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ON THE STRUCTURE OF REAL TRANSITIVE LIE ALGEBRAS

BY

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ABSTRACT. In this paper, we examine some of the ways in which abstract algebraic objects in a transitive Lie algebra L are expressed geometrically in the action of each transitive Lie pseudogroup Γ associated to L. We relate those chain decompositions of Γ which result from considering Γ-invariant foliations to Jordan-Hölder sequences (in the sense of Cartan and Guillemin) for L. Local coordinates are constructed which display the nature of the partial differential equations defining Γ; in particular, locally homogeneous pseudocomplex structures (also called CR-structures) are associated to the nonabelian quotients of complex type in a Jordan-Hölder sequence for L.

Introduction. From its classical origins in the works of Sophus Lie and Élie Cartan, the study of pseudogroups of local diffeomorphisms preserving geometric structures on manifolds has attracted many authors. It now is known **[7]** that the local structure of a transitive analytic pseudogroup is determined by an associated topological Lie algebra, known as a transitive Lie algebra, in much the same way that the local structure of a finite-dimensional Lie group is determined by its Lie algebra of invariant vector fields. In this paper, we shall examine some of the ways in which abstract algebraic objects in transitive Lie algebras are expressed geometrically. We shall be interested particularly in the decompositions of a transitive Lie algebra which result from considering invariant foliations.

The work of H. Goldschmidt and D. C. Spencer **[10]** allows us to study the integrability problem for transitive pseudogroup structures through a study of the structure and realizations of closed ideals in transitive Lie algebras. Our work was undertaken with this goal in mind, and our results, particularly those which describe the structure of nonabelian minimal closed ideals, will play a role in the forthcoming solution by H. Goldschmidt **[9]** of the integrability problem for all pseudogroups which contain the translations. A simple example which illustrates the way in which the pseudocomplex structures we associate to nonabelian minimal closed ideals of complex type affect the integrability problem can be found in **[4]**.

Viewed abstractly, a transitive Lie algebra over a ground field K (which we shall assume to be either the real number field $\mathbf{R}$ or the field $\mathbf{C}$ of complex numbers, and endowed with the discrete topology) is a topological Lie algebra L whose underlying topological vector space is linearly compact, and which possesses a fundamental subalgebra, that is, an open subalgebra L^0 which contains no ideals of L except $\{0\}$.

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From the standpoint of formal geometry, transitive Lie algebras arise in the following way: Form the ring $F = K[[x_1, \dots, x_n]]$ of formal power series in n indeterminates over K, and consider the Lie algebra $\mathrm{Der}(F)$ of derivations of F, that is, expressions of the form

$$\xi = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial x_j}, \qquad \xi_j \in F,$$

which we view as formal vector fields; the Lie bracket of two derivations ξ, $\eta \in \mathrm{Der}(F)$ is written, then, as

$$[\xi, \eta] = \sum_{1 \leqslant i, j \leqslant n} \left(\xi_i \frac{\partial \eta_j}{\partial x_i} - \eta_i \frac{\partial \xi_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

For each $p \geqslant 0$, let $\mathrm{Der}^p(F)$ denote the subalgebra made up of those formal vector fields ξ whose coefficients ξ_j all vanish to order p at the origin. In the topology defined by taking $\{\mathrm{Der}^p(F)\}_{p \geqslant 0}$ as a fundamental system of neighborhoods of 0, the space $\mathrm{Der}(F)$ is a linearly compact topological Lie algebra. A subalgebra L of $\mathrm{Der}(F)$ is called transitive if it is topologically closed in $\mathrm{Der}(F)$ and satisfies the relation

$$\mathrm{Der}(F) = L + \mathrm{Der}^0(F).$$

Each transitive subalgebra L of $\mathrm{Der}(F)$ is, in the induced topology, an abstract transitive Lie algebra, and the isotropy subalgebra

$$L^0 = L \cap \mathrm{Der}^0(F)$$

of L is fundamental. Conversely, a theorem of Guillemin and Sternberg **[15]** asserts that, given a transitive Lie algebra L and a fundamental subalgebra $L^0 \subset L$, one can realize L as a transitive subalgebra of formal vector fields in such a way that L^0 is realized as the isotropy subalgebra of L; such a realization of (L, L^0) is unique, up to the action of a formal change of coordinates. We shall concentrate here upon the way in which such realizations of a transitive Lie algebra L reflect the structure of L as an abstract topological Lie algebra.

Closed ideals are among the most important algebraic objects attached to a transitive Lie algebra. In **[12]**, Guillemin proved that a linearly compact topological Lie algebra L is transitive if and only if L satisfies the descending chain condition on its closed ideals. Using this result, Guillemin went on to establish the existence, for each closed ideal I in a transitive Lie algebra L, of a Jordan-Hölder decomposition for (L, I). This consists of a finite descending chain

$$I = I_0 \supset I_1 \supset \cdots \supset I_p = \{0\}$$

of closed ideals of L, such that, for each index l with $0 \leqslant l \leqslant p - 1$, one of the following alternatives holds:

(i) The quotient I_l/I_{l+1} is abelian.

(ii) The quotient I_l/I_{l+1} is nonabelian, and there are no closed ideals of L properly contained between I_l and I_{l+1}.

The number and isomorphism classes of quotients of type (ii), both as topological Lie algebras and topological L-modules, were shown by Guillemin to be the same in any two Jordan-Hölder sequences for (L, I). In the category of transitive pseudogroups, it had been conjectured by É. Cartan that such a decomposition would always exist.

The geometric setting in which we study closed ideals and Jordan-Hölder sequences is described below. As before, set $F = K[[x_1,\dots,x_n]]$, and let L be a transitive subalgebra of $\mathrm{Der}(F)$ with isotropy subalgebra L^0. A class of geometric objects naturally associated to L is the collection of invariant formal foliations. By an invariant foliation, we mean a unitary subalgebra $G \subset F$ which remains invariant under the action of L, and which can be expressed, after a formal change of coordinates $(x_1,\dots,x_n) \mapsto (y_1,\dots,y_m, z_1,\dots,z_{n-m})$, as

$$G = K[[y_1,\dots,y_m]] \subset K[[y_1,\dots,y_m, z_1,\dots,z_{n-m}]] = F,$$

for some integer m with $0 \leqslant m \leqslant n$. In terms of such coordinates, each element ξ of L takes the form

$$\xi = \sum_{j=1}^{m} \xi_j(y_1,\dots,y_m)\frac{\partial}{\partial y_j} + \sum_{j=1}^{n-m} \xi_{m+j}(y_1,\dots,y_m; z_1,\dots,z_{n-m})\frac{\partial}{\partial z_j};$$

that is, the first m coefficients $\xi_1,\dots,\xi_m$ of ξ lie in G. By considering the action of L on G, we define a homomorphism π_G: $L \to \mathrm{Der}(G)$ whose image $\pi_G(L)$ is a transitive subalgebra of $\mathrm{Der}(G)$. The collection of formal vector fields ξ in L for which $\xi_1,\dots,\xi_m$ vanish at the origin is a subalgebra N_G of L containing L^0, and N_G is the inverse image in L of the isotropy subalgebra of $\pi_G(L)$. Those elements ξ of L such that $\xi_1 = \cdots = \xi_m = 0$ make up the kernel I_G of π_G; this ideal I_G is the union of all ideals of L contained in N_G. We remark that, given an abstract transitive Lie algebra L and a closed ideal $I \subset L$, there exists a realization of L as a transitive subalgebra of $\mathrm{Der}(F)$, for some formal power series ring F, such that there is an L-invariant foliation $G \subset F$ with $I = I_G$. Let H denote the formal power series ring $K[[z_1,\dots,z_{n-m}]]$. Substituting $y_1 = \cdots = y_m = 0$ in the coefficients $\xi_1,\dots,\xi_n$ of a formal vector field ξ in N_G, we obtain an element

$$\tilde{\xi} = \sum_{j=1}^{n-m} \xi_{m+j}(0,\dots,0; z_1,\dots,z_{n-m})\frac{\partial}{\partial z_j}$$

of $\mathrm{Der}(H)$; if U is a subspace of N_G, we set $\tilde{U} = \{\tilde{\xi} | \xi \in U\}$. This mapping $\xi \mapsto \tilde{\xi}$ is a homomorphism $\tilde{\pi}$: $N_G \to \mathrm{Der}(H)$ whose image $\tilde{N}_G$ is a transitive subalgebra of $\mathrm{Der}(H)$, with isotropy subalgebra $\tilde{L}^0$. Phrased geometrically, the simultaneous level sets of $y_1,\dots,y_m$ define a foliation $\mathscr{G}$ of K^n, and N_G consists of those formal vector fields ξ in L which are tangent to the leaf of $\mathscr{G}$ which passes through the origin; the restriction of ξ to this leaf is what we have called $\tilde{\xi}$. We may express each element of L in the form

$$\begin{aligned}\xi &= \sum_{j=1}^{m} \xi_j(y_1,\dots,y_m)\frac{\partial}{\partial y_j} + \sum_{\alpha\in\mathbf{N}^m} \xi_\alpha y^\alpha \\ &= \pi_G(\xi) + \xi_H(y_1,\dots,y_m),\end{aligned}$$

by rewriting the terms $\sum_{j=1}^{n-m}\xi_{m+j}(y_1,\dots,y_m; z_1,\dots,z_{n-m})(\partial/\partial z_j)$ of ξ as a formal series expansion ξ_H in $y_1,\dots,y_m$ with coefficients $\xi_\alpha \in \mathrm{Der}(H)$; for $\xi \in N_G$, the leading term of ξ_H is $\tilde{\xi}$. The formal coordinates $z_1,\ldots,z_{n-m}$ can be chosen (cf. Corollary 2.1) so that all coefficients ξ_α of each $\xi \in L$ lie in $\tilde{N}_G$, and if I is a closed ideal of L contained in I_G, then each $\xi \in I$ has its coefficients ξ_α lying in the closed ideal $\tilde{I}$ of $\tilde{N}_G$. These series expansions ξ_H reflect the nature of the linear partial differential equations defining I_G; we shall characterize them much more precisely in several special cases of interest.

Given a transitive subalgebra L of $\mathrm{Der}(F)$ and an L-invariant formal foliation $G \subset F$, there is a composition series for (L, I_G) suggested naturally by the action of L on F: Choose a chain of L-invariant foliations

$$G = F_0 \subset F_1 \subset \cdots \subset F_p = F$$

such that each F_j is maximal among the invariant foliations contained in F_{j+1}, for $0 \leqslant j \leqslant p-1$, and consider the sequence

$$I_G = I_{F_0} \supset I_{F_1} \supset \cdots \supset I_{F_p} = \{0\}$$

of closed ideals of L defined by these foliations; it is evident that such a chain must be finite. In an unpublished work, Guillemin **[14]** described how this composition series could be refined to produce a Jordan-Hölder sequence for (L, I_G). Using an important lemma from **[14]** and the classification of primitive Lie algebras, we present, in Theorem 4.1, an explicit decomposition of each quotient $I_{F_j}/I_{F_{j+1}}$, which yields a Jordan-Hölder series for (L, I_G). Following this, in Theorems 4.2–4.6, we show how our decomposition of $I_{F_j}/I_{F_{j+1}}$ is portrayed geometrically, by introducing formal coordinates which display clearly the structure of those ideals which make up our decomposition, and the action of L upon them. A summary of these results appears below; to avoid repetition, we assume, unless otherwise stated, that all Lie algebras considered in our outline are defined over the real number field $\mathbf{R}$.

A few preliminary observations will be helpful. Each quotient $I_{F_j}/I_{F_{j+1}}$ is an ideal in the transitive subalgebra $\pi_{F_{j+1}}(L) \simeq L/I_{F_{j+1}}$ of $\mathrm{Der}(F_{j+1})$, and is defined by the maximal invariant foliation F_j in F_{j+1}. Thus, in studying these quotients $I_{F_j}/I_{F_{j+1}}$, we shall suppose, without loss of generality, that G is a maximal L-invariant foliation in F. This is equivalent to assuming that the transitive subalgebra $\tilde{N}_G$ of $\mathrm{Der}(H)$ is primitive; that is, the action of $\tilde{N}_G$ leaves no formal foliation invariant in H, except for the scalars K and H itself. Under this assumption, the closed ideal $\tilde{I}_G$ is of finite codimension in $\tilde{N}_G$, and is itself a transitive subalgebra of $\mathrm{Der}(H)$ unless $I_G = \{0\}$. A complete classification is known for the primitive Lie algebras; we shall categorize I_G according to the type of $\tilde{N}_G$ in this classification. In our accompanying decriptions of the realization of L and I_G in appropriate coordinate systems, we employ an extension of Schur's lemma which appears in **[12]**: If R is a simple transitive Lie algebra over K, then the commutator ring K_R of R, that is, the algebra of K-linear mappings $R \to R$ which commute with the adjoint representation of R, is a (commutative) field which is a finite algebraic extension of K, and R is a transitive Lie algebra over K_R. Having stipulated that K is the field $\mathbf{R}$ of real numbers, we must have K_R equal to $\mathbf{R}$ or $\mathbf{C}$. For notational convenience, if L' is a closed Lie subalgebra of $\mathrm{Der}(H)$ and $f_1,\ldots,f_l$ are functionally independent formal power series in G which generate a formal foliation $G' \subset G$, then we write $L' \hat{\otimes}_{\mathbf{R}} G'$ for the collection of formal power series

$$\eta = \sum_{\beta \in \mathbf{N}^l} \eta_\beta f^\beta$$

in the variables $f_1, \ldots, f_l$ with coefficients $\eta_\beta \in L'$. Furthermore, if L' has a complex structure and $g_1, \ldots, g_s$ are (complex) functionally independent elements of the complex power series ring $G_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} G$, which generate a formal foliation $G'' \subset G_{\mathbf{C}}$, we write $L' \hat{\otimes}_{\mathbf{R}} G''$ for the complex space of formal power series in $g_1, \ldots, g_s$ with coefficients lying in L'. By expressing $f_1, \ldots, f_l$ and $g_1, \ldots, g_s$ in terms of $y_1, \ldots, y_m$, both $L' \hat{\otimes}_{\mathbf{R}} G'$ and $L' \hat{\otimes}_{\mathbf{C}} G''$ are identified with closed Lie subalgebras of $\mathrm{Der}(F)$. If G' and G'' are invariant under the action of L, then $\pi_G(L)$, viewed as a subalgebra of $\mathrm{Der}(F)$, leaves $L' \hat{\otimes}_{\mathbf{R}} G'$ and $L' \hat{\otimes}_{\mathbf{C}} G''$ stable by its adjoint action; moreover, if L' is a transitive subalgebra of $\mathrm{Der}(H)$, then the sums

$$(L' \hat{\otimes}_{\mathbf{R}} G') \oplus \pi_G(L), \quad (L' \hat{\otimes}_{\mathbf{C}} G'') \oplus \pi_G(L)$$

are transitive subalgebras of $\mathrm{Der}(F)$ in which $L' \hat{\otimes}_{\mathbf{R}} G'$ and $L' \hat{\otimes}_{\mathbf{C}} G''$ form closed ideals.

We now begin our outline of Theorems 4.1–4.6.

If the primitive Lie algebra $\tilde{N}_G$ is of infinite dimension, then $I_G \neq \{0\}$ and the commutator $I' = [I_G, I_G]$ is a nonabelian minimal closed ideal of L. Thus, the chain $I_G \supset I' \supset \{0\}$ is a Jordan-Hölder sequence for (L, I_G), with abelian quotient I_G/I'. In $\mathrm{Der}(H)$, a transitive subalgebra is formed by $R = [\tilde{N}_G, \tilde{N}_G]$, which is a simple Lie algebra. Every derivation of R is induced by a unique element of the normalizer of R in $\mathrm{Der}(H)$; therefore, we write $\mathrm{Der}(R)$ for this normalizer, which is evidently a transitive subalgebra of $\mathrm{Der}(H)$. Each derivation of R is K_R-linear, so $\mathrm{Der}(R)$ is a transitive Lie algebra over K_R; the quotient $\mathrm{Der}(R)/R$ is abelian, and has dimension $\leqslant 1$ over K_R. According to whether K_R is equal to $\mathbf{R}$ or $\mathbf{C}$, there is an L-invariant formal foliation $G' \subset G$, or a complex L-invariant formal foliation $G' \subset G_{\mathbf{C}}$, such that

$$L \subset \big(\mathrm{Der}(R) \hat{\otimes}_{K_R} G'\big) \oplus \pi_G(L), \quad I_G = L \cap \big(\mathrm{Der}(R) \hat{\otimes}_{K_R} G'\big), \quad I' = R \hat{\otimes}_{K_R} G',$$

for a suitable choice of the formal coordinates $z_1, \ldots, z_{n-m}$.

If $\tilde{N}_G$ is finite dimensional and semisimple, then it is either a simple Lie algebra g or the direct sum $g_1 \oplus g_2$ of two isomorphic simple ideals. When $\tilde{N}_G$ is simple, the closed ideal I_G is nonabelian and minimal in L, unless it is reduced to $\{0\}$; corresponding to the cases $K_g = \mathbf{R}$ and $K_g = \mathbf{C}$, there exist either an L-invariant foliation $G' \subset G$ or a complex L-invariant foliation $G' \subset G_{\mathbf{C}}$, such that, in suitable coordinates, we have

$$L \subset \big(g \hat{\otimes}_{K_g} G'\big) \oplus \pi_g(L), \quad I_G = L \cap \big(g \hat{\otimes}_{K_g} G'\big),$$

$$I_G = g \hat{\otimes}_{K_g} G' \quad (\text{for } I_G \neq \{0\}).$$

When $\tilde{N}_G$ is semisimple but not simple, either I_G is the direct sum $I_1 \oplus I_2$ of two nonabelian minimal closed ideals of L, or I_G is itself nonabelian and minimal in L, or $I_G = \{0\}$. Write K_g for the commutator field of the isomorphic simple Lie algebras g_1 and g_2. Again, according to whether K_g is equal to $\mathbf{R}$ or $\mathbf{C}$, there exist two L-invariant foliations G_1 and G_2 in G, or two complex L-invariant foliations G_1 and G_2 in $G_{\mathbf{C}}$ such that

$$L \subset \left(\left(g_1 \hat{\otimes}_{K_g} G_1\right) \oplus \left(g_2 \hat{\otimes}_{K_g} G_2\right)\right) \oplus \pi_G(L),$$
$$I_G = L \cap \left(\left(g_1 \hat{\otimes}_{K_g} G_1\right) \oplus \left(g_2 \hat{\otimes}_{K_g} G_2\right)\right),$$

and, if $I_G = I_1 \oplus I_2$, then

$$I_1 = g_1 \hat{\otimes}_{K_g} G_1, \quad I_2 = g_2 \hat{\otimes}_{K_g} G_2,$$

while if I_G is a minimal closed ideal, then $I_G = g_1 \hat{\otimes}_{K_g} G_1$, all in an appropriate system of formal coordinates.

The most intricate decomposition occurs when the primitive Lie algebra $\tilde{N}_G$ is finite dimensional, but not semisimple. (For brevity, we have included the abelian case, where $\tilde{N}_G$ is one dimensional, in this part of our outline.) In this instance, there is an abelian ideal V of $\tilde{N}_G$ complementary to $\tilde{L}^0$, and $\tilde{L}^0$ acts faithfully and irreducibly on V. Thus $\tilde{L}^0$ is a reductive Lie algebra, and $\tilde{N}_G$ is a semidirect product

$$\tilde{N}_G = V \oplus \left(g_0 \oplus g_1 \oplus \cdots \oplus g_p\right),$$

where g_0 is the center of $\tilde{L}^0$ and $g_1, \ldots, g_p$ are the simple ideals which make up its semisimple part. There exist closed ideals $I_V, I_0, \ldots, I_p$ of L such that

$$I_V \subset I_j \subset I_G \quad \text{for } 0 \leqslant j \leqslant p,$$
$$I_G/I_V = \left(I_0/I_V\right) \oplus \left(I_1/I_V\right) \oplus \cdots \oplus \left(I_p/I_V\right);$$

moreover, the ideal I_V is abelian, as is the quotient I_0/I_V, while each quotient I_j/I_V, with $1 \leqslant j \leqslant p$, is either reduced to $\{0\}$ or is a nonabelian minimal closed ideal of L/I_V. The appearance of this decomposition in a suitable coordinate system is distinctive. There are two L-invariant foliations G_V and G_0 in G, and, for each index j with $1 \leqslant j \leqslant p$, there is either an L-invariant foliation $G_j \subset G$ or a complex L-invariant foliation $G_j \subset G_{\mathbf{C}}$, according to whether K_{g_j} is equal to $\mathbf{R}$ or $\mathbf{C}$, such that

$$L \subset \left(\left(V \hat{\otimes}_{\mathbf{R}} G_V\right) \oplus \left(g_0 \hat{\otimes}_{\mathbf{R}} G_0\right) \oplus \bigoplus_{j=1}^{p} \left(g_j \hat{\otimes}_{K_{g_j}} G_j\right)\right) \oplus \pi_G(L),$$
$$I_G = L \cap \left(\left(V \hat{\otimes}_{\mathbf{R}} G_V\right) \oplus \left(g_0 \hat{\otimes}_{\mathbf{R}} G_0\right) \oplus \bigoplus_{j=1}^{p} \left(g_j \hat{\otimes}_{K_{g_j}} G_j\right)\right),$$
$$I_V = L \cap \left(V \hat{\otimes}_{\mathbf{R}} G_V\right),$$
$$I_0 = L \cap \left(\left(V \hat{\otimes}_{\mathbf{R}} G_V\right) \oplus \left(g_0 \hat{\otimes}_{\mathbf{R}} G_0\right)\right),$$
$$I_j = L \cap \left(\left(V \hat{\otimes}_{\mathbf{R}} G_V\right) \oplus \left(g_j \hat{\otimes}_{K_{g_j}} G_j\right)\right),$$

for $1 \leqslant j \leqslant p$, and whenever I_j/I_V is not reduced to $\{0\}$, we have

$$I_j + \left(V \hat{\otimes}_{\mathbf{R}} G_V\right) = \left(V \hat{\otimes}_{\mathbf{R}} G_V\right) \oplus \left(g_j \hat{\otimes}_{K_{g_j}} G_j\right).$$

An easy application of the Jacobson-Bourbaki density theorem (see the discussion following Theorem 4.5) shows that for each index j with $1 \leqslant j \leqslant p$ and $I_j/I_V \neq \{0\}$, the abelian ideal I_V is a subspace of $V \hat{\otimes}_{\mathbf{R}} G_V$ which is stable under right multiplication by G_j, when $K_{g_j} = \mathbf{R}$, or by one of the subrings G_j, $\overline{G}_j$, $G_j \cdot \overline{G}_j$, $\mathrm{Re}(G_j)$ of $G_{\mathbf{C}}$, when $K_{g_j} = \mathbf{C}$, depending on the nature of the representation of g_j on V. It is possible that $I_G \neq \{0\}$ and yet $I_V = \{0\}$, but in this case I_G is a nonabelian minimal closed ideal of L, so $I_G = I_j$ for some j with $1 \leqslant j \leqslant p$.

To achieve these realizations of L and I_G which we have reported, we can employ any arbitrary set of formal coordinates $y_1, \ldots, y_m$ generating the maximal L-invariant foliation G; only the formal coordinates $z_1, \ldots, z_{n-m}$ are altered in our construction. Therefore, our results may be invoked inductively in any chain of invariant foliations

$$F_0 \subset F_1 \subset \cdots \subset F_p = F$$

such that each F_j is maximal in F_{j+1}, for $0 \leqslant j \leqslant p-1$. To construct coordinates in which our stated realizations of the quotients $I_{F_j}/I_{F_{j+1}}$ are obtained simultaneously, we apply our results to each transitive subalgebra $\pi_{F_{j+1}}(L)$ of $\mathrm{Der}(F_{j+1})$ and maximal invariant foliation F_j in F_{j+1}, beginning with $j = p-1$ and proceeding down the chain.

This completes our outline of Theorems 4.1–4.6. A noteworthy consequence of Theorem 4.1 concerns the length of Jordan-Hölder sequences: There can be at most $\dim(L/L^0)$ nonabelian quotients in any Jordan-Hölder sequence for (L, L). This result, which is contained in Corollary 4.2, pertains to the classical problem of determining the space of lowest dimension upon which a given pseudogroup can act transitively. In Proposition 4.1, we have indicated a few simplifications of Theorem 4.1 which occur in the (geometrically important) case where there exists an abelian subalgebra $A \subset L$ with $L = L^0 \oplus A$.

Closely related to Theorems 4.2–4.6 is a description of the structure of nonabelian minimal closed ideals; such ideals make up the nonabelian quotients in Jordan-Hölder decompositions of transitive Lie algebras. As above, we conduct a summary of this result, Theorem 3.1, over the ground field $\mathbf{R}$. Let L be a linearly compact real Lie algebra and I a nonabelian minimal closed ideal of L. The centralizer of I in L is then a closed ideal Z such that $Z \cap I = \{0\}$ and L/Z is a transitive Lie algebra; therefore, replacing L by L/Z and I by its image in L/Z, we shall suppose without loss of generality that L is transitive and acts faithfully on I. There exists a unique maximal closed ideal J of I; the normalizer of J in L forms an open subalgebra N of L, and the quotient $R = I/J$ is a simple real transitive Lie algebra. From the adjoint action of N on I, one obtains a structure of a topological N-module on R. Let R^0 be a maximal fundamental subalgebra of R; then the stabilizer of R^0 in N is a fundamental subalgebra L^0 of L such that $N = L^0 + I$. Set $n = \dim(L/L^0)$ and $F = \mathbf{R}[[x_1, \ldots, x_n]]$; choose a realization of L as a transitive subalgebra of $\mathrm{Der}(F)$ with isotropy subalgebra L^0. To complete our description of (L, I), we employ once again the notation we developed in discussing invariant foliations. There exist formal coordinates $(y_1, \ldots, y_m, z_1, \ldots, z_{n-m})$ in terms of which $G = \mathbf{R}[[y_1, \ldots, y_m]]$ forms a maximal L-invariant foliation in F, and N is equal to N_G. As before, set $H = \mathbf{R}[[z_1, \ldots, z_{n-m}]]$ and consider the projection $\tilde{\pi}\colon N \to \mathrm{Der}(H)$; then J is equal to $\ker(\tilde{\pi}|_I)$, and $\tilde{\pi}|_I$ identifies (R, R^0) with a transitive subalgebra of $\mathrm{Der}(H)$. Every derivation of R is K_R-linear, and is induced by a unique element of the normalizer in $\mathrm{Der}(H)$ of R; we write $\mathrm{Der}(R)$ for this normalizer, which is evidently a transitive subalgebra of $\mathrm{Der}(H)$. Moreover, $\mathrm{Der}(R)$ is a transitive Lie algebra over K_R and R forms a closed ideal in $\mathrm{Der}(R)$ of codimension $\leqslant 1$ over K_R. We shall say that the nonabelian minimal closed ideal I is of real or complex type according to whether K_R is equal to $\mathbf{R}$ or $\mathbf{C}$. When I is of real type, then we can choose the formal coordinates $(z_1, \dots, z_{n-m})$ so that

$$L \subset (\mathrm{Der}(R) \mathbin{\hat{\otimes}}_{\mathbf{R}} G) \oplus \pi_G(L), \quad I = R \mathbin{\hat{\otimes}}_{\mathbf{R}} G.$$

If I is of complex type, there exist $p \leqslant m$ complex functionally independent elements $g_1, \dots, g_p$ of $G_{\mathbf{C}}$ which generate an L-invariant complex foliation $G' \subset G_{\mathbf{C}}$ such that $G' \cdot \bar{G}' = G_{\mathbf{C}}$ and, for a suitable choice of $(z_1, \dots, z_{n-m})$, we have

$$L \subset (\mathrm{Der}(R) \mathbin{\hat{\otimes}}_{\mathbf{R}} G') \oplus \pi_G(L), \quad I = R \mathbin{\hat{\otimes}}_{\mathbf{C}} G'.$$

These results extend the work of Guillemin **[12]**, who showed that a nonabelian minimal closed ideal I is isomorphic, as an abstract Lie algebra, to the algebra $R[[x_1, \dots, x_q]]$ of R-valued formal power series in $q \geqslant 0$ indeterminates (see also **[5]**).

Given a real formal power series ring $F = \mathbf{R}[[x_1, \dots, x_n]]$ and a real transitive subalgebra $L \subset \mathrm{Der}(F)$, a geometric interpretation is attached to each L-invariant complex foliation G' in $F_{\mathbf{C}}$. Let $g_1, \dots, g_p$ be complex functionally independent generators for G'; then this collection specifies a formal power series mapping φ: $\mathbf{R}^n \to \mathbf{C}^m$ with $\varphi(0) = 0$, which is, formally, the expansion of a submersion of a neighborhood of 0 in $\mathbf{R}^n$ onto a generic real submanifold S of $\mathbf{C}^m$. Moreover, every formal vector field $\xi \in L$ is φ-projectable, and $\varphi_*(L)$ comprises a transitive subalgebra of the formal vector fields on S at $0 \in S$. Each element $\varphi_*(\xi)$, for $\xi \in L$, is the restriction to S of a formal holomorphic vector field on $\mathbf{C}^m$ at 0 which is everywhere tangent to S, and so the pseudocomplex structure on S induced by its embedding in $\mathbf{C}^m$ is locally homogeneous with respect to the pseudogroup of local biholomorphic transformations of $\mathbf{C}^m$ which preserve S. Evidently, by choosing a different set of functionally independent generators for G' we replace φ by $\Phi \circ \varphi$, where Φ: $\mathbf{C}^m \to \mathbf{C}^m$ is a formal biholomorphic mapping with $\Phi(0) = 0$. The rank of φ, that is, the dimension of S, is given by the number of complex functionally independent elements required to generate $G' \cdot \bar{G}'$ in $F_{\mathbf{C}}$. Because such complex invariant foliations appear prominently in our descriptions of nonabelian minimal closed ideals and of ideals defined by maximal invariant foliations, we have devoted the latter part of §3 to a discussion of the formal geometry summarized above, in the slightly more abstract context of open subalgebras contained in the complexification of a real linearly compact Lie algebra. We remark that the pseudocomplex geometry associated to a nonabelian minimal closed ideal I in a real transitive Lie algebra L is determined by the structure of I as an L-module; therefore, we encounter the same pseudocomplex structures as we proceed through the nonabelian quotients of complex type in any two Jordan-Hölder decompositions of a closed ideal in L.

The abelian quotients which appear as we decompose a closed ideal in a transitive Lie algebra depend strongly on which Jordan-Hölder sequence we choose to follow; in decomposing closed ideals defined by maximal invariant foliations, we have described above one set of abelian quotients which can occur. For completeness, we give also, in the discussion following Corollary 2.1, a description of abelian quotients which appears, essentially, in Cartan **[3]**: Let L be a transitive Lie algebra and I be a closed ideal of L. Then, there exists a Jordan-Hölder sequence for (L, I) such that, for each abelian quotient I_j/I_{j+1}, there is a formal power series ring $F = K[[x_1,\dots,x_n]]$ and a realization of L/I_j as a transitive subalgebra of $\mathrm{Der}(F)$ which leaves the formal foliation $G = K[[x_1,\dots,x_{n-1}]]$ invariant, and every element $\xi \in I_j/I_{j+1}$ takes the form

$$\xi = g(x_1,\dots,x_{n-1})\frac{\partial}{\partial x_n},$$

with $g \in G$. These coefficients g are the formal solutions of a linear homogeneous partial differential equation $Dg = 0$, where D is an overdetermined operator which is invariant under the action of $\pi_G(L/I_{j+1})$.

