

ON CERTAIN BOOLEAN ALGEBRAS $\mathcal{P}(\omega)/I$

BY

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ABSTRACT. We consider possible isomorphisms between algebras of the form $\mathcal{P}(\omega)/I$, assuming CH. In particular, the solution of a problem of Erdős and Ulam is given. We include some remarks on the completeness of such algebras.

0. Notation and summary of results. We use standard set-theoretical terminology. By Fin we denote the ideal of finite subsets of ω —the set of natural numbers. We consider Boolean algebras of the form $\mathcal{P}(\omega)/I$, where I is an ideal containing Fin . Throughout this paper I and J will denote such ideals.

The algebra $\mathcal{P}(\omega)/\text{Fin}$ plays an important role in general topology, particularly in the study of $\beta\omega \setminus \omega$. The following ideals are also of certain significance, especially in number theory:

$$I_0 = \left\{ a \subset \omega : \sum_{n \in a} \frac{1}{n} < \infty \right\},$$

$$I_1 = \left\{ a \subset \omega : \lim_{n \rightarrow \infty} \frac{|a \cap n|}{n} = 0 \right\} \text{—the ideal of sets of density 0,}$$

$$I_2 = \left\{ a \subset \omega : \lim_{m \rightarrow \infty} \frac{\sum_{n \in a \cap m} (n+1)^{-1}}{\log m} = 0 \right\} \text{—the ideal of sets}$$

of logarithmic density 0.

Notice that we identify each natural number with the set of its predecessors. Hence $a \cap n = \{k : k < n \text{ \& } k \in a\}$.

Assuming the Continuum Hypothesis (CH) holds, Erdős and Monk proved that $\mathcal{P}(\omega)/I_0$ and $\mathcal{P}(\omega)/\text{Fin}$ are isomorphic (written $\mathcal{P}(\omega)/I_0 \simeq \mathcal{P}(\omega)/\text{Fin}$). Erdős and Ulam raised the question whether $\mathcal{P}(\omega)/I_1$ and $\mathcal{P}(\omega)/I_2$ are isomorphic if CH is assumed (see [1, Question 48; 2, pp. 38–39]).

Our main purpose is to formulate conditions on I and J implying $\mathcal{P}(\omega)/I \simeq \mathcal{P}(\omega)/J$. All results of this kind were obtained under the assumption of CH. We do not know whether it is possible to prove or show the independence of these theorems without the assumption of CH.

Our first criterion is a topological one. We identify each subset of ω with its representing function. Then $\mathcal{P}(\omega)$ carries in a natural way the topology of the Cantor set, the basic sets being of the form $U_s = \{a \in \mathcal{P}(\omega) : a \upharpoonright \text{Dom}(s) = s\}$,

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where s is some function mapping a finite subset of ω into $\{0, 1\}$. Since each U_s is a clopen set, Fin and I_0 are F_σ subsets of $\mathcal{P}(\omega)$, while I_1 and I_2 are $F_{\sigma\delta}$ subsets.

In view of this observation the above-mentioned result of Erdős and Monk becomes a corollary of

THEOREM 1 (CH). *If I and J are F_σ ideals then $\mathcal{P}(\omega)/I \approx \mathcal{P}(\omega)/J$.*

Unfortunately, the analogous statement for $F_{\sigma\delta}$ ideals is false (see Corollary 3.2 and Problem C). This fact has been known for a long time (see [2]), but our proof is probably new.

So in order to solve the problem of Erdős and Ulam we have to consider more subtle properties of I_1 and I_2 .

By a/I we denote the equivalence class for the relation \sim_I , where $a \sim_I b$ iff $a \Delta b \in I$, and by R^+ the set of all positive reals.

We consider functions f such that

- (i) $f: \omega \rightarrow R^+$,
- (ii) $\sum_{n \in \omega} f(n) = \infty$,
- (iii) $\lim_{m \rightarrow \infty} (f(m) / \sum_{n \leq m} f(n)) = 0$.

Functions satisfying (i)–(iii) will be called *EU-functions*.

For an *EU-function* f and for $a \subset \omega$, let

$$d_f(a) = \limsup_{m \rightarrow \infty} \frac{\sum_{n \in a \cap m} f(n)}{\sum_{n < m} f(n)}$$

and $I_f = \{a \subset \omega: d_f(a) = 0\}$. Obviously, I_f is an ideal containing Fin .

In $\mathcal{P}(\omega)/I_f$ we define a metric by

$$\rho_f(a/I_f, b/I_f) = d_f(a \Delta b).$$

We shall prove in §3 that $(\mathcal{P}(\omega)/I_f, \rho_f)$ is a complete metric space.

THEOREM 3 (CH). *Let f and h be EU-functions. There exists a $\varphi: \mathcal{P}(\omega)/I_f \rightarrow \mathcal{P}(\omega)/I_h$ being simultaneously an isomorphism of Boolean algebras and an isometry.*

If $f(n) = 1$ and $h(n) = (n + 1)^{-1}$ for all n , then $I_1 = I_f$ and $I_2 = I_h$ because $\sum_{n < m} (n + 1)^{-1}$ approximates $\log m$. Hence Theorem 3 yields a positive solution of the problem of Erdős and Ulam.

Now we introduce some notation used throughout this paper. Let $A \subset \mathcal{P}(\omega)$, $a \in \mathcal{P}(\omega)$. We define $A/I = \{a/I: a \in A\}$. By $[A, a]_I$ or $[A/I, a/I]$ we denote the subalgebra of $\mathcal{P}(\omega)/I$ generated by A/I and a/I .

With each ideal $I \subset \mathcal{P}(\omega)$ we associate a natural projection $\Pi_I: \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)/I$ defined by $\Pi_I(a) = a/I$.

Let $A \subseteq \mathcal{P}(\omega)/I$ be a subalgebra. The image of a choice function, i.e. a function $S: A \rightarrow \mathcal{P}(\omega)$ such that $S(a) \in a$ for all $a \in A$, is called a selector.

Note that there is a unique way to define Boolean operations on the image of a choice function S to make S an isomorphism. We shall denote those operations (likewise the operations on $\mathcal{P}(\omega)/I$) by $\wedge, \vee, -$; the Boolean ordering will be denoted by $<$.

It is well known that $\mathcal{P}(\omega)/\text{Fin}$ contains an antichain of power 2^{\aleph_0} . It follows from Proposition 4.1 and Lemma 4.3 that $\mathcal{P}(\omega)/I$ contains an uncountable antichain whenever I is a Borel subset of $\mathcal{P}(\omega)$. Hence for Borel ideals I there is no selector S such that the operations $\wedge, \vee, -$ are the set-theoretical ones. For our purpose it is helpful to have selectors such that $\wedge, \vee, -$ differ as little as possible from the set-theoretical operations. Therefore we introduce the concept of a split. If $\Pi: A \rightarrow B$ is an epimorphism of Boolean algebras then a monomorphism $\pi: B \rightarrow A$ is called a split of Π iff $\Pi \circ \pi = \text{id}_B$.

As we observed above, the natural projection Π_I usually has no split. However, for a large class of ideal I —those having property Δ defined below—there exists a split for the natural projection of $\mathcal{P}(\omega)/\text{Fin}$ onto $\mathcal{P}(\omega)/I$. Subsequently, we use the word “split” in the following sense:

DEFINITION 0.1. Let A be a subalgebra of $\mathcal{P}(\omega)/I$. By a split of A we understand a monomorphism $\pi: A \rightarrow \mathcal{P}(\omega)/\text{Fin}$ such that $\pi(a) \subset a$ for all $a \in A$.

Note that if S is a selector for the image of a split of $\mathcal{P}(\omega)/I$ then for all $a, b \in S$ we have $(a \wedge b) \Delta (a \cap b) \in \text{Fin}$, etc.

DEFINITION 0.2. An ideal $I \subseteq \mathcal{P}(\omega)$ has property Δ iff for any $\{a_i: i \in \omega\} \subset I$ there exist $\{b_i: i \in \omega\} \subset I$ such that $\bigcup_{i \in \omega} b_i \in I$ and $a_i \setminus b_i \in \text{Fin}$ for all $i \in \omega$.

