently, we can write the dimension formula for the cusp forms of degree two and weight k ($k \geq 4$) as

$$\dim_{\mathbf{C}} S(k; \mathrm{Sp}(2, \mathbf{Z})) = N_1 + N_2 + N_3 + N_4.$$

REFERENCES

1. Hermann Boerner, *Representations of groups*, North-Holland, Amsterdam, 1970.
2. Minking Eie, *Dimension formulas for the vector spaces of Siegel's cusp forms of degree two and degree three*, Dissertation, University of Chicago, December 1982.
3. R. Godement, *Généralités sur les formes modulaires*. I, II, Séminaire Henri Cartan, 10e années, 1957, 1958.

4. L. K. Hua, *On the theory of functions of several complex variables*. I, II, III, Amer. Math. Soc. Transl. **32** (1962), 163–263.
5. ———, *Inequalities involving determinants*, Amer. Math. Soc. Transl. **32** (1962), 265–272.
6. Hans Maass, *Siegel's dual forms and Dirichlet series*, Lecture Notes in Math., vol. 216, Springer-Verlag, Berlin and New York, 1971, pp. 181–196.
7. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
8. George W. Mackey, *Unitary group representation in physics, probability and number theory*, Benjamin, New York, 1978.
9. Hideo Shimizu, *On discontinuous groups operating on the product of the upper half plane*, Math. Ann. **177** (1963), 33–71.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

Current address: Institute of Mathematics, Academia Sinica, Nankang, Taipei, Taiwan, Republic of China