The basic tool which we employ to construct realizations of closed ideals is Theorem 2.1, which is an abstract result on embeddings of filtered extensions of transitive Lie algebras. This theorem can be applied to discuss questions of the following sort: Suppose that $F = K[[y_1,\dots,y_m, z_1,\dots,z_{n-m}]]$ is a formal power series ring and that L and L' are transitive subalgebras of $\mathrm{Der}(F)$ which preserve the formal foliation $G = K[[y_1,\dots,y_m]]$. Assume, moreover, that the inclusion $\pi_G(L) \subset \pi_G(L')$ holds in $\mathrm{Der}(G)$ and that for $H = K[[z_1,\dots,z_{n-m}]]$, we have $\tilde{N}_G \subset \tilde{N}'_G$ in $\mathrm{Der}(H)$. By replacing $z_1,\dots,z_{n-m}$ by a new set of formal coordinates $z'_1,\dots,z'_{n-m}$ such that $z'_j = z_j \bmod((y_1,\dots,y_m))$, where $((y_1,\dots,y_m))$ is the ideal in F generated by $y_1,\dots,y_m$, we define an automorphism φ: $F \to F$ such that $\varphi|_G = \mathrm{id}_G$, and our hypotheses are still satisfied if we replace L by $\varphi_*(L)$. Then we ask: When does such an automorphism φ exist for which $\varphi_*(L) \subset L'$? Theorem 2.1 includes a solution to this problem.

We conclude with a few remarks on the organization and presentation of our work. In §1, we give a compendium of those results on transitive and linearly compact Lie algebras which we shall require. Very few proofs are given in this section, since they can be found in **[5, 12, 13, 15, 23]**; we also direct the reader to these sources for a more systematic introduction to the study of transitive Lie algebras. In §2, we present Theorem 2.1 and obtain its most immediate corollaries. Our results on the structure of nonabelian minimal closed ideals, together with a discussion of the pseudocomplex geometry associated to complex invariant foliations, make up §3; there is some overlapping between this section and our monograph **[5]**. The final section of our work, §4, is devoted to the study of ideals defined by maximal invariant foliations. For simplicity, and because our interest in transitive Lie algebras stems from their differential-geometric applications, we have chosen to state results only over the ground fields **R** and **C**. The substitution of an algebraically closed field of characteristic zero would cause no difficulties, and §2 could be conducted over any field of characteristic zero. Most of the results in §§3 and 4 depend on the classification of infinite-dimensional primitive Lie algebras; this classification has been published only for algebraically closed fields, and for the real number field **R**.

During the course of my work, I received constant encouragement from Hubert Goldschmidt and Donald C. Spencer; the result owes much to their suggestions and trenchant criticism. My debt to Victor Guillemin, who generously provided an unpublished manuscript which formed the starting point for §4 of this paper, is evident.

1. Preliminaries. Throughout this paper, we shall operate over a ground field K which may be taken to be either the real number field $\mathbf{R}$ or the field $\mathbf{C}$ of complex numbers, unless otherwise indicated. If V is a vector space over $\mathbf{R}$, we write $V_{\mathbf{C}}$ for the complex vector space $V_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} V$ obtained from V by complexification; if $v = v_1 + \sqrt{-1}\, v_2$ with $v_1, v_2 \in V$ is an element of $V_{\mathbf{C}}$, we write $\bar{v} = v_1 - \sqrt{-1}\, v_2$ for the complex conjugate of v. We identify V with the real subspace $\{ v \in V_{\mathbf{C}} | v = \bar{v} \}$ made up of all real vectors in $V_{\mathbf{C}}$. If U is a complex subspace of $V_{\mathbf{C}}$, we denote by

$$\bar{U} = \{ v \in V_{\mathbf{C}} | \bar{v} \in U \}$$

the complex subspace conjugate to U. If V and W are real vector spaces and ϕ: $V \to W$ is a linear mapping, we write $\phi_{\mathbf{C}}$: $V_{\mathbf{C}} \to W_{\mathbf{C}}$ for the complex-linear mapping $\mathbf{1}_{\mathbf{C}} \otimes \phi$ obtained from ϕ.

If U is a vector space over K, we denote by

$$\otimes(U) = \bigoplus_{p \in \mathbf{Z}} (\otimes^p(U)), \quad S(U) = \bigoplus_{p \in \mathbf{Z}} (S^p(U)), \quad \Lambda(U) = \bigoplus_{p \in \mathbf{Z}} (\Lambda^p(U)),$$

the tensor, symmetric and exterior algebras of U, respectively; in order to express these as $\mathbf{Z}$-graded algebras, we have set

$$\otimes^{-p}(U) = S^{-p}(U) = \Lambda^{-p}(U) = \{0\} \quad \text{for } p > 0.$$

Endow K with the discrete topology, and let V be a Hausdorff topological vector space over K. Then V is said to be *linearly compact* if

(i) V is complete and

(ii) there exists a fundamental system $\{ V_\alpha \}$ of neighborhoods of 0 in V such that each V_α is a vector subspace of finite codimension in V.

If W is a finite-dimensional vector space over K, then W is linearly compact in the discrete topology; conversely, the discrete topology provides the only linearly compact structure on W, by virtue of condition (ii) above. In the sequel, a finite-dimensional vector space often will be endowed implicitly with the discrete topology. We shall present more illuminating examples momentarily.

We list below those properties of linearly compact spaces required in our work; proofs of these assertions may be found in Chapter 2 of the book **[18]** of Köthe, or in the first section of Guillemin's paper **[12]**.

PROPOSITION 1.1. *Let V and W be linearly compact vector spaces over K.*

(i) *If V' is a closed subspace of V, then both V' and V/V' are linearly compact.*

(ii) *A subspace V' of V is open if and only if V' is closed and of finite codimension in V.*

(iii) *If V' and V'' are closed subspaces of V, then the sum $V' + V''$ is closed in V.*

(iv) *The topological direct sum $V \oplus W$ is linearly compact.*

(v) *If* φ: $V \to W$ *is a continuous linear mapping, then the image* $\varphi(V)$ *is a closed subspace of* W, *and* φ *is an open mapping of* V *onto* $\varphi(V)$. *In particular, any continuous linear bijection* φ: $V \to W$ *is a topological isomorphism.*

(vi) *If* V' *is a closed subspace of* V, *then there exists a closed subspace* $V'' \subset V$ *such that* $V = V' \oplus V''$.

(vii) (*Closed graph theorem*) *A linear mapping* φ: $V \to W$ *is continuous if and only if the graph of* φ *is closed in* $V \times W$.

(viii) *If* $\{E_\alpha\}$ *is a family of linearly compact vector spaces, then the product* $\Pi_\alpha E_\alpha$ *is linearly compact. If* $\{(E_\alpha, f_{\alpha\beta})\}$ *is an inverse system of linearly compact vector spaces and continuous linear mappings, then the projective limit* $\varprojlim E_\alpha$ *is linearly compact.*

(ix) (*Chevalley's theorem*) *Let* $V_1 \supset V_2 \supset \cdots \supset V_l \supset \cdots$ *be a descending chain of closed subspaces of* V *such that* $\bigcap_{j \geq 1} V_j = \{0\}$. *Then, for each open subspace* U *of* V *there exists an integer* $p \geq 1$ *such that* $V_p \subset U$. *In particular, if each* V_j *is an open subspace of* V, *then the family* $\{V_j\}_{j \geq 1}$ *is a fundamental system of neighborhoods of* 0 *in* V.

Let V and W be linearly compact vector spaces over K, and suppose that $\{V_\alpha\}$ (resp. $\{W_\beta\}$) is a collection of open subspaces which forms a fundamental system of neighborhoods of 0 in V (resp. W). We define a structure of topological vector space on the tensor product $V \otimes_K W$ by taking as a fundamental system of neighborhoods of 0 the subspaces

$$V_\alpha \otimes_K W + V \otimes_K W_\beta$$

of $V \otimes_K W$, where V_α (resp. W_β) ranges over all elements of $\{V_\alpha\}$ (resp. $\{W_\beta\}$). The topology thus defined depends only on the topologies of V and W; however, this topology on $V \otimes_K W$ is, in general, not complete. The Hausdorff completion of $V \otimes_K W$ is a linearly compact topological vector space which we denote by $V \hat{\otimes}_K W$, and call the *completed tensor product* of V and W. If at least one of the spaces V and W is finite dimensional, then the ordinary tensor product $V \otimes_K W$ is complete, and thus coincides with $V \hat{\otimes}_K W$. In general, the completed tensor product $V \hat{\otimes}_K W$ may be identified with the projective limit

$$\varinjlim (V/V_\alpha) \otimes_K (W/W_\beta)$$

of finite-dimensional discrete spaces, and with either of the projective limits

$$\varinjlim (V/V_\alpha) \otimes_K W, \quad \varinjlim V \otimes_K (W/W_\beta)$$

of linearly compact spaces. As a special case of what we have said above, given a real linearly compact vector space V, there is a linearly compact structure defined naturally on the complexification $V_{\mathbf{C}}$ of V. Further information regarding the completed tensor product can be found in the exercises to the chapter on completions of filtered structures in Bourbaki [2].

Let U be a vector space of finite dimension over K. Then we can form the completion of the symmetric algebra $S(U^*)$, obtaining in this way a local algebra

$$F\{U^*\} = \prod_{p \in \mathbf{Z}} S^p(U^*)$$

which we call the algebra of *formal power series on U*. For each $l \in \mathbf{Z}$, the space

$$F^l\{U^*\} = \left(\prod_{p \leqslant l} \{0\}_p \times \left(\prod_{p > l} S^p(U^*)\right)\right)$$

of formal power series which vanish to order l at the origin is an ideal of $F\{U^*\}$, and we have

$$(1.1) \qquad (F^r\{U^*\}) \cdot (F^s\{U^*\}) = F^{r+s+1}\{U^*\}$$

for all $r, s \geqslant -1$; note that $F^l\{U^*\} = F\{U^*\}$ when $l < 0$. The space $F^0\{U^*\}$ is the unique maximal ideal of $F\{U^*\}$, and, for each $l \geqslant 0$, the ideal $F^l\{U^*\}$ is the $(l+1)$st power of $F^0\{U^*\}$. As the product of finite-dimensional discrete spaces, the algebra $F\{U^*\}$ is endowed with a linearly compact topology in which the ideals $\{F^l\{U^*\}\}_{l \geqslant -1}$ comprise a fundamental system of neighborhoods of 0. From (1.1), we see that multiplication in $F\{U^*\}$ is continuous with respect to this topology; thus $F\{U^*\}$ is a linearly compact topological algebra over K. Suppose that $\dim(U) = n$, and choose elements $f_1, \ldots, f_n$ of $F\{U^*\}$ whose images form a basis for the vector space

$$F^0\{U^*\}/F^1\{U^*\} \simeq U^*.$$

Then, there is a unique isomorphism

$$\varphi\colon F\{U^*\} \to K[[x_1, \ldots, x_n]]$$

of $F\{U^*\}$ onto the algebra of formal power series over K in n indeterminates $x_1, \ldots, x_n$ such that $\varphi(f_j) = x_j$ for $1 \leqslant j \leqslant n$. We shall call $f_1, \ldots, f_n$ a *formal coordinate system* for U. If U is a real vector space, it is evident that the complexification $F_{\mathbf{C}}\{U^*\}$ of $F\{U^*\}$ is identified naturally with the algebra $F\{U_{\mathbf{C}}^*\}$ of formal power series on the complexification $U_{\mathbf{C}}$ of U.

For brevity, we shall call a local algebra F a ring of formal power series when F is isomorphic to $F\{U^*\}$ for some finite-dimensional vector space U; we then write F^l for the $(l+1)$st power of the maximal ideal F^0 of F, for all $l \geqslant 0$, and we set $F^l = F$ when $l < 0$. Unless otherwise stated, we always assume that a formal power series ring F is given the linearly compact topology in which the ideals $\{F^j\}_{j \geqslant 0}$ comprise a fundamental system of neighborhoods of 0, as discussed above; this structure often is called the Krull topology on F.

If A is an algebra (not necessarily associative) over K, we denote by $\mathrm{Der}(A)$ the Lie algebra of derivations of A, that is, the space of K-linear mappings $D\colon A \to A$ such that, for all $a, b \in A$

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b).$$

If D_1 and D_2 lie in $\mathrm{Der}(A)$, then their commutator

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

is also a derivation of A; this defines the Lie bracket in $\mathrm{Der}(A)$.

In this paper, we shall study those topological Lie algebras L which have the property that the topological vector space underlying L is linearly compact; for simplicity, we say then that L is a *linearly compact Lie algebra*. An example is provided by any finite-dimensional Lie algebra L, endowed with the discrete topology, over K. To construct a linearly compact Lie algebra in which the topology plays a more essential role, let F be the algebra of formal power series $F = F\{U^*\}$ on a finite-dimensional vector space U, and consider the Lie algebra $\mathrm{Der}(F)$ of derivations of F. By choosing a formal system of coordinates for U, identify F with the algebra $K[[x_1,\dots,x_n]]$, where $n = \dim(U)$. For each derivation D of F, we see from (1.1) that

$$D(F^{2p+1}) = D(F^p \cdot F^p) \subset F^p \cdot D(F^p) \subset F^p,$$

for all $p \geqslant 0$, and thus D is continuous with respect to the Krull topology on F. The algebra $K[x_1,\dots,x_n]$ of polynomials is dense in F, and any derivation of F must annihilate the scalars K; thus, each derivation of F is uniquely determined by its action on the monomials $x_1,\dots,x_n$ of degree one. It follows that $\mathrm{Der}(F)$ is identified with the Lie algebra of formal vector fields, that is, expressions of the form

$$\xi = \sum_{1 \leqslant j \leqslant n} \xi_j \frac{\partial}{\partial x_j}$$

with coefficients $\xi_j \in F$. The action of such a formal vector field ξ on F is given by

$$\xi \cdot f = \sum_{1 \leqslant j \leqslant n} \xi_j \frac{\partial f}{\partial x_j} \quad \text{for } f \in F,$$

and thus the Lie bracket in $\mathrm{Der}(F)$ is expressed as

$$[\xi, \eta] = \sum_{1 \leqslant i,\, j \leqslant n} \left(\xi_i \frac{\partial \eta_j}{\partial x_i} - \eta_i \frac{\partial \xi_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

for all ξ, $\eta \in F$. We define a descending chain of subspaces of $\mathrm{Der}(F)$ by setting $\mathrm{Der}^l(F) = \{ D \in \mathrm{Der}(F) | D(F) \subset F^l \}$ for all $l \in \mathbf{Z}$; a formal vector field ξ lies in $\mathrm{Der}^l(F)$ if and only if each of its coefficients ξ_j lies in F^l. From our expression of the Lie bracket in $\mathrm{Der}(F)$, we see at once that

$$[\mathrm{Der}^l(F), \mathrm{Der}^m(F)] \subset \mathrm{Der}^{l+m}(F),$$

for all l, $m \in \mathbf{Z}$, and so we have placed a filtered Lie algebra structure on $\mathrm{Der}(F)$. Each subalgebra $\mathrm{Der}^l(F)$ of this filtration is of finite codimension in $\mathrm{Der}(F)$, and we have $\bigcap_{l \in \mathbf{Z}} \mathrm{Der}^l(F) = \{0\}$. Thus, the subalgebras $\{\mathrm{Der}^l(F)\}_{l \geqslant 0}$ constitute a fundamental system of neighborhoods of 0 for a unique structure of Hausdorff topological vector space on $\mathrm{Der}(F)$, and the Lie bracket in $\mathrm{Der}(F)$ is continuous with respect to this structure. Moreover, it is evident that $\mathrm{Der}(F)$ is complete in this topology. We have, thus, defined a linearly compact Lie algebra structure on $\mathrm{Der}(F)$. We should also note that $\mathrm{Der}^l(F)(F^m) \subset F^{l+m}$, for all l, $m \in \mathbf{Z}$, which shows that F is a topological module over $\mathrm{Der}(F)$. It is also useful to express the action of $\mathrm{Der}(F)$ without specifying formal coordinates for U, in the following way: Each element $u \in U$ induces, by contraction with the symmetric algebra $S(U^*)$, a derivation δ_u: $S(U^*) \to S(U^*)$. Because $S(U^*)$ forms a dense subalgebra in F and δ_u is continuous with respect to the induced topology, we may extend δ_u uniquely to a continuous derivation δ_u: $F \to F$. As we saw above, each derivation of F is uniquely determined by its action on the subspace $S^1(U^*) = U^*$ of F, since this subspace and the scalars K generate the dense subalgebra $S(U^*)$. Hence, we have a natural identification

$$\mathrm{Der}(F) = F \otimes_K U = \mathrm{Hom}_K(U^*, F).$$

The action of $\mathrm{Der}(F)$ on F, under this identification, is expressed on decomposable elements by $(g \otimes u) \cdot f = g\delta_u(f)$, and the Lie bracket in $\mathrm{Der}(F)$ is given by

$$[f \otimes u, g \otimes v] = (f\delta_u(g)) \otimes v - (g\delta_v(f)) \otimes u,$$

for all $f, g \in F$ and $u, v \in U$. Thus, our filtration of $\mathrm{Der}(F)$ assumes the form

$$\mathrm{Der}^l(F) = F^l \otimes_K U \quad \text{for } l \in \mathbf{Z};$$

the resulting natural identifications

$$\mathrm{Der}^l(F)/\mathrm{Der}^{l+1}(F) = S^{l+1}(U^*) \otimes_K U \quad \text{for } l \in \mathbf{Z},$$

are often convenient.

Let L be a linearly compact Lie algebra. An open subalgebra L^0 of L is said to be *fundamental* when L^0 contains no ideals of L except $\{0\}$. If L possesses a fundamental subalgebra, then L is called a *transitive Lie algebra*. Sometimes, we say that (L, L^0) is a transitive Lie algebra, to indicate a particular choice of fundamental subalgebra L^0 for L. We note that any finite-dimensional discrete Lie algebra L is transitive, since $\{0\}$ is a fundamental subalgebra of L. If F is a formal power series ring over K, then $(\mathrm{Der}(F), \mathrm{Der}^0(F))$ is a transitive Lie algebra. Indeed, if L is a closed subalgebra of $\mathrm{Der}(F)$ such that $\mathrm{Der}(F) = L + \mathrm{Der}^0(F)$, then it is easy to check that the open subalgebra $L^0 = L \cap \mathrm{Der}^0(F)$ of L contains no ideals of L except $\{0\}$, and is thus fundamental. We shall, then, say that L is a *transitive subalgebra of* $\mathrm{Der}(F)$.

It is evident that if L is a real linearly compact Lie algebra, then the complexification $L_{\mathbf{C}}$ of L is a linearly compact Lie algebra over $\mathbf{C}$; moreover, if L^0 is a fundamental subalgebra of L, then the open subalgebra $L^0_{\mathbf{C}}$ is fundamental in $L_{\mathbf{C}}$.

As was proved by Guillemin [**12**, Theorem 3], transitive Lie algebras also may be characterized by the finiteness of all descending chains of closed ideals lying within them (see also Theorem 14.1 of [**10**]).

THEOREM 1.1. *Let L be a linearly compact Lie algebra. Then, the following are equivalent.*

(i) *L is a transitive Lie algebra.*

(ii) *L satisfies the descending chain condition on closed ideals.*

COROLLARY 1.1. *Let L be a transitive Lie algebra and I a closed ideal of L. Then, the quotient L/I is a transitive Lie algebra.*

Let L be a transitive Lie algebra and I a closed ideal of L. A *Jordan-Hölder sequence for* (L, I) consists of a properly nested descending chain

$$I = I_0 \supset I_1 \supset \cdots \supset I_n = \{0\}$$

of closed ideals of L contained in I, such that for each l with $0 \leqslant l \leqslant n-1$, one of the following alternatives holds:

(i) The quotient I_l/I_{l+1} is abelian.

(ii) The quotient I_l/I_{l+1} is nonabelian and there are no closed ideals of L properly contained between I_l and I_{l+1}.

Theorem 1.1 ensures that any such descending chain will be finite. When the ideal I is equal to L, we call this composition series a *Jordan-Hölder sequence for L*.

The following result was conjectured by É. Cartan; its proof is due to Guillemin [**12**, Theorem 6.1].

THEOREM 1.2. *Let L be a transitive Lie algebra, and I a closed ideal of L. Then, there exists a Jordan-Hölder sequence for (L, I). Suppose that*

$$I = I_0 \supset I_1 \supset \cdots \supset I_n = \{0\}, \quad I = I'_0 \supset I'_1 \supset \cdots \supset I'_p = \{0\},$$

are two Jordan-Hölder sequences for (L, I), and denote by $\mathcal{Q}$ (resp. $\mathcal{Q}'$) the subset of $\{0,\dots,n-1\}$ (resp. $\{0,\dots,p-1\}$) consisting of those l for which the quotient I_l/I_{l+1} (resp. I'_l/I'_{l+1}) is nonabelian. Then there exists a bijection $f\colon \mathcal{Q} \to \mathcal{Q}'$, and, for all $l \in \mathcal{Q}$, mappings

$$\lambda_l\colon (I_l/I_{l+1}) \to (I'_{f(l)}/I'_{f(l)+1}),$$

which are isomorphisms both of topological Lie algebras and of topological L-modules.

We shall refer, loosely, to the last part of Theorem 1.2 as saying that all Jordan-Hölder sequences for (L, I) have the same number and type of nonabelian quotients. When $I = L$, we may view Theorem 1.2 as a weak analogue of the Levi decomposition for finite-dimensional Lie algebras. To elaborate on this remark, let g be a finite-dimensional Lie algebra, and r the radical of g. Then, the quotient $s = g/r$ is semisimple, and is thus the direct sum $s = s_1 \oplus \cdots \oplus s_q$ of simple ideals $\{s_j\}_{1 \leqslant j \leqslant q}$. Write $\pi\colon g \to s$ for the natural projection, and let

$$r = r_0 \supset r_1 \supset \cdots \supset r_p = \{0\}$$

be the ideals of the derived series of r, defined inductively by setting $r_0 = r$, $r_{j+1} = [r_j, r_j]$; each term r_j is evidently an ideal of g, and, because r is solvable its derived series descends to $\{0\}$. A Jordan-Hölder sequence for g is, then, given by the chain

$$\begin{aligned} g = \pi^{-1}(s_1 \oplus \cdots \oplus s_q) &\supset \pi^{-1}(s_1 \oplus \cdots \oplus s_{q-1}) \\ \supset \cdots \supset \pi^{-1}(s_1) &\supset r \supset r_1 \supset \cdots \supset r_p = \{0\}. \end{aligned}$$

The first q quotients of this sequence are nonabelian, and isomorphic to the simple Lie algebras $s_1,\dots,s_q$; the p remaining quotients are the abelian Lie algebras $r_0/r_1,\dots,r_{p-1}/r_p$. By contrast, when L is a transitive Lie algebra of infinite dimension, the nonabelian quotients of a Jordan-Hölder sequence for L need not be simple Lie algebras, and it will not be possible, in general, to construct a Jordan-Hölder sequence for L in which all abelian quotients are obtained from the radical of L.

Next, we introduce certain filtrations of linearly compact Lie algebras. Let L be a Lie algebra, and S be a subspace of L. Then, the *derived subspace* $D_L(S)$ is defined as

$$D_L(S) = \{\eta \in S \,|\, [L, \eta] \subset S\}.$$

We iterate this construction to define inductively a descending chain $\{D^p(S)\}_{p \geqslant 0}$ of subspaces of S, by setting $D_L^0(S) = S$ and $D_L^{p+1}(S) = D_L(D_L^p(S))$; for convenience, we take $D_L^p(S) = L$, for $p < 0$. We record the following properties of the derived subspace (see, for example, Proposition 1.2 of [**5**]).

LEMMA 1.1. *Let L be a Lie algebra, and S a subspace of L. Then*:

(i) *The derived subspace $D_L(S)$ is a subalgebra of L contained in S. Moreover, if S is a subalgebra of L, then $D_L(S)$ is an ideal of S.*

(ii) *If I is an ideal of L contained in S, then $I \subset D_L(S)$. Thus, the intersection $D_L^\infty(S) = \bigcap_{p \geqslant 0} D_L^p(S)$ is an ideal of L contained in S, and contains all ideals of L contained in S.*

Assume, in addition, that the Lie algebra L is linearly compact. Then:

(iii) *If S is an open (resp. closed) subspace of L, then the subalgebra $D_L(S)$ is open (resp. closed) in L.*

From part (iii) of the preceding lemma, we see that a linearly compact Lie algebra L is transitive if and only if there exists an open neighborhood $\mathcal{O}$ of zero in L which contains no ideals of L except $\{0\}$. For, if $\mathcal{O}$ is such an open neighborhood, then $\mathcal{O}$ contains an open subspace S, whose derived subspace $D_L(S)$ is a fundamental subalgebra of L by (iii); the converse is obvious. We also remark that if M is an open subalgebra of a linearly compact Lie algebra L, then $D_L^\infty(M)$ is a closed ideal of L contained in M, and the quotient $L/D_L^\infty(M)$ is a transitive Lie algebra, with fundamental subalgebra $M/D_L^\infty(M)$; in particular, the open subalgebra M is fundamental for L if and only if $D_L^\infty(M) = \{0\}$. These observations are immediate consequences of parts (ii) and (iii) of the lemma.

By a simple Lie algebra we mean, as is usual, a nonabelian Lie algebra L in which the only ideals are $\{0\}$ and L itself. From what we have said above, it is apparent that any simple linearly compact Lie algebra is transitive. Moreover, in the work of Guillemin [**12**], it is proved that a linearly compact nonabelian Lie algebra L is simple if the only *closed* ideals of L are $\{0\}$ and L.

When we impose upon S more algebraic structure than merely that of a subspace, the filtration $\{D_L^l(S)\}_{l \in \mathbf{Z}}$ acquires some important properties (see, for example, Proposition 1.3 and Lemma 1.2 of [**5**]).

LEMMA 1.2. (i) *Let L be a Lie algebra and M a subalgebra of L. Then, the subalgebras $\{D_L^p(M)\}_{p \in \mathbf{Z}}$ endow L with a structure of filtered Lie algebra*; *that is, for all $p, q \in \mathbf{Z}$,*

$$[D_L^p(M), D_L^q(M)] \subset D_L^{p+q}(M).$$

(ii) *Let L be a Lie algebra. Suppose that I is an ideal of L, and J is an ideal of I. Define a filtration $\{I_p\}_{p \in \mathbf{Z}}$ of I by setting $I_p = I$ for $p \leqslant 0$, $I_1 = J$ and $I_p = D_L^{p-1}(J)$ for $p \geqslant 2$. Then, for all $p, q \in \mathbf{Z}$,*

$$[I_p, I_q] \subset I_{p+q}.$$

In particular, the spaces $\{I_p\}_{p \in \mathbf{Z}}$ are ideals of I.

When L^0 is a fundamental subalgebra of a transitive Lie algebra L, the subalgebras $\{D_L^p(L^0)\}_{p\in\mathbf{Z}}$, which give the filtration described in (i), comprise a fundamental system of neighborhoods of 0 in L; this follows at once from parts (ii) and (iii) of Lemma 1.1 and Chevalley's theorem (part (ix) of Proposition 1.1). For example, when F is a formal power series ring and (L, L^0) is the transitive Lie algebra $(\mathrm{Der}(F), \mathrm{Der}^0(F))$, it is easy to check that

$$D^p_{\mathrm{Der}(F)}(\mathrm{Der}^0(F)) = \mathrm{Der}^p(F),$$

for all $p \in \mathbf{Z}$; we have noted previously that these spaces make up a fundamental system of neighborhoods of 0 in $\mathrm{Der}(F)$, and define a filtered Lie algebra structure on $\mathrm{Der}(F)$ as described in part (i) of the preceding lemma.

Our next result is immediately verified.

LEMMA 1.3. *Let (L, L^0) be a transitive Lie algebra and M a closed subalgebra of L with $L = M + L^0$. Then M is a transitive Lie algebra, with fundamental subalgebra $M^0 = M \cap L^0$; indeed $D_M^l(M^0) = M \cap D_L^l(L^0)$ for all $l \geqslant 0$.*

For brevity, we often refer to the situation described in Lemma 1.3 by saying that (M, M^0) is a *transitive subalgebra of* (L, L^0); when F is a formal power series ring, we have already abbreviated this a bit further by calling a transitive subalgebra of $(\mathrm{Der}(F), \mathrm{Der}^0(F))$ simply a transitive subalgebra of $\mathrm{Der}(F)$.

A few more observations concerning the derived subspace, which we shall require in the body of this paper, appear below (see, for example, Lemmas 4.1 and 4.2 of [5]). Part (iii) is a special case of (ii), and is given separate mention only for convenience.

LEMMA 1.4. (i) *Let I be a closed ideal of a linearly compact Lie algebra L. If S is an open subspace of I, then the normalizer $N_L(S)$ of S in L is an open subalgebra of L.*

(ii) *Let I be an ideal of a Lie algebra L and S a subalgebra of I. Then, the inclusion $N_L(S) \subset N_L(D_I(S))$ holds between the normalizers in L of S and $D_I(S)$.*

(iii) *Let L be a Lie algebra. Suppose that S is a subalgebra of L, and J an ideal of S. Then $D_L(J)$ is an ideal of S.*

Next, we consider various types of morphisms and realizations of linearly compact and transitive Lie algebras. By a morphism of linearly compact Lie algebras, we mean, of course, a continuous linear mapping $\lambda\colon L \to L'$ which is a Lie algebra homomorphism between two linearly compact Lie algebras L and L'. A *morphism of transitive Lie algebras* consists of a morphism of linearly compact Lie algebras

$$\lambda\colon (L, L^0) \to (L', L'^0)$$

between transitive Lie algebras (L, L^0) and (L', L'^0) such that

$$L' = \lambda(L) + L'^0, \quad L^0 = \lambda^{-1}(L'^0);$$

that is, $(\lambda(L), \lambda(L^0))$ forms a transitive subalgebra of (L', L'^0). When such a morphism λ is injective, we sometimes call it an *embedding of transitive Lie algebras*.

Let F be a formal power series ring and L a linearly compact Lie algebra. By a *transitive representation of L on F*, we mean a morphism of linearly compact Lie algebras $\lambda\colon L \to \mathrm{Der}(F)$ whose image $\lambda(L)$ is a transitive subalgebra of $\mathrm{Der}(F)$. The open subalgebra $M = \lambda^{-1}(\mathrm{Der}^0(F))$ of L is then called the *isotropy subalgebra* of λ, and we sometimes write $\lambda\colon (L, M) \to \mathrm{Der}(F)$ to indicate that a particular subalgebra M of L is the isotropy subalgebra of λ. As is seen at once, the formal power series ring F on which L is transitively represented is determined, up to isomorphism, by the choice of isotropy subalgebra M; indeed, the ring F must be isomorphic to $F\{(L/M)^*\}$. We observe also that, for all $p \in \mathbf{Z}$,

$$D_L^p(M) = \lambda^{-1}(\mathrm{Der}^p(F));$$

that is, a transitive representation λ is filtration-preserving with respect to the filtrations $\{D_L^p(M)\}_{p\in\mathbf{Z}}$ and $\{\mathrm{Der}^p(F)\}_{p\in\mathbf{Z}}$ of L and $\mathrm{Der}(F)$. This is an immediate consequence of Lemma 1.3. In particular, the kernel of λ must be equal to $D_L^\infty(M)$.