Note that the ideals Fin, I_0, I_f have property Δ .

A modification of the following theorem is used in the proof of Theorem 3.

THEOREM 2 (CH). *If I has property Δ then there exists a split of $\mathcal{P}(\omega)/I$.*

§ i for $i \leq 3$ is devoted to the proof of Theorem i . In §4 we derive some other properties of ideals and algebras of the form $\mathcal{P}(\omega)/I$. In particular, we prove that if I is an analytic set, then the algebra $\mathcal{P}(\omega)/I$ is not complete, and, moreover, if I possesses property Δ , then this algebra is not \aleph_1 -complete. This generalizes Sierpiński's result asserting the nonexistence of an analytic ultrafilter. In §5 we state some open questions.

Now we are going to explain the general method which allows us to derive Theorems 1–3 from the lemmas proved in the appropriate sections.

We construct the isomorphism (respectively split) piece by piece. First we fix orderings of $\mathcal{P}(\omega)/I$ and $\mathcal{P}(\omega)/J$ of type ω_1 . Then we define the isomorphism by the back-and-forth method in the following way. Assume $\varphi: A \rightarrow B$ is already defined, where $A \subset \mathcal{P}(\omega)/I$ and $B \subset \mathcal{P}(\omega)/J$ are countable subalgebras. Let $a/I \in \mathcal{P}(\omega)/I$ and $b/J \in \mathcal{P}(\omega)/J$ be the first elements in $\mathcal{P}(\omega)/I \setminus A$ and $\mathcal{P}(\omega)/J \setminus B$ respectively. We extend φ to $\varphi_1: [A, a/I] \rightarrow B_1$ and then we find $\varphi_2^{-1}: [B_1, b/J] \rightarrow A_1$ extending φ_1^{-1} .

If at every stage we can choose φ_2 to be an isomorphism (and an isometry), then after ω_1 successive extensions the required isomorphism will be constructed.

For the construction of a split it suffices to fix an ordering of $\mathcal{P}(\omega)/I$ of type ω_1 and then to extend inductively splits defined on countable subalgebras of $\mathcal{P}(\omega)/I$.

So it remains to show in §§1–3 how to extend an isomorphism (monomorphism) $\varphi: A \rightarrow B$ to an isomorphism (monomorphism) ψ such that $\text{Dom}(\psi) = [A, a/I]$ for

a given a/I . Note that

$$[A, a/I] = \{(a/I \wedge c/I) \vee (-a/I \wedge d/I) : c/I, d/I \in A\}.$$

LEMMA 0.3. Let $\varphi: A \rightarrow B$ be an isomorphism (monomorphism), $a/I \in \mathcal{P}(\omega)/I$ and $b/J \in \mathcal{P}(\omega)/J$ such that:

$$(1)_J \quad \begin{aligned} \varphi(c/I) - b/J &= 0 && \text{for all } c/I \text{ such that } c/I \leq a/I, \\ b/J - \varphi(c/I) &= 0 && \text{for all } c/I \text{ such that } a/I \leq c/I, \\ b/J - \varphi(c/I) &\neq 0 && \text{for all } c/I \text{ such that } a/I - c/I \neq 0, \\ \varphi(c/I) - b/J &\neq 0 && \text{for all } c/I \text{ such that } c/I - a/I \neq 0. \end{aligned}$$

Then $\psi: [A, a/I] \rightarrow [B, b/J]$, defined by $\psi((a/I \wedge c/I) \vee (-a/I \wedge d/I)) = (b/J \wedge \varphi(c/I)) \vee (-b/J \wedge \varphi(d/I))$, is an isomorphism (monomorphism).

The verification that ψ is a well-defined function whenever b/J satisfies $(1)_J$ is straightforward. Moreover, it follows immediately from its definition that ψ is an isomorphism (monomorphism) of Boolean algebras.

Now in order to find the required extension, we need only solve the system $(1)_J$ with respect to b/J . First we transform $(1)_J$ into an equivalent system more convenient for our purposes.

Since B is assumed to be countable, we can find sequences $\{c_n/J, d_n/J, e_n/J : n \in \omega\} \subseteq B$ such that, for all $n, k \in \omega$, $c_n/J \leq c_{n+1}/J \leq \dots \leq d_{n+1}/J \leq d_n/J$, $e_n/J - c_k/J \neq 0$, $d_k/J - e_n/J \neq 0$ and $(1)_J$ is equivalent to the system

$$(2)_J \quad \begin{aligned} c_n/J &\leq x \leq d_n/J \\ e_n/J - x &\neq 0 && \text{for all } n. \\ x - e_n/J &\neq 0 \end{aligned}$$

(The free variable is denoted by x instead of b/J for convenience.)

Up to now we have been considering functions defined on subalgebras of $\mathcal{P}(\omega)/I$, but it is much more convenient to deal with elements of $\mathcal{P}(\omega)$ than with objects of the form a/I . Hence we shall consider functions $\varphi: A \rightarrow B$, where $A, B \subset \mathcal{P}(\omega)$ are selectors for A/I and B/J , respectively. A function of this type will be frequently called an isomorphism (monomorphism) of selectors (for A/I and B/J) iff there exists a (unique) isomorphism (monomorphism) $\bar{\varphi}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \Pi_I \downarrow & & \downarrow \Pi_J \\ A/I & \xrightarrow{\bar{\varphi}} & B/J \end{array}$$

Notice that φ as above is actually an isomorphism (monomorphism) if we remember that A and B carry Boolean operations $\wedge, \vee, -$.

Now we may reformulate our task as follows: Given an isomorphism (monomorphism) $\varphi: A \rightarrow B$ of selectors for A/I and B/J and an element $\hat{a}/I \in \mathcal{P}(\omega)/I$, find

a representative $a \in \hat{a}/I$ and an isomorphism (monomorphism) of selectors ψ extending φ on a .

The following definition will be very convenient.

DEFINITION 0.4. Let $\{a_n, b_n, c_n: n \in \omega\}$ be sequences of subsets of ω and let $J \subset \mathcal{P}(\omega)$ be an ideal. We call these sequences J -regular iff $a_n \subset a_{n+1} \subset \dots \subset b_{n+1} \subset b_n, c_k \setminus a_n \notin J$, and $b_n \setminus c_n \notin J$ for all $k, n \in \omega$.

LEMMA 0.5. Assume $J \in \mathcal{P}(\omega)$ is an ideal and for $n \in \omega$, let \hat{a}_n, \hat{b}_n be such that

$$\dots \leq \hat{a}_n/J \leq \hat{a}_{n+1}/J \leq \dots \leq \hat{b}_{n+1}/J \leq \hat{b}_n/J \leq \dots.$$

There exist $a_n \in \hat{a}_n/J$ and $b_n \in \hat{b}_n/J$ for $n \in \omega$ such that

$$\dots \subset a_n \subset a_{n+1} \subset \dots \subset b_{n+1} \subset b_n \subset \dots.$$

PROOF. Put $a_0 = \hat{a}_0, b_0 = \hat{b}_0$. Assume a_n and b_n are already defined and put $b_{n+1} = (\hat{b}_{n+1} \cup a_n) \cap b_n$ and $a_{n+1} = a_n \cup (\hat{a}_{n+1} \cap b_{n+1})$. An easy inductive argument yields the conclusion. \square

By Lemma 0.5 we can find J -regular sequences $\{a_n, b_n, c_n: n \in \omega\}$ such that x/J will be a solution for $(2)_J$ whenever x is a solution of the following system:

$$(3)_J \quad \begin{array}{ll} a_n < x < b_n \\ c_n \setminus x \notin J & \text{for } n \in \omega. \\ x \setminus c_n \notin J \end{array}$$

Adopting terminology from model theory, we say the Boolean algebra $\mathcal{P}(\omega)/I$ is \aleph_1 -saturated iff every system of the form $(3)_J$ has a solution.

1. F_σ -ideals. In this section we prove

THEOREM 1'. Let I, J be F_σ -ideals, $A \subset \mathcal{P}(\omega)/I, B \subset \mathcal{P}(\omega)/J, |A| \leq \aleph_0, a \in \mathcal{P}(\omega)/I$ and $\varphi: A \rightarrow B$ an isomorphism. Then there exist $b \in \mathcal{P}(\omega)/J$ and $\psi \supset \varphi$ such that $\psi: [A, a] \rightarrow [B, b]$ is an isomorphism.