It is evident that if L is a real linearly compact Lie algebra, and $\lambda\colon (L, M) \to \mathrm{Der}(F)$ is a transitive representation of L on a real formal power series ring F with isotropy subalgebra M, then the complexification $\lambda_{\mathbf{C}}$ of λ is a transitive complex representation

$$\lambda_{\mathbf{C}}\colon (L_{\mathbf{C}}, M_{\mathbf{C}}) \to \mathrm{Der}(F_{\mathbf{C}})$$

of $L_{\mathbf{C}}$ on the complex power series ring $F_{\mathbf{C}}$ with isotropy subalgebra $M_{\mathbf{C}}$.

Concerning the existence and uniqueness of transitive representations we have the following basic result.

THEOREM 1.3. *Let L be a linearly compact Lie algebra.*

(i) *Let M be an open subalgebra of L, and set $U = L/M$, $F = F\{U^*\}$. Then, there exists a transitive representation $\lambda\colon (L, M) \to \mathrm{Der}(F)$ of L on F with isotropy subalgebra M.*

(ii) *Let M' and M'' be open subalgebras of L such that $M' \supset M''$ and let*

$$\lambda'\colon (L, M') \to \mathrm{Der}(F'), \quad \lambda''\colon (L, M'') \to \mathrm{Der}(F'')$$

be transitive representations of L on formal power series rings F' and F'' having isotropy subalgebras M' and M'', respectively. Then, there exists a unique unitary monomorphism $\varphi\colon F' \to F''$ such that, for all $\xi \in L$ and $f \in F'$,

$$\varphi(\lambda'(\xi)\cdot f) = \lambda''(\xi)\cdot\varphi(f).$$

Moreover, the mapping φ is filtration-preserving, that is, $F'^l = \varphi^{-1}(F''^l)$ for all $l \in \mathbf{Z}$. In particular, if $M' = M''$ then φ is an isomorphism.

Part (i) of this theorem appears in the foundational paper [**15**, Theorem III] of V. W. Guillemin and S. Sternberg. Part (ii) is an extension of the uniqueness statement given in conjunction with (i) by Guillemin and Sternberg; a proof of (ii) can be found, for example in [**5**, Proposition 5.2]. As a consequence of (i), any transitive Lie algebra (L, L^0) can be realized as a transitive subalgebra of $\mathrm{Der}(F\{(L/L^0)^*\})$; such a realization is determined, up to the action of an automorphism of $F\{(L/L^0)^*\}$, by the choice of a fundamental subalgebra L^0 to serve as isotropy subalgebra, according to the last statement of (ii).

We also can give a formal geometric interpretation to part (ii) of the preceding theorem. Let n and m be the codimensions in L of the open subalgebras M' and M''

appearing in (ii), and view $\lambda'(L)$ and $\lambda''(L)$ as transitive subalgebras of formal vector fields at the origin in K^n and K^m, respectively. Then the monomorphism φ given by (ii) is the pullback mapping

$$\Phi^*: K[[x_1,\ldots,x_n]] \to K[[x_1,\ldots,x_m]]$$

associated to a formal power series submersion Φ: $K^m \to K^n$, with $\Phi(0) = 0$. Each formal vector field $\lambda''(\xi)$, for $\xi \in L$, is Φ-related to $\lambda'(\xi)$; that is,

$$\lambda'(\xi) \circ \Phi = \Phi_* \lambda''(\xi).$$

H. Goldschmidt [7] has proved that these constructions can be made analytic: we can identify F' and F'' with $K[[x_1,\ldots,x_n]]$ and $K[[x_1,\ldots,x_m]]$ in such a way that the formal submersion Φ converges in a neighborhood of the origin, and those elements of $\lambda'(L)$ and $\lambda''(L)$ which are convergent comprise dense subalgebras in $\lambda'(L)$ and $\lambda''(L)$.

It is convenient to recast part (ii) of Theorem 1.3 in terms of the closed unitary subalgebras invariant under a transitive representation. In this discussion we follow the unpublished notes of V. W. Guillemin [**14**].

Let F be a formal power series ring. Then a transitive subalgebra L of $\mathrm{Der}(F)$ leaves no ideals of F invariant except $\{0\}$ and F itself. This statement, which is trivial to verify, suggests an apparent extension of the notion of a transitive representation to arbitrary linearly compact local algebras. Let A be a unitary local algebra which is linearly compact in the topology defined by taking as a fundamental system of neighborhoods of 0 the powers $A^l = (A^0)^{l+1}$ of the maximal ideal A^0 of A, and such that A/A^0 is isomorphic to our ground field K. If L is a linearly compact Lie algebra, then an appropriate candidate for a transitive representation of L on A is a homomorphism λ: $L \to \mathrm{Der}(A)$ such that the subalgebra $M = \{\xi \in L | \lambda(\xi)(A^0) \subset A^0\}$ is open in L, and no ideals of A except $\{0\}$ and A itself remain invariant under $\lambda(L)$. For such a representation λ to exist, however, the algebra A actually must be a formal power series ring, and λ must be a transitive representation of L in our usual sense. One way to see this is to choose a transitive representation μ: $(L, M) \to \mathrm{Der}(F)$ of L with isotropy subalgebra M on a formal power series ring F; the existence of such a representation is guaranteed by (i) of Theorem 1.3. The proof given for Proposition 5.2 of [**5**] applies without any essential change to show that there exists a unitary monomorphism φ: $A \to F$ such that $\varphi^{-1}(F^0) = A^0$, and, for all $\xi \in L$ and $a \in A$,

$$\mu(\xi) \cdot \varphi(a) = \varphi(\lambda(\xi) \cdot a). \tag{1.2}$$

Because φ preserves multiplication, we must have $\varphi(A^l) \subset F^l$ for all $l \geqslant 1$, and thus φ is continuous; from (v) of Proposition 1.1 we see now that $\varphi(A)$ is closed in F. Therefore, to show that φ is an isomorphism it will suffice to prove that $\varphi(A)$ is dense in F. We shall, in fact, demonstrate that

$$\varphi(A^0) + F^1 = F^0; \tag{1.3}$$

since any complement to F^1 in F^0 generates a dense subalgebra of F (together with the scalars K), this will suffice to complete the proof. If (1.3) were false, there would exist an element ξ of L not lying in M such that

$$\mu(\xi)\cdot\varphi(A^0)\subset F^0,$$

because (F^0/F^1) acts as $(L/M)^*$ under the representation μ. From (1.2), we could conclude that

$$\varphi(\lambda(\xi)\cdot A^0)\subset F^0,$$

and thus $\lambda(\xi)\cdot A^0\subset A^0$, which contradicts our assumption that $\xi\notin M$. We have shown, therefore, that A is isomorphic to a formal power series ring F under φ, and that under this isomorphism, the mapping λ is a transitive representation of L on A in our previous sense.

Our work above enables us to study those closed unitary subalgebras which remain invariant under a transitive representation. Let F be a formal power series ring, and G a closed unitary subalgebra of F. Then G is a local algebra; indeed the maximal ideal G^0 of G is the intersection

$$G^0 = G\cap F^0. \tag{1.4}$$

To verify this statement, we observe that each element g of G not lying in G^0 can be written in the form $g = c\cdot(1-g_0)$, where $c\in K$ is a nonzero scalar and g_0 lies in G^0. Because $g\notin F^0$, the inverse g^{-1} exists in F; moreover, we can express g^{-1} as a convergent series

$$g^{-1}=\frac{1}{c}\sum_{p=0}^{\infty}(g_0)^p$$

in F. All partial sums of this series lie in G; therefore, because G is closed, the inverse g^{-1} also lies in G. This proves that G is local. Now let L be a linearly compact Lie algebra, and let $\lambda\colon (L, M)\to \mathrm{Der}(F)$ be a transitive representation of L on F whose isotropy subalgebra we denote by M. Suppose that G is invariant under $\lambda(L)$; then λ induces a homomorphism $\mu\colon L\to\mathrm{Der}(G)$, and, by (1.4), the subalgebra $N_G=\{\xi\in L\,|\,\mu(\xi)(G^0)\subset G^0\}$ contains M, and is therefore open in L. The action of $\mu(L)$ on G does not leave invariant any nontrivial ideal of G, for, if I were an invariant ideal of G not equal to $\{0\}$ or G, then I would generate in F a nontrivial ideal, by (1.4), invariant under $\lambda(L)$; this is impossible, because $\lambda(L)$ is a transitive subalgebra of $\mathrm{Der}(F)$. From our last paragraph, we conclude that G is a formal power series ring, and that μ is a transitive representation of L on G with isotropy subalgebra N_G. Because $N_G\supset M$, we see from the definition of μ that the inclusion mapping $G\to F$ is the unique monomorphism φ associated to μ and λ by (ii) of Theorem 1.3; also, by that result, we have $G^l = G\cap F^l$ for all $l\in\mathbf{Z}$. Finally, we shall establish that each open subalgebra N of L containing M is the isotropy subalgebra N_H of exactly one closed unitary subalgebra H of F invariant under $\lambda(L)$. To this end, given an open subalgebra $N\supset M$, choose a transitive representation

$$\lambda'\colon (L, N)\to\mathrm{Der}(F')$$

of L on a formal power series ring F', with isotropy subalgebra N. Such a representation exists by part (i) of Theorem 1.3, and, according to (ii) of that theorem, there exists a unique monomorphism φ: $F' \to F$ which intertwines the actions of λ' and λ. The image $H = \varphi(F')$ is therefore a closed unitary subalgebra of F invariant under $\lambda(L)$, and its isotropy subalgebra N_H is equal to N. Moreover, by the last statement in (ii) of Theorem 1.3, the representation λ' is uniquely determined up to the action of an automorphism of F'; therefore, the invariant subalgebra H is determined by N, and does not depend on λ'.

For reference, we record the observations made above.

THEOREM 1.4. *Let L be a linearly compact Lie algebra and M an open subalgebra of L. Suppose that λ: $(L, M) \to \mathrm{Der}(F)$ is a transitive representation of L with isotropy subalgebra M on a formal power series ring F. Let G be a closed unitary subalgebra of F which is invariant under $\lambda(L)$. Then G is itself a formal power series ring, with*

$$G^l = G \cap F^l \tag{1.5}$$

for all $l \in \mathbf{Z}$; moreover, the Lie algebra L is transitively represented on G by λ, and the isotropy subalgebra

$$N_G = \{\xi \in L | \lambda(\xi)(G^0) \subset G^0\}$$

of this representation contains M. For each open subalgebra N of L containing M, there exists one and only one invariant closed unitary subalgebra H of F such that $N = N_H$.

We often call N_G the isotropy subalgebra of the λ-invariant subring G, in the sequel.

In a formal power series ring F, a closed unitary subalgebra G which is itself a formal power series ring satisfying (1.5) is said to be a *formal foliation*. For each formal foliation G, we can find a formal system of coordinates $(f_1, \ldots, f_n)$ for F such that, in the resulting identification of F with $K[[x_1, \ldots, x_n]]$, the subring G is identified with $K[[x_1, \ldots, x_m]]$, for some integer m with $0 \leqslant m \leqslant n$.

Let L be a transitive Lie algebra. A fundamental subalgebra L^0 of L is called a *primitive subalgebra* if L^0 is maximal among the proper open subalgebras of L, and a transitive Lie algebra which possesses a primitive subalgebra is called a *primitive Lie algebra*. According to Theorem 1.4, a transitive Lie algebra L is primitive if and only if L can be realized as a transitive subalgebra of $\mathrm{Der}(F)$, for some formal power series ring F, in such a way that L preserves no nontrivial formal foliation of F; this is the classical definition of primitivity. Any simple linearly compact Lie algebra L is primitive, because any maximal proper open subalgebra of L is also fundamental.

The finite-dimensional primitive Lie algebras were classified by Morozov [**19**]; a treatment in English appears in [**11**]. Enumeration of the primitive subalgebras of a simple Lie algebra is quite involved (for a discussion, with references, see [**11**]); we shall need only the classification of primitive subalgebras for finite-dimensional primitive Lie algebras which are not simple, which also is due to Morozov.

THEOREM 1.5. *Let L be a finite-dimensional Lie algebra.*

(i) *If L is abelian, then L is primitive if and only if the dimension of L is equal to one; in this case, $\{0\}$ is the only primitive subalgebra of L.*

(ii) *If L is simple, then L is primitive.*

(iii) *Assume that L is semisimple but not simple. Then L is primitive if and only if it is the direct sum $L = g_1 \oplus g_2$ of two isomorphic simple ideals g_1 and g_2. In this case, for each isomorphism $\varphi: g_1 \to g_2$ a primitive subalgebra is given by the graph*

$$L_\varphi^0 = \{\xi + \varphi(\xi) | \xi \in g_1\}$$

of φ, and there is obtained in this way a bijective correspondence between the set of isomorphisms $g_1 \to g_2$ and the set of primitive subalgebras of L.

(iv) *Assume that L is nonabelian but not semisimple. Then L is primitive if and only if L is the direct sum $L = V \oplus g$ of a nonzero abelian ideal V and a nonzero subalgebra g which acts faithfully and irreducibly on V. In this case, the subalgebra g is reductive and is a primitive subalgebra of L, and V is both the unique minimal ideal and the only nonzero abelian ideal of L. Moreover, the elements of V are in a bijective correspondence with the primitive subalgebras of L; this correspondence is obtained by associating to each element $v \in V$ the primitive subalgebra of L*

$$g_v = \{[v, \xi] + \xi | \xi \in g\}$$

which is conjugate to g under the inner automorphism induced by v.

It is easy to determine all derivations of a finite-dimensional primitive Lie algebra L. Referring to Theorem 1.5, if L is abelian, then any linear mapping of L is a derivation; if L is of either of the semisimple types (ii) or (iii), then it is a classical result **[17]** that every derivation of L is inner, i.e., comes from the adjoint representation of L. When L is of the affine type (iv), we can use the faithful action of g on V to identify L with a Lie subalgebra of the semidirect extension $V \oplus \mathrm{End}_K(V)$ of $\mathrm{End}_K(V)$ by the abelian ideal V. Then every derivation of L comes from the normalizer of L in $V \oplus \mathrm{End}_K(V)$.

LEMMA 1.5. *Let L be a finite-dimensional primitive Lie algebra of the affine type described in* (iv) *of Theorem* 1.5, *and identify L with a Lie subalgebra of $V \oplus \mathrm{End}_K(V)$ by viewing g as a Lie subalgebra of* $\mathrm{End}_K(V)$. *Then* $\mathrm{Der}(L)$ *is realized faithfully as the normalizer of L in $V \oplus \mathrm{End}_K(V)$. Decompose the reductive Lie algebra g as $g = g_0 \oplus s$, where g_0 is the center of g and s is its maximal semisimple ideal. Write Z for the centralizer of g in* $\mathrm{End}_K(V)$. *Then,*

$$V \oplus (Z \oplus s) = \mathrm{Der}(L)$$

is the normalizer in $V \oplus \mathrm{End}_K(V)$ of L.

REMARK. When $K = \mathbf{C}$, the space Z consists of all scalar multiples of the identity transformation of V, by Schur's lemma. In the real case, however, the Lie algebra Z need not be abelian; this space, which is naturally identified with $\mathrm{End}_g(V)$, may be quaternionic.

PROOF. Let D be a derivation of L; then there exists an element $v \in V$ such that

$$(D - \mathrm{ad}(v))(g) \subset g. \tag{1.6}$$

If g is semisimple, this statement follows from Whitehead's first lemma **[17]**; otherwise, there exists a nonzero transformation $c \in g_0$. Such an element c must be an invertible linear transformation of V, by Schur's lemma, and we take v to be the unique vector such that $c(v) = D(x) \bmod g$, which is to say that $(D - \mathrm{ad}(v))(c)$ lies in g. Because $D - \mathrm{ad}(v)$ is a derivation of L and $[c, g] = \{0\}$, we have, for all $\xi \in g$,

$$[c, (D - \mathrm{ad}(v))(\xi)] = 0 \mod g,$$

from which we conclude that $(D - \mathrm{ad}(v))(\xi)$ lies in g, since c acts invertibly on V. This establishes (1.6). Next, we remark that V must be invariant under all derivations of L, since V is the only nonzero abelian ideal in L. Define a linear transformation $T \in \mathrm{End}_K(V)$ by setting $T = (D - \mathrm{ad}(v))|_V$. Then T lies in the normalizer of g; indeed, for all $\xi \in g$,

$$[\xi, T] = (D - \mathrm{ad}(v))(\xi),$$

as follows from the definition of T. By construction, the element

$$\zeta = (v - T) \in V \oplus \mathrm{End}_K(V)$$

lies in the normalizer of L, and satisfies

$$(D - \mathrm{ad}(\zeta))(V) = \{0\}, \quad (D - \mathrm{ad}(\zeta))(g) \subset g.$$

Because $[V, g] \subset V$, it now follows, for all $\xi \in g$ and $v \in V$, that

$$[v, (D - \mathrm{ad}(\zeta))(\xi)] = 0;$$

we conclude that $D = \mathrm{ad}(\zeta)$, since g acts faithfully on V. We thus have shown that every derivation of L comes from the normalizer of L in $V \oplus \mathrm{End}_K(V)$, and it is trivial to check that this normalizer acts faithfully on L. Clearly, the normalizer of L takes the form $V \oplus N$, where N is the normalizer of g in $\mathrm{End}_K(V)$. Moreover, the action of N on V is irreducible, because $g \subset N$; therefore, the adjoint representation of N is completely reducible **[17]**. Because g is an ideal in N, it follows that there exists an ideal of N complementary to (and commuting with) g. This completes the proof.

Before we can discuss the structure of infinite-dimensional primitive Lie algebras, we require some information concerning the endomorphism ring of a simple transitive Lie algebra. If L is a Lie algebra over K, then the *endomorphism* (or *commutator*) *ring* K_L of L is the subring of $\mathrm{End}_K(L)$ consisting of all K-linear mappings $c\colon L \to L$ such that, for all $\xi, \eta \in L$,

$$c([\xi, \eta]) = [\xi, c(\eta)].$$

Rather obviously, the scalars K are embedded in K_L. Moreover, if $[L, L] = L$ then K_L is a commutative ring, since

$$(c_1c_2)([\xi, \eta]) = c_1([\xi, c_2(\eta)]) = [c_1(\xi), c_2(\eta)] = (c_2c_1)([\xi, \eta]),$$

for all ξ, $\eta \in L$ and c_1, $c_2 \in K_L$. In **[12]**, Guillemin gave an extension to simple transitive Lie algebras of Schur's lemma, and a representation-theoretic consequence of that result.

PROPOSITION 1.2. *Let R be a simple transitive Lie algebra over K. Then the endomorphism ring K_R of R is a field which is a finite algebraic extension of K, and each element c of K_R is a linear homeomorphism of R. Thus, R has a natural structure of simple transitive Lie algebra over the field K_R.*

Let V be a vector space of finite dimension over K, and write γ for the representation of R on $R \otimes_K V$ whose action is given by

$$\gamma(\xi)(\eta \otimes v) = [\xi, \eta] \otimes v,$$

for all $\xi, \eta \in R$ and $v \in V$. Form the K_R-vector space $K_R \otimes_K V$, and write

$$\nu: R \otimes_{K_R} (K_R \otimes_K V) \to R \otimes_K V$$

for the natural K-linear isomorphism defined by

$$\nu(\xi \otimes (c \otimes v)) = (c(\xi)) \otimes v,$$

for all $\xi \in R$, $c \in K_R$ and $v \in V$. Then, for each γ-invariant subspace W of $R \otimes_K V$, there exists a K_R-subspace U of $K_R \otimes_K V$ such that $W = \nu(R \otimes_{K_R} U)$.

In view of Proposition 1.2, we often say that K_R is the *commutator field* of the simple transitive Lie algebra R.

We describe below those aspects of the structure of infinite-dimensional primitive Lie algebras to which we make later reference. An explicit and complete classification of such Lie algebras is known; see, for example, **[13]** for a classification valid over any algebraically closed field of characteristic zero, and **[22]** for the list which applies over the real numbers **R**.

THEOREM 1.6. (i) *Let L be a primitive Lie algebra of infinite dimension. Then L possesses a unique primitive subalgebra L^0.*

(ii) *Let R be a simple transitive Lie algebra and R^0 a primitive subalgebra of R. Then every derivation of R is continuous and linear over K_R. Moreover, the space* $\mathrm{Der}(R)$ *of derivations of R is a transitive Lie algebra over K_R, and*

$$\mathrm{Der}^0(R) = \{D \in \mathrm{Der}(R) | D(R^0) \subset R^0\}$$

is a primitive subalgebra of $\mathrm{Der}(R)$. *The natural injection $R \to \mathrm{Der}(R)$, which identifies R with the space of inner derivations, is a topological embedding of (R, R^0) as a transitive closed ideal of* $(\mathrm{Der}(R), \mathrm{Der}^0(R))$; *if R is finite-dimensional, then* $\mathrm{Der}(R) = R$. *The codimension (over K_R) of R in* $\mathrm{Der}(R)$ *is $\leqslant 1$; thus, the quotient* $\mathrm{Der}(R)/R$ *is abelian, and there exists a closed abelian subalgebra of* $\mathrm{Der}(R)$ *complementary to R. The linearly compact topology possessed by* $\mathrm{Der}(R)$ *as a transitive Lie algebra is the same as the weak topology* $\mathrm{Der}(R)$ *inherits as a space of linear operators on R.*

(iii) *Let L be an infinite-dimensional primitive Lie algebra and L^0 the unique primitive subalgebra of L. Then the derived ideal of L, $R = [L, L]$, is a simple transitive Lie algebra of infinite dimension. Further, the action of L on R is faithful and induces an embedding*

$$(L, L^0) \to (\mathrm{Der}(R), \mathrm{Der}^0(R))$$

of (L, L^0) as a transitive closed subalgebra (over K) of $(\mathrm{Der}(R), \mathrm{Der}^0(R))$ *which contains R; here* $\mathrm{Der}(R)$ *is given the primitive Lie algebra structure described in* (ii). *There exists a finite-dimensional subspace V of L such that $[V, L] = R$.*

Proofs of parts (i) and (iii) of Theorem 1.6 can be found in the classification paper of Guillemin **[13]**; part (ii) is essentially due to C. Freifeld **[6]**. The last statement in (ii) is an easy consequence of Chevalley's theorem (see Lemma 2.6 of **[5]**).

A formal model for Lie algebras of vector fields depending on parameters will be needed in the sequel; for this, we employ the completed tensor product. Let L be a linearly compact Lie algebra and F be a formal power series ring. Then, the ordinary tensor product $L \otimes_K F$ carries a natural Lie algebra structure, whose action on decomposable elements is expressed by the formula

$$[\xi \otimes f, \eta \otimes g] = [\xi, \eta] \otimes (fg),$$

for all $\xi, \eta \in L$ and $f, g \in F$. It is verified easily that there exists a unique structure of a linearly compact Lie algebra on the completed tensor product $L \hat{\otimes}_K F$ which extends the structure we defined above on the dense subspace $L \otimes_K F$ of $L \hat{\otimes}_K F$. By means of a formal system of coordinates, identify F with the power series ring $K[[x_1, \ldots, x_n]]$; then elements of $L \hat{\otimes}_K F$ are identified with formal sums $\sum_{\alpha \in \mathbf{N}^n} \xi_\alpha x^\alpha$ whose coefficients ξ_α lie in L. Here, the usual multi-index notation

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n}$$

has been employed. Given two elements of $L \hat{\otimes}_K F$,

$$\xi = \sum_{\alpha \in \mathbf{N}^n} \xi_\alpha x^\alpha, \quad \eta = \sum_{\beta \in \mathbf{N}^n} \eta_\beta x^\beta,$$

the Lie bracket in $L \hat{\otimes}_K F$ is expressed as

$$[\xi, \eta] = \sum_{\gamma \in \mathbf{N}^n} \Big(\sum_{\alpha + \beta = \gamma} [\xi_\alpha, \eta_\beta] \Big) x^\gamma,$$

as it is easy to check.

The Lie algebra $\operatorname{Der}(F)$ is represented naturally by derivations on $L \otimes_K F$; this action is expressed, on decomposable elements, by

$$\zeta \cdot (\xi \otimes f) = \xi \otimes (\zeta \cdot f),$$

for all $\xi \in L$, $f \in F$ and $\zeta \in \operatorname{Der}(F)$. Again, this action extends uniquely to yield a structure of a linearly compact topological $\operatorname{Der}(F)$-module on the completed tensor product $L \hat{\otimes}_K F$. Since $\operatorname{Der}(F)$ acts by derivations of the Lie algebra structure of $L \hat{\otimes}_K F$, we can form, for any closed subalgebra A of $\operatorname{Der}(F)$, the semidirect product $(L \hat{\otimes}_K F) \oplus A$, which is a linearly compact topological Lie algebra. When L is a transitive Lie algebra and A is a transitive subalgebra of $\operatorname{Der}(F)$, this construction results in a transitive Lie algebra structure on the semidirect product (see, for example, [**5**]).

LEMMA 1.6. *Let (L, L^0) be a transitive Lie algebra and F a formal power series ring. Suppose that (A, A^0) is a transitive Lie subalgebra of* $\operatorname{Der}(F)$. *Then, the semidirect product*

$$S = (L \hat{\otimes}_K F) \oplus A$$

is a transitive Lie algebra, and the subalgebra

$$S^0 = \big((L^0 \hat{\otimes}_K F) + (L \hat{\otimes}_K F^0) \big) \oplus A^0$$

is a fundamental for S.

In constructing homomorphisms and realizations of transitive Lie algebras, an essential role is played by the cohomology of a complex introduced into the study of transitive Lie pseudogroups by D. C. Spencer [**24**]. Let V be a finite-dimensional vector space. Over the symmetric algebra $S(V)$, we consider those modules $G = \bigoplus_{p\in\mathbf{Z}} G_p$ which are *graded*, in the sense that $S^q(V)\cdot G_p \subset G_{p-q}$ for all $p, q \in \mathbf{Z}$. If U is a vector space over K, then $\operatorname{Hom}_K(S(V), U)$ has a natural $S(V)$-module structure obtained from multiplication in $S(V)$,

$$(a\cdot T)(b) = T(a\cdot b)$$

for all $a, b \in S(V)$ and $T \in \operatorname{Hom}_K(S(V), U)$. The tensor product

$$U\otimes_K S(V^*) = \bigoplus_{p\in\mathbf{Z}} (U\otimes_K S^p(V^*))$$

is identified with a submodule, graded in the sense above, of $\operatorname{Hom}_K(S(V), U)$. This graded module and its submodules will be encountered frequently in the sequel.

In a graded $S(V)$-module G, multiplication by $V\subset S(V)$ defines a natural mapping

$$\delta\colon G_p \to G_{p-1}\otimes_K \Lambda^1 V^* \simeq \operatorname{Hom}_K(V, G_{p-1}),$$

for all $p\in\mathbf{Z}$. We extend this definition to make $G\otimes_K\Lambda(V^*)$ a bigraded complex, defining

$$\delta\colon G_p\otimes_K \Lambda^q V^* \to G_{p-1}\otimes_K\Lambda^{q+1}V^*$$

in bidegree (p, q) by specifying that, for all $g\in G_p$ and $\omega\in\Lambda^q V^*$,

$$\delta(g\otimes\omega) = (\delta(g))\wedge\omega.$$

For each $v\in V$, write $\delta_v\colon G_p\to G_{p-1}$ for multiplication by v in the $S(V)$-module G. Then the definition of δ just given amounts to

$$\delta(\beta)(v_0\wedge\cdots\wedge v_q) = \sum_{j=0}^{q}(-1)^j\delta_{v_j}\big(\beta(v_0\wedge\cdots\wedge\hat{v}_j\wedge\cdots\wedge v_q)\big),$$

for all $\beta\in G_p\otimes_K\Lambda^q V^*$ and $v_0,\ldots,v_q\in V$; the circumflex $\hat{}$ in this formula indicates that the underlying term has been deleted. A trivial computation shows now that $\delta\circ\delta = 0$; that is, the sequence

$$G_{p+1}\otimes_K\Lambda^{q-1}V^* \xrightarrow{\delta} G_p\otimes_K\Lambda^q V^* \xrightarrow{\delta} G_{p-1}\otimes_K\Lambda^{q+1}V^*$$

is a complex, for all $p, q\in\mathbf{Z}$. We call the bigraded complex $\{G_p\otimes\Lambda^q V^*, \delta\}_{p,q\in\mathbf{Z}}$ the *Spencer complex* of the graded module G, and we denote by $H^{p,q}(G)$ the cohomology of this complex at $G_p\otimes_K\Lambda^q V^*$. This bigraded cohomology $\{H^{p,q}(G)\}_{p,q\in\mathbf{Z}}$ is called the *Spencer cohomology* of G.

The computation of Spencer cohomology, for those cases which arise later in this paper, is summarized below. These results are well known (see, for example, [**24**]).

LEMMA 1.7. (i) *Let V be a finite-dimensional vector space and W^* a subspace of the dual space V^*. Suppose that U is a (not necessarily finite-dimensional) vector space, and form the graded $S(V)$-module $U\otimes_K S(W^*)$. Then, for all $j>0$ and $l\geqslant 0$,*

$$H^{j,l}(U\otimes_K S(W^*)) = \{0\}.$$

Write V_0 for the annihilator $(W^*)^\perp$ of W^* in V. Then, for all $l > 0$, a $(0, l)$-cochain $\omega \in U \otimes_K \Lambda^l V^*$ is a cocycle; the corresponding cohomology class

$$[\omega] \in H^{0,l}(U \otimes_K S(W^*))$$

vanishes if and only if, for all $v_1, \dots, v_l \in V_0$,

$$\omega(v_1 \wedge v_2 \wedge \cdots \wedge v_l) = 0.$$

In particular, if $W^* = V^*$, then the cohomology groups $H^{j,l}(U \otimes_K S(V^*))$ vanish in all bidegrees (j, l) except $(0,0)$.