If CH holds then one can deduce Theorem 1 from Theorem 1' by applying the back-and-forth method as described in §0.

Throughout this section we denote by I an F_σ -ideal. Moreover, we assume $I = \bigcup_{n \in \omega} F_n$, where F_n are closed sets and $F_n \subset F_{n+1}$ for $n \in \omega$.

In order to prove Theorem 1' it suffices to show that $\mathcal{P}(\omega)/I$ is an \aleph_1 -saturated Boolean algebra, i.e. that every system of the form $(3)_I$ has a solution.

LEMMA 1.1. Let $k, n \in \omega, b \notin I$ and $s \in 2^k$. Then there exists an $m > k$ such that $U_{s \cap b \upharpoonright [k, m]} \subset 2^\omega \setminus F_n$.

PROOF. Put $c = s \cap b \upharpoonright (\omega \setminus k)$. Then $c \notin I$, so $c \in 2^\omega \setminus F_n$ and by openness of $2^\omega \setminus F_n$ there exists an $m > k$ such that $U_{c \upharpoonright m} \subset (2^\omega \setminus F_n)$. \square

LEMMA 1.2. Let $\{a_n, b_n, c_n: n \in \omega\}$ be I -regular sequences of subsets of ω . There exists an element d such that $a_l \setminus d \in \text{Fin}, d \setminus b_l \in \text{Fin}, c_l \setminus d \notin I$ and $d \setminus c_l \notin I$ for all $l \in \omega$.

PROOF. By Lemma 1.1 we can inductively define sequences u_n^k and v_n^k for $k < n \in \omega$ such that for all $k < n$:

- (i) u_n^k, v_n^k are finite sequences of 0's and 1's;
- (ii)

$$\begin{aligned} \text{Dom}(u_n^k) \cap \text{Dom}(u_n^{k'}) &= \text{Dom}(v_n^k) \cap \text{Dom}(v_n^{k'}) \\ &= \text{Dom}(v_n^k) \cap \text{Dom}(u_n^{k'}) = \emptyset \quad \text{for } \langle k, n \rangle \neq \langle k', n' \rangle; \end{aligned}$$

- (iii) $u_n^k \subset c_k \setminus a_n, v_n^k \subset b_n \setminus c_k$;

- (iv) if $s = \bar{s}_{k,n} = \bigcup_{k < m \leq n} u_m^k$ or $s = \underline{s}_{k,n} = \bigcup_{k < m \leq n} v_m^k$, then $U_s \subset 2^\omega \setminus F_n$.

Put

$$d = \left(\bigcup_{n \in \omega} a_n \setminus \bigcup_{n \in \omega} \bigcup_{k < n} u_n^k \right) \cup \bigcup_{n \in \omega} \bigcup_{k < n} v_n^k.$$

Then for arbitrary l we have:

$$\begin{aligned} a_l \setminus d &\subset a_l \cap \bigcup_{n \in \omega} \bigcup_{k < n} u_n^k \subset \bigcup_{n < l} \bigcup_{k < n} u_n^k \in \text{Fin}, \\ d \setminus b_l &\subset \bigcup_{n \in \omega} \bigcup_{k < n} v_n^k \setminus b_l \subset \bigcup_{n < l} \bigcup_{k < n} v_n^k \in \text{Fin}, \\ c_1 \setminus d &\supset c_1 \cap \bigcup_{n \in \omega} \bigcup_{k < n} u_n^k \supset c_l \cap \bigcup_{l < n \in \omega} u_n^l = \bigcup_{l < n \in \omega} u_n^l, \\ d \setminus c_l &\supset \bigcup_{n \in \omega} \bigcup_{k < n} v_n^k \setminus c_l \supset \bigcup_{l < n \in \omega} v_n^l \setminus c_l = \bigcup_{l < n \in \omega} v_n^l. \end{aligned}$$

By (iv), if $n > l$ and $s = \bar{s}_{l,n} = \bigcup_{l < m \leq n} u_m^l$, then

$$\bigcup_{l < n \in \omega} u_n^l \in U_s \subseteq 2^\omega \setminus F_n.$$

Hence

$$\bigcup_{l < n \in \omega} u_n^l \in \bigcap_{l < n} (2^\omega \setminus F_n) = 2^\omega \setminus I,$$

so $c_l \setminus d \notin I$. The same argument shows that $d \setminus c_l \notin I$. \square

Obviously, the element d satisfying Lemma 1.2 is a solution of the system $(3)_I$. Theorem 1' is now an immediate consequence of Lemma 0.3. \square

2. Split. For the concept of a split and property Δ recall Definitions 0.1 and 0.2. In this section we prove

THEOREM 2'. *If I has property Δ , $A \subset \mathcal{P}(\omega)/I$, $|A| \leq \aleph_0$, $a \in \mathcal{P}(\omega)/I$ and π splits A , then there exists a $\bar{\pi} \supset \pi$ splitting $[A, a]$.*

If CH holds, then Theorem 2 is easily deduced from Theorem 2' by the method described in §0. We apply Theorem 2' itself in §3.

In this section I denotes an ideal having property Δ . We need

LEMMA 2.1. *Let $a_n \subset a_{n+1} \subset b_{n+1} \subset b_n$, $a_n \setminus d \in I$ and $d \setminus b_n \in I$ for $n \in \omega$. Then there exists a \bar{d} such that $d \Delta \bar{d} \in I$, $a_n \setminus \bar{d} \in \text{Fin}$ and $\bar{d} > b_n \in \text{Fin}$ for $n \in \omega$.*

PROOF. By property Δ we can find sets $u_n \subset (a_n \setminus d)$ and $v_n \subset (d \setminus b_n)$ for $n \in \omega$ such that $(a_n \setminus d) \setminus u_n \in \text{Fin}$, $(d \setminus b_n) \setminus v_n \in \text{Fin}$ and $\bigcup_{n \in \omega} u_n, \bigcup_{n \in \omega} v_n \in I$. Put

$$\bar{d} = \left(d \cup \bigcup_{n \in \omega} u_n \right) \setminus \bigcup_{n \in \omega} v_n.$$

Obviously, $d \Delta \bar{d} \in I$ and

$$\bigcup_{n \in \omega} v_n \cap a_m = \bigcup_{n \in \omega} u_n \setminus b_m = \emptyset$$

for $m \in \omega$. Hence

$$a_m \setminus \bar{d} = a_m \setminus \left(d \cup \bigcup_{n \in \omega} u_n \right) \subset a_m \setminus (d \cup u_m) = (a_m \setminus d) \setminus u_m \in \text{Fin}$$

and

$$\bar{d} \setminus b_m = \left(d \setminus \bigcup_{n \in \omega} v_n \right) \setminus b_m \subset (d \setminus v_m) \setminus b_m = (d \setminus b_m) \setminus v_m \in \text{Fin}. \quad \square$$

PROOF OF THEOREM 2'. As shown in §0 it is sufficient to prove that, having a solution d of system $(3)_I$, one can find a solution \bar{d} of system $(3)_{\text{Fin}}$ such that $d \Delta \bar{d} \in I$. Now observe that if $\{a_n, b_n, c_n: n \in \omega\}$ are I -regular sequences and d is a solution of $(3)_I$, then \bar{d} satisfying the conclusion of Lemma 2.1 is a solution of $(3)_{\text{Fin}}$. \square

3. f -ideals. We introduce the following convention: if $f, g, h: \omega \rightarrow \omega$ and $a \subset \omega$ then $F(a) = \sum_{n \in a} f(n)$, $G(a) = \sum_{n \in a} g(n)$ and $H(a) = \sum_{n \in a} h(n)$. Notice that for any natural number $k = \{0, \dots, k-1\}$ we have $F(k) = \sum_{n < k} f(n)$ and $F(\{k\}) = f(k)$.

Lemma 3.1 and Corollary 3.2 are interesting on their own and are not used to prove Theorem 3.