(ii) *Let V be a finite-dimensional real vector space and W^* a complex subspace of the complex dual space $(V_{\mathbf{C}})^*$. Suppose that U is a (not necessarily finite-dimensional) complex vector space and form the graded $S(V)$-module $U \otimes_{\mathbf{C}} S(W^*)$. Then, for all $j > 0$ and $l \geqslant 0$,*

$$H^{j,l}(U \otimes_{\mathbf{C}} S(W^*)) = \{0\}.$$

Write $(V_{\mathbf{C}})_0$ for the annihilator $(W^*)^\perp$ of W^* in $V_{\mathbf{C}}$. For all $l > 0$, each $(0, l)$-cochain $\omega \in U \otimes_{\mathbf{R}} \Lambda^l V^*$ is a cocycle and regarded as an alternating l-linear map $V \times \cdots \times V \to U$, determines a unique complexification

$$\omega_{\mathbf{C}} \in U \otimes_{\mathbf{C}} \Lambda^l (V_{\mathbf{C}})^*;$$

the cohomology class of ω, $[\omega] \in H^{0,l}(U \otimes_{\mathbf{C}} S(W^*))$, vanishes if and only if, for all $v_1, \dots, v_l \in (V_{\mathbf{C}})_0$,

$$\omega_{\mathbf{C}}(v_1 \wedge \cdots \wedge v_l) = 0.$$

Let G be a graded $S(V)$-module in which there exists a smallest (possibly negative) integer m with $G_m \neq \{0\}$; shifting the graduation of G, we can suppose that m is equal to 0. Assume that

$$\delta \colon G_q \to G_{q-1} \otimes_K \Lambda^1 V^*$$

is injective in each bidegree $(q, 0)$ with $q > 0$. Then, a simple induction shows that multiplication by $S^q(V)$ in G defines an injective mapping

$$\psi_q \colon G_q \to G_0 \otimes_K S^q(V^*) \simeq \operatorname{Hom}_K(S^q(V), G_0),$$

for each $q \geqslant 0$. We can sum these maps to obtain an embedding

$$\psi \colon G \to G_0 \otimes_K S(V^*) \tag{1.7}$$

of G as a graded $S(V)$-submodule of $G_0 \otimes_K S(V^*)$. We call such a graded module G *p-acyclic*, for an integer $p \geqslant 0$, if the Spencer cohomology groups $H^{r,s}(G)$ are reduced to $\{0\}$ for all $r \geqslant 1$ and $0 \leqslant s \leqslant p$. Note that any graded module of the type considered in this paragraph is 0-acyclic and that any graded morphism between two such modules is completely determined by its homogeneous term of degree zero. We close this preliminary section with

LEMMA 1.8. *Let V be a finite-dimensional vector space and*

$$G = \bigoplus_{p \in \mathbf{Z}} G_p, \quad G' = \bigoplus_{p \in \mathbf{Z}} G'_p$$

graded $S(V)$-modules with $G_{-p} = G'_{-p} = \{0\}$ for $p > 0$. Suppose that δ is injective in both modules in all bidegrees $(p, 0)$ with $p > 0$, and that G' is 1-acyclic. Let μ_0: $G_0 \to G'_0$ be a linear map. Then, there exists a unique graded homomorphism of $S(V)$-modules μ: $G \to G'$, whose zeroth homogeneous term is μ_0, if and only if the linear mapping

$$(\mu_0 \otimes \mathrm{id}) \circ \delta \colon G_1 \to G'_0 \otimes_K \Lambda^1 V^*$$

satisfies

$$(\mu_0 \otimes \mathrm{id})(\delta(G_1)) \subset \delta(G'_1).$$

The proof of this lemma is straightforward.

2. Extensions of transitive Lie algebras. We begin by defining a filtered Lie algebra structure on closed ideals of linearly compact Lie algebras. Essentially, this construction was introduced by Guillemin in [**12**].

LEMMA 2.1. *Let L be a linearly compact Lie algebra and I a closed ideal of L. Suppose that J is a closed ideal of I; then, if the quotient I/J is a transitive Lie algebra, the normalizer of J in L, $N = N_L(J)$, is an open subalgebra of L. Assume that N is open. Define a filtration*

$$\mathscr{F} = \mathscr{F}_L(I, J) = \{I_j\}_{j \in \mathbf{Z}} \tag{2.1}$$

of I by setting $I_j = I$ for $j \leqslant 0$, $I_1 = J$ and $I_{j+1} = D_L^j(J)$ for $j \geqslant 1$. Then I is endowed with a structure of filtered Lie algebra; that is, $[I_j, I_l] \subset I_{j+l}$ for all $j, l \in \mathbf{Z}$. Moreover, $[N, I_j] \subset I_j$ for all $j \in \mathbf{Z}$; that is, the normalizer N preserves $\mathscr{F}$. Let M be an open subalgebra of L contained in N and write V for the finite-dimensional vector space L/M. For each integer j the bilinear mapping $L \times I_j \to I_{j-1}$, given by the Lie bracket, factors, since M preserves $\mathscr{F}$, to a map

$$\rho_j \colon V \times (I_j/I_{j+1}) \to (I_{j-1}/I_j).$$

Form the graded Lie algebra

$$\mathrm{gr}(I, \mathscr{F}) = \bigoplus_{j \in \mathbf{Z}} \mathrm{gr}^j(I, \mathscr{F}) = \bigoplus_{j \in \mathbf{Z}} (I_j/I_{j+1});$$

then the sum of the family $\{\rho_j\}_{j \in \mathbf{Z}}$ is naturally identified with a linear mapping

$$\rho \colon V \to \mathrm{End}_K(\mathrm{gr}(I, \mathscr{F})).$$

Regard V as an abelian Lie algebra. Then ρ is a representation of V on $\mathrm{gr}(I, \mathscr{F})$; thus, identifying the universal enveloping algebra of V with the symmetric algebra $S(V)$, there results a structure of $S(V)$-module on $\mathrm{gr}(I, \mathscr{F})$. Furthermore, this module structure is graded, in the sense that, for all $j, l \in \mathbf{Z}$,

$$S^j(V) \cdot \mathrm{gr}^l(I, \mathscr{F}) \subset \mathrm{gr}^{l-j}(I, \mathscr{F}),$$

and $\mathrm{gr}^{-l}(I, \mathscr{F}) = \{0\}$, for all $l > 0$. In the Spencer complex of $\mathrm{gr}(I, \mathscr{F})$, the operator

$$\delta \colon \mathrm{gr}^l(I, \mathscr{F}) \to \mathrm{gr}^{l-1}(I, \mathscr{F}) \otimes_K \Lambda^1 V^*$$

is injective for all $l > 0$. *The monomorphism of* $S(V)$*-modules*

$$\psi\colon \operatorname{gr}(I, \mathscr{F}) \to (I/J) \otimes_K S(V^*) \tag{2.2}$$

defined in (1.7) *is also a monomorphism of graded Lie algebras. Endow* $\operatorname{gr}(I, \mathscr{F})$ *and* $S(V)$ *with the topologies they inherit as subspaces of the linearly compact products*

$$\prod_{l \in \mathbf{Z}} \operatorname{gr}^l(I, \mathscr{F}), \quad \prod_{l \in \mathbf{Z}} S^l(V^*),$$

respectively. Then $\operatorname{gr}(I, \mathscr{F})$ *is a graded topological Lie algebra and topological* $S(V)$*-module; therefore, each operator of the Spencer complex*

$$\delta\colon \operatorname{gr}^j(I, \mathscr{F}) \otimes_K \Lambda^l V^* \to \operatorname{gr}^{j-1}(I, \mathscr{F}) \otimes_K \Lambda^{l+1} V^*$$

is continuous, for all $j, l \in \mathbf{Z}$.

PROOF. We shall be fairly brief since a detailed description of a construction very similar to the present one exists in [**12**, §6 and **5**, §4].

Assume first that I/J is a transitive Lie algebra, and let S be the preimage in I of a fundamental subalgebra of I/J. Then,

$$J = D_I^\infty(S) = \bigcap_{l \geq 0} D_I^l(S).$$

The normalizer $N_L(S)$ of S in L is an open subalgebra of L, by part (i) of Lemma 1.4; moreover, according to part (ii) of that lemma each of the normalizers $N_L(D_I^l(S))$ contains $N_L(S)$. Thus,

$$N = N_L(J) \supset \bigcap_{l \geq 0} N_L\big(D_I^l(S)\big) \supset N_L(S),$$

which shows that N is an open subalgebra of L.

For the remainder of the proof, we do not assume that I/J is transitive. Part (ii) of Lemma 1.2 and part (iii) of Lemma 1.4 show that the filtration $\mathscr{F}$, as we have defined it in (2.1), endows I with a structure of filtered Lie algebra, and that each closed ideal I_j of the filtration $\mathscr{F}$ is stable under the adjoint action of N. Assuming the existence of an open subalgebra M of L contained in N, the definition of the $S(V)$-module structure of $\operatorname{gr}(I, \mathscr{F})$ outlined in the lemma is, for the most part, self-explanatory; we note only that the reason that ρ is a representation of V, as an abelian Lie algebra, lies in Jacobi's identity. To prove that the embedding ψ, defined in (2.2), is a Lie algebra homomorphism, we remark first that each mapping $\rho(v)$, for $v \in V$, is a derivation of the Lie algebra $\operatorname{gr}(I, \mathscr{F})$, again by Jacobi's identity. In the $S(V)$-module structure of $(I/J) \otimes_K S(V^*)$, the space V also acts by derivations of the Lie algebra structure, as is seen at once. From these observations, we prove that, for all $p, q \geq 0$ and $a \in \operatorname{gr}^p(I, \mathscr{F})$ and $b \in \operatorname{gr}^q(I, \mathscr{F})$,

$$[\psi(a), \psi(b)] = \psi([a, b]),$$

by induction on $p + q$. The case $p + q = 0$ holds because ψ is the identity mapping $\operatorname{gr}^0(I, \mathscr{F}) \to I/J$ in degree zero. If the result has been established for $p + q - 1$, then because ψ is a morphism of $S(V)$-modules and V acts by derivations of degree

-1 in both graded Lie algebras involved, we have, for all $a \in \mathrm{gr}^p(I, \mathscr{F})$, $b \in \mathrm{gr}^q(I, \mathscr{F})$ and $v \in V$,

$$\begin{aligned} v \cdot \psi([a,b]) &= \psi(\rho(v)[a,b]) = \psi([\rho(v)a, b] + [a, \rho(v)b]) \\ &= [\psi(\rho(v)a), \psi(b)] + [\psi(a), \psi(\rho(v)b)] \\ &= [v \cdot \psi(a), \psi(b)] + [\psi(a), v \cdot \psi(b)] \\ &= v \cdot [\psi(a), \psi(b)]. \end{aligned}$$

This is equivalent to saying that

$$\delta(\psi([a,b])) = \delta([\psi(a), \psi(b)]);$$

since δ is injective in $(I/J) \otimes_K S(V^*)$ in bidegree $(p+q, 0)$, our result is established by induction on $p+q$. The topological $S(V)$-module structure of $\mathrm{gr}(I, \mathscr{F})$ is obtained immediately from continuity of the Lie bracket in L. This concludes the proof.

In Theorem 2.1, which appears below, we consider essentially the following question: Let (L, L^0) be a transitive Lie algebra, and suppose that

$$0 \to I' \to L' \overset{\pi'}{\to} L \to 0,$$

$$0 \to I'' \to L'' \overset{\pi''}{\to} L \to 0$$

are linearly compact extensions of L by closed ideals I' and I''. Write M' and M'' for the preimages in L' and L'' of L^0, and let J' and J'' be closed ideals of I' and I'' such that

$$[M', J'] \subset J', \quad [M'', J''] \subset J''.$$

Using the construction in Lemma 2.1, form the filtrations $\mathscr{F}' = \mathscr{F}_{L'}(I', J')$ and $\mathscr{F}'' = \mathscr{F}_{L''}(I'', J'')$ of I' and I'', respectively, and obtain from these the associated graded Lie algebras and $S(V)$-modules $\mathrm{gr}(I', \mathscr{F}')$ and $\mathrm{gr}(I'', \mathscr{F}'')$, where $V = L/L^0$. Suppose now that μ: $\mathrm{gr}(I', \mathscr{F}') \to \mathrm{gr}(I'', \mathscr{F}'')$ is a homomorphism of graded Lie algebras and $S(V)$-modules. By a morphism of extensions, we mean a morphism ϕ: $L' \to L''$ of topological Lie algebras such that the diagram

$$\begin{array}{ccc} L' & \overset{\phi}{\to} & L'' \\ \pi' \downarrow & & \downarrow \pi'' \\ L & \overset{\mathrm{id}}{\to} & L \end{array}$$

is commutative; such a mapping necessarily satisfies $\phi(I') \subset I''$. Then we ask: When does a morphism of extensions ϕ exist whose restriction to I',

$$\phi|_{I'}: I' \to I'',$$

is filtration preserving, and has associated graded mapping

$$\mathrm{gr}(\phi|_{I'}): \mathrm{gr}(I', \mathscr{F}') \to \mathrm{gr}(I'', \mathscr{F}'')$$

equal to μ? Assuming that $\mathrm{gr}(I'', \mathscr{F}'')$ is 2-acyclic, we offer a complete solution to our question in Theorem 2.1. This result is an analogue, for extensions of transitive

Lie algebras, of the embedding theorem for transitive Lie algebras given by D. S. Rim [**21**]. An existence theorem for extensions analogous to the result proved for truncated Lie algebra structures by Guillemin and Sternberg [**15**, Theorem II] also can be formulated; we prefer to postpone its presentation to another occasion.

THEOREM 2.1. *Let L and L' be linearly compact Lie algebras. Suppose that M and M' are open subalgebras of L and L', respectively, and set*

$$I = D_L^\infty(M), \quad I' = D_{L'}^\infty(M');$$

write $\pi\colon L \to L/I$ and $\pi'\colon L' \to L'/I'$ for the corresponding projections. Assume that there exists an embedding of transitive Lie algebras

$$\lambda\colon (L/I, M/I) \to (L'/I', M'/I').$$

Identify the finite-dimensional vector spaces L/M and L'/M' under the isomorphism $L/M \to L'/M'$ induced by λ, and write V for the common vector space so obtained; choose an injective linear mapping $\Theta\colon V \to L$ such that, for all $v \in V$, $\Theta(v) = v$ mod *M. Let J' be a closed ideal of I' such that*

$$M' \subset N_{L'}(J'), \quad D_{L'}^\infty(J') = \{0\}. \tag{2.3}$$

Assume that $\phi_0\colon L \to L'$ is a continuous linear mapping such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\phi_0} & L' \\ \pi\downarrow & & \downarrow\pi' \\ L/I & \xrightarrow[\lambda]{} & L'/I' \end{array} \tag{2.4}$$

is commutative. Set, for all $\xi, \eta \in L$,

$$\alpha_0(\xi, \eta) = [\phi_0(\xi), \phi_0(\eta)] - \phi_0([\xi, \eta]);$$

then $\alpha_0\colon L \times L \to I'$ is a continuous bilinear antisymmetric map. Suppose that, for all $\xi, \eta \in M$,

$$\alpha_0(\xi, \eta) = 0 \mod J'. \tag{2.5}$$

Then the inverse image $J = \phi_0^{-1}(J')$ is a closed ideal of I such that $M \subset N_L(J)$, and the restriction $\phi_0|_I$ induces a monomorphism of topological Lie algebras $\mu_0\colon I/J \to I'/J'$. Define, by (2.1), *filtrations*

$$\mathscr{F} = \mathscr{F}_L(I, J) = \{I_j\}_{j \in \mathbf{Z}}, \quad \mathscr{F}' = \mathscr{F}_{L'}(I', J') = \{I'_j\}_{j \in \mathbf{Z}},$$

of I and I', respectively; form the associated graded Lie algebras and $S(V)$-modules $\mathrm{gr}(I, \mathscr{F})$ *and* $\mathrm{gr}(I', \mathscr{F}')$. *In the Spencer cohomology of* $\mathrm{gr}(I', \mathscr{F}')$, *assume that $H^{j,1}(\mathrm{gr}(I', \mathscr{F}')) = \{0\}$ for all $j \geqslant 1$. If the mapping*

$$(\mu_0 \otimes \mathrm{id}) \circ \delta\colon \mathrm{gr}^1(I, \mathscr{F}) \to \mathrm{gr}^0(I', \mathscr{F}') \otimes_K \Lambda^1 V^*$$

between the Spencer complexes of $\mathrm{gr}(I, \mathscr{F})$ *and* $\mathrm{gr}(I', \mathscr{F}')$ *satisfies*

$$((\mu_0 \otimes \mathrm{id}) \circ \delta)(\mathrm{gr}^1(I, \mathscr{F})) \subset \delta(\mathrm{gr}^1(I', \mathscr{F}')), \tag{2.6}$$

then there exists a unique monomorphism of graded Lie algebras and $S(V)$-modules

$$\mu\colon \mathrm{gr}(I, \mathscr{F}) \to \mathrm{gr}(I', \mathscr{F}')$$

whose graded term of degree zero is equal to μ_0. *For* $j = 1, 2$, *write* $\rho_j\colon L' \to L'/I'_j$ *for the vector space projections of* L' *onto* L'/I'_1 *and* L'/I'_2, *respectively. Define a continuous linear mapping*

$$b_0\colon M \to \mathrm{gr}^0(I', \mathscr{F}') \otimes_K \Lambda^1 V^*$$

and an element

$$c_0 \in \mathrm{gr}^0(I', \mathscr{F}') \otimes_K \Lambda^2 V^*$$

by setting, for all $\xi \in M$ *and* $v, w \in V$,

$$(b_0(\xi))(v) = \alpha_0(\Theta(v), \xi) \mod J',$$
$$c_0(v \wedge w) = \alpha_0(\Theta(v), \Theta(w)) \mod J'.$$

Then b_0 *is independent of the choice of linear splitting* Θ; *moreover, in the Spencer cohomology of the graded* $S(V)$*-module* $\mathrm{gr}(I', \mathscr{F}')$, *the cohomology classes*

$$[b_0(\xi)] \in H^{0,1}(\mathrm{gr}(I', \mathscr{F}'))$$

of $b_0(\xi)$, *for all* $\xi \in M$ *and* $[c_0] \in H^{0,2}(\mathrm{gr}(I', \mathscr{F}'))$ *of* c_0 *depend on* ϕ_0 *only through its projection* $\rho_1 \circ \phi_0$. *Assume that* $[b_0(\xi)]$ *vanishes for all* $\xi \in M$; *then* (2.6) *is satisfied, and the cohomology class* $[c_0]$ *is also independent of* Θ. *Moreover, if*

$$e_0\colon M \to \mathrm{gr}^1(I', \mathscr{F}')$$

denotes the unique continuous linear mapping such that $\delta(e_0(\xi)) = b_0(\xi)$, *for all* $\xi \in M$, *then the difference*

$$\phi_1 = ((\rho_2 \circ \phi_0)|_M - e_0)\colon M \to L'/I'_2$$

depends only on $\rho_1 \circ \phi_0$. *Suppose, in addition, that the cohomology class* $[c_0]$ *also vanishes, and that, for all* $j \geqslant 1$,

$$H^{j,2}(\mathrm{gr}(I', \mathscr{F}')) = \{0\}.$$

Then, there exists a morphism of topological Lie algebras $\phi\colon L \to L'$ *such that, for all* $\xi \in L$,

$$\phi(\xi) = \phi_0(\xi) \mod J', \tag{2.7}$$

and, thus, the diagram

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L' \\ \pi\downarrow & & \downarrow\pi' \\ L/I & \xrightarrow[\lambda]{} & L'/I' \end{array} \tag{2.8}$$

is commutative. When restricted to M, *this morphism* ϕ *satisfies*

$$\phi|_M(\xi) = \phi_1(\xi) \mod I'_2 \tag{2.9}$$

for all $\xi \in M$. *Moreover,* $I = \phi^{-1}(I')$ *and the restriction* $\phi|_I$ *is filtration-preserving; that is, for all* $j \in \mathbf{Z}$,

$$I_j = \phi^{-1}(I'_j). \tag{2.10}$$

In particular, the kernel of ϕ *is* $D_L^\infty(J)$. *The graded morphism associated to* ϕ,

$$\operatorname{gr}(\phi)\colon \operatorname{gr}(I, \mathscr{F}) \to \operatorname{gr}(I', \mathscr{F}'),$$

is equal to μ.

Furthermore, suppose that $\tilde{I}$ *and* $I^{\#}$ *are closed ideals of L and L′ contained in I and I′, respectively. Write*

$$\tilde{\mathscr{F}} = \{\tilde{I}_j\}_{j\in\mathbf{Z}}, \quad \mathscr{F}^{\#} = \{I_j^{\#}\}_{j\in\mathbf{Z}}$$

for the filtrations of $\tilde{I}$ *and* $I^{\#}$ *obtained from the intersections*

$$\tilde{I}_j = \tilde{I} \cap I_j, \quad I_j^{\#} = I^{\#} \cap I_j',$$

for all $j \in \mathbf{Z}$; *then* $\operatorname{gr}(\tilde{I}, \tilde{\mathscr{F}})$ *and* $\operatorname{gr}(I^{\#}, \mathscr{F}^{\#})$ *are canonically identified with graded Lie subalgebras and* $S(V)$*-submodules of* $\operatorname{gr}(I, \mathscr{F})$ *and* $\operatorname{gr}(I', \mathscr{F}')$, *respectively. Assume that, for all* $j \geqslant 1$, $H^{j,1}(\operatorname{gr}(I^{\#}, \mathscr{F}^{\#})) = \{0\}$. *If*

$$\phi_1(\tilde{I}) \subset I^{\#}/I_2^{\#}, \tag{2.11}$$

then $\phi(\tilde{I}) \subset I^{\#}$. *In this case, the restriction* $\phi|_{\tilde{I}}$ *is filtration-preserving, i.e.,* $\tilde{I}_j = (\phi|_{\tilde{I}})^{-1}(I_j^{\#})$, *for all* $j \in \mathbf{Z}$, *and the associated graded morphism*

$$\operatorname{gr}(\phi|_{\tilde{I}})\colon \operatorname{gr}(\tilde{I}, \tilde{\mathscr{F}}) \to \operatorname{gr}(I^{\#}, \mathscr{F}^{\#})$$

coincides with the restriction to $\operatorname{gr}(\tilde{I}, \tilde{\mathscr{F}})$ *of* μ.

Proof. Because (2.4) is a commutative diagram, the values of α_0 lie in I', as we stated. From (2.3) and (2.5), we see that ϕ_0 induces a morphism of topological Lie algebras $M \to M'/J'$, and because the diagram (2.4) is commutative, we have $I = \phi_0^{-1}(I')$. The preimage $\phi_0^{-1}(J')$ is, therefore, a closed ideal J of M contained in I, and ϕ_0 induces a monomorphism $M/J \to M'/J'$; we have written μ_0 for the restriction of this map to I/J. Because $\operatorname{gr}(I', \mathscr{F}')$ is 1-acyclic, it follows from Lemmas 1.8 and 2.1 that whenever (2.6) holds there exists a unique monomorphism of graded $S(V)$-modules

$$\mu\colon \operatorname{gr}(I, \mathscr{F}) \to \operatorname{gr}(I', \mathscr{F}')$$

whose graded term of degree zero is equal to μ_0. Moreover, we then have a commutative diagram

$$\begin{array}{ccc} (I/J)\otimes_K S(V^*) & \xrightarrow{\mu_0\otimes(\mathrm{id})} & (I'/J')\otimes_K S(V^*) \\ \psi\uparrow & & \uparrow\psi' \\ \operatorname{gr}(I, \mathscr{F}) & \xrightarrow[\mu]{} & \operatorname{gr}(I', \mathscr{F}') \end{array}$$

where ψ and ψ' are the monomorphisms defined in (2.2). The map $\mu_0 \otimes (\mathrm{id})$ is obviously a monomorphism of graded Lie algebras, as are ψ and ψ', by Lemma 2.1. We thus conclude that μ is also a monomorphism of graded Lie algebras, as asserted. To check that the cohomology classes $[b_0(\xi)]$, for $\xi \in M$, and $[c_0]$ depend on ϕ_0 only through its projection $\rho_1 \circ \phi_0$, choose another continuous linear mapping $\phi_0'\colon L \to L'$ such that $\rho_1 \circ \phi_0 = \rho_1 \circ \phi_0'$. Set

$$e' = \rho_2 \circ (\phi_0' - \phi_0)\colon L \to \operatorname{gr}^1(I', \mathscr{F}'),$$

and define an element $f' \in \mathrm{gr}^1(I', \mathscr{F}') \otimes_K \Lambda^1 V^*$ by setting, for all $v \in V$,

$$f'(v) = e'(\Theta(v)).$$

Write α_0': $L \times L \to I'$ for the antisymmetric bilinear map obtained by substituting ϕ_0' for ϕ_0 in the definition of α_0. Then, for all $\xi, \eta \in L$,

$$\begin{aligned}\alpha_0'(\xi,\eta) - \alpha_0(\xi,\eta) &= [\phi_0'(\xi),\phi_0'(\eta)] - [\phi_0(\xi),\phi_0(\eta)] + \phi_0'([\xi,\eta]) - \phi_0([\xi,\eta]) \\ &= [\phi_0'(\xi),\phi_0'(\eta)] - [\phi_0(\xi),\phi_0(\eta)] \mod J' \\ &= [(\phi_0' - \phi_0)(\xi),\phi_0'(\eta)] + [\phi_0(\xi),(\phi_0' - \phi_0)(\eta)] \mod J'.\end{aligned}$$

Because (2.4) is a commutative diagram, we have, for all $v \in V$,

$$\phi_0'(\Theta(v)) = \phi_0(\Theta(v)) = v \mod M'$$

and $(\phi_0')^{-1}(M') = (\phi_0)^{-1}(M') = M$. From the definition of the $S(V)$-module structure of $\mathrm{gr}(I', \mathscr{F}')$ and (2.3), we see that, for all $\xi \in M$ and $v \in V$,

$$\begin{aligned}&\alpha_0'(\Theta(v),\xi) - \alpha_0(\Theta(v),\xi) \\ &\quad = [(\phi_0' - \phi_0)(\Theta(v)),\phi_0'(\xi)] + [\phi_0(\Theta(v)),(\phi_0' - \phi_0)(\xi)] \mod J' \\ &\quad = [\phi_0(\Theta(v)),(\phi_0' - \phi_0)(\xi)] \mod J' \\ &\quad = (\delta(e'(\xi)))(v) \mod J'.\end{aligned}$$

This shows that the effect of replacing ϕ_0 by ϕ_0' in the definition of b_0 is to replace b_0 by $b_0 + \delta \circ (e'|_M)$. Similarly, for all $v, w \in V$,

$$\begin{aligned}&\alpha_0'(\Theta(v),\Theta(w)) - \alpha_0(\Theta(v),\Theta(w)) \\ &\quad = [(\phi_0' - \phi_0)(\Theta(v)),\phi_0'(\Theta(w))] + [\phi_0(\Theta(v)),(\phi_0' - \phi_0)(\Theta(w))] \mod J' \\ &\quad = -(\delta(f'(v)))(w) + (\delta(f'(w)))(v) \mod J' \\ &\quad = (\delta f')(v \wedge w) \mod J',\end{aligned}$$

which shows that c_0 is replaced by $c_0 + \delta f'$ when ϕ_0 is replaced by ϕ_0'. Now we check the effect of changing the linear splitting Θ. Let Θ': $V \to L$ be another such linear splitting; then, by (2.5) we have, for all $\xi \in M$ and $v \in V$,

$$\alpha_0((\Theta'(v) - \Theta(v)),\xi) = 0 \mod J'.$$

This shows that the linear mapping b_0 does not depend on Θ. Now assume that the cohomology class $[b_0(\xi)]$ vanishes, for all $\xi \in M$. We showed above that this condition depends only on $\rho_1 \circ \phi_0$; in fact, in terms of the notation employed above, if e_0': $M \to \mathrm{gr}^1(I', \mathscr{F}')$ is the continuous linear mapping which is obtained by replacing ϕ_0' for ϕ_0 in the definition of e_0, then, because δ is injective in bidegree $(1,0)$, $e_0' = e_0 + e'$. Thus, by the definition of e',

$$(\rho_2 \circ \phi_0')|_M - e_0' = (\rho_2 \circ \phi_0)|_M - e_0 \mod I_2',$$

which shows that ϕ_1 depends only on $\rho_1 \circ \phi_0$. To check the effect of changing Θ upon $[c_0]$, under these hypotheses, set

$$h = e_0 \circ (\Theta' - \Theta) \in \mathrm{gr}^1(I', \mathscr{F}') \otimes_K \Lambda^1 V^*.$$

We have already proved that b_0 is independent of Θ; thus, for all $v, w \in V$,

$$\begin{aligned}\alpha_0(\Theta'(v),\Theta'(w)) - \alpha_0(\Theta_0(v),\Theta(w)) \\ &= \alpha_0(\theta'(v),(\Theta'-\Theta)(w)) + \alpha_0((\Theta'-\Theta)(v),\Theta(w)) \\ &= b_0((\Theta'-\Theta)(w))(v) - b_0((\Theta'-\Theta)(v))(w) \mod J' \\ &= (\delta(h(w)))(v) - (\delta(h(v)))(w) \mod J' \\ &= (\delta h)(v \wedge w) \mod J',\end{aligned}$$

and so the effect of replacing Θ by Θ' is to replace c_0 by $c_0 + \delta h$. To show that our hypotheses imply (2.6), write π_2 and π_2' for the projections of J and J' onto $\mathrm{gr}^1(I, \mathscr{F})$ and $\mathrm{gr}^1(I', \mathscr{F}')$, respectively. Then, for all $\xi \in J$ and $v \in V$,

$$\begin{aligned}b_0(\xi)(v) &= [\phi_0(\Theta(v)),\phi_0(\xi)] - \phi_0([\Theta(v),\xi]) \\ &= \delta((\pi_2' \circ \phi_0|_J)(\xi))(v) - \mu_0(\delta(\pi_2(\xi))(v)) \mod J';\end{aligned}$$

therefore, in terms of cohomology classes we have, for all $\xi \in J$,

$$[b_0(\xi)] = -[((\mu_0 \otimes \mathrm{id}) \circ \delta)(\pi_2(\xi))].$$

Thus, the condition (2.6) is equivalent to requiring that $[b_0(\xi)]$ vanish for all $\xi \in J$; this is surely true when $[b_0(\xi)]$ vanishes for all $\xi \in M$, as we have assumed.

We now take up the construction of ϕ. From the mapping ϕ_0, we shall exhibit a sequence of continuous linear mappings ϕ_l: $L \to L'$ for all integers $l \geqslant 0$, which is obtained by induction on l, and has the following properties:

(i) For all $l \geqslant 0$ and $\xi \in L$, we have $\phi_{l+1}(\xi) = \phi_l(\xi) \mod I_l'$, and therefore the diagram

$$\text{(2.12)} \qquad \begin{array}{ccc} L & \xrightarrow{\phi_l} & L \\ \pi\downarrow & & \downarrow\pi' \\ L/I & \xrightarrow[\lambda]{} & L'/I' \end{array}$$

is commutative.

(ii) Define a continuous bilinear antisymmetric mapping α_l: $L \times L \to I'$, for each $l \geqslant 0$, by setting, for all $\xi, \eta \in L$,

$$\alpha_l(\xi,\eta) = [\phi_l(\xi),\phi_l(\eta)] - \phi_l([\xi,\eta]).$$

Then, for all $\xi, \eta \in L$ and $l \geqslant 0$, we have $\alpha_l(\xi, \eta) = 0 \mod I_l'$.

(iii) For each integer j with $0 \leqslant j \leqslant l+1$, the relation $I_j = \phi_l^{-1}(I_j')$ is satisfied by ϕ_l, for all $l \geqslant 0$.

Prior to constructing the sequence $\{\phi_l\}_{l \geqslant 0}$, we now indicate how the morphism ϕ will be obtained from this sequence. The filtration $\mathscr{F}'$ consists of a descending chain $\{I_l'\}_{l \geqslant 0}$ of closed ideals of I' which, according to the second part of (2.3), satisfies $\bigcap_{l \geqslant 0} I_l' = \{0\}$. We may therefore apply Chevalley's theorem to conclude from (i) that the sequence $\{\phi_l\}_{l \geqslant 0}$ converges uniformly to a continuous linear mapping $\phi = \lim_{l \to \infty} \phi_l$: $L \to L'$ such that (2.8) forms a commutative diagram, and from (ii) that ϕ is a Lie algebra homomorphism. Clearly, then,

$$\phi(\xi) = \phi_l(\xi) \qquad \mod I_{l+1}',$$

for all $\xi \in L$ and $l \geqslant 0$; thus we have (2.7). Moreover, when we define ϕ_1 it will be such that $\phi_1|_M$ agrees, modulo I'_2, with what we have called ϕ_1 in the statement of our theorem, and therefore (2.9) will be satisfied. The filtration-preserving property (2.10) of ϕ is a direct consequence of (iii). By (2.9), the associated monomorphism of graded Lie algebras $\mathrm{gr}(\phi)$ has μ_0 for its graded term of degree zero. Moreover, because the diagram (2.8) is commutative we have, for all $v \in V$,

$$\phi(\Theta(v)) = v \mod M'.$$

Hence, by the definition of the $S(V)$-module structures of $\mathrm{gr}(I, \mathscr{F})$ and $\mathrm{gr}(I', \mathscr{F}')$, the equation

$$\phi([\Theta(v), \xi]) = [\phi(\Theta(v)), \phi(\xi)],$$

for all $v \in V$ and $\xi \in I$, implies that $\mathrm{gr}(\phi)$ is also a morphism of $S(V)$-modules. Since μ is the unique such morphism with zeroth graded term μ_0, it must be equal to $\mathrm{gr}(\phi)$.

As indicated above, our construction of the sequence $\{\phi_l\}_{l \geqslant 0}$ proceeds from ϕ_0 by induction on l. Assume that ϕ_l has been constructed. Then, according to (ii) we may define continuous linear mappings

$$a_l\colon M \times M \to \mathrm{gr}^l(I', \mathscr{F}'), \quad b_l\colon M \to \mathrm{gr}^l(I', \mathscr{F}') \otimes_K \Lambda^1 V^*,$$

and an element

$$c_l \in \mathrm{gr}^l(I', \mathscr{F}') \otimes_K \Lambda^2 V^*,$$

by setting, for all $\xi, \eta \in M$ and $v, w \in V$,

$$a_l(\xi, \eta) = \alpha_l(\xi, \eta) \mod I'_{l+1},$$
$$(b_l(\xi))(v) = \alpha_l(\Theta(v), \xi) \mod I'_{l+1},$$
$$c_l(v \wedge w) = \alpha_l(\Theta(v), \Theta(w)) \mod I'_{l+1}.$$

Observe that b_0 and c_0 coincide with the objects we have so named in the statement of our theorem. We now claim that both $b_l(\xi)$, for all $\xi \in M$, and c_l are cocycles in the Spencer complex of $\mathrm{gr}(I', \mathscr{F}')$; this is trivial for $l = 0$, since $\mathrm{gr}^{-1}(I', \mathscr{F}') = \{0\}$. Furthermore, we claim that the mapping a_l vanishes identically; for $l = 0$ this is our hypothesis (2.5), while for $l \geqslant 1$ it is equivalent to asserting that $a_l(\xi, \eta)$ is also a cocycle in the Spencer complex, for all $\xi, \eta \in M$, because δ is injective in bidegree $(l, 0)$. We may assume, therefore, that $l \geqslant 1$ in establishing our claims. To give a convenient expression for the δ-operator, we define a trilinear antisymmetric map

$$\beta_l\colon L \times L \times L \to I'$$

by setting, for all $\xi_1, \xi_2, \xi_3 \in L$,

$$\beta_l(\xi_1, \xi_2, \xi_3) = \sum_{(i,j,k)} [\phi_l(\xi_i), \alpha_l(\xi_j, \xi_k)],$$

where the sum extends over all cyclic permutations (i, j, k) of $(1, 2, 3)$. Because (2.12) is a commutative diagram, we have $\phi_l(\Theta(v)) = v \mod M'$ for all $v \in V$, and

$$[\phi_l(M), I'_l] \subset [M', I'_l] \subset I'_l.$$

From this and the antisymmetry of α_l, we obtain the following expressions for the δ-operator, for all $\xi, \eta \in M$ and $u, v, w \in V$:

$$\delta(a_l(\xi,\eta))(v) = \beta_l(\Theta(v),\xi,\eta) \mod I'_l,$$
$$\delta(b_l(\xi))(v \wedge w) = \beta_l(\Theta(v),\Theta(w),\xi) \mod I'_l,$$
$$\delta c_l(u \wedge v \wedge w) = \beta_l(\Theta(u),\Theta(v),\Theta(w)) \mod I'_l.$$

Hence, to prove our claims it will certainly suffice to show that, for all $\xi_1, \xi_2, \xi_3 \in L$,

$$\beta_l(\xi_1,\xi_2,\xi_3) = 0 \mod I'_l.$$

However, applying our inductive hypothesis (ii) we see that, for all $\xi_1, \xi_2, \xi_3 \in L$,

$$\begin{aligned}\beta_l(\xi_1,\xi_2,\xi_3) &= \sum_{(i,j,k)} \left[\phi_l(\xi_i),\left[\phi_l(\xi_j),\phi_l(\xi_k)\right]\right] - \sum_{(i,j,k)} \left[\phi_l(\xi_i),\phi_l(\left[\xi_j,\xi_k\right])\right] \\ &= \sum_{(i,j,k)} \left[\phi_l(\xi_i),\left[\phi_l(\xi_j),\phi_l(\xi_k)\right]\right] - \sum_{(i,j,k)} \phi_l(\left[\left[\xi_i,\xi_j\right],\xi_k\right])\end{aligned}$$

and both of these last cyclic sums vanish, by Jacobi's identity; this establishes our claims. We have assumed that $\mathrm{gr}(I', \mathscr{F}')$ is 2-acyclic, and that the cohomology classes $[b_l(\xi)]$, for all $\xi \in M$, and $[c_0]$ both vanish. By our cocycle claims established above and Lemma 2.1, together with parts (v) and (vi) of Proposition 1.1, there exists a continuous linear mapping e_l: $M \to \mathrm{gr}^{l+1}(I', \mathscr{F}')$, and an element $f_l \in \mathrm{gr}^{l+1}(I', \mathscr{F}') \otimes_K \Lambda^1 V^*$, such that, for all $\xi \in M$,

$$\delta(e_l(\xi)) = b_l(\xi), \quad \delta f_l = c_l; \tag{2.13}$$

because δ is injective in bidegree $(l+1, 0)$, the mapping e_l is unique. Observe that e_0 is the same map given that name in the theorem. Choose a continuous linear mapping ε which splits the exact sequences of topological vector spaces

$$0 \to I'_{l+2} \to I'_{l+1} \underset{\varepsilon}{\overset{\to}{\leftarrow}} \mathrm{gr}^{l+1}(I', \mathscr{F}') \to 0;$$

such a splitting exists by (vi) of Proposition 1.1. To define ϕ_{l+1}, we set

$$\phi_{l+1}(\xi) = \phi_l(\xi) - (\varepsilon \circ e_l)(\xi),$$

for all $\xi \in M$, and

$$\phi_{l+1}(\Theta(v)) = \phi_l(\Theta(v)) - (\varepsilon \circ f_l)(v),$$

for all $v \in V$. Since $\Theta(V)$ is a finite-dimensional vector space complementary to M in L, and e_l is continuous and M is open in L, a well-defined continuous linear mapping ϕ_{l+1}: $L \to L'$ is obtained, which obviously satisfies (i). To see that (ii) holds for ϕ_{l+1}, we first note that, for all $\xi, \eta \in L$,

$$\begin{aligned}&\alpha_{l+1}(\xi,\eta) - \alpha_l(\xi,\eta) \\ &\quad = \left[\phi_{l+1}(\xi),(\phi_{l+1} - \phi_l)(\eta)\right] + \left[(\phi_{l+1} - \phi_l)(\xi),\phi_l(\eta)\right] \\ &\qquad - (\phi_{l+1} - \phi_l)(\left[\xi,\eta\right]) \\ &\quad = \left[\phi_{l+1}(\xi),(\phi_{l+1} - \phi_l)(\eta)\right] + \left[(\phi_{l+1} - \phi_l)(\xi),\phi_l(\eta)\right] \mod I'_{l+1}.\end{aligned}$$

Therefore, for all $\xi, \eta \in M$ and $v, w \in V$, we have

$$(\alpha_{l+1} - \alpha_l)(\xi, \eta) = 0 \mod I'_{l+1},$$
$$(\alpha_{l+1} - \alpha_l)(\Theta(v), \xi) = -\delta(e_l(\xi))(v) \mod I'_{l+1},$$
$$(\alpha_{l+1} - \alpha_l)(\Theta(v), \Theta(w)) = -\delta f_l(v \wedge w) \mod I'_{l+1}.$$

Recalling that a_l vanishes identically, we see from (2.13) and the definitions of a_l, b_l and c_l that ϕ_{l+1} meets our requirement (ii). Observe that that the restriction $\phi_1|_M$ coincides with what we have called ϕ_1 in the theorem, as we promised that it would. To verify (iii) for $l+1$, we first remark that the only case in question is $j = l+2$, since (i) holds for $l+1$, and (iii) holds for l. The image $\phi_{l+1}(L)$ satisfies $\phi_{l+1}(L) + M' = L'$, since (2.12) commutes. Therefore, applying the definitions of the filtrations $\mathscr{F}$ and $\mathscr{F}'$, and (i) for $l+1$ and (iii) for $j = l+1$, we see that

$$\begin{aligned}\phi_{l+1}^{-1}(I'_{l+2}) &= \{\xi \in I_{l+1} | [L', \phi_{l+1}(\xi)] \subset I'_{l+1}\}\\ &= \{\xi \in I_{l+1} | [\phi_{l+1}(L), \phi_{l+1}(\xi)] \subset I'_{l+1}\}\\ &= \{\xi \in I_{l+1} | \phi_{l+1}([L, \xi]) \subset I'_{l+1}\}\\ &= \{\xi \in I_{l+1} | [L, \xi] \subset I_{l+1}\} = I_{l+2},\end{aligned}$$

which proves (iii) for $j = l+2$, and thus completes the inductive construction of $\{\phi_l\}_{l \geqslant 0}$.

It only remains to demonstrate the final paragraph of our theorem. Because $I^{\#}$ is a closed ideal of L', it will suffice, by Chevalley's theorem, to show that any closed ideal I of L which meets the requirement (2.11) satisfies

$$\phi(\tilde{I}) \subset I^{\#} + I'_l \tag{2.14}$$

for all $l \geqslant 0$. Moreover, according to (2.9), the condition (2.11) is equivalent to requiring that (2.14) holds for $l = 2$, which is therefore where we begin our induction. Suppose that (2.14) holds for some integer $l \geqslant 2$. Write

$$\rho_{l+1}: I'_l \to \mathrm{gr}^l(I', \mathscr{F}')$$

for the natural projection. Given $\xi \in \tilde{I}$, choose $\eta_l \in I^{\#}$ such that $\phi(\xi) = \eta_l \mod I'_l$ by our inductive hypothesis. Because (2.8) is a commutative diagram, we have $\phi(\Theta(v)) = v \mod M'$ for all $v \in V$, and therefore, in the Spencer complex of $\mathrm{gr}(I', \mathscr{F}')$,

$$\begin{aligned}\delta(\rho_{l+1}(\phi(\xi) - \eta_l))(v) &= [\phi(\Theta(v)), \phi(\xi) - \eta_l] \mod I'_l\\ &= \phi([\Theta(v), \xi]) - [\phi(\Theta(v)), \eta_l] \mod I'_l.\end{aligned}$$

But $[\Theta(v), \xi] \in \tilde{I}$ and $[\phi(\Theta(v)), \eta_l] \in I^{\#}$, whence, by our inductive hypothesis,

$$\delta(\rho_{l+1}(\phi(\xi) - \eta_l)) \in \mathrm{gr}^{l-1}(I^{\#}, \mathscr{F}^{\#}) \otimes_K \Lambda^1 V^*.$$

Because $\mathrm{gr}(I^{\#}, \mathscr{F}^{\#})$ is 1-acyclic and $l - 1 \geqslant 1$, we conclude that there exists an $\eta \in I_l^{\#}$ such that

$$\phi(\xi) = \eta_l + \eta \mod I'_{l+1},$$

which proves (2.14) for $l+1$, and thus, by induction, for all $l \geqslant 0$. This completes the proof of Theorem 2.1.

In most of our applications of Theorem 2.1, we exploit the presence of specific conditions on the structure of $\mathrm{gr}(I, \mathscr{F})$. Without further restrictions, however, we can achieve the following realization of I.

COROLLARY 2.1. *Let (L, L^0) be a transitive Lie algebra. Suppose that M is an open subalgebra of L containing L^0 and define a transitive Lie algebra*

$$(\tilde{M}, \tilde{L}^0) = \big(M/\big(D_M^\infty(L^0)\big), L^0/\big(D_M^\infty(L^0)\big)\big).$$

Set $U = L/M$, $F = F\{U^*\}$, and choose a transitive representation $\lambda\colon (L, M) \to \mathrm{Der}(F)$; write $\varepsilon\colon L/M \to U = \mathrm{Der}(F)/\mathrm{Der}^0(F)$ for the linear isomorphism induced by λ. Form the semidirect product

$$S = (\tilde{M} \hat{\otimes}_K F) \oplus \mathrm{Der}(F),$$

which is a transitive Lie algebra with fundamental subalgebra

$$S^0 = \big((\tilde{L}^0 \hat{\otimes}_K F) + (\tilde{M} \hat{\otimes}_K F^0)\big) \oplus \mathrm{Der}^0(F);$$

let $\rho\colon S \to \mathrm{Der}(F)$ be the natural projection. Then, there exists an embedding of transitive Lie algebras $\phi\colon (L, L^0) \to (S, S^0)$ such that $\rho \circ \phi = \lambda$, and

$$D_L^\infty(M) = \phi^{-1}(\tilde{M} \hat{\otimes}_K F),$$
$$D_L^\infty(M) \cap D_M^\infty(L^0) = \phi^{-1}(\tilde{M} \hat{\otimes}_K F^0).$$

Furthermore, suppose that I is a closed ideal of L contained in M, and set $J = I \cap D_M^\infty(L^0)$; then J is a closed ideal of M contained in I, and the quotient $\tilde{I} = I/J$ is identified with a closed ideal of $\tilde{M}$. Under these assumptions, the space

$$\tilde{I} \hat{\otimes}_K F \subset \tilde{M} \hat{\otimes}_K F$$

forms a closed ideal of S, and $\phi(I) \subset \tilde{I} \hat{\otimes}_K F$. Write $\mathscr{F} = \{I_l\}_{l\in\mathbf{Z}}$ for the filtration $\mathscr{F}_L(I, J)$ defined in (2.1) *and form the associated graded Lie algebra and $S(U)$-module* $\mathrm{gr}(I, \mathscr{F})$. *Then, for all $l \in \mathbf{Z}$,*

$$I_l = (\phi|_I)^{-1}(\tilde{I} \hat{\otimes}_K F^{l-1}),$$

and the associated monomorphism of graded Lie algebras $\mathrm{gr}(\phi)\colon \mathrm{gr}(I, \mathscr{F}) \to \tilde{I} \otimes_K S(U^*)$ is related to the morphism ψ defined in (2.2) *by $\mathrm{gr}(\phi) = (\mathrm{id} \otimes S(\varepsilon^{-1*})) \circ \psi$.*

PROOF. For convenience, we shall assume that the linear isomorphism ε induced by λ is equal to the identity; the general case is obtained trivially from this. We shall apply Theorem 2.1, with

$$L' = (\tilde{M} \hat{\otimes}_K F) \oplus \lambda(L), \quad M' = (\tilde{M} \hat{\otimes}_K F) \oplus \lambda(M),$$
$$I' = \tilde{M} \hat{\otimes}_K F, \quad J' = \tilde{M} \hat{\otimes}_K F^0, \quad V = U;$$

then $I'_l = \tilde{M} \hat{\otimes}_K F^l$, for $l \in \mathbf{Z}$, are the elements of the filtration $\mathscr{F}'$ of I', and thus $\mathrm{gr}(I', \mathscr{F}') = \tilde{M} \otimes_K S(U^*)$. Therefore, in all bidegrees $(p, q) \neq (0,0)$, we have, by Lemma 1.7,

$$H^{p,q}(\mathrm{gr}(I', \mathscr{F}')) = \{0\},$$

and so the only cocycle requirement we shall be obliged to meet with our initial map ϕ_0 is (2.5). Choose an injective linear mapping Θ: $U \to L$ such that, for all $u \in U$,

$$\Theta(u) = u \mod M,$$

write π: $M \to \tilde{M}$ for the natural projection, and identify $\tilde{M}$ with the closed subspace $\tilde{M} \otimes_K K$ of $\tilde{M} \hat{\otimes}_K F$; then $\tilde{M}$ is complementary to $\tilde{M} \hat{\otimes} F^0$. Define a continuous linear mapping ϕ_0: $L \to S$ by setting, for all $\xi \in M$ and $u \in U$,

$$\phi_0(\xi) = \pi(\xi) + \lambda(\xi), \quad \phi_0(\Theta(u)) = \lambda(\Theta(u)).$$

Evidently, we have $\rho \circ \phi_0 = \lambda$, and because λ and π are homomorphisms whose images $\lambda(L)$ and $\pi(M)$ commute in S, the restriction $\phi_0|_M$ is a homomorphism, and thus ϕ_0 meets our requirement (2.5). Moreover, for all $\xi \in D_L^\infty(M)$ and $u \in U$, the definition of $b_0(\xi)$ becomes

$$(b_0(\xi))(u) = -\pi([\Theta(u), \xi]),$$

and so we see that $b_0(\xi) \in \tilde{I} \otimes_K \Lambda^1 U^*$. In the filtration of $\tilde{I} \hat{\otimes}_K F$ induced from that of $\tilde{M} \hat{\otimes}_K F$, we have

$$\mathrm{gr}(\tilde{I} \hat{\otimes}_K F) = \tilde{I} \otimes_K S(U^*),$$

so again, in all bidegrees $(p, q) \neq (0,0)$,

$$H^{p,q}(\mathrm{gr}(\tilde{I} \hat{\otimes}_K F)) = \{0\}.$$

Therefore, by the injectivity of δ in degree zero, we must have

$$\phi_1(I) \subset \tilde{I} \hat{\otimes}_K F \mod \tilde{M} \hat{\otimes}_K F^1.$$

It now follows at once from Theorem 2.1 that the morphism ϕ obtained there from ϕ_0 satisfies all of our assertions.

We shall make a few observations concerning the structure of abelian ideals based on Corollary 2.1. To apply that corollary, we require a preliminary result.

LEMMA 2.2. *Let I be a nonzero closed abelian ideal of a linearly compact Lie algebra L. Choose a proper open subspace J of I. Then J is a closed ideal of I and there exists an open subalgebra M of L such that*

$$M \cap I = J, \quad M \subset N_L(J). \tag{2.15}$$

PROOF. The subspace J is obviously a closed ideal of I, because I is abelian. Moreover, by Lemma 1.4, the normalizer $N_L(J)$ is an open subalgebra of L. Thus, there exists a descending chain $\{N_j\}_{j\geq 0}$ of open subalgebras of L contained in $N_L(J)$ such that $\bigcap_{j\geq 0} N_j = \{0\}$. Each intersection $N_j \cap I$ forms an open subalgebra in I, and the chain $\{N_j \cap I\}_{j\geq 0}$ again descends to $\{0\}$. Thus, by Chevalley's theorem, there exists an integer $l \geq 0$ such that $N_l \cap I \subset J$. Because $N_l \subset N_L(J)$, the sum $M = N_l + J$ is an open subalgebra of L contained in $N_L(J)$, and we have $M \cap I = J$, by construction.

As in the lemma, we now assume that I is a nonzero abelian ideal of a linearly compact Lie algebra L. Let J be an open subspace of I and write N for the normalizer $N_L(J)$. Denote by M an open subalgebra of L satisfying (2.15); then we see at once that $I \cap D_N^\infty(M) = J$. Thus, in the transitive Lie algebra

$$(\tilde{N}, \tilde{M}) = (N/(D_N^\infty(M)), M/(D_N^\infty(M))),$$

the quotient I/J is identified with a finite-dimensional abelian ideal V such that $V \cap \tilde{M} = \{0\}$. Set $U = L/N$ and $F = F\{U^*\}$. The semidirect product

$$S = (\tilde{N} \hat{\otimes}_K F) \oplus \operatorname{Der}(F)$$

is a transitive Lie algebra in which $V \hat{\otimes}_K F$ forms a closed abelian ideal, and

$$S^0 = \left((\tilde{M} \hat{\otimes}_K F) + (\tilde{N} \hat{\otimes}_K F^0)\right) \oplus \operatorname{Der}^0(F)$$

is a fundamental subalgebra of S. Applying Corollary 2.1 to the transitive Lie algebra $(L/(D_L^\infty(M)), M/(D_L^\infty(M)))$, we obtain a morphism μ: $L \to S$ of topological Lie algebras such that $\mu(I) \subset V \hat{\otimes}_K F$; the image $\mu(L)$ is a transitive subalgebra of (S, S^0), the kernel of μ is $D_L^\infty(M)$, and the projection of μ onto $\operatorname{Der}(F)$, $\rho \circ \mu$: $L \to \operatorname{Der}(F)$, is a transitive representation of (L, N). The kernel of $\mu|_I$ is $D_L^\infty(J)$, which is properly contained in I, and if $\mathscr{F} = \{I_j\}_{j \in \mathbf{Z}}$ denotes the filtration $\mathscr{F}_L(I, J)$ defined in (2.1), then, for all $l \in \mathbf{Z}$,

$$I_l = (\mu|_I)^{-1}\left(V \hat{\otimes}_K F^{l-1}\right).$$

Thus, we have realized the abelian quotient $I/(D_L^\infty(J))$ as a space of formal V-valued functions on U and L acts on $I/(D_L^\infty(J))$ by linear differential operators, in terms of this realization. Our choice of open subspace $J \subsetneq I$ was completely arbitrary; therefore, we may choose J to be of codimension one in I, and obtain a realization of $I/(D_L^\infty(J))$ as a space of formal scalar-valued functions on U.

This construction may be applied inductively to obtain a properly nested chain

$$I = I^0 \supsetneq I^1 \supsetneq I^2 \supsetneq I^3 \supsetneq \cdots$$

of closed abelian ideals $\{I^j\}$ of L, and, for each j, an open subspace J^j of codimension one in I^j, such that $I^{j+1} = D_L^\infty(J^j)$; thus, we obtain a decomposition of I into quotients which can be realized as formal scalar-valued functions. If L is transitive, then this chain is finite, by Theorem 1.1, terminating in $I^p = \{0\}$ for some $p > 0$. Therefore, we may apply this procedure to the abelian quotients in any Jordan-Hölder sequence for a transitive Lie algebra L, obtaining a finite refinement in which all of the abelian quotients can be realized as formal scalar-valued functions. É. Cartan made a corresponding statement concerning the abelian quotients which appear in Jordan-Hölder decompositions of transitive Lie pseudogroups in his paper [**3**, pp. 144–146]. Given Lemma 2.2, these results could also be obtained from the study of abelian ideals in transitive Lie algebras made by H. Goldschmidt and D. C. Spencer in [**10**].

3. Nonabelian minimal closed ideals. We begin our study of closed ideals which are nonabelian and minimal with a discussion of graded structures associated to such ideals, following Guillemin [**12**].

PROPOSITION 3.1. *Let L be a linearly compact topological Lie algebra and I a closed ideal of L. Suppose that J is a closed ideal of I for which the quotient $R = I/J$ is a simple transitive Lie algebra. Then the normalizer $N = N_L(J)$ is an open subalgebra of L. Let M be an open subalgebra of L contained in N and set $U = L/M$. Write $\mathscr{F} = \{I_j\}_{j \in \mathbf{Z}}$ for the filtration $\mathscr{F}_L(I, J)$ of I defined in* (2.1); *form the graded Lie* *algebra and* $S(U)$*-module* $\mathrm{gr}(I, \mathscr{F})$ *and the monomorphism*

$$\psi\colon \mathrm{gr}(I, \mathscr{F}) \to R \otimes_K S(U^*)$$

of graded Lie algebras and $S(U)$*-modules defined in* (2.2). *Set* $V = K_R \otimes_K U$, *which is a finite-dimensional vector space over the commutator field* K_R, *and make the natural identification of graded Lie algebras and* $S(U)$*-modules*

$$R \otimes_K S(U^*) \simeq R \otimes_{K_R} S(V^*).$$

Then under this identification, there exists a K_R*-subspace* W^* *of* V^* *such that*

$$\psi(\mathrm{gr}(I, \mathscr{F})) = R \otimes_{K_R} S(W^*);$$

if $K_R = K$, *then* $V = U$ *and* W^* *is the annihilator of* (N/M) *in* U^*. *Set* $I_\infty = D_L^\infty(J) = \bigcap_{j\in\mathbf{Z}} I_j$. *Then, in the filtered Lie algebra* I, *the relations*

$$I_{i+j} = [I_i, I_j] + I_\infty \tag{3.1}$$

hold for all $i, j \geqslant 0$. *If* J' *is a closed ideal of* I *such that* $I \supsetneq J' \supset I_\infty$, *then* $J \supset J'$. *In particular, the quotient* I/I_∞ *is a nonabelian minimal closed ideal of* L/I_∞ *and* J/I_∞ *is the unique maximal closed ideal of* I/I_∞.

PROOF. That N is an open subalgebra of L follows immediately from Lemma 2.1. Set

$$\mathscr{I} = \psi(\mathrm{gr}(I, \mathscr{F})) \subset R \otimes_K S(U^*);$$

then $\mathscr{I}$ is a graded Lie subalgebra and $S(U)$-submodule of $R \otimes_K S(U^*)$, and the zeroth graded term $\mathscr{I}_0$ of $\mathscr{I}$ is equal to R, by the definition of ψ. Considering the action of $\mathscr{I}_0$ on each of the graded terms $\mathscr{I}_p$ of $\mathscr{I}$, we see from Proposition 1.2 that there exists a graded K_R-subspace $\mathscr{W} = \bigoplus_{p\geqslant 0} \mathscr{W}_p$ of $K_R \otimes_K S(U^*)$ such that $\mathscr{I} = R \otimes_{K_R} \mathscr{W}$, with $\mathscr{W}_0 = K$. Set $W^* = \mathscr{W}_1$. Because $[R, R] = R$, it follows that $\mathscr{W} \supset S(W^*)$; however, because $\mathscr{I}$ is an $S(V)$-submodule of $R \otimes_{K_R}(K_R \otimes_K S(V^*))$, we have

$$\delta(\mathscr{W}_j) \subset \mathscr{W}_{j-1} \otimes_K \Lambda^1 V^*,$$

for all $j \geqslant 1$. A simple induction now establishes that $\mathscr{W} \subset S(W^*)$, and thus $\mathscr{I} = R \mathbin{\hat{\otimes}}_{K_R} S(W^*)$. When $K_R = K$, a vector $v \in V$ annihilates W^* if and only if any representative ξ for v in L satisfies $[\xi, J] \subset J$, as is seen at once from the definition of $\mathscr{F}$; hence, the annihilator in V of W^* is N/M, as asserted. Again because $[R, R] = R$, we have

$$\left[R \otimes_{K_R} S^p(W^*), R \otimes_{K_R} S^q(W^*)\right] = R \otimes_{K_R} S^{p+q}(W^*);$$

that is, $[\mathscr{I}_p, \mathscr{I}_q] = \mathscr{I}_{p+q}$ for all $p, q \geqslant 0$. The relations (3.1) now follow at once from part (iii) of Theorem 1.6 and Chevalley's theorem. Let J' be a closed ideal containing I_∞. If J' is not contained in J, then, because I/J is simple, we have $I = J' + J$. This is the case $l = 1$ of a formula $I = J' + I^l$; the general case follows by induction, since if this result is known for an integer l, we have, by (3.1) and the case $l = 1$,

$$I = [I, I] + I_\infty = \left[(J' + J), (J' + I^l)\right] + I_\infty \subset J' + I^l.$$

Since J' contains I_∞, we conclude that $J' = I$ by Chevalley's theorem. This completes the proof.

COROLLARY 3.1. *Let L be a linearly compact Lie algebra and I a closed nonabelian minimal closed ideal of L. Then, there exists a unique maximal closed ideal J of I, and the quotient $R = I/J$ is a simple transitive Lie algebra.*

PROOF. According to (iii) of Lemma 1.1, open subalgebras exist in I; since any such subalgebra is of finite codimension there exists a maximal open subalgebra A of I. Then $I/(D_I^\infty(A))$ is a primitive Lie algebra, and thus it contains a maximal closed ideal $\tilde{J}$ by (iii) of Theorem 1.6. The preimage J of $\tilde{J}$ in I is a maximal closed ideal of I. Moreover, the quotient I/J is nonabelian, since otherwise, because J is closed, the inclusion $\overline{[I, I]} \subset J$ would apply to the closure of $[I, I]$; because I is a minimal closed ideal of L, we could conclude that $[I, I] = \{0\}$, contradicting our hypothesis that I is nonabelian. Thus $R = I/J$ is simple and our result follows at once from Proposition 3.1.

According to Proposition 1.2, the commutator field K_R of a simple real transitive Lie algebra R must be a finite extension of the real ground field $\mathbf{R}$; therefore, K_R is equal to $\mathbf{R}$ or $\mathbf{C}$. A familiar argument shows that $K_R = \mathbf{C}$ if and only if the complexification $R_{\mathbf{C}}$ of R is not simple; in that case, we can express $R_{\mathbf{C}}$ as the direct sum $R_{\mathbf{C}} = R'' \oplus \overline{R''}$ of two conjugate closed ideals which are simple complex transitive Lie algebras isomorphic to R. Let I be a nonabelian minimal closed ideal of a linearly compact real Lie algebra L, and let J be the unique maximal closed ideal of I. Then we shall say that I is of *real* or *complex type* according to whether the commutator field K_R of the simple real transitive Lie algebra $R = I/J$ is equal to $\mathbf{R}$ or $\mathbf{C}$. The splitting behavior of nonabelian minimal closed ideals under complexification mimics that of simple transitive Lie algebras:

LEMMA 3.1. *Let L be a real linearly compact Lie algebra and I a nonabelian minimal closed ideal of L. Write J for the unique maximal closed ideal of I, and R for the simple transitive quotient I/J.*

(i) *If I is of real type, then its complexification $I_{\mathbf{C}}$ is a nonabelian minimal closed ideal of $L_{\mathbf{C}}$.*

(ii) *If I is of complex type, then its complexification is the direct sum*

$$I_{\mathbf{C}} = I'' \oplus \overline{I''} \tag{3.2}$$

of two conjugate nonabelian minimal closed ideals I'' and $\overline{I''}$ of $L_{\mathbf{C}}$, and these two ideals are the only nonzero ideals of $L_{\mathbf{C}}$ properly contained in $I_{\mathbf{C}}$. Moreover, the intersections

$$J'' = I'' \cap J_{\mathbf{C}}, \quad \overline{J''} = \overline{I''} \cap J_{\mathbf{C}}$$

are the unique maximal closed ideals of I'' and $\overline{I''}$, respectively. Write $\pi''\colon L_{\mathbf{C}} \to L_{\mathbf{C}}/\overline{I''}$ for the natural projection, and identify I'' with its image $\pi''(I'')$. Then the restriction $\pi''|_L\colon L \to L''/\overline{I''}$ is a monomorphism of real topological Lie algebras, and $\pi''|_I\colon I \to I''$ is an isomorphism. Set

$$N = N_L(J), \quad N'' = N_{L_{\mathbf{C}}}(J'').$$

Then N and N'' are open subalgebras of L and $L_{\mathbf{C}}$, respectively, and satisfy the inclusions

$$N'' \supset N_{\mathbf{C}}, \quad N'' \cap L = N. \tag{3.3}$$

Thus, there exist precisely two complex structures of complex vector space on I for which I becomes a linearly compact topological Lie algebra and each derivation $\mathrm{ad}(\xi)|_I$, $\xi \in L$, *is complex-linear. These two complex structures result from the identification of I with I'' by $\pi''|_I$, or with $\overline{I''}$ by conjugation of that map. In terms of each complex structure, the maximal ideal J forms a complex ideal of I, and thus the entire filtration $\mathscr{F} = \mathscr{F}_L(I, J)$ defined in* (2.1) *is made up of complex ideals of I. Fix on I the complex structure which corresponds to its identification with I'', and extend the adjoint representation of L on I complex-linearly to $L_{\mathbf{C}}$. Then I becomes a complex topological $L_{\mathbf{C}}$-module, and the stabilizer of J in $L_{\mathbf{C}}$ is N''. Let M be an open subalgebra of L contained in N, and set $U = L/M$. Endow the graded Lie algebra* $\mathrm{gr}(I, \mathscr{F})$, *including* $R = \mathrm{gr}^0(I, \mathscr{F})$, *with the complex structure it inherits from I. Then, the isomorphism of graded Lie algebras and $S(U)$-modules*

$$\psi\colon \mathrm{gr}(I, \mathscr{F}) \to R \otimes_{\mathbf{C}} S(W^*)$$

defined in Proposition 3.1 *is complex-linear, and $W^* \subset (U_{\mathbf{C}})^*$ is the annihilator of the complex subspace $(N''/M_{\mathbf{C}})$ of $U_{\mathbf{C}}$.*

PROOF. Because I is a minimal closed ideal of L, there can be no ideal of L in J except $\{0\}$; thus, the complexification $J_{\mathbf{C}}$ of J also contains no ideal of $L_{\mathbf{C}}$ except $\{0\}$. If I is of real type, then $R_{\mathbf{C}} = I_{\mathbf{C}}/J_{\mathbf{C}}$ is a simple complex transitive Lie algebra. Part (i) of the lemma now follows from Proposition 3.1.