LEMMA 3.1. *Let h be an EU-function. Then $\langle \mathcal{P}(\omega)/I_h, \rho_h \rangle$ is a complete metric space.*

PROOF. It is easy to verify that ρ_h is a well-defined metric.

We show that any Cauchy sequence in the sense of ρ_h has a limit.

Let $\{a_n/I_h: n \in \omega\}$ be a Cauchy sequence in $\langle \mathcal{P}(\omega)/I_h, \rho_h \rangle$ and let $\{a_n: n \in \omega\}$ be a corresponding sequence of arbitrary representatives. There exists a subsequence $\{a_{m_n}: n \in \omega\}$ such that $\rho_h(a_{m_n}/I_h, a_k/I_h) < 2^{-n}$ for all $m_n < k < \omega$. Since any limit of $\{a_{m_n}: n \in \omega\}$ is simultaneously a limit of $\{a_n: n \in \omega\}$, we may assume without loss of generality that $\rho_h(a_n/I_h, a_k/I_h) < 2^{-n}$ for all $n < k < \omega$.

Thus by the definition of ρ_h the following holds:

$$d_h(a_n \Delta a_k) = \limsup_{m \rightarrow \infty} \frac{H(m \cap (a_n \Delta a_k))}{H(m)} < 2^{-n} \quad \text{for all } n < k < \omega.$$

We fix a sequence $\{m_n: n \in \omega\}$ such that

- (1) $m_0 = 0$,
- (2) $H(m \cap (a_k \Delta a_l))/H(m) < 2^{-k}$ for all $k < l \leq n$ and $m \geq m_n$,
- (3) $H(m_n)/H(m_{n+1}) < 2^{-n}$ for all $n \in \omega$.

Now we define $a \subset \omega$ by

$$a \upharpoonright [m_n, m_{n+1}) = a_n \upharpoonright [m_n, m_{n+1}) \quad \text{for all } n \in \omega.$$

It remains to show that a is the limit of $\{a_n: n \in \omega\}$. For all $n \in \omega$ we have

$$\begin{aligned} \rho_h(a_n/I_h, a/I_h) &= \limsup_{m \rightarrow \infty} \frac{H(m \cap (a_n \Delta a))}{H(m)} \\ &\leq \sup_{m \geq m_{n+1}} \frac{H(m \cap (a_n \Delta a))}{H(m)} \\ &\leq \frac{H(m_n) + H([m_n, m_{n+1}) \cap (a_n \Delta a_n)) + H([m_{n+1}, m_{n+2}) \cap (a_n \Delta a_{n+1})) + \dots + H([m_{n+k}, m) \cap (a_n \Delta a_{n+k}))}{H(m)} \\ &\leq \frac{H(m_n)}{H(m_{n+1})} + \frac{H(m \cap (a_n \Delta a_{n+1}))}{H(m)} + \dots + \frac{H(m \cap (a_n \Delta a_{n+1}))}{H(m)} \\ &\leq 2^{-n} + \sum_{k=1}^{\infty} 2^{-(n+k)} = 2^{-n+1}. \end{aligned}$$

Hence $\lim_{n \rightarrow \infty} \rho_h(a_n/I_h, a/I_h) = 0$ and therefore a/I_h is the limit of $\{a_n/I_h: n \in \omega\}$. \square

COROLLARY 3.2. *Let h be as above. The Boolean algebra $\mathcal{P}(\omega)/I_h$ is not \aleph_1 -saturated. Hence $\mathcal{P}(\omega)/I_h$ and $\mathcal{P}(\omega)/\text{Fin}$ are not isomorphic.*

PROOF. Let $\{a_n/I_h: n \in \omega\}$ be an increasing Cauchy sequence and let a/I_h be its limit. If $\mathcal{P}(\omega)/I_h$ were \aleph_1 -saturated, there would be a b/I_h such that $a_n/I_h < b/I_h < a/I_h$ for all n . But then

$$\rho_h(a/I_h, b/I_h) \leq \rho_h(a/I_h, a_n/I_h) \quad \text{for all } n;$$

hence $\rho_h(a/I_h, b/I_h) = 0$. A contradiction. \square

The proof of Corollary 3.2 yields the idea of constructing an isomorphism which is simultaneously an isometry. But it is not hard to show that not every isometric isomorphism of subalgebras of $\mathcal{P}(\omega)/I_f$ and $\mathcal{P}(\omega)/I_h$ may be extended over an arbitrary element of $\mathcal{P}(\omega)/I_f$ to an isometric isomorphism. One can actually easily construct 8-element subalgebras and an element b such that no isometric isomorphism is extendable to b . Hence we must look for more precise characterizations of elements of $\mathcal{P}(\omega)/I_h$.

The idea becomes much clearer if we consider a “continuous” case instead of a “discrete” one. Now we are going to explain this “continuous” case.

Unless stated otherwise, let $f, h: R^+ \rightarrow R^+$ denote continuous functions such that $\int_{R^+} f = \int_{R^+} h = \infty$. By Bor we denote the Boolean algebra of all Borel subsets of R^+ and define

$$d_f(a) = \limsup_{x \rightarrow \infty} \frac{\int_{[0, x)} \chi_a \cdot f}{\int_{[0, x)} f} \quad \text{for every } a \in \text{Bor}$$

and

$$I_f = \{a \in \text{Bor} : d_f(a) = 0\}.$$

Analogously we define $d_h(a)$ and I_h .

THEOREM 3.3. *Let f and h be as above. Then Bor/I_f and Bor/I_h are isomorphic.*

There is a very short proof of Theorem 3.3 making no use of CH, but it gives no idea how to prove Theorem 3. We shall sketch here a more sophisticated proof of Theorem 3.3 under the assumption of CH, which we shall “translate” afterwards into a proof of Theorem 3.

SKETCH OF THE PROOF OF THEOREM 3.3. The following definition is crucial in our proof.

DEFINITION. (a) A family $X \subseteq \mathcal{P}(R^+)$ is called an f -partition iff

(i) $X = \{X_m : m \in \omega\}$, $X_0 = [0, x_0)$, $X_{m+1} = [x_m, x_{m+1})$ for $m \in \omega$, and

(ii) $w_m^f = \int_{X_m} f / \int_{[0, x_m)} f \rightarrow 0$.

(b) Let X be an f -partition and $a \in \text{Bor}$. We define a function $B_{f,X}(a) : \omega \rightarrow [0, 1]$, called the f -behaviour of a with respect to X , by

$$B_{f,X}(a)(m) = \int_{X_m} \chi_a f / \int_{X_m} f.$$

Let X be an f -partition.

OBSERVATION 1. For every $a, b \in \text{Bor}$, if $B_{f,X}(a) = B_{f,X}(b)$ then $d_f(a) = d_f(b)$.

OBSERVATION 2. For any f, h there exists an f -partition X and an h -partition Y such that $\int_{X_m} f = \int_{Y_m} h$ for all $m \in \omega$.

Until the end of this sketch we fix f, h, X, Y satisfying the conclusion of Observation 2.

OBSERVATION 3. If $a, b \in \text{Bor}$ and $B_{f,X}(a)(m) = B_{h,Y}(b)(m)$ for all but finitely many $m \in \omega$, then $d_f(a) = d_h(b)$.

OBSERVATION 4. If $P = \{p_m : m \in \omega\}$ is any sequence of reals such that $0 \leq p_m \leq 1$ for all $m \in \omega$, then there exists an $a \in \text{Bor}$ such that $B_{f,X}(a)(m) = p_m$ for all $m \in \omega$. In other words, if P is any possible behaviour, there exists an $a \in \text{Bor}$ f -behaving with respect to X exactly like P .

DEFINITION. Let $A, B \subseteq \text{Bor}$ and $\varphi : A \rightarrow B$ a function. We say that φ is well behaved if it preserves behaviour, i.e. if $B_{f,X}(a)(m) = B_{h,Y}(\varphi(a))(m)$ for all $a \in A$ and all but finitely many m .