If I is of complex type, then $R_{\mathbf{C}}$ splits as a direct sum $R_{\mathbf{C}} = R'' \oplus \overline{R''}$ of conjugate simple closed ideals. The preimages in $I_{\mathbf{C}}$ of R'' and $\overline{R''}$ are distinct maximal closed ideals of $I_{\mathbf{C}}$; because $I_{\mathbf{C}}$ is nonabelian, we infer from Corollary 3.1 that $I_{\mathbf{C}}$ is not a minimal closed ideal of $L_{\mathbf{C}}$. Thus, we may choose a nonzero closed ideal I'' of $L_{\mathbf{C}}$ properly contained in $I_{\mathbf{C}}$. To prove that $I_{\mathbf{C}}$ can be expressed as the direct sum (3.2), we first consider the space $I'' \cap I$ of real vectors in I''. This space forms a closed ideal of L in I, and cannot be equal to I because I'' is a proper complex subspace of $I_{\mathbf{C}}$; therefore, we have $I'' \cap I = \{0\}$. If an element ξ of $I_{\mathbf{C}}$ lies in $I'' \cap \overline{I''}$, then so does $\bar{\xi}$, from which we see that the real and imaginary parts of ξ lie in $I'' \cap \overline{I''}$. Because I'' contains no real vectors, we conclude that $I'' \cap \overline{I''} = \{0\}$. The sum $I'' \oplus \overline{I''}$ also contains the real and imaginary parts of all of its elements. therefore, we have $(I'' \oplus \overline{I''}) \cap I = I$, because this intersection is a nonzero closed ideal of L lying in I; the expression (3.2) of $I_{\mathbf{C}}$ now follows at once. Because $I_{\mathbf{C}}$ is nonabelian, so are its summands I'' and $\overline{I''}$. We chose I'' subject only to the condition that it be a closed ideal of $L_{\mathbf{C}}$ lying properly between $I_{\mathbf{C}}$ and $\{0\}$. From this, it follows easily that I'' and $\overline{I''}$ are nonabelian minimal closed ideals, for, if A were a closed ideal of $L_{\mathbf{C}}$ lying properly between I'' and $\{0\}$, we could draw the absurd conclusion

$$I_{\mathbf{C}} = A \oplus \bar{A} \subsetneqq I'' \oplus \overline{I''} = I_{\mathbf{C}}.$$

To see that I'' and $\overline{I''}$ are the only nonzero closed ideals of $L_{\mathbf{C}}$ properly contained in $I_{\mathbf{C}}$, we use a similar argument: if A is another such closed ideal, then we can conclude that A is nonabelian and minimal, and $I_{\mathbf{C}} = A \oplus \bar{A}$, because we chose I'' arbitrarily. But, if A is distinct from both of the nonabelian minimal closed ideals I'' and $\overline{I''}$, then the intersections $A \cap I''$ and $A \cap \overline{I''}$ are both reduced to $\{0\}$. In that case, we have

$$[A, I_{\mathbf{C}}] = [A, I''] \oplus [A, \overline{I''}] \subset (A \cap I'') \oplus (A \cap \overline{I''}) = \{0\},$$

which is impossible, because A is nonabelian.

From the decomposition (3.2) of $I_{\mathbf{C}}$, it follows that there exists a unique linear isomorphism $\sigma\colon I \to I$ such that $\sigma^2 = -\mathrm{id}_I$ and

$$I'' = \{ \xi - i\sigma(\xi) | \xi \in I \};$$

by considering the action of L on I'', we see that

$$[\xi, \sigma(\eta)] = \sigma([\xi, \eta]),$$

for all $\xi \in L$ and $\eta \in I$. Because I'' is closed in $I_{\mathbf{C}}$, we infer from the closed graph theorem (part (vii) of Proposition 1.1) that σ is continuous. From the relation $[I, J] = J$ given by Proposition 3.1, we obtain

$$J = [\sigma(I), J] = \sigma([I, J]) = \sigma(J),$$

which shows that J is a complex ideal of I, in the complex structure defined by σ; because L acts complex-linearly on I, the derived spaces $D_L^j(J)$ which make up $\mathscr{F}_L(I, J)$ are also complex ideals of I. Each linearly compact complex structure on I consistent with the adjoint action of L induces a splitting of $I_{\mathbf{C}}$ by conjugate closed ideals, and is characterized by that splitting; thus σ and $-\sigma$ define the only such structures. To see that $\pi''|_L\colon L \to L_{\mathbf{C}}/\overline{I''}$ is injective, we recall that there are no real vectors in $\overline{I''}$. Moreover, for all $\xi \in I$,

$$\xi = \xi - i\sigma(\xi) \mod \overline{I''},$$

which shows that $\pi''|_I$ is a complex-linear isomorphism of I onto I''. Because J is the unique maximal closed ideal of I, the space

$$\pi''(J) = \{ \xi - i\sigma(\xi) | \xi \in J \} = J''$$

is the unique maximal closed ideal of I''. With respect to the $L_{\mathbf{C}}$-module structure of I given by its complex structure, it is verified at once that $\pi''|_I\colon I \to I''$ is an isomorphism of $L_{\mathbf{C}}$-modules; if we impose upon I'' the filtration $\mathscr{F}'' = \mathscr{F}_{L_{\mathbf{C}}}(I'', J'')$, then $\pi''|_I$ also is filtration-preserving with respect to $\mathscr{F}$ and $\mathscr{F}''$. The normalizers N and N'' are open subalgebras of L and $L_{\mathbf{C}}$, respectively, by Lemma 2.1; the relations (3.3) apply because $\tau''|_I$ is an isomorphism of $L_{\mathbf{C}}$-modules. Form the graded mapping

$$\mathrm{gr}(\tau''|_I)\colon \mathrm{gr}(I, \mathscr{F}) \to \mathrm{gr}(I'', \mathscr{F}'')$$

associated to $\tau''|_I$; this mapping is a complex-linear isomorphism of $S(U)$-modules, again because $\tau''|_I$ is an isomorphism of $L_{\mathbf{C}}$-modules. Applying Proposition 3.1 to the nonabelian minimal closed ideal I'' of $L_{\mathbf{C}}$, there exists an isomorphism

$$\psi''\colon \mathrm{gr}(I'', \mathscr{F}) \to R \otimes_{\mathbf{C}} S(W''^*)$$

of graded Lie algebras and $S(U_{\mathbf{C}})$-modules, with $W''^* = (N''/M_{\mathbf{C}})^{\perp}$. Because the morphism ψ is determined by its homogeneous term of degree zero, we can place ψ

in a commutative diagram

$$\begin{array}{ccc} \operatorname{gr}(I'', \mathscr{F}'') & \overset{\operatorname{gr}(\tau'')^{-1}}{\rightarrow} & \operatorname{gr}(I, \mathscr{F}) \\ \psi'' \downarrow & & \downarrow \psi \\ R \otimes_{\mathbf{C}} S(W''^*) & \overset{\Theta}{\rightarrow} & R \otimes_{\mathbf{C}} S(U_{\mathbf{C}}^*) \end{array}$$

where Θ: $R \otimes_{\mathbf{C}} S(W^*) \to R \otimes_{\mathbf{C}} S(U_{\mathbf{C}}^*)$ is the natural injection. This shows that $W^* = W''^*$, and concludes the proof.

It was proved by Guillemin in [**12**, Theorem 7.1] that a nonabelian minimal closed ideal I is isomorphic, as an abstract Lie algebra, to the completion of its associated graded algebra $\operatorname{gr}(I, \mathscr{F})$; an analogous statement, in the category of Lie pseudogroups, had been made by É. Cartan [**3**, Theorem XII]. We present below an extension of these results (see also [**5**]).

THEOREM 3.1. *Let L be a linearly compact Lie algebra and I a nonabelian minimal closed ideal of L. Write J for the unique maximal closed ideal of I, and R for the simple transitive Lie algebra I/J. The normalizer $N = N_L(J)$ is, then, an open subalgebra of L; let M denote an open subalgebra of L such that $N \supset M \supset I$. Let R^0 be a primitive subalgebra of R, and* $\operatorname{Der}^0(R)$ *be the primitive subalgebra of* $\operatorname{Der}(R)$ *formed by the stabilizer of R^0 (cf. Theorem* 1.6). *Identify (R, R^0) with the transitive closed ideal of inner derivations in* $(\operatorname{Der}(R), \operatorname{Der}^0(R))$. *Set $U = L/M$, $F = \{U^*\}$, and choose a transitive representation λ: $(L, M) \to$* $\operatorname{Der}(F)$. *Moreover, establish the following notation, according to whether the commutator field K_R coincides with K:*

(a) *If $K_R = K$, let G denote the λ-invariant unitary subring of F having isotropy subalgebra N (cf. Theorem* 1.4). *Let σ: $N \to$* $\operatorname{Der}(R) = \operatorname{Der}(I/J)$ *be the Lie algebra homomorphism induced by the adjoint action of N on I.*

(b) *If $K_R \neq K$, that is, if L is a real topological Lie algebra and I is of complex type, choose a complex structure on I compatible with the adjoint action of L, by Lemma* 3.1. *Let N'' be the stabilizer in $L_{\mathbf{C}}$ of J, in terms of this complex structure; then N'' is an open complex subalgebra of $L_{\mathbf{C}}$ containing $N_{\mathbf{C}}$, a fortiori $M_{\mathbf{C}}$. Write G for the $\lambda_{\mathbf{C}}$-invariant unitary complex subalgebra of $F_{\mathbf{C}}$ having isotropy subalgebra N'' (cf. Theorem* 1.4). *Let*

$$\sigma: N'' \to \operatorname{Der}(R) = \operatorname{Der}(I/J)$$

be the complex Lie algebra homomorphism induced by the action of N'' on I. Under these notations, the stabilizer of G in $\operatorname{Der}(F)$ *is a transitive subalgebra (A_G, A_G^0) of* $\operatorname{Der}(F)$ *containing $\lambda(L)$, and the semidirect product*

$$S = \left(\operatorname{Der}(R) \mathbin{\hat{\otimes}}_{K_R} G\right) \oplus A_G$$

is a transitive Lie algebra over K, with fundamental subalgebra

$$S^0 = \left(\left(\operatorname{Der}^0(R) \mathbin{\hat{\otimes}}_{K_R} G\right) + \left(\operatorname{Der}(R) \mathbin{\hat{\otimes}}_{K_R} G^0\right)\right) \oplus A_G^0.$$

Write ρ: $S \to A_G$ for the natural projection, and continue to denote by λ the induced mapping $L \to A_G$. Identify $\operatorname{Der}(R)$ *with the closed subalgebra* $\operatorname{Der}(R) \mathbin{\hat{\otimes}}_{K_R} K_R$ *of* $\operatorname{Der}(R) \mathbin{\hat{\otimes}}_{K_R} G$.

Then, the mapping σ is a morphism of topological Lie algebras whose image P is a transitive closed ideal of $\mathrm{Der}(R)$ *over K_R containing R. There exists a morphism of topological Lie algebras ϕ: $L \to S$ such that the image $\phi(I)$ is equal to $R \hat{\otimes}_{K_R} G$, and whose restriction $\phi|_I$: $I \to R \hat{\otimes}_{K_R} G$ is an isomorphism. Moreover, the subalgebra $\phi(L)$ is transitive in (S, S^0), and ϕ satisfies*

$$\phi(L) \subset \left(P \hat{\otimes}_{K_R} G\right) \oplus A_G, \quad \rho \circ \phi = \lambda, \quad \phi^{-1}\left(\mathrm{Der}(R) \hat{\otimes}_{K_R} G\right) = D_L^{\infty}(M),$$

$$\phi^{-1}(S^0) = M \cap \sigma^{-1}\left(\mathrm{Der}^0(R)\right), \quad \phi(M) \subset \left(\sigma(M) \oplus \left(P \hat{\otimes}_{K_R} G^0\right)\right) \oplus A_G^0.$$

Write Z for the centralizer of I in L; then $\ker(\phi) = Z \cap D_L^{\infty}(M)$.

PROOF. When $K_R = K$, the mapping σ is induced by the structure of a topological N-module possessed by I/J, while when $K_R \neq K$, it is the structure of a complex topological N''-module on I/J which induces σ. Therefore, the map σ is continuous with respect to the weak topology of $\mathrm{Der}(R)$; from (ii) of Theorem 1.6, we conclude that σ is a morphism of linearly compact topological Lie algebras, as indicated. Because $I \subset N$, the image P of σ must contain R; since R forms a transitive closed ideal in $\mathrm{Der}(R)$ and $\mathrm{Der}(R)/R$ is abelian, we conclude that P is a transitive closed ideal of $\mathrm{Der}(R)$.

In constructing ϕ, we shall assume that the linear map

$$\varepsilon: L/M \to U = \mathrm{Der}(F)/\mathrm{Der}^0(F)$$

induced by λ is equal to the identity; passage to the general case is achieved trivially from this. To construct ϕ, we shall apply Theorem 2.1, with

$$L' = \left(P \hat{\otimes}_{K_R} G\right) \oplus A_G, \quad M' = \left(P \hat{\otimes}_{K_R} G\right) \oplus A_G^0,$$

$$I' = P \hat{\otimes}_{K_R} G, \quad J' = P \hat{\otimes}_{K_R} G^0, \quad V = U.$$

The ideals which comprise the filtration $\mathscr{F}' = \mathscr{F}_{L'}(I', J')$ defined in (2.1) are

$$I'_j = P \hat{\otimes}_{K_R} G^{j-1}, \qquad j \in \mathbf{Z};$$

thus, the associated graded Lie algebra and $S(U)$-module $\mathrm{gr}(I', \mathscr{F}')$ is given by

$$\mathrm{gr}(I', \mathscr{F}') = P \otimes_{K_R} S(W^*),$$

where W^* is the annihilator

$$W^* = (N/M)^{\perp} \subset U^* \quad \text{when } K_R = K,$$

$$W^* = (N''/M_{\mathbf{C}})^{\perp} \subset (U_{\mathbf{C}})^* \quad \text{when } K_R \neq K.$$

From Lemma 1.7, we see that the Spencer cohomology of $\mathrm{gr}(I', \mathscr{F}')$, as an $S(U)$-module, satisfies

$$H^{j,l}(\mathrm{gr}(I', \mathscr{F}')) = \{0\}, \qquad j > 0, l \geqslant 0.$$

If $K_R = K$, choose a linear mapping τ_0: $L \to P$ such that $\tau_0|_N = \sigma$; if $K_R \neq K$, let τ_0: $L_{\mathbf{C}} \to P$ be a $\mathbf{C}$-linear map with $\tau_0|_{N''} = \sigma$. Then τ_0 is continuous, since it agrees with the continuous linear map σ on an open neighborhood of 0. Viewing P as the closed subspace $P \otimes_{K_R} K_R$ of $P \hat{\otimes}_{K_R} G$, we define an initial approximation

$$\phi_0 \colon L \to \left(P \hat{\otimes}_{K_R} G\right) \oplus A_G$$

to ϕ by setting $\phi_0(\xi) = \tau_0(\xi) + \lambda(\xi)$ for all $\xi \in L$. Because A_G commutes with P in $(P \hat{\otimes}_{K_R} G) \oplus A_G$, and λ is a Lie algebra homomorphism, we have, for all $\xi, \eta \in L$,

$$(3.4) \qquad \left[\phi_0(\xi), \phi_0(\eta)\right] - \phi_0([\xi, \eta]) = \left[\tau_0(\xi), \tau_0(\eta)\right] - \tau_0([\xi, \eta]).$$

We have already verified that $\operatorname{gr}(I', \mathscr{F}')$ is 2-acyclic, and, because σ is a Lie algebra homomorphism, we see from (3.4) and Lemma 1.7 that the cocycle requirements of Theorem 2.1 are satisfied. Applying Theorem 2.1, we obtain a morphism of topological Lie algebras $\phi \colon L \to (P \hat{\otimes}_{K_R} G) \oplus A_G$, such that $\rho \circ \phi = \lambda$, and thus

$$\phi^{-1}\left(P \hat{\otimes}_{K_R} G\right) = D_L^{\infty}(M) \supset I.$$

In proving that $\phi(I) \subset R \hat{\otimes}_{K_R} G$, we can avoid the general mechanism given in the last part of Theorem 2.1. Instead, we observe that, because P/R is abelian, we have

$$\left[\left(P \hat{\otimes}_{K_R} G\right), \left(P \hat{\otimes}_{K_R} G\right)\right] \subset R \hat{\otimes}_{K_R} G,$$

and, because I is a nonabelian minimal closed ideal, we have $[I, I] = I$, by Proposition 3.1. Therefore,

$$\phi(I) = \phi([I, I]) = \left[\phi(I), \phi(I)\right] \subset \left[\left(P \hat{\otimes}_{K_R} G\right), \left(P \hat{\otimes}_{K_R} G\right)\right] \subset R \hat{\otimes}_{K_R} G,$$

which is the desired inclusion. Since I/J is nonabelian and simple, we have

$$J = \ker(\sigma|_I) = I \cap \phi_0^{-1}\left(P \hat{\otimes}_{K_R} G^0\right).$$

Form the filtration $\mathscr{F} = \mathscr{F}_L(I, J) = \{I_j\}_{j \in \mathbf{Z}}$ of I defined in (2.1). Then, according to Theorem 2.1, we have

$$I_j = \left(\phi|_I^{-1}\right)\left(R \hat{\otimes}_{K_R} G^{j-1}\right)$$

for all $j \in \mathbf{Z}$, and the associated graded mapping

$$\operatorname{gr}(\phi|_I) \colon \operatorname{gr}(I, \mathscr{F}) \to R \otimes_{K_R} S(W^*)$$

is a morphism of graded Lie algebras and $S(U)$-modules; moreover, the zeroth graded term of $\operatorname{gr}(\phi|_I)$ is the mapping $I/J \to R$ induced by $\phi_0|_I = \sigma$, and is therefore equal to the identity map. From Proposition 3.1 and Lemma 3.1, we deduce that $\operatorname{gr}(\phi|_I)$ is an isomorphism. Because $\{I_j\}_{j \in \mathbf{Z}}$ and $\{R \hat{\otimes}_{K_R} G^{j-1}\}_{j \in \mathbf{Z}}$ are separated filtrations of I and $R \hat{\otimes}_{K_R} G$, respectively, and $\phi(I)$ is closed in $R \hat{\otimes}_{K_R} G$, it follows at once from Chevalley's theorem that $\phi|_I \colon I \to R \hat{\otimes}_{K_R} G$ is an isomorphism. To determine the kernel of ϕ, we first remark that the centralizer of $R \hat{\otimes}_{K_R} G$ in $\operatorname{Der}(R) \hat{\otimes}_{K_R} G$ is, evidently, equal to $\{0\}$. Because ϕ maps I isomorphically onto $R \hat{\otimes}_{K_R} G$ and we have

$$\phi^{-1}\left(\operatorname{Der}(R) \hat{\otimes}_{K_R} G\right) = D_L^{\infty}(M),$$

it follows that $\ker(\phi)$ is the centralizer of I in $D_L^{\infty}(M)$, as asserted. The remaining statements of our theorem are immediate consequences of Theorem 2.1. This concludes the proof.

COROLLARY 3.2. *Let I be a nonabelian minimal closed ideal of a linearly compact Lie algebra L. Then, there exists a closed subalgebra L' of L such that $L = I \oplus L'$.*

PROOF. We shall apply Theorem 3.1; we retain the notation of that theorem, taking $M = N_L(J)$ for definiteness. According to Theorem 1.6, there exists in Der(R) a closed abelian subalgebra C which is complementary to R; thus, $B = (C \hat{\otimes}_{K_R} G) \oplus A_G$ is a closed subalgebra of S such that $S = (R \hat{\otimes}_{K_R} G) \oplus B$. Because the morphism ϕ constructed in Theorem 3.6 maps I isomorphically onto $R \hat{\otimes}_{K_R} G$, we need only take $L' = \phi^{-1}(B)$ to obtain a closed subalgebra of L complementary to I, as asserted.

Implicit in Theorem 3.1 are certain structures associated to the choice of an open complex subalgebra in a real linearly compact Lie algebra. The remainder of this section is devoted to a discussion of these objects.

Let L be a linearly compact real topological Lie algebra, and M'' be an open complex subalgebra of $L_{\mathbf{C}}$. Set $M = M'' \cap L$; then M is an open subalgebra of L whose complexification is, clearly,

$$M_{\mathbf{C}} = M'' \cap \overline{M''}. \tag{3.5}$$

The quotients $U = L/M$ and $E'' = M''/M_{\mathbf{C}}$ are finite-dimensional vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively; moreover, from (3.5) we see that E'' forms a complex subspace of $U_{\mathbf{C}}$ such that the intersection of E'' and its complex conjugate $E' = \overline{E''}$ is reduced to $\{0\}$. The direct sum $E' \oplus E''$ is a conjugation-invariant subspace of $U_{\mathbf{C}}$; thus, there is a unique subspace E of U such that $E_{\mathbf{C}} = E' \oplus E''$. We may also consider U as a real subspace of the complex finite-dimensional vector space $W = L_{\mathbf{C}}/M''$; then E is the largest complex subspace of W contained in U. We now define a mapping λ: $E_{\mathbf{C}} \times E_{\mathbf{C}} \to (U/E)_{\mathbf{C}}$, which we shall call the *Levi form of* (L, M''), as follows: given $u, v \in E_{\mathbf{C}}$, choose $\xi, \eta \in L_{\mathbf{C}}$ such that $u = \xi \bmod M_{\mathbf{C}}$ and $v = \eta \bmod M_{\mathbf{C}}$; then the value $\lambda(u, v)$ is given by

$$\lambda(u, v) = i[\xi, \bar{\eta}] \quad \operatorname{mod}(M'' + \overline{M''}). \tag{3.6}$$

Because both M'' and $\overline{M''}$ are stable under the adjoint action of $M_{\mathbf{C}}$, our expression for $\lambda(u, v)$ does not depend upon the choice of representatives ξ, η; thus, the Levi form is well defined. From the bilinearity and skew-symmetry of the Lie bracket, we also see that λ is Hermitian; that is, the mapping λ is $\mathbf{R}$-bilinear and satisfies

$$\lambda(cu, v) = c\lambda(u, v), \qquad \lambda(u, v) = \overline{\lambda(v, u)},$$

for all $c \in \mathbf{C}$ and $u, v \in E_{\mathbf{C}}$. Since M'' and $\overline{M''}$ are both subalgebras of $L_{\mathbf{C}}$, we have

$$\lambda(E'' \times E') = \lambda(E' \times E'') = \{0\};$$

moreover, for all $u, v \in E_{\mathbf{C}}$, we see from the definition of λ that $\lambda(u, v) = \overline{-\lambda(\bar{u}, \bar{v})}$. Thus, the Levi form λ is determined by either of its restrictions

$$\lambda' \colon E' \times E' \to (U/E)_{\mathbf{C}}, \quad \lambda'' \colon E'' \times E'' \to (U/E)_{\mathbf{C}},$$

which are Hermitian mappings related by

$$\lambda'(u, v) = \overline{-\lambda''(\bar{u}, \bar{v})} \quad \text{for } u, v \in E'.$$

It is clear from the definitions that the Levi forms of (L, M'') and $(L, \overline{M''})$ are the same object, and that the Levi form of (L, M'') vanishes identically if and only if $M'' + \overline{M''}$ is a subalgebra of $L_{\mathbf{C}}$.

We pause to sketch the geometry which underlies these considerations; a more complete account of the formal geometry outlined below appears in §5 of our monograph [**5**]. Set

$$n = \dim_{\mathbf{C}}(W), \quad m = \dim_{\mathbf{R}}(U),$$

and choose transitive representations

$$\mu\colon (L, M) \to \mathrm{Der}(F\{U^*\}), \quad \rho\colon (L_{\mathbf{C}}, M'') \to \mathrm{Der}(F\{W^*\}).$$

Because $M'' \supset M_{\mathbf{C}}$, we may apply Theorem 1.3 to $\mu_{\mathbf{C}}$ and ρ, and conclude that there exists a unique monomorphism of complex local algebras $\varphi\colon F\{W^*\} \to F\{U_{\mathbf{C}}^*\}$, such that, for all $f \in F\{W^*\}$ and $\xi \in L$,

$$\varphi(\rho(\xi)\cdot f) = \mu(\xi)\cdot\varphi(f). \tag{3.7}$$

Choosing formal coordinate systems, we make the identifications

$$F\{U^*\} \simeq \mathbf{R}[[x_1,\ldots,x_m]], \qquad F\{W^*\} \simeq \mathbf{C}[[z_1,\ldots,z_n]].$$

Then φ is identified with the pullback mapping on formal functions associated to a formal power series mapping $\Phi\colon \mathbf{R}^m \to \mathbf{C}^n$, with $\Phi(0) = 0$. Viewing $\mu(L)$ and $\rho(L_{\mathbf{C}})$ as Lie algebras of formal vector fields at the origins of $\mathbf{R}^m$ and $\mathbf{C}^n$, respectively, we identify U with the tangent space $T_0(\mathbf{R}^m)$ and W with $T_0(\mathbf{C}^n)$. Then we see from (3.7) that the differential of Φ at the origin,

$$\Phi_*|_0\colon T_0(\mathbf{R}^m) \to T_0(\mathbf{C}^n),$$

is identified with the inclusion mapping of U as a real subspace of W. Thus Φ is a formal immersion, and is *generic*, that is,

$$\mathrm{span}_{\mathbf{C}}(\Phi_*|_0(T_0(\mathbf{R}^m))) = T_0(\mathbf{C}^n).$$

For each $\xi \in L$, the formal vector field $\mu(\xi)$ on $\mathbf{R}^m$ is Φ-related to the formal holomorphic vector field $\rho(\xi)$ on $\mathbf{C}^n$; this is simply a restatement of (3.7). Phrased geometrically, then, the mapping Φ is a formal embedding of a neighborhood of $0 \in \mathbf{R}^m$ as a generic real m-dimensional submanifold S of $\mathbf{C}^n$; for each $\xi \in L$, the formal holomorphic vector field $\rho(\xi)$ is tangent to S, and the restriction of $\rho(\xi)$ to S is $\mu(\xi)$. Hence, the real submanifold $S \subset \mathbf{C}^n$ is *locally homogeneous*; that is, the holomorphic formal vector fields on $\mathbf{C}^n$ which are everywhere tangent to S restrict to form a transitive Lie algebra of formal vector fields on S. Because μ and ρ are unique up to the action of isomorphisms of $F\{U^*\}$ and $F\{W^*\}$, respectively, the submanifold S is determined by the choice of M'', up to the action of a formal biholomorphic transformation of $\mathbf{C}^n$ which preserves the origin. The subspace E of U is identified with the maximal complex subspace of the tangent space $T_0(S) \subset T_0(\mathbf{C}^n)$, and E' and E'' are identified with the spaces of complex tangent vectors of types (1,0) and (0,1), respectively, in $\mathbf{C} \otimes_{\mathbf{R}} T_0(S)$. Under these identifications, the Levi form of S at 0 (in the differential-geometric sense) is what we have called λ' (see pp. 188–194 of [**5**]). Using techniques similar to those employed by Goldschmidt in [**7**], it can be shown that Φ is convergent in a neighborhood of $0 \in \mathbf{R}^n$ when expressed in terms of appropriate formal systems of coordinates for U and W; thus Φ determines the germ at 0 of a generic, locally homogeneous real-analytic submanifold of $\mathbf{C}^n$, which is uniquely associated to (L, M'') up to a local biholomorphic transformation of $\mathbf{C}^n$. An account of this analytic construction will appear in a separate publication. The Levi form of (L, M'') vanishes identically if and only if the mapping Φ is linear in terms of appropriate formal coordinates for U and W; this formal analogue of the complex Frobenius theorem of L. Nirenberg [**20**] is obtained easily from (ii) of Theorem 1.3 (see Proposition 5.4 of [**5**]).

Let I be a nonabelian minimal closed ideal of complex type in L, and let J be the maximal closed ideal of I. According to Lemma 3.1, there are two complex structures on I which are compatible with the adjoint action of L on I; these structures are conjugate, and J forms a complex ideal of I in each of them. The stabilizers in $L_{\mathbf{C}}$ of J, with respect to these two complex structures, are conjugate open complex subalgebras N'', $\overline{N''} \subset L_{\mathbf{C}}$. As we have already observed, the Levi forms of (L, N'') and $(L, \overline{N''})$ coincide; thus, this Hermitian mapping λ, which we call the *Levi form of I*, is canonically associated to the real topological Lie algebra and topological L-module structures of I. (In §5 of [**5**], we called the restrictions λ' and λ'' of λ the Levi forms of I; our present exposition was suggested by Goldschmidt.)

We now assume that L is a real transitive Lie algebra, and let I be an arbitrary closed ideal of L. Choose a Jordan-Hölder sequence

$$\{0\} = I_0 \subset I_1 \subset \cdots \subset I_r = I$$

for (L, I), and let $\mathscr{C}$ be the subset of $\{1, \ldots, r\}$ consisting of those indices j for which the quotient I_j/I_{j-1} is a nonabelian minimal closed ideal of complex type in L/I_{j-1}. Write J_j for the maximal closed ideal of I_j/I_{j-1}, for each $j \in \mathscr{C}$. From our discussion above, we see that there are two conjugate open complex subalgebras N_j'', $\overline{N_j''} \subset L_{\mathbf{C}}$, which are the stabilizers in $L_{\mathbf{C}}$ of J_j with respect to the two complex structures of I_j/I_{j-1}; moreover, these subalgebras determine a Hermitian mapping λ_j which is the Levi form of I_j/I_{j+1}. The set $\{N_j'', \overline{N_j''}, \lambda_j\}$ depends only upon the real topological Lie algebra and topological L-module structures of I_j/I_{j-1}. Therefore, by Theorem 1.2, the collection $\{N_j'', \overline{N_j''}, \lambda_j\}_{j \in \mathscr{C}}$, counting multiplicities, is independent of the choice of Jordan-Hölder sequences for (L, I).

4. Primitive invariant foliations. Let (L, L^0) be a transitive Lie algebra. In this section, we consider composition series for L made up of closed ideals defined by the formal foliations which remain invariant under a transitive realization of (L, L^0). Our starting point is a result which appears in the unpublished notes [**14**] of V. W. Guillemin.