OBSERVATION 5. Let $A, B \subseteq \text{Bor}$ be countable, $\varphi : A \rightarrow B$ a well-behaved function, and $\bar{\varphi} : A/I_f \rightarrow B/I_h$ an isomorphism such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \Pi_{I_f} \downarrow & & \downarrow \Pi_{I_h} \\ A/I_f & \xrightarrow{\bar{\varphi}} & B/I_h \end{array}$$

Furthermore, let $a \in \text{Bor}$ be such that $a/I_f \notin A/I_f$. Then there exist $\hat{A} \supset A$ and $\hat{B} \supset B$, a well-behaved function $\psi \supset \varphi$ and an isomorphism $\bar{\psi}: \hat{A}/I_f \rightarrow \hat{B}/I_h$ such that $a \in \hat{A}$ and the following diagram commutes:

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\psi} & \hat{B} \\ \Pi_{I_f} \downarrow & & \downarrow \Pi_{I_h} \\ \hat{A}/I_f & \xrightarrow{\bar{\psi}} & \hat{B}/I_h \end{array}$$

Now let us return to our essential task. Until the end of this section the letters f, g, h will denote *EU*-functions.

As indicated earlier, we shall “translate” the proof of Theorem 3.3. However, the “translation” is not as literal as one might desire and we would like to explain the main differences.

— We shall not redefine the concept of behaviour, for our proof makes no explicit use of it.

— Lemma 3.5 expresses essentially the same fact as Observation 1. We chose a different formulation for technical reasons.

— It is usually not possible for a given behaviour P to find an $a \subset \omega$ behaving “exactly like P ”. So we must look for elements behaving “similarly”. Under this modification Observation 4 may be regarded as a special case of Lemma 3.8.

— The most serious difficulties arise in the translation of Observation 2. Even if we are looking for partitions X and Y such that $\lim_{m \rightarrow \infty} |F(X_m) - H(Y_m)| = 0$, we might be unsuccessful. The following trick helps us in overcoming these difficulties: We fix a function g' and show that $\mathcal{P}(\omega)/I_f \approx \mathcal{P}(\omega)/I_{g'}$ for all f under consideration. This suffices for proving Theorem 3. Moreover, for any fixed f we may choose a function g and a g -partition Y such that $F(X_m) = G(Y_m)$ for all $m \in \omega$, and $d_g(a) = d_{g'}(a)$ for every $a \subset \omega$. Notice that $\mathcal{P}(\omega)/I_g = \mathcal{P}(\omega)/I_{g'}$. Hence we will be done if we prove $\mathcal{P}(\omega)/I_f \approx \mathcal{P}(\omega)/I_g$.

PROOF OF THEOREM 3.

DEFINITION 3.4. A family $X \subseteq \mathcal{P}(\omega)$ is called an h -partition iff:

- (i) $X = \{X_m: m \in \omega\}$, $X_0 = [0, x_0]$, $X_{m+1} = [x_m, x_{m+1})$ for $m \in \omega$;
- (ii) $w_m^h = F(X_m)/F(x_m) \rightarrow 0$;
- (iii) $d_m^h = \max_{n \in X_m} [h(n)/H(X_m)] \rightarrow 0$.

We write w_m and d_m rather than w_m^h and d_m^h if no confusion arises.

Let f be fixed.

LEMMA 3.5. If X is an f -partition then

$$d_f(a) = \limsup_{m \rightarrow \infty} \frac{F(a \cap x_m)}{F(x_m)} \quad \text{for all } a \subset \omega.$$

PROOF. Let $x_m \leq n < x_{m+1}$. Then

$$\begin{aligned} \frac{F(a \cap n)}{F(n)} &= \frac{F(a \cap x_m)}{F(n)} + \frac{F(a \cap [x_m, n))}{F(n)} \\ &\leq \frac{F(a \cap x_m)}{F(x_m)} + \frac{F([x_m, n)) + F([n, x_{m+1}))}{F(n) + F([n, x_{m+1}))} \\ &\leq \frac{F(a \cap x_m)}{F(x_m)} + \frac{F(x_{m+1})}{F(x_{m+1})} = \frac{F(a \cap x_m)}{F(x_m)} + w_{m+1}. \end{aligned}$$

Hence

$$\limsup_{m \rightarrow \infty} \frac{F(a \cap x_m)}{F(x_m)} \geq d_f(a) = \limsup_{n \rightarrow \infty} \frac{F(a \cap n)}{F(n)}. \quad \square$$

LEMMA 3.6. Let $g'(n) = (n + 1)^{-1}$. There exist an f -partition $X = \{X_m; m \in \omega\}$, a g' -partition $Y = \{Y_m; m \in \omega\}$ and a function g such that

- (a) Y is a g -partition,
- (b) $d_g = d_{g'}$,
- (c) $F(X_m) = G(Y_m)$ for all $m \in \omega$.

PROOF.

Stage A: Construction of X . For each natural number $l > 0$ we fix an N_l such that $f(k)/F(k) < 1/2l^2$ for all $k > N_l$. X will be constructed inductively. We put $X_0 = [0, x_0)$ for some x_0 satisfying $F(x_0) > 20$ and $x_0 > N_5$. Having constructed $X_m = [x_{m-1}, x_m)$, we define

$$\begin{aligned} l_m &= \max\{l': N_{l'} < x_m \text{ \& } F(x_m) > 4l'\}, \\ x_{m+1} &= \min\{k: F([x_m, k))/F(k) > 1/2l_m\} \text{ and} \\ X_{m+1} &= [x_m, x_{m+1}). \end{aligned}$$

Stage B: X is an f -partition. Point (i) of Definition 3.4 is trivially satisfied. We check (ii) and (iii). Let $\varepsilon > 0$ be arbitrary. For sufficiently large m the following holds:

$$\begin{aligned} 1/\varepsilon &< l_m \text{ and} \\ w_{m+1} &= \frac{F(X_{m+1})}{F(x_{m+1})} < \frac{F(X_{m+1}) - f(x_{m+1})}{F(x_{m+1} - 1)} + \frac{f(x_{m+1})}{F(x_{m+1})} < \frac{1}{2l_m} + \frac{1}{2l_m^2} < \varepsilon, \\ d_{m+1} &= \max_{n \in X_{m+1}} \frac{f(n)}{F(X_{m+1})} = \max_{n \in X_{m+1}} \frac{f(n)}{F(x_{m+1})} \cdot \frac{F(x_{m+1})}{F(X_{m+1})} < \frac{1}{2l_m^2} \cdot 2l_m < \varepsilon. \end{aligned}$$

Thus $w_m \rightarrow 0$ and $d_m \rightarrow 0$, proving (ii) and (iii).

Stage C: Construction of Y . Let $y_{-1} = 0$, $y_m = \max\{k: \sum_{y_{m-1} \leq n < k} (n + 1)^{-1} \leq F(X_m)\}$ and $Y_m = [y_{m-1}, y_m)$ for all $m \geq 0$.

Stage D: Y is a g' -partition. First note that for all $m \in \omega$,

$$F(X_{m+1}) > \frac{F(x_{m+1})}{2l_m} > \frac{F(x_m)}{2l_m} > \frac{4l_m}{2l_m} = 2.$$

Hence, for all $m \in \omega \setminus \{0\}$,

$$G'(Y_m) > F(X_m) - \frac{1}{y_m + 1} > F(X_m) - 1 > 1.$$

Thus

$$w_m^{g'} = \frac{G'(Y_m)}{G'(y_m)} \leq 2 \frac{F(X_m)}{F(x_m)} = 2w_m^f \rightarrow 0$$

and

$$d_{m+1}^{g'} = \max_{n \in Y_{m+1}} \frac{g'(n)}{G'(Y_{m+1})} = \frac{g'(n)}{G'(Y_{m+1})} \leq \frac{1}{y_m + 1} \rightarrow 0.$$

Stage E: Definition of g satisfying (a) and (c). We define

$$g(n) = \begin{cases} g'(n) + F(X_m) - G'(Y_m) & \text{if } n = y_{m-1} \text{ for some } m \geq 0, \\ g'(n) & \text{otherwise.} \end{cases}$$

g satisfies (c) by definition; we show that it satisfies (a):

$$\begin{aligned} w_m^g &= \frac{G(Y_m)}{G(y_m)} = \frac{F(X_m)}{F(x_m)} = w_m^f \rightarrow 0; \\ d_m^g &= \max_{n \in Y_m} \frac{g(n)}{G(Y_m)} \leq \frac{g'(y_{m-1})}{G(Y_m)} + \frac{F(X_m) - G'(Y_m)}{G(Y_m)} \\ &< \frac{1}{y_{m-1} + 1} + \frac{1}{y_m + 1} \rightarrow 0. \end{aligned}$$

Stage F: g satisfies (b). Let $a = \{y_m : m \geq -1\}$. First we show that $d_g(a) = d_{g'}(a) = 0$. Fix $\varepsilon > 0$ and let M_0, M_1 be such that for all $k \geq M_0$,

$$\frac{g(y_k)}{G(Y_{k+1})} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{G(y_{M_0})}{G(y_{M_1})} < \frac{\varepsilon}{2}.$$

For $m > M_1$ the following holds:

$$\begin{aligned} \frac{G(a \cap y_m)}{G(y_m)} &\leq \frac{G(y_{M_0})}{G(y_{M_1})} + \frac{g(y_{M_0}) + \cdots + g(y_{m-1})}{G(Y_{M_0+1}) + \cdots + G(Y_m)} \\ &\leq \frac{\varepsilon}{2} + \max_{M_0 \leq k < m} \frac{g(y_k)}{G(Y_{k+1})} \leq \varepsilon. \end{aligned}$$

By Lemma 3.5 $d_g(a) = 0$. An analogous proof shows that $d_{g'}(a) = 0$.

Now let b be such that $b \cap a = \emptyset$. We shall be done if we show that $d_g(b) = d_{g'}(b)$ for such b . First we notice that $G(b \cap Y_m) = G'(b \cap Y_m)$ for every m . Hence

$$\begin{aligned} \frac{G(b \cap y_m)}{G(y_m)} - \frac{G'(b \cap y_m)}{G'(y_m)} &= G(b \cap y_m) \frac{G'(y_m) - G(y_m)}{G(y_m) \cdot G'(y_m)} \\ &= G(b \cap y_m) \frac{G'(a \cap y_m) - G(a \cap y_m)}{G(y_m) \cdot G'(y_m)} \\ &\leq \frac{G(b \cap y_m)}{G(y_m)} \cdot \frac{G'(a \cap y_m)}{G'(y_m)} + \frac{G'(b \cap y_m)}{G'(y_m)} \cdot \frac{G(a \cap y_m)}{G(y_m)} \\ &\leq \frac{G'(a \cap y_m)}{G'(y_m)} + \frac{G(a \cap y_m)}{G(y_m)} \rightarrow d_{g'}(a) + d_g(a) = 0. \end{aligned}$$

For an arbitrary $c \subseteq \omega$ we have

$$d_g(c) = d_g(c \setminus a) = d_{g'}(c \setminus a) = d_{g'}(c).$$

This concludes the proof of Lemma 3.6. \square

For the rest of the proof we fix g , $X = \{X_m : m \in \omega\}$ and $Y = \{Y_m : m \in \omega\}$, satisfying the conclusion of Lemma 3.6.

LEMMA 3.7. *Let $a, b \subset \omega$ be such that*

$$\lim_{m \rightarrow \infty} \left(\frac{F(a \cap X_m)}{F(X_m)} - \frac{G(b \cap Y_m)}{G(Y_m)} \right) = 0.$$

Then $d_f(a) = d_g(b)$.

PROOF. We fix $\varepsilon > 0$ and choose $N \in \omega$ large enough that

$$\left| \frac{F(a \cap X_m)}{F(X_m)} - \frac{G(b \cap Y_m)}{G(Y_m)} \right| < \frac{\varepsilon}{2} \quad \text{for all } m > N.$$

Further, we choose $M \in \omega$ such that $F(x_N)/F(x_M) < \varepsilon/2$.

Then for $m > M$ the following holds:

$$\begin{aligned} \left| \frac{F(a \cap x_m)}{F(x_m)} - \frac{G(b \cap y_m)}{G(y_m)} \right| &= \frac{|F(a \cap x_m) - G(b \cap y_m)|}{F(x_m)} \\ &\leq \frac{|F(a \cap x_N) - G(b \cap y_N)|}{F(x_M)} \\ &\quad + \frac{|F(a \cap X_{N+1}) - G(b \cap Y_{N+1})| + \cdots + |F(a \cap X_m) - G(b \cap Y_m)|}{F(X_{N+1}) + \cdots + F(X_m)} \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $d_f(a) = d_g(b)$ by Lemma 3.5. \square

LEMMA 3.8. Let b_0, \dots, b_{k-1} be pairwise almost disjoint subsets of ω , Z an h -partition, u_m^l nonnegative reals such that

$$\lim_{m \rightarrow \infty} \left| \frac{H(b_l \cap Z_m)}{H(Z_m)} - u_m^l \right| = 0 \quad \text{for all } l < k.$$

Moreover, let $0 \leq v_m^l \leq u_m^l$ for $l < k$ and $m \in \omega$. Then there exists a $b \subset \omega$ such that

$$\lim_{m \rightarrow \infty} \left| \frac{H(b \cap b_l \cap Z_m)}{H(Z_m)} - v_m^l \right| = 0,$$

and if $v_m^l = 0$, then $b \cap b_l \cap Z_m = \emptyset$.

PROOF. We choose $M \in \omega$ large enough that the sets $b_1 \cap (\omega \setminus Z_M), \dots, b_k \cap (\omega \setminus Z_M)$ are pairwise disjoint and put $b \cap Z_M = \emptyset$. Then we define successively $b \cap Z_m$ for $m > M$ so that $|H(b \cap b_l \cap Z_m)/H(Z_m) - v_m^l|$ is as small as possible for all $l < k$. It is easily seen that for $m > M$, $l < k$:

$$\left| \frac{H(b \cap b_l \cap Z_m)}{H(Z_m)} - v_m^l \right| \leq \left| \frac{H(b_l \cap Z_m)}{H(Z_m)} - u_m^l \right| + \max_{n \in Z_m} \frac{h(n)}{H(n)}.$$

By Definition 3.4, b is as required. \square

Now we are going to define a well-behaved isomorphism. As indicated in §0, we shall not construct the isomorphism on elements of $\mathcal{P}(\omega)/I_f$, but on a selector.

DEFINITION 3.9. (a) We call $A \subset \mathcal{P}(\omega)$ a splitting subset for I_h and write $A \in SSI_h$ iff A is a selector for A/I_h and there exists a split $\pi: A/I_h \rightarrow A/\text{Fin}$.

(b) Let $A \in SSI_f$ and $B \in SSI_g$. We say that a function $\varphi: A \rightarrow B$ is well behaved iff

$$\lim_{m \rightarrow \infty} \left| \frac{F(a \cap X_m)}{F(X_m)} - \frac{G(\varphi(a) \cap X_m)}{G(X_m)} \right| = 0 \quad \text{for all } a \in A.$$

(c) A well-behaved function φ is called a homomorphism (isomorphiss) iff there exist homomorphisms (isomorphisms) $\bar{\varphi}$ and $\hat{\varphi}$ such that the following diagram commutes:

$$\begin{array}{ccc} A/I_f & \xrightarrow{\bar{\varphi}} & B/I_g \\ \pi_A \downarrow & & \downarrow \pi_B \\ A/\text{Fin} & \xrightarrow{\hat{\varphi}} & B/\text{Fin} \\ S_A \downarrow & & \downarrow S_B \\ A & \xrightarrow{\varphi} & B \end{array}$$

Here π_A, π_B denote splits, and S_A, S_B choice functions.

REMARK 3.10. In the previous definition $\pi, \bar{\varphi}$ and $\hat{\varphi}$ are unique. Hence in point (c) it suffices to require the existence of $\bar{\varphi}$.

Claim 3.11. Let $\varphi: A \rightarrow B$ be a well-behaved homomorphism. Then φ is a monomorphism and an isometry.

PROOF. By Lemma 3.7, $d_f(a) = d_g(\varphi(a))$ for every $a \in A$. Hence $\text{Ker } \varphi = 0$. \square

As indicated in §0, Theorem 3 would be proved if we show how to extend a well-behaved isomorphism between countable $A, B \subset \mathcal{P}(\omega)$ over any given $a/I_f \in \mathcal{P}(\omega)/I_f$ to a well-behaved isomorphism. One may doubt whether we should prove additionally that φ^{-1} may be extended over any given b/I_g , for our assumptions about f and g are not symmetric. But the proof of the following lemma makes no use of additional properties of g .