LEMMA 4.1. *Let L be a linearly compact Lie algebra and I a closed ideal of L. Suppose that J is a closed ideal of I, and J contains no ideals of L except $\{0\}$. Assume, moreover, that the quotient I/J can be decomposed as a direct sum of closed ideals*

$$I/J = g_0 \oplus g_1 \oplus \cdots \oplus g_p$$

(possibly with $p = 0$), such that g_0 is abelian and each g_j with $j > 0$ is a simple transitive Lie algebra. Let $\pi: I \to I/J$ be the natural projection, and in each preimage $I_j' = \pi^{-1}(g_j)$, with $0 \leqslant j \leqslant p$, denote the largest closed ideal of L by $I_j = D_L^{\infty}(I_j')$. Then I_0 is abelian, and each I_j with $j > 0$ is a nonabelian minimal closed ideal of L

whose unique maximal closed ideal J_j is equal to $I_j \cap J$. Moreover, I can be expressed as the direct sum

$$I = I_0 \oplus I_1 \oplus \cdots \oplus I_p.$$

PROOF. If I is an integer with $0 \leqslant i \leqslant p$, then

$$\pi\left(I_i \cap \left(\sum_{j \neq i} I_j\right)\right) \subset g_i \cap \left(\bigoplus_{j \neq i} g_j\right) = \{0\}.$$

Thus, the intersection $I_i \cap (\Sigma_{j \neq i} I_j)$ is reduced to $\{0\}$, because it is an ideal of L contained in J. This shows that the sum of the family $\{I_i\}_{0 \leqslant i \leqslant p}$ is direct. Next, let j be an integer with $1 \leqslant j \leqslant p$. We shall prove that the corresponding ideal I_j is not reduced to $\{0\}$. Write

$$\mathscr{F} = \mathscr{F}_L(I, I_j') = \{\mathscr{I}_l\}_{l \in \mathbf{Z}}$$

for the filtration of I associated to L and I_j' which we defined in (2.1); then,

$$I_j = \bigcap_{l \in \mathbf{Z}} \mathscr{I}_l = D_L^{\infty}(I_j'),$$

by definition. Now suppose that $I_j = \{0\}$. Choose a fundamental subalgebra h_j of g_j. Then, the preimage $\pi^{-1}(h_j)$ is an open subalgebra of $\mathscr{I}_1 = I_j'$, and each ideal $\mathscr{I}_l$ of the descending chain $\{\mathscr{I}_l\}_{l \geqslant 1}$ is closed in L; therefore, we may apply Chevalley's theorem to conclude that $\mathscr{I}_m \subset \pi^{-1}(h_j)$, for some $m \geqslant 1$. As an immediate consequence, we have the inclusion $\mathscr{I}_m \subset J$, because $\pi(\mathscr{I}_m)$ is an ideal of g_j contained in the fundamental subalgebra h_j, and is thus reduced to $\{0\}$. Recall from Lemma 2.1 that $\mathscr{F}$ makes I a filtered Lie algebra, that is, $[\mathscr{I}_r, \mathscr{I}_s] \subset \mathscr{I}_{r+s}$ for all $r, s \in \mathbf{Z}$. From the inclusion $\mathscr{I}_m \subset J$, we can thus infer that the term of length m

$$\mathscr{C}^m(\mathscr{I}_1) = [\mathscr{I}_1, [\mathscr{I}_1, \ldots, [\mathscr{I}_1, \mathscr{I}_1], \ldots]]$$

in the lower central series of $\mathscr{I}_1$ is contained in J. Because $g_j = \mathscr{I}_1/J$, we conclude that $\mathscr{C}^m(g_j) = \{0\}$; thus g_j is nilpotent. This contradicts our assumption that g_j is nonabelian and simple; hence I_j cannot be reduced to $\{0\}$. Because J contains no ideals of L except $\{0\}$, the image $\pi(I_j)$ must be a nonzero closed ideal in g_j. Thus, $\pi(I_j) = g_j$, because g_j is simple; moreover, the kernel $J_j = I_j \cap J$ of $\pi|_{I_j}$ contains no ideals of L except $\{0\}$. We may now apply Proposition 3.1 to conclude that I_j is a nonabelian minimal closed ideal of L and that J_j is the unique maximal closed ideal of I_j.

To see that I_0 is abelian, we simply observe that

$$\pi([I_0, I_0]) \subset [g_0, g_0] = \{0\},$$

because $I_0 \subset \pi^{-1}(g_0)$ and g_0 is abelian. Thus $[I_0, I_0]$ is an ideal of L contained in J; by hypothesis, the only such ideal is $\{0\}$, so I_0 is abelian.

It only remains to prove that I is equal to the direct sum of the I_l, for $0 \leqslant l \leqslant p$. The case $p = 0$ is trivial. We now assume that $p > 0$, and prove the result first under the additional assumption that $g_0 = \{0\}$. The case $p = 1$ is now trivial. Inductively, assume the result is known for p equal to some integer $p_0 \geqslant 1$. To treat the case $p = p_0 + 1$, observe that

$$\tilde{I} = D_L^{\infty}\left(\pi^{-1}(g_1 \oplus \cdots \oplus g_{p_0})\right)$$

is a closed ideal of L and contains each I_j for $1 \leqslant j \leqslant p_0$; if we define a closed ideal $\tilde{J} = J \cap \tilde{I}$ of $\tilde{I}$, then $\tilde{J}$ contains no ideals of L except $\{0\}$, and we have, because $I_j \subset \tilde{I}$ for $1 \leqslant j \leqslant p_0$,

$$\tilde{I}/\tilde{J} = g_1 \oplus \cdots \oplus g_{p_0}.$$

From our inductive hypothesis it follows that $\tilde{I} = I_1 \oplus \cdots \oplus I_{p_0}$. Moreover, because

$$g_{p_0+1} = I/\big(\pi^{-1}(g_1 \oplus \cdots \oplus g_{p_0})\big)$$

is a simple transitive Lie algebra, it follows from Proposition 3.1 that there are no closed ideals of L between I and $\tilde{I}$. Thus, from the relation

$$I \supset I_1 \oplus \cdots \oplus I_{p_0+1} \supsetneqq \tilde{I},$$

we conclude that $I = I_1 \oplus \cdots \oplus I_{p_0+1}$. By induction, we have proved the direct sum decomposition of I under the assumption that $g_0 = \{0\}$.

Finally, to treat the case where g_0 may not be reduced to $\{0\}$, set $J_0 = \pi^{-1}(g_0)$, and let $\rho: L \to L/I_0$ be the natural projection. Then $\rho(I)$ is a closed ideal of L/I_0, and $\rho(J_0)$ is a closed ideal of $\rho(I)$ such that

$$\rho(I)/\rho(J_0) = g_1 \oplus \cdots \oplus g_p.$$

Moreover, there are no ideals of L/I_0 contained in $\rho(J_0)$ except $\{0\}$, because I_0 contains every ideal of L in J_0. Let $\sigma: \rho(I) \to \rho(I)/\rho(J_0)$ be the natural projection; then by the case we proved above,

$$\rho(I) = \tilde{I}_1 \oplus \cdots \oplus \tilde{I}_p,$$

where each $\tilde{I}_j$ is a nonabelian minimal closed ideal of L/I_0 obtained as the largest ideal of L/I_0 in $\sigma^{-1}(g_j)$. We clearly have $\tilde{I}_j \supset \rho(I_j) \supsetneqq \{0\}$, so $\rho(I_j) = \tilde{I}_j$, since $\tilde{I}_j$ is minimal. This implies $\rho(I) = \rho(I_1 \oplus \cdots \oplus I_p)$, hence $I = I_0 \oplus \cdots \oplus I_p$, which concludes the proof.

Given a transitive realization of a transitive Lie algebra (L, L^0), we can now describe the structure of those closed ideals in L defined by an invariant formal foliation along which L acts primitively.

THEOREM 4.1. *Let (L, L^0) be a transitive Lie algebra and M be minimal among the open subalgebras of L which properly contain L^0. Then $\tilde{M} = M/(D_M^\infty(L^0))$ is a primitive Lie algebra, with primitive subalgebra $\tilde{L}^0 = L^0/(D_M^\infty(L^0))$. Let I be the largest ideal of L in M, $I = D_L^\infty(M)$, and define a closed ideal J of M in I by setting $J = I \cap D_M^\infty(L^0)$; then J contains no ideals of L except $\{0\}$. Write $\pi: M \to \tilde{M}$ for the natural projection. The kernel of $\pi|_I$ is J, and, if I is not reduced to $\{0\}$, the image $\tilde{I} = \pi(I) \simeq I/J$ is a transitive closed ideal of $(\tilde{M}, \tilde{L}^0)$. Furthermore, the ideal I admits one of the following descriptions*:

(i) *If the dimension of $\tilde{M}$ is infinite, then $I \neq \{0\}$, and $I' = [I, I]$ is a nonabelian minimal closed ideal of L. The unique maximal closed ideal of I' is*

$$J' = I' \cap D_M^\infty(L^0),$$

which is a closed ideal of M; *the quotient* I'/J' *is isomorphic to the simple transitive closed ideal* $R = [\tilde{M}, \tilde{M}]$ *of* $(\tilde{M}, \tilde{L}^0)$ *in* $\tilde{I}$. *The chain of closed ideals* $I \supset I' \supset \{0\}$ *is thus a Jordan-Hölder sequence for* (L, I); *the ideals* I *and* I' *coincide if and only if* $\tilde{I}$ *is equal to* R. *The intersection of* I *with the centralizer of* I' *in* L *is* $\{0\}$.

(ii) *If* $\tilde{M}$ *is finite dimensional and simple, then either* $I = \{0\}$ *or* I *is a nonabelian minimal closed ideal of* L. *In the latter case* J *is the unique maximal closed ideal of* I.

(iii) *If* $\tilde{M}$ *is semisimple but not simple, then* $\tilde{M}$ *is the direct sum* $\tilde{M} = g_1 \oplus g_2$ *of two isomorphic simple ideals* g_1 *and* g_2. *Set*

$$I_j = D_L^\infty\big(I \cap \pi^{-1}(g_j)\big),$$

for $j = 1, 2$; *then each* I_j *is a closed ideal of* L, *and* I *is the direct sum* $I = I_1 \oplus I_2$. *Moreover, unless the ideal* I_j *is reduced to* $\{0\}$, *it is a nonabelian minimal closed ideal of* L; *the unique maximal closed ideal of* I_j *is then*

$$J_j = I_j \cap D_M^\infty(L^0)$$

which is a closed ideal of M, *and the projection* $\pi(I_j)$ *is equal to* g_j.

(iv) *If* $\tilde{M}$ *is finite dimensional and nonabelian but not semisimple, then* $\tilde{M}$ *is the direct sum* $\tilde{M} = V \oplus \tilde{L}^0$ *of an abelian ideal* V *and the primitive subalgebra* $\tilde{L}^0$, *which is faithfully and irreducibly represented on* V. *Thus, the subalgebra* $\tilde{L}^0$ *is the direct sum*

$$\tilde{L}^0 = g_0 \oplus g_1 \oplus \cdots \oplus g_p$$

(*possibly with* $p = 0$) *of simple ideals* $g_1, \ldots, g_p$ *and the center* g_0 *of* $\tilde{L}^0$. *Write*

$$\pi_V \colon M \to M/\big(\pi^{-1}(V)\big) \simeq \tilde{L}^0$$

for the natural projection, and define closed ideals of L *in* I *by setting*

$$I_V = D_L^\infty\big(\pi^{-1}(V)\big), \quad I_j = D_L^\infty\big(\pi_V^{-1}(g_j)\big),$$

for $0 \leqslant j \leqslant p$. *Then* I_V *is abelian, and for any two distinct integers* i, j *with* $0 \leqslant i, j \leqslant p$, *the relation* $I_i \cap I_j = I_V$ *holds. Unless* I_V *is reduced to* $\{0\}$, $\pi(I_V) = V$. *The quotient* I_0/I_V *is abelian; for* $j > 0$, *each quotient* I_j/I_V, *when not reduced to* $\{0\}$, *is a nonabelian minimal closed ideal of* L/I_V. *The closed ideal* I *of* L *can be expressed as the sum*

$$I = I_0 + I_1 + \cdots + I_p,$$

and the quotient I/I_V *as the direct sum*

$$I/I_V = (I_0/I_V) \oplus (I_1/I_V) \oplus \cdots \oplus (I_p/I_V).$$

Each intersection $J_j = I_j \cap \pi^{-1}(V)$ *is a closed ideal of* M *containing* I_V; *if* $I_j/I_V \neq \{0\}$, *for some* $j > 0$, *then* J_j/I_V *is the unique maximal closed ideal of* I_j/I_V, *and the projection* $\pi(I_j)$ *is then equal to* $V \oplus g_j$. *If* I_V *is reduced to* $\{0\}$, *then* $I_0 = \{0\}$, *and at most one of the ideals* $I_1, \ldots, I_p$ *is nonzero; the ideal* I *is, then, either zero or a nonabelian minimal closed ideal of* L.

(v) *If* $\tilde{M}$ *is abelian, then it is one dimensional, and* I *is an abelian ideal of* L.

PROOF. The ideal $D_M^\infty(L^0)$ of M contains all ideals of M in L^0, and, because M is minimal among the subalgebras of L which contain L^0, the subalgebra L^0 is maximal in M. It follows at once that $\tilde{L}^0$ is a maximal and fundamental, which is to say primitive, subalgebra of $\tilde{M}$. By definition, the ideal I of L is contained in M. If $I \neq \{0\}$, then the fundamental subalgebra L^0 cannot contain I, and so we must have $I + L^0 = M$, because $I + L^0$ is a subalgebra of M which properly contains L^0, and L^0 is a maximal subalgebra of M. Upon passage to the quotient, we infer that, when $I \neq \{0\}$, its projection $\tilde{I}$ is a transitive closed ideal of $(\tilde{M}, \tilde{L}^0)$, as asserted in the theorem. By its definition, the closed ideal J of M is the kernel of $\pi|_I$; moreover, J lies in L^0, and thus contains no ideal of L except $\{0\}$, because L^0 is a fundamental subalgebra of L. We have now verified all of the statements preceding the numbered assertions (i)–(v); to see that in (i)–(v) we have exhausted all possibilities for the primitive Lie algebra $(\tilde{M}, \tilde{L}^0)$, we direct the reader to the classifications listed in Theorem 1.5 and to Theorem 1.6.

It remains to show that in categories (i)–(v) the closed ideal I of L admits the description asserted. Assume that $\tilde{M}$ is of the infinite-dimensional type (i). To see that $I \neq \{0\}$, we argue by contradiction. If the intersection

$$I = \bigcap_{l \geqslant 0} D_L^l(M)$$

were reduced to $\{0\}$, there would exist, by Chevalley's theorem, an integer $r > 0$ such that $D_L^r(M) \subset L^0$. But, according to Lemma 1.2, each $D_L^l(M)$ forms in M an ideal of finite codimension. Therefore, if I were reduced to $\{0\}$, the infinite-dimensional Lie algebra $\tilde{M}$ would be a quotient of the finite-dimensional Lie algebra $M/D_L^r(M)$, which is an obvious contradiction. Thus $I \neq \{0\}$, as asserted. Next, we observe that, since $\tilde{I}$ is not reduced to $\{0\}$, $\tilde{I} \supset R$ by Theorem 1.6; thus

$$R = [\tilde{M}, \tilde{M}] \supset [\tilde{I}, \tilde{I}] \supset [R, R] = R,$$

and because $\pi(I') = [\tilde{I}, \tilde{I}]$, we have $\pi(I') = R$. The closure $\overline{I'}$ of I' still satisfies $\pi(\overline{I'}) = R$, because π is continuous and R is closed in $\tilde{M}$; moveover, the kernel of $\pi|_{\overline{I'}}$,

$$\overline{I'} \cap J = \overline{I'} \cap D_M^\infty(L^0),$$

is closed in $\overline{I'}$ and contains no ideals of L except $\{0\}$, since J has this property. We conclude from Proposition 3.1 that $\overline{I'}$ is a nonabelian minimal closed ideal of L, and $\overline{I'} \cap J$ is the unique maximal closed ideal of I'. But $\overline{I'} \subset I$, because I is closed; thus, by Proposition 3.1,

$$\overline{I'} = [\overline{I'}, \overline{I'}] \subset [I, I] = I'.$$

We see, then, that I' is itself a nonabelian minimal closed ideal of L, and J' is the unique maximal closed ideal of I'; by its definition J' is an ideal of M, and is equal to the kernel of $\pi|_{I'}$, whence $I'|J'$ is isomorphic to R. Obviously, the quotient $I/I' = I/[I, I]$ is abelian, so $\{0\} \subset I' \subset I$ is a Jordan-Hölder sequence for (L, I). In the event that $\tilde{I} = R$, the ideal I must itself be nonabelian and minimal, by Proposition 3.1; this shows that the ideals I and I' coincide if and only if $\tilde{I} = R$. To complete the proof of (i), write C for the centralizer of I' in L. Then the projection $\pi(C \cap I)$ commutes with $R = \pi(I')$ in $\tilde{M}$, whence, by Theorem 1.6, $\pi(C \cap I) = \{0\}$; that is, the intersection $C \cap I$ is contained in J. Since J contains no ideals of L except $\{0\}$, we see that $C \cap I$ is reduced to $\{0\}$, as asserted.

To treat case (ii), we simply observe that if I is not reduced to $\{0\}$, then we must have $\tilde{I} = \tilde{M}$, since $\tilde{I}$ is a nonzero ideal of the simple Lie algebra $\tilde{M}$. Since J contains no ideals of L except $\{0\}$, and I/J is isomorphic to $\tilde{M}$, and is thus simple, Proposition 3.1 implies that I is a nonabelian minimal closed ideal of L.

The proof of (iii) is also brief. If I is not reduced to $\{0\}$, its projection $\tilde{I}$ is a transitive closed ideal of $\tilde{M} = g_1 \oplus g_2$, and must therefore be equal to one of the summands g_1 or g_2, or to $\tilde{M}$ itself. Since J contains no ideals of L except $\{0\}$, Lemma 4.1 applies to complete the proof of (iii).

To treat the more involved case (iv), we first observe that $\pi(I_V)$ is an ideal of $\tilde{M}$ contained in V, which is to say that $\pi(I_V)$ is an invariant subspace of V under the action of $\tilde{L}^0$. Because $\tilde{L}^0$ acts irreducibly on V, it follows that $\pi(I_V)$ is equal to $\{0\}$ or to V. If $\pi(I_V) = \{0\}$, then $I_V \subset J$; since J contains no ideals of L except $\{0\}$, this forces $I_V = \{0\}$. So we see that the projection $\pi(I_V)$ must be equal to V, unless I_V is reduced to $\{0\}$. To verify that I_V is abelian we note that

$$\pi([I_V, I_V]) \subset [V, V] = \{0\};$$

therefore $[I_V, I_V]$ is an ideal of L contained in J, and is thus equal to $\{0\}$. If $I \neq \{0\}$, its projection $\tilde{I}$ must be a transitive closed ideal of $(\tilde{M}, \tilde{L}^0)$, and thus, by Theorem 1.5, $V \subset \tilde{I}$. Because $\tilde{L}^0$ is reductive, it follows that, after renumbering the simple ideals $g_1, \dots, g_p$, we may express $\tilde{I}$ as

$$\tilde{I} = V \oplus h_0 \oplus g_1 \oplus \cdots \oplus g_q$$

for some integer q with $0 \leqslant q \leqslant p$, and some ideal h_0 contained in the center g_0 of $\tilde{L}^0$. In the quotient $L^\# = L/I_V$, consider the closed ideal $I^\# = I/I_V$. By definition, the ideal I_V of L contains every ideal of L which lies in $I \cap \pi^{-1}(V)$; passing to the quotient, we see that

$$J^\# = \big(I \cap \pi^{-1}(V)\big)/I_V$$

is the closed ideal of $I^\#$ and contains no ideal of $L^\#$ except $\{0\}$. Set $M^\# = M/I_V$; since $\pi^{-1}(V) \supset I_V$, we have a natural projection

$$\rho\colon M^\# \to M/\big(\pi^{-1}(V)\big) \simeq g_0 \oplus g_1 \oplus \cdots \oplus g_p.$$

Clearly, the kernel of $\rho|_{I^\#}$ is $J^\#$. Moreover, if I is not reduced to $\{0\}$, the quotient

$$I^\#/J^\# \simeq \rho(I^\#) \simeq \tilde{I}/V$$

can be identified with $h_0 \oplus g_1 \oplus \cdots \oplus g_q$. We can then apply Lemma 4.1 to write $I^\#$ as the direct sum

$$I^\# = I_0^\# \oplus I_1^\# \oplus \cdots \oplus I_p^\#$$

of the closed ideals of $L^\#$ defined by setting, for $0 \leqslant j \leqslant p$,

$$I_j^\# = D_{L^\#}^\infty\big(I^\# \cap \rho^{-1}(g_j)\big).$$

The ideal $I_0^\#$ is abelian, and $\rho(I_0^\#) = h_0$; for $1 \leqslant j \leqslant q$, the ideal $I_j^\#$ is nonabelian and minimal in $L^\#$, its projection is $\rho(I_j^\#) = g_j$, and $J_j^\# = I_j^\# \cap J^\#$ is the unique maximal closed ideal of $I_j^\#$. For $j > q$, the ideal $I_j^\#$ is reduced to $\{0\}$. Write σ: $L \to L/I_V$ for the natural projection. Then, examining the definitions of the various projections, we see that $\pi_V = \rho \circ (\sigma|_M)$. It follows at once that the ideals I_j of L defined in the statement of (iv) are related to the ideals $I_j^{\#}$ of $L^{\#}$ by $I_j = \sigma^{-1}(I_j^{\#})$, and we also have $J_j = \sigma^{-1}(J_j^{\#})$, for $0 \leqslant j \leqslant p$. From this, we obtain at once all of the remaining assertions of (iv), except the final statement: If $I_V = \{0\}$, then $I_0 = \{0\}$, and at most one of the ideals $I_1, \ldots, I_p$ is nonzero. Suppose that I_j is a nonzero ideal in $\{I_j\}_{0 \leqslant j \leqslant p}$; then its projection $\pi(I_j)$ is a nonzero ideal of $\tilde{M}$, since the kernel $D_M^{\infty}(L^0)$ contains no ideal of L except $\{0\}$. Thus, by Theorem 1.5, $V \subset \pi(I_j)$. Now suppose, in addition, that $I_V = \{0\}$. Then $\pi^{-1}(V)$ cannot contain the nonzero ideal I_j, so we have

$$\pi(I_j) \cap \tilde{L}^0 \neq \{0\};$$

in particular, since $\tilde{L}^0$ acts faithfully on V, the projection $\pi(I_j)$ is nonabelian. But we already know that the quotient $I_0/I_V = I_0$ is abelian, so we infer that j cannot be 0, i.e., $I_0 = \{0\}$. Finally, assume that there exists an integer $l \neq j$ for which I_l is nonzero. Then,

$$\left[\pi(I_j), \pi(I_l)\right] \supset \left[V, \pi(I_l) \cap \tilde{L}^0\right] \neq \{0\},$$

because $\tilde{L}^0$ acts faithfully on V. But we have also

$$\left[I_j, I_l\right] \subset I_j \cap I_l = I_V = \{0\}.$$

This contradiction shows that at most one of the ideals $\{I_j\}_{1 \leqslant j \leqslant p}$ is nonzero, and ends the proof of (iv).

In the final case (v), we simply observe that, because $\tilde{M}$ is abelian, we have $[I, I] \subset L^0$ and thus $[I, I] = \{0\}$, because L^0 is a fundamental subalgebra of L. This completes the proof of Theorem 4.1.

Corollary 4.1. *Let (L, L^0) be a transitive Lie algebra, let M be minimal among the open subalgebras of L which properly contain L^0, and let I be the largest ideal $I = D_L^{\infty}(M)$ of L in M. Then, there exists a Jordan-Hölder sequence*

$$\{0\} = I^0 \subset I^1 \subset \cdots \subset I^m = I$$

for (L, I) with the following properties:

(a) *If I^{l+1}/I^l is a nonabelian quotient, and J^{l+1} is the unique maximal closed ideal of I^{l+1}/I^l, then the stabilizer $N_L(J^{l+1})$ of J^{l+1} in L contains M.*

(b) *The sequence contains at most two abelian quotients; the number of nonabelian quotients is less than or equal to the dimension of the vector space M/L^0.*

Proof. If I is reduced to $\{0\}$, there is nothing to prove, so we assume $I \neq \{0\}$ and apply Theorem 4.1. If our situation is covered by case (i) of the theorem, then we have a Jordan-Hölder sequence $\{0\} \subset I' \subset I$ for (L, I), with one nonabelian quotient I', and an abelian quotient I/I'; the latter may be reduced to $\{0\}$. The maximal ideal J' of I' is an ideal of M, according to (i) of the theorem. Thus, (a) and (b) are satisfied. In case (ii), the ideal I is itself nonabelian and minimal in L, and its maximal ideal J is an ideal of M; our corollary is obviously true in this case. If case (iii) of the theorem applies to M and L^0, then either I is nonabelian and minimal, and J is the maximal ideal of I, or I is the direct sum $I = I_1 \oplus I_2$ of two nonabelian minimal closed ideals of L; it is noted in (iii) that the maximal ideals J_1 and J_2 of I_1 and I_2 are ideals of M, as is J. Moreover, according to Theorem 1.5, we have

$$\dim(M/L^0) = \dim(\tilde{M}/\tilde{L}^0) = \dim(g_1),$$

and the Lie algebra g_1, being simple, has dimension $\geqslant 3$. We have thus verified (a) and (b) for case (iii), and case (v) is trivial. If the remaining case (iv) of the theorem applies to our situation, then $\tilde{I}$ is a transitive closed ideal of $(\tilde{M}, \tilde{L}^0)$. After possibly renumbering the simple ideals $g_1, \dots, g_p$ of $\tilde{L}^0$, we see from Theorem 1.5 that $\tilde{I}$ is of the form

$$\tilde{I} = V \oplus h_0 \oplus g_1 \oplus \cdots \oplus g_q$$

for some integer q with $0 \leqslant q \leqslant p$, and some ideal h_0 of the center g_0 of $\tilde{L}^0$. Thus, according to (iv),

$$\{0\} \subset I_V \subset I_0 \subset (I_0 + I_1) \subset \cdots \subset (I_0 + I_1 + \cdots + I_q) = I$$

is a Jordan-Hölder sequence for (L, I), with two abelian quotients I_V, I_0/I_V (one or both of which may be reduced to $\{0\}$), and q nonabelian quotients

$$(I_0 + I_1 + \cdots + I_l)/(I_0 + I_1 + \cdots + I_{l-1}) \simeq I_l/I_V,$$

for $1 \leqslant l \leqslant q$. The preimage in I_l of the unique maximal closed ideal of I_l/I_V was identified in (iv) of the theorem as an ideal J_l of M, for $1 \leqslant l \leqslant q$. Thus, we see that (a) and the first part of (b) are satisfied; it only remains to relate the number q of nonabelian quotients to the dimension of the vector space V, which is isomorphic to L/M. Assume, for the moment, that the ground field K is equal to $\mathbf{C}$. Because $\tilde{L}^0$ acts faithfully on V, we may view $\tilde{L}^0$ as a Lie subalgebra of the linear transformations of V. Let

$$S = g_1 \oplus \cdots \oplus g_p$$

denote the semisimple part of $\tilde{L}^0$, and choose a Cartan subalgebra H of S; then the number p of simple ideals of S cannot exceed the dimension of H. Because we have assumed $K = \mathbf{C}$, there exists a basis for V with respect to which all elements of H are simultaneously diagonal; moreover, because S is semisimple, we have $[S, S] = S$, and every element of S has, therefore, trace zero. It follows that

$$q \leqslant p \leqslant \dim(H) \leqslant \dim(V) - 1$$

(this is a very poor estimate, but will serve our purposes); this verifies (b) when $K = \mathbf{C}$. To treat the case $K = \mathbf{R}$, we simply note that the complexification $S_{\mathbf{C}}$ of the semisimple part S of $\tilde{L}^0$ acts faithfully on $V_{\mathbf{C}}$; moreover, $S_{\mathbf{C}}$ is semisimple and has $\geqslant p$ simple summands. Therefore, the argument above applies to complete the proof.

Before we proceed, we present an example which shows that in case (iv) of Theorem 4.1, we may indeed have $I \neq \{0\}$ and $I_V = \{0\}$. Choose a finite-dimensional simple Lie algebra g over K; denote by W a one-dimensional vector space, and write F for the local algebra $F\{W^*\}$. For our example, we take as L the semidirect product

$$L = (g \mathbin{\hat{\otimes}_K} F) \oplus \operatorname{Der}(F).$$

If we consider g to be embedded in $g \hat{\otimes}_K F$ as the subalgebra $g \otimes_K K$, then the subalgebra

$$L^0 = g \oplus \left(g \hat{\otimes}_K F^1\right) \oplus \mathrm{Der}^0(F)$$

is fundamental for L, and

$$M = (g \hat{\otimes}_K F) \oplus \mathrm{Der}^0(F)$$

is minimal among the subalgebras of L which properly contain L^0. The largest ideal of L contained in M is $I = g \hat{\otimes}_K F$, which is nonabelian and minimal in L. One checks easily that

$$D_M^\infty(L^0) = \left(g \hat{\otimes}_K F^1\right) \oplus \mathrm{Der}^0(F);$$

thus, the primitive Lie algebra $\tilde{M}$ of our example is

$$\tilde{M} = M/D_M^\infty(L^0) = (g \hat{\otimes}_K F)/\left(g \hat{\otimes}_K F^1\right),$$

and its fundamental subalgebra $\tilde{L}^0$ is

$$\tilde{L}^0 = \left(g \oplus \left(g \hat{\otimes}_K F^1\right)\right)/\left(g \hat{\otimes}_K F^1\right).$$

Therefore, our primitive Lie algebra $\tilde{M}$ is a semidirect extension $\tilde{M} = V \oplus g$ of g by an abelian ideal V; this ideal V is the projection $(g \hat{\otimes}_K F^0)/(g \hat{\otimes}_K F^1)$ of $g \hat{\otimes}_K F^0$; and so the action of g on V is equivalent to the adjoint representation of g, which is faithful and irreducible. Thus, our example is of the type covered by case (iv) of the theorem; the abelian ideal I_V of our example must be equal to $\{0\}$, because I is a nonabelian minimal closed ideal of L. Our next proposition shows that in an important class of transitive Lie algebras, this phenomenon cannot occur.

PROPOSITION 4.1. *Let (L, L^0) be a transitive Lie algebra, and suppose that there exists an abelian subalgebra A of L^0 such that $L = L^0 \oplus A$. Identify A with the vector space L/L^0, and write $\gamma\colon L^0 \to gl(A)$ for the representation of L^0 on A induced by the Lie bracket in L. Then, there is a bijective correspondence between the set of γ-invariant subspaces of A and the set of subalgebras of L which contain L^0; this correspondence associates to each γ-invariant subspace W of A the subalgebra $M_W = L^0 \oplus W$ of L. Moreover, if W is a γ-invariant subspace of A, then $W \subset D_L^\infty(M_W)$. In particular, the subalgebra L^0 is maximal among the fundamental subalgebras of L. Let W be a nonzero irreducible γ-invariant subspace of A; then M_W is minimal among the subalgebras of L which properly contain L^0, and, conversely, any such minimal subalgebra is obtained in this way. Adopt the notation of Theorem 4.1, with $M = M_W$. Then $W \subset I \neq \{0\}$, and, if the primitive Lie algebra $\tilde{M}$ is of type* (iv) *of that theorem, then $I_V \neq \{0\}$.*

PROOF. The bijective correspondence set forth in the first part of our proposition is trivial. To verify that $W \subset D_L^\infty(M_W)$ holds for all γ-invariant subspaces $W \subset A$, we prove that, for all $l \geq 0$,

$$W \subset D_L^l(M_W), \tag{4.1}$$

by induction on l; the case $l = 0$ is obvious. According to (i) of Lemma 1.2, each derived subspace $D_L^l(M_W)$ remains stable under the adjoint action of M_W. Therefore, because A is abelian, if (4.1) holds for a given integer l we have

$$[L, W] = [(M_W + A), W] \subset [M_W, D_L^l(M_W)] + [A, W] \subset D_L^l(M_W).$$

This proves (4.1) for $l + 1$, and thus, by induction, for all $l \geqslant 0$. The bijective correspondence asserted between nonzero irreducible γ-invariant subspaces of A and subalgebras minimal among the subalgebras of L which properly contain L^0 is evident. If W is a nonzero irreducible subspace of A under γ, then we have already proved that $W \subset I$, a fortiori $I \neq \{0\}$, above. It only remains to prove that when the primitive Lie algebra $\tilde{M}$ is of the affine type (iv), then I_V is not reduced to $\{0\}$. Assume that $I_V = \{0\}$; then, because $I \neq \{0\}$, we see from Theorem 4.1 that I is a nonabelian minimal closed ideal of L. After possibly renumbering the simple ideals $\{g_j\}_{1 \leqslant j \leqslant p}$ of $\tilde{L}^0$, we have $I = I_1$, $\pi(I) = V \oplus g_1$, and $J_1 = I \cap \pi^{-1}(V)$ is the unique maximal closed ideal of I. Furthermore, because V is abelian,

$$[J_1, J_1] \subset \pi^{-1}([V, V]) = D_M^\infty(L^0);$$

we conclude from Proposition 3.1 that

$$D_L(J_1) \subset D_M^\infty(L^0) = \ker(\pi).$$

The quotient $I/J_1 = g_1$ acts on $J_1/(D_L(J_1))$, giving a structure of g_1-module to this latter space which is, by Proposition 3.1, a direct sum of simple g_1-submodules isomorphic to the adjoint module of g_1. We have just shown that V can be identified with a g_1-submodule of $J_1/(D_L(J_1))$, and V is, therefore, also the direct sum of such simple adjoint submodules. Because g_1 is simple, its Killing form is nondegenerate; hence, there is a nondegenerate symmetric bilinear form β on V which is g_1-invariant. If we identify g_1 with a subspace of $V \otimes_K S^1(V^*)$ via its action on V, then, in the Spencer complex of $V \otimes_K S(V^*)$, the first prolongation of g_1,

$$g_1^{(1)} = \delta^{-1}(g_1 \otimes_K \Lambda^1 V^*) \subset V \otimes_K S^2 V^*,$$

is reduced to $\{0\}$; this follows from the existence of β and is a classical result [23]. We shall now deduce that

$$\pi(W) = V, \tag{4.2}$$

and obtain from this a contradiction. Because $W \cap L^0 = \{0\}$ and $W \subset I$, the projection $\pi(W)$ is an abelian subalgebra of $\pi(I)$ complementary to g_1; hence, there is a linear mapping $c\colon V \to g_1$ such that

$$\pi(W) = \{v + c(v) | v \in V\}.$$

Regard c as an element of $g_1 \otimes_K \Lambda^1 V^*$. Then, because $\pi(W)$ is abelian, we have $\delta c = 0$, as is seen at once. However, the module $V \otimes_K S(V^*)$ is acyclic in all nonnegative bidegrees save $(0, 0)$; since the first prolongation $g_1^{(1)}$ vanishes, we conclude that $c = 0$, which is (4.2). To complete the proof, we show that, for all $l \geqslant 0$,

$$W \subset D_L^l(\pi^{-1}(V));$$

the case $l = 0$ is implied by (4.2). From (iii) of Lemma 1.4, we see that each derived subspace $D_L^l(\pi^{-1}(V))$ remains stable under the adjoint action of M_W; the inductive step can therefore be carried out in the same way as in the proof of (4.1). We thus have

$$W \subset \bigcap_{l \geqslant 0} D_L^l(\pi^{-1}(V)) = I_V.$$

This contradicts our hypothesis $I_V = \{0\}$ and ends the proof.

We now examine some further consequences of Theorem 4.1. Let (L, L^0) be a transitive Lie algebra and choose a chain of subalgebras of L,

$$L^0 = M_0 \subset M_1 \subset \cdots \subset M_n = L,$$

such that there are no subalgebras of L contained properly between M_j and M_{j+1}, for $0 \leqslant j \leqslant n - 1$. Corresponding to this choice, we have a chain

$$\{0\} = I_{M_0} \subset I_{M_1} \subset \cdots \subset I_{M_n} = L$$

of closed ideals of L, with $I_{M_j} = D_L^\infty(M_j)$, for $0 \leqslant j \leqslant n$. By construction, in each Lie algebra $L_j^\# = L/I_{M_j}$, the subalgebras

$$\left(L_j^\#\right)^0 = M_j/I_{M_j}, \quad M_j^\# = M_{j+1}/I_{M_j}$$

are such that $(L_j^\#)^0$ is fundamental, and $M_j^\#$ is minimal among the subalgebras of $L_j^\#$ which properly contain $(L_j^\#)^0$. Moreover, the quotient $I_j^\# = I_{M_{j+1}}/I_{M_j}$ is the largest ideal of $L_j^\#$ contained in $M_j^\#$. Therefore, Theorem 4.1 describes the structure of $I_j^\#$; from Corollary 4.1, we obtain a Jordan-Hölder sequence for $(L_j^\#, I_j^\#)$,

$$\{0\} = I^0 \subset I^1 \subset \cdots \subset I^m = I_j^\#,$$

having the following properties:

(a) If I^{l+1}/I^l is a nonabelian quotient and J^{l+1} denotes the unique maximal closed ideal of I^{l+1}/I^l, then the stabilizer $N_{L_j^\#}(J^{l+1})$ of J^{l+1} in $L_j^\#$ contains $M_j^\#$.

(b) The sequence contains at most two abelian quotients; the number of nonabelian quotients is less than or equal to the dimension of the vector space M_{j+1}/M_j.

By taking, for each j, the preimages in $I_{M_{j+1}}$ of the ideals in our Jordan-Hölder decomposition of $(L_j^\#, I_j^\#)$, we obtain in all a Jordan-Hölder sequence for L. The number and type of the nonabelian quotients must be the same in all Jordan-Hölder sequences for L, and we have a telescoping sum

$$\dim(L/L^0) = \sum_{j=1}^{n} \dim(M_j/M_{j-1}),$$

for the dimensions of the vector spaces M_j/M_{j-1}. Therefore, we have proved

COROLLARY 4.2. *Let L be a transitive Lie algebra and L^0 a fundamental subalgebra of L. Set $r = \dim(L/L^0)$; then any Jordan-Hölder sequence for L has at most r nonabelian quotients and there exists a Jordan-Hölder sequence with at most $2r$ abelian quotients. Suppose that*

$$\{0\} = I_0 \subset I_1 \subset \cdots \subset I_m = L$$

is a Jordan-Hölder sequence for L; denote by $\mathcal{Q}$ the subset of $\{1,\ldots,m\}$ consisting of those integers j for which the quotient I_j/I_{j-1} is nonabelian. For each $j \in \mathcal{Q}$, let J_j be the unique maximal closed ideal of I_j/I_{j-1}, and $N_L(J_j)$ the stabilizer of J_j in L^0. Then the intersection $\bigcap_{j\in\mathcal{Q}} N_L(J_j)$ properly contains L^0. Indeed, the inclusion

$$M \subset \bigcap_{j\in\mathcal{Q}} N_L(J_j)$$

holds for any subalgebra M which is minimal in the collection of open subalgebras of L which properly contain L^0.

The following proposition is one of the results which permits a solution to the integrability problem for all transitive Lie pseudogroups acting on $\mathbf{R}^n$ which contain the translations (see the outline given in [**8**] and the forthcoming work [**9**]). We gave a proof of this result, under an additional hypothesis on B, in [**5**]; our current methods allow a simpler proof here.

PROPOSITION 4.2. *Let L be a real transitive Lie algebra and I a closed ideal of L. Denote by B the centralizer of I in L. Suppose that there exists a fundamental subalgebra L^0 and an abelian subalgebra A of L such that $L = L^0 + A + B$. Let*

$$\{0\} = I_0 \subset I_1 \subset \cdots \subset I_r = I$$

be a Jordan-Hölder sequence for (L, I). If I_l/I_{l-1} is a nonabelian quotient of complex type, with $1 \leqslant l \leqslant r$, then the Levi form of I_l/I_{l-1} vanishes identically.

PROOF. Let I_l/I_{l-1} be a nonabelian quotient of complex type in our Jordan-Hölder sequence for (L, I). If J_l is the maximal closed ideal of I_l/I_{l-1}, then the stabilizer $N_L(J_l)$ of J_l in L obviously satisfies $B \subset N_L(J_l)$, because B commutes with I in L. Moreover, the fundamental subalgebra L^0 must be contained in $N_L(J_l)$, by Corollary 4.2. Thus, we have $L = N_L(J_l) + A$; since A is abelian, our result now follows directly from the definition (3.6) of the Levi form.

Retaining the notation of Theorem 4.1, we present below, in Theorems 4.2–4.6, a list of transitive embeddings of (L, L^0) which depict the structure of the closed ideal I. These are, essentially, special realizations of (L, L^0) as transitive subalgebras of formal vector fields, and may be regarded as normal forms which can be achieved by formal changes of coordinates in any transitive realization of (L, L^0). Wherever it is convenient, we assume that $I \neq \{0\}$, since otherwise we have nothing to say beyond Corollary 2.1. One obtains Theorems 4.2 and 4.3 by simple modifications of Theorem 3.1, and Theorem 4.6 is an easy consequence of Corollary 2.1. Therefore, we give only the proof of Theorem 4.5, which corresponds to case (iv) of Theorem 4.1; the proof of Theorem 4.4, which corresponds to case (iii), proceeds along the same lines, but is somewhat easier.

For brevity, when R is a simple transitive Lie algebra and F is a ring of formal power series over K, for each closed unitary subalgebra G of $K_R \hat{\otimes}_K F$ we have identified $R \hat{\otimes}_{K_R} G$ with a closed subalgebra of $R \hat{\otimes}_K F$, in the theorems stated below.

THEOREM 4.2. *Adopt the notation and hypotheses of case* (*i*) *of Theorem* 4.1. *Endow* $\mathrm{Der}(R)$ *with a structure of transitive Lie algebra over the commutator field* K_R, *by Theorem* 1.6, *and identify* $\tilde{M}$ *with its image in* $\mathrm{Der}(R)$ *under the adjoint representation of* $\tilde{M}$ *on* R. *Denote by* $\tilde{I}_c$ *the ideal of* $\mathrm{Der}(R)$ *spanned over* K_R *by* $\tilde{I}$. *Set* $U = L/M$, $F = F\{U^*\}$, *and choose a transitive representation* $\lambda\colon (L, M) \to \mathrm{Der}(F)$. *Write* N *for the normalizer* $N_L(J')$; *then* N *is an open subalgebra of* L *containing* M. *Furthermore, according to whether the ground field* K *coincides with* K_R, *establish the following notation*:

(a) *If* $K_R = K$, *let* G *be the* λ-*invariant unitary subring of* F *which has isotropy subalgebra* N (*cf. Theorem* 1.4).

(b) *If* $K_R \neq K$, *that is, if* L *is a real transitive Lie algebra and* I' *is a nonabelian minimal closed ideal of complex type in* L, *choose a complex structure on* I' *compatible with the adjoint action of* L, *by Lemma* 3.1. *Let* N'' *denote the stabilizer in* $L_{\mathbf{C}}$ *of* J', *in terms of this complex structure*; *then* N'' *is an open complex subalgebra of* $L_{\mathbf{C}}$ *containing* $N_{\mathbf{C}}$, *a fortiori* $M_{\mathbf{C}}$. *Write* G *for the* $\lambda_{\mathbf{C}}$-*invariant unitary complex subalgebra of* $F_{\mathbf{C}}$ *having isotropy subalgebra* N'' (*cf. Theorem* 1.4).

Under these notations, the stabilizer of G *in* $\mathrm{Der}(F)$ *is a transitive subalgebra* (A_G, A_G^0) *of* $\mathrm{Der}(F)$ *containing* $\lambda(L)$, *and the semidirect product*

$$S = \left(\mathrm{Der}(R)\, \hat{\otimes}_{K_R} G\right) \oplus A_G$$

is a transitive Lie algebra over K, *with fundamental subalgebra*

$$S^0 = \left(\left(\mathrm{Der}^0(R)\, \hat{\otimes}_{K_R} G\right) + \left(\mathrm{Der}(R)\, \hat{\otimes}_{K_R} G^0\right)\right) \oplus A_G^0.$$

Write $\rho\colon S \to A_G$ *for the natural projection, and continue to denote by* λ *the induced mapping* $L \to A_G$. *Identify* $\mathrm{Der}(R)$ *with the subalgebra*

$$\mathrm{Der}(R)\, \hat{\otimes}_{K_R} K_R \subset \mathrm{Der}(R)\, \hat{\otimes}_{K_R} G.$$

Then, there exists an embedding of transitive Lie algebras $\phi\colon (L, L^0) \to (S, S^0)$ *such that* $\rho \circ \phi = \lambda$, *and*

$$\phi(I') = R\, \hat{\otimes}_{K_R} G, \quad \phi(I) \subset \tilde{I}_c\, \hat{\otimes}_{K_R} G, \quad \phi(M) \subset \left(\tilde{M} \oplus \left(\mathrm{Der}(R)\, \hat{\otimes}_{K_R} G^0\right)\right) \oplus A_G^0,$$
$$M = \phi^{-1}\left(\left(\mathrm{Der}(R)\, \hat{\otimes}_{K_R} G\right) \oplus A_G^0\right), \quad I = \phi^{-1}\left(\mathrm{Der}(R)\, \hat{\otimes}_{K_R} G\right).$$

THEOREM 4.3. *Adopt the notation and hypotheses of case* (ii) *of Theorem* 4.1 *and assume that* $I \neq \{0\}$. *Set* $U = L/M$, $F = F\{U^*\}$ *and choose a transitive representation* $\lambda\colon (L, M) \to \mathrm{Der}(F)$. *Write* N *for the normalizer* $N_L(J)$, *which is an open subalgebra of* L *containing* M. *Furthermore, according to whether* K *coincides with the commutator field* $K_{\tilde{M}}$ *of the simple Lie algebra* $\tilde{M}$, *establish the following notation*:

(a) *If* $K_{\tilde{M}} = K$, *let* G *be the* λ-*invariant unitary subring of* F *having isotropy subalgebra* N (*cf. Theorem* 1.4).

(b) *If* $K_{\tilde{M}} \neq K$, *that is, if* L *is a real transitive Lie algebra and* I *is a nonabelian minimal closed ideal of complex type in* L, *choose a complex structure on* I *compatible with the adjoint action of* L, *by Lemma* 3.1. *Let* N'' *be the stabilizer of* J *in* $L_{\mathbf{C}}$, *in terms of this complex structure*; *then* N'' *is an open complex subalgebra of* $L_{\mathbf{C}}$ *containing* $N_{\mathbf{C}}$, *a fortiori* $M_{\mathbf{C}}$. *Write* G *for the* $\lambda_{\mathbf{C}}$*-invariant unitary complex subalgebra of* $F_{\mathbf{C}}$ *which has isotropy subalgebra* N'' (*cf. Theorem* 1.4).

Under these notations, the stabilizer of G *in* Der(F) *is a transitive subalgebra* (A_G, A_G^0) *of* Der(F) *containing* $\lambda(L)$, *and the semidirect product*

$$S = (\tilde{M} \hat{\otimes}_{K_{\tilde{M}}} G) \oplus A_G$$

is a transitive Lie algebra over K, *with fundamental subalgebra*

$$S^0 = \left(\left(\tilde{L}^0 \hat{\otimes}_{K_{\tilde{M}}} G \right) + \left(\tilde{M} \hat{\otimes}_{K_{\tilde{M}}} G^0 \right) \right) \oplus A_G^0.$$

Write $\rho\colon S \to A_G$ *for the natural projection, and continue to denote by* λ *the induced mapping* $L \to A_G$.

Then, there exists an embedding of transitive Lie algebras $\phi\colon (L, L^0) \to (S, S^0)$ *such that* $\rho \circ \phi = \lambda$, *and*

$$\phi(I) = \tilde{M} \hat{\otimes}_{K_{\tilde{M}}} G, \quad M = \phi^{-1}\left((\tilde{M} \hat{\otimes}_{K_{\tilde{M}}} G) \oplus A_G^0 \right).$$

THEOREM 4.4. *Adopt the notation and hypotheses of case* (iii) *of Theorem* 4.1, *and write* K_g *for the abstract field which is isomorphic to the commutator fields of the isomorphic simple ideals* g_1 *and* g_2 *of* $\tilde{M}$. *Set* $U = L/M$, $F = F\{U^*\}$ *and choose a transitive representation* $\lambda\colon (L, M) \to$ Der(F). *For* $j = 1, 2$, *if the corresponding ideal* I_j *is not reduced to* $\{0\}$, *write* N_j *for the normalizer* $N_L(J_j)$; *then* N_j *is an open subalgebra of* L *containing* M. *If* $I_j = \{0\}$, *set, for convenience,* $G_j = K_g \otimes_K F$. *In addition, for each nonzero ideal* I_j *establish the following notation, according to whether* K_g *coincides with* K:

(a) *If* $K_g = K$, *let* G_j *be the* λ*-invariant unitary subring of* F *which has isotropy subalgebra* N_j (*cf. Theorem* 1.4).

(b) *If* $K_g \neq K$, *then each nonzero ideal* I_j *is nonabelian, minimal, and of complex type in* L, *which is a real transitive Lie algebra. In this case, endow* I_j *with a complex structure compatible with the adjoint action of* L, *by Lemma* 3.1, *and write* N_j'' *for the stabilizer in* $L_{\mathbf{C}}$ *of* J_j. *Then* N_j'' *is an open complex subalgebra of* $L_{\mathbf{C}}$ *containing* $N_{\mathbf{C}}$, *a fortiori* $M_{\mathbf{C}}$. *Let* G_j *denote the* $\lambda_{\mathbf{C}}$*-invariant unitary complex subalgebra of* $F_{\mathbf{C}}$ *which has isotropy subalgebra* N_j'' (*cf. Theorem* 1.4).

Under these notations, the intersection of the stabilizers of G_1 *and* G_2 *in* Der(F) *is a transitive subalgebra* $(A_{G_1,G_2}, A_{G_1,G_2}^0)$ *of* Der(F), *and the semidirect product*

$$S = \left(\left(g_1 \hat{\otimes}_{K_g} G_1 \right) \oplus \left(g_2 \hat{\otimes}_{K_g} G_2 \right) \right) \oplus A_{G_1,G_2}$$

is a transitive Lie algebra. Identify $\tilde{M}$ *with the subalgebra*

$$\left(\left(g_1 \otimes_{K_g} K_g \right) \oplus \left(g_2 \otimes_{K_g} K_g \right) \right) \subset \left(\left(g_1 \hat{\otimes}_{K_g} G_1 \right) \oplus \left(g_2 \hat{\otimes}_{K_g} G_2 \right) \right);$$

then

$$S^0 = \left(\tilde{L}^0 \oplus \left(g_1 \hat{\otimes}_{K_g} G_1^0 \right) \oplus \left(g_2 \hat{\otimes}_{K_g} G_2^0 \right) \right) \oplus A_{G_1,G_2}^0$$

is a fundamental subalgebra of S. *Write* $\rho\colon S \to A_{G_1,G_2}$ *for the natural projection, and continue to denote by* λ *the induced mapping* $L \to A_{G_1,G_2}$.

Then, there exists an embedding of transitive Lie algebras $\phi\colon (L, L^0) \to (S, S^0)$ *such that* $\rho \circ \phi = \lambda$, *and*

$$I = \phi^{-1}\Big(\big(g_1 \hat{\otimes}_{K_g} G_1\big) \oplus \big(g_2 \hat{\otimes}_{K_g} G_2\big)\Big),$$

$$M = \phi^{-1}\Big(\big(g_1 \hat{\otimes}_{K_g} G_1\big) \oplus \big(g_2 \hat{\otimes}_{K_g} G_2\big) \oplus A^0_{G_1, G_2}\Big);$$

moreover, for each ideal I_j *which is not reduced to* $\{0\}$, $\phi(I_j) = g_j \hat{\otimes}_{K_g} G_j$.

THEOREM 4.5. *Adopt the notation and hypotheses of case* (iv) *of Theorem* 4.1, *and assume that* $I \neq \{0\}$. *Set* $U = L/M$, $F = F\{U^*\}$ *and choose a transitive representation* $\lambda\colon (L, M) \to \mathrm{Der}(F)$.

Denote by Z the centralizer of $g_0 \oplus g_1 \oplus \cdots \oplus g_p$ *in* $\mathrm{End}_K(V)$. *Make the identification*

$$\mathrm{Der}(\tilde{M}) = V \oplus (Z \oplus g_1 \oplus \cdots \oplus g_p)$$

permitted by Lemma 1.5, *and identify* $\tilde{M}$ *with the transitive ideal*

$$\tilde{M} = V \oplus (g_0 \oplus g_1 \oplus \cdots \oplus g_p)$$

of inner derivations in $\mathrm{Der}(\tilde{M})$; *then* $g_0 \subset Z$. *Define open subalgebras of L by setting*

$$N = N_L(M) \cap N_L\big(D^\infty_M(L^0)\big), \quad N_j = N_L(J_j),$$

for $1 \leqslant j \leqslant p$; *then the inclusions* $N_j \supset N \supset M$ *are satisfied. Let G denote the* λ*-invariant unitary subring of F having isotropy subalgebra N* (*cf. Theorem* 1.4). *In addition, for each index* $j > 0$ *for which* $I_j/I_V \neq \{0\}$, *establish the following notation, according to whether the commutator field* K_{g_j} *coincides with* K:

(a) *If* $K_{g_j} = K$, *write* G_j *for the* λ*-invariant unitary subring of F whose isotropy subalgebra is equal to* N_j; *then* $G_j \subset G \subset F$.

(b) *If* $K_{g_j} \neq K$, *that is, if L is a real transitive Lie algebra and* I_j/I_V *is a nonabelian minimal closed ideal of complex type in* L/I_V, *fix on* I_j/I_V *a complex structure compatible with its structure as an L-module, by Lemma* 3.1. *In terms of this complex structure, let* N_j'' *be the stabilizer in* $L_{\mathbf{C}}$ *of* J_j; *then* N_j'' *is an open complex subalgebra of* $L_{\mathbf{C}}$ *containing* $N_{\mathbf{C}}$, *a fortiori* $M_{\mathbf{C}}$. *Write* G_j *for the* $\lambda_{\mathbf{C}}$*-invariant unitary complex subalgebra of* $F_{\mathbf{C}}$ *whose isotropy subalgebra is* N_j''; *then the inclusions* $G_j \subset G_{\mathbf{C}} \subset F_{\mathbf{C}}$ *are valid.*

For the remaining indices $j > 0$ *for which* I_j/I_V *is reduced to* $\{0\}$, *set* $G_j = K_{g_j} \otimes_K G$, *for convenience of notation. The simultaneous stabilizer*

$$\big\{\xi \in \mathrm{Der}(F) \,|\, \xi(G) \subset G \text{ and } \xi(G_j) \subset G_j \text{ for } 1 \leqslant j \leqslant p\big\}$$

is, then, a transitive subalgebra (A, A^0) *of* $\mathrm{Der}(F)$ *containing* $\lambda(L)$, *and the semidirect product*

$$S = \Big((V \hat{\otimes}_K G) \oplus (Z \hat{\otimes}_K G) \oplus \bigoplus_{1 \leqslant j \leqslant p} \big(g_j \hat{\otimes}_{K_{g_j}} G_j\big)\Big) \oplus A$$

is a transitive Lie algebra. Identify $\mathrm{Der}(\tilde{M})$ *with the subalgebra*

$$(V \otimes_K K) \oplus (Z \otimes_K K) \oplus \bigoplus_{1 \leqslant j \leqslant p} \big(g_j \otimes_{K_{g_j}} K_{g_j}\big)$$

of S; then the space

$$S^0 = \Big(\tilde{L}^0 \oplus \big(V \hat{\otimes}_K G^0\big) \oplus \big(Z \hat{\otimes}_K G^0\big) \oplus \bigoplus_{1 \leqslant j \leqslant p} \big(g_j \hat{\otimes}_{K_{g_j}} G_j^0\big)\Big) \oplus A^0$$

forms a fundamental subalgebra in S. Define closed ideals of S by setting

$$I' = ((V \oplus Z) \hat{\otimes}_K G) \oplus \bigoplus_{1 \leqslant j \leqslant p} \big(g_j \hat{\otimes}_{K_{g_j}} G_j\big),$$

$$I_V^{\#} = V \hat{\otimes}_K G, \quad I_0^{\#} = (V \oplus g_0) \hat{\otimes}_K G, \quad I_j^{\#} = (V \hat{\otimes}_K G) \oplus \big(g_j \hat{\otimes}_{K_{g_j}} G_j\big),$$

for $1 \leqslant j \leqslant p$. *Write* $\rho\colon S \to A$ *for the natural projection and continue to denote by* λ *the induced mapping* $L \to A$.

Then, there exists an embedding of transitive Lie algebras $\phi\colon (L, L^0) \to (S, S^0)$ *such that* $\rho \circ \phi = \lambda$, *and*

$$\phi(L) \subset \tilde{M} + S^0, \quad I = \phi^{-1}(I'), \quad I_V = \phi^{-1}\big(I_V^{\#}\big),$$

$$I_0 = \phi^{-1}\big(I_0^{\#}\big), \quad I_j = \phi^{-1}\big(I_j^{\#}\big)$$

for $1 \leqslant j \leqslant p$, *and if* I_j/I_V *is not reduced to* $\{0\}$, *then* $I_j^{\#} = I_V^{\#} + \phi(I_j)$.

PROOF. Any assertion made in the theorem prior to the statement that such a morphism ϕ exists is trivial. To construct ϕ, we shall apply Theorem 2.1, with

$$L' = I' \oplus \lambda(L), \quad M' = I' \oplus \lambda(M),$$

$$J' = \big((V \oplus Z) \hat{\otimes}_K G^0\big) \oplus \bigoplus_{1 \leqslant j \leqslant p} \big(g_j \hat{\otimes}_{K_{g_j}} G_j\big),$$

and, therefore, the filtration $\mathscr{F}'$ employed in that theorem is expressed as

$$I_l' = \big((V \oplus Z) \hat{\otimes}_K G^{l-1}\big) \oplus \bigoplus_{1 \leqslant j \leqslant p} \big(g_j \hat{\otimes}_{K_{g_j}} G_j^{l-1}\big),$$

for all $l \in \mathbf{Z}$. For convenience, we renumber the simple ideals $\{g_j\}_{1 \leqslant j \leqslant p}$ of $\tilde{L}^0$ so that there are two integers $q \geqslant r \geqslant 0$ with the following property: among the indices j with $1 \leqslant j \leqslant p$, we have $I_j/I_V \neq \{0\}$ precisely when $1 \leqslant j \leqslant q$, and, among these latter indices, we have $K_{g_j} \neq K$ precisely when $1 \leqslant j \leqslant r$. We now set

$$W_j^* = \big(L_{\mathbf{C}}/N_j''\big)^*, \qquad 1 \leqslant j \leqslant r,$$

$$W_j^* = \big(L/N_j\big)^*, \qquad r+1 \leqslant j \leqslant q,$$

$$W_j^* = W^* = (L/N)^*, \qquad q+1 \leqslant j \leqslant p,$$

thus defining subspaces of U^* or $(U_{\mathbf{C}})^*$. From our expression for $\mathscr{F}'$, we see that $\mathrm{gr}(I', \mathscr{F}')$ is the graded Lie algebra and $S(U)$-module

$$\mathrm{gr}(I', \mathscr{F}') = ((V \oplus Z) \otimes_K S(W^*)) \oplus \bigoplus_{j=1}^{r} \big(g_j \otimes_{\mathbf{C}} S(W_j^*)\big) \oplus \bigoplus_{j=r+1}^{p} \big(g_j \otimes_K S(W^*)\big).$$

According to Lemma 1.7, then, the Spencer cohomology groups of $\mathrm{gr}(I', \mathscr{F}')$ vanish in all bidegrees (l, m) with $l > 0$. We now seek a suitable choice for ϕ_0. The adjoint representation of N on M induces a morphism of topological Lie algebras

$$\sigma\colon N \to \mathrm{Der}(\tilde{M}) = \mathrm{Der}\big(M/D_M^{\infty}(L^0)\big)$$

such that $\sigma|_M = \pi$. Any finite-dimensional simple Lie algebra g is naturally identified, via its adjoint representation, with $\mathrm{Der}(g)$, as is well known **[17]**. Thus, we obtain, as above, morphisms

$$\sigma_j'': N_j'' \to g_j = \mathrm{Der}\big(I_j/J_j\big), \qquad 1 \leqslant j \leqslant r,$$
$$\sigma_j: N_j \to g_j = \mathrm{Der}\big(I_j/J_j\big), \qquad r+1 \leqslant j \leqslant q;$$

we remark that each of the maps σ_j'' is C-linear, with respect to the complex structure on g_j induced by that fixed on I_j/I_V. In terms of the direct sum decomposition

$$\mathrm{Der}(\tilde{M}) = V \oplus Z \oplus \bigoplus_{j=1}^{p} g_j,$$

let π_j denote the projection $\mathrm{Der}(\tilde{M}) \to g_j$, for $1 \leqslant j \leqslant p$. Then, we see from the definitions of the various maps involved that

$$\pi_i \circ \sigma = \sigma_i''|_N, \quad \pi_j \circ \sigma = \sigma_j|_N, \tag{4.3}$$

for $1 \leqslant i \leqslant r$ and $r+1 \leqslant j \leqslant q$. Now choose a continuous linear mapping τ_0: $L \to \mathrm{Der}(\tilde{M})$ subject to the requirements for $r+1 \leqslant j \leqslant q$,

$$\tau_0|_N = \sigma, \quad (\pi_j \circ \tau_0)|_{N_j} = \sigma_j \tag{4.4}$$

and, in terms of the complex-linear extension of $\pi_i \circ \tau_0$ to $L_{\mathbf{C}}$, for $1 \leqslant i \leqslant r$,

$$(\pi_i \circ \tau_0)|_{N_i''} = \sigma_i''. \tag{4.5}$$

It is clear from (4.3) that such a map τ_0 exists. We now define our initial approximation ϕ_0: $L \to L'$ to ϕ by setting, for all $\xi \in L$, $\phi_0(\xi) = \tau_0(\xi) + \lambda(\xi)$; recall that we have identified $\mathrm{Der}(\tilde{M})$ with a closed subalgebra of I' complementary to J'. Because $\lambda(L)$ commutes with $\mathrm{Der}(\tilde{M})$ in L', we have, for all $\xi, \eta \in L$,

$$\phi_0([\xi,\eta]) - [\phi_0(\xi), \phi_