LEMMA 3.12. Assume $A, B \subset \mathcal{P}(\omega)$, $\varphi: A \rightarrow B$ is a well-behaved isomorphism, $|A| \leq \aleph_0$, and $\hat{a} \in \mathcal{P}(\omega)$. Then there exist an $a \in \hat{a}/I_f$, countable $\hat{A}, \hat{B} \subset \mathcal{P}(\omega)$ and a well-behaved isomorphism $\psi: \hat{A} \rightarrow \hat{B}$ such that $a \in \hat{A}$, $A \subset \hat{A}$, $B \subset \hat{B}$ and $\psi \upharpoonright A = \varphi$.

PROOF. By Definition 3.9(a), $A \in \text{SSI}_f$ and $B \in \text{SSI}_g$. Hence there exist splits π_A and π_B and choice functions S, S', S_A, S_B such that the following diagrams commute:

$$\begin{array}{ccc} A/I_f & \xrightarrow{S} & A \\ \pi_A \downarrow & & \uparrow S_A \\ A/\text{Fin} & \xrightarrow{S_A} & A \end{array} \quad \begin{array}{ccc} B/I_g & \xrightarrow{S'} & B \\ \pi_B \downarrow & & \uparrow S_B \\ B/\text{Fin} & \xrightarrow{S_B} & B \end{array}$$

Furthermore, let $\bar{\varphi}, \hat{\varphi}$ denote isomorphisms such that the following diagram commutes:

$$\begin{array}{ccc} A/I_f & \xrightarrow{\bar{\varphi}} & B/I_g \\ \pi_A \downarrow & & \downarrow \pi_B \\ A/\text{Fin} & \xrightarrow{\hat{\varphi}} & B/\text{Fin} \\ S_A \downarrow & & \downarrow S_B \\ A & \xrightarrow{\varphi} & B \end{array}$$

Let $A' = [A, \hat{a}]_{I_f}$. By Theorem 2' there exists a split $\tau_A: A' \rightarrow \mathcal{P}(\omega)/\text{Fin}$ extending π_A . We choose choice functions T_A and T extending S_A and S , respectively, such that the following diagram commutes:

$$\begin{array}{ccc} A' & \xrightarrow{T} & \mathcal{P}(\omega) \\ \tau_A \downarrow & & \uparrow T_A \\ \mathcal{P}(\omega)/\text{Fin} & \xrightarrow{T_A} & \mathcal{P}(\omega) \end{array}$$

and put $\hat{A} = T(A')$, $a = T(\hat{a}/I_f)$.

Let $A/I_f = \bigcup_{i \in \omega} \bar{A}_i$, where $\bar{A}_i \subset \bar{A}_{i+1}$ and the \bar{A}_i 's are finite subalgebras. Let $\bar{a}_0^i, \dots, \bar{a}_{k(i)}^i$ be the atoms of \bar{A}_i . For $i \in \omega, j \leq k(i)$ we denote

$$a_j^i = S(\bar{a}_j^i), \quad b_j^i = \varphi(a_j^i).$$

By our assumption, for any given i the sets $a_0^i, \dots, a_{k(i)}^i$ (respectively $b_0^i, \dots, b_{k(i)}^i$) are pairwise almost disjoint.

By Lemma 3.8 we may choose b_i such that for $j \leq k(i)$:

$$(1) \quad O_m^{i,j} = \left| \frac{F(a \cap a_j^i \cap X_m)}{F(X_m)} - \frac{G(b_i \cap b_j^i \cap Y_m)}{G(Y_m)} \right| \xrightarrow{m \rightarrow \infty} 0,$$

$$(2) \quad a \cap a_j^i \cap X_m = \emptyset \Rightarrow b_i \cap b_j^i \cap Y_m = \emptyset.$$

Since φ is well-behaved, we have also

$$(3) \quad \bar{O}_m^{i,j} = \left| \frac{F(-a \cap a_j^i \cap X_m)}{F(X_m)} - \frac{G(-b_i \cap b_j^i \cap Y_m)}{G(Y_m)} \right| \xrightarrow{m \rightarrow \infty} 0.$$

Now let M_i be large enough that for $m \geq M_i$ and $j \leq k(i)$ the following holds:

$$O_m^{i,j} < \frac{1}{i \cdot k(i)}, \quad \bar{O}_m^{i,j} < \frac{1}{i \cdot k(i)}$$

and the operations of $T([A_i, a]_{I_f})$ and B_i restricted to $\omega \setminus x_{M_i}$ (respectively $\omega \setminus y_{M_i}$) coincide with the set-theoretical operations (i.e. $c \vee d \upharpoonright (\omega \setminus x_{M_i}) = c \upharpoonright (\omega \setminus x_{M_i}) \cup d \upharpoonright (\omega \setminus x_{M_i})$, etc.).

Let $b = \bigcup_{i \in \omega} b_i \cap [y_{M_i}, y_{M_{i+1}})$ and $B' = [B, b]_{I_g}$. By Theorem 2' and (2) there exist a split τ_B extending π_B and choice functions T', T_B such that $T'(b/I_g) = b$ and the following diagram commutes:

$$\begin{array}{ccc} B' & & \\ \tau_B \downarrow & \searrow T' & \\ \mathcal{P}(\omega)/\text{Fin} & \xrightarrow{T_B} & \mathcal{P}(\omega) \end{array}$$

We define $\hat{B} = T'(B')$. Now for every $c \in \hat{A}$ we fix $a_1, a_2 \in A$ such that $c = (a_1 \wedge a) \vee (a_2 \wedge -a)$ and define

$$\psi(c) = (\varphi(a_1) \wedge b) \vee (\varphi(a_2) \wedge -b).$$

SUBLEMMA 3.13. *For every $c \in A$,*

$$\lim_{m \rightarrow \infty} \left| \frac{F(c \cap a \cap X_m)}{F(X_m)} - \frac{G(\varphi(c) \cap b \cap Y_m)}{G(Y_m)} \right| = 0.$$

PROOF. Let $c \in A$ and $\varepsilon > 0$. There is an l such that $1/l < \varepsilon$ and $c \in A_l$.

For $m > M_l$ there exists an $i \geq l$ such that $M_i < m \leq M_{i+1}$. Obviously, $c \in A_i$. Hence $c = d_0 \vee \dots \vee d_u$, where $d_j = a_k^i$ for some $k \leq k(i)$ and $u \leq k(i)$. By the definition of M_i and b we have $(d_j \wedge a) \cap X_m = d_j \cap a \cap X_m$, $(\varphi(d_j) \wedge b) \cap Y_m = \varphi(d_j) \cap b_i \cap Y_m$ and

$$\begin{aligned} & \left| \frac{F(c \cap a \cap X_m)}{F(X_m)} - \frac{G(\varphi(c) \cap b \cap Y_m)}{G(Y_m)} \right| \\ & \leq \sum_{j=0}^u \left| \frac{F(d_j \cap a \cap X_m)}{F(X_m)} - \frac{G(\varphi(d_j) \cap b \cap Y_m)}{G(Y_m)} \right| \\ & < \sum_{j=0}^u \frac{1}{i \cdot k(i)} \leq \frac{1}{i} < \frac{1}{l} < \varepsilon. \end{aligned}$$

This concludes the proof of Sublemma 3.13. \square

COROLLARY 3.14. For every $c \in A$,

$$\lim_{m \rightarrow \infty} \left| \frac{F(c \cap -a \cap X_m)}{F(X_m)} - \frac{G(\varphi(c) \cap -b \cap Y_m)}{G(Y_m)} \right| = 0. \quad \square$$

COROLLARY 3.15. For every $c \in \hat{A}$, $a_1, a_2, \bar{a}_1, \bar{a}_2 \in A$, if $c = (a_1 \wedge a) \vee (a_2 \wedge -a) = (\bar{a}_1 \wedge a) \vee (\bar{a}_2 \wedge -a)$, then

$$(\varphi(a_1) \wedge b) \vee (\varphi(a_2) \wedge -b) = (\varphi(\bar{a}_1) \wedge b) \vee (\varphi(\bar{a}_2) \wedge -b). \quad \square$$

Thus by Lemma 0.3 ψ as defined above is a homomorphism.

COROLLARY 3.16. For every $c \in \hat{A}$,

$$\lim_{m \rightarrow \infty} \left| \frac{F(c \cap X_m)}{F(X_m)} - \frac{G(\psi(c) \cap Y_m)}{G(Y_m)} \right| = 0. \quad \square$$

Thus ψ is well behaved and by Claim 3.11 ψ is a well-behaved monomorphism. Moreover, ψ is an epimorphism by the definition of \hat{B} . Lemma 3.12 is therefore proved. \square

We conclude this section with an interesting corollary of Theorem 3.

COROLLARY 3.17 (CH). For every EU-function f the algebra $\mathcal{P}(\omega)/I_f$ is homogeneous.

PROOF. Let $h(n) = 1$ for all n . Obviously, $\mathcal{P}(\omega)/I_h$ is homogeneous. By Theorem 3, $\mathcal{P}(\omega)/I_f$ is homogeneous for every EU-function f . \square

4. Remarks on completeness. In this section we prove that an algebra $\mathcal{P}(\omega)/I$ is incomplete or even not \aleph_1 -complete provided that the ideal I has some nice properties.

The following two propositions were proved by Sierpiński (see [3]). We include their proofs for completeness. By I^* we denote the filter dual to the ideal I .

PROPOSITION 4.1. If I has the property of Baire, then I is of first category.

PROOF. Suppose the contrary. Then there exists a basic set U_s such that $U_s \setminus I$ is of first category. By $\text{Fin} \subset I$ this implies $\mathcal{P}(\omega) \setminus I$ is of first category. On the other hand there exists a homeomorphism $\varphi: \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ such that $\mathcal{P}(I) = I^* \subset \mathcal{P}(\omega) \setminus I$. Hence I is of first category, contradiction. \square

PROPOSITION 4.2. Assume m is the measure generated by $m(U_s) = 2^{-lh(s)}$ for $s \in 2^{<\omega}$. If I is m -measurable, then $m(I) = 0$.

PROOF. Let $m(I) = x$. By $\text{Fin} \subset I$ we have $m(U_s \cap I) = x \cdot m(U_s) = x \cdot 2^{-lh(s)}$ for $s \in 2^{<\omega}$. This easily implies $m(V \cap I) = x \cdot m(V)$ for all open V . Let $\{V_i\}_{i \in \omega}$ be a sequence of open supersets of I such that $\lim_{i \rightarrow \infty} m(V_i) = m(I) = x$. Then

$$x = \lim_{i \rightarrow \infty} m(I \cap V_i) = x \cdot \lim_{i \rightarrow \infty} m(V_i) = x^2.$$

On the other hand, $x \leq \frac{1}{2}$ by $m(I) = m(I^*)$ and $I \cap I^* = \emptyset$. Hence $x = 0$. \square

The interesting question is: does there exist an analytic ideal on ω such that the algebra $\mathcal{P}(\omega)/I$ is complete? As we will show the answer is *no*.

The following lemma may be regarded as a corollary of Théorème 2.1(ii) in [4]. The reader interested in measurable ideals (filters) and such having the Baire property may find more interesting facts about them in that paper.

LEMMA 4.3. *Let I be an ideal of first category. Then there exist an element $c \in \mathcal{P}(\omega)$ and a monomorphism $h: \mathcal{P}(\omega)/\text{Fin} \rightarrow \mathcal{P}(c)/I$.*

PROOF. Let $I \subset \bigcup_n Y_n$ be such that $Y_n \subset Y_{n+1}$ for all $n \in \omega$ and the Y_n 's are closed nowhere dense subsets of $\mathcal{P}(\omega)$. By an easy induction one can construct sequences $\{x_n\}_{n \in \omega}$ and $\{d_n\}_{n \in \omega}$ satisfying the following conditions:

- (i) $x_0 = 0, x_n \in \omega$;
- (ii) $d_n \subset [x_n, x_{n+1})$;
- (iii) $d_n \neq \emptyset$;
- (iv) $U_{(s \cup d_n) \upharpoonright x_{n+1}} \subset \mathcal{P}(\omega) \setminus Y_n$ for every $s \subset [0, x_n)$.

Now put $c = \bigcup_{n \in \omega} d_n$ and $h(a/\text{Fin}) = (\bigcup_{n \in a} d_n)/I$. Clearly h is a well-defined homomorphism of Boolean algebras. By (iv), if a is infinite, then $h(a/\text{Fin}) \notin Y_n$ for any n ; hence $h(a/\text{Fin}) \notin I$. It turns out that h is a monomorphism. \square

THEOREM 4.4. *If I is of first category, then $\mathcal{P}(\omega)/I$ is not complete.*

PROOF. Let A be a set of pairwise incompatible elements of $\mathcal{P}(\omega)/\text{Fin}$ of power 2^{\aleph_0} . Take h and c as in the conclusion of Lemma 4.3. The set $B = h(A)$ has power 2^{\aleph_0} . The elements of B are incompatible in $\mathcal{P}(c)/I$, hence they are incompatible in $\mathcal{P}(\omega)/I$. By the cardinality argument there exists a subset of B without a supremum in $\mathcal{P}(\omega)/I$. \square

THEOREM 4.5. *If I has property Δ and is of first category, then $\mathcal{P}(\omega)/I$ is not σ -complete.*

PROOF. We keep the notation from the proof of Lemma 4.3. Take a sequence $\{a_n\}_{n \in \omega}$ such that for $n \in \omega$ we have $a_n \subset a_{n+1}$, $a_{n+1} \setminus a_n \notin I$ and a_n is a union of d_m 's. Let $a_n \setminus b \in I$ for each $n \in \omega$. By Lemma 2.1 there exists a representative \bar{b} of b/I such that $a_n \setminus \bar{b} \in \text{Fin}$ for each $n \in \omega$. Now take d_{k_n} such that $d_{k_n} \subset (a_{n+1} \setminus a_n) \cap \bar{b}$. Obviously $b' = \bar{b} \setminus \bigcup_{n \in \omega} d_{k_n}$ is such that $a_n \setminus b' \in I$ and $b \setminus b' \notin I$. Hence the sequence $\{a_n\}_{n \in \omega}$ has no supremum. \square

5. Problems. We conclude with a list of open problems.

A. Can one remove CH from the assumptions of Theorems 1, 2 and 3?

We suggest that the answer is *no*. The following seems to be easier.

B. Reprove Theorems 1 and 3 using MA instead of CH.

Notice that at least in the case of Theorem 1 some new ideas are necessary, because by the existence of so-called $\omega_1 \omega_1^*$ -gaps, $\mathcal{P}(\omega)/\text{Fin}$ is not ω_2 -saturated.

It is easy to construct an ideal $I \in F_{\sigma\delta} \setminus F_\sigma$ such that the algebra $\mathcal{P}(\omega)/I$ is not homogeneous. By Corollary 3.17, $\mathcal{P}(\omega)/I \neq \mathcal{P}(\omega)/I_f$ for any EU-function f , at least if CH holds.

C. Is it true that if $I, J \in F_{\sigma\delta} \setminus F_\sigma$ and $\mathcal{P}(\omega)/I, \mathcal{P}(\omega)/J$ are homogeneous, then $\mathcal{P}(\omega)/I \simeq \mathcal{P}(\omega)/J$?

D. Is the assumption that the ideal I has property Δ necessary in Theorem 4.5?

REFERENCES

1. E. K. van Douwen, J. D. Monk and M. Rubin, *Some questions about Boolean algebras*, Algebra Universalis **11** (1980), 220–243.
2. P. Erdős, *My Scottish book "problems"*, The Scottish Book (D. Mauldin, ed.), Birkhäuser, Boston, Mass., 1981.
3. W. Sierpiński, *Hypothèse du continu*, Monografie Mat., Vol. 4, Warszawa-Lwów, 1934.
4. M. Talagrand, *Compacts de fonctions mesurables et filtres non mesurables*, Studia Math. **67** (1980), 13–43.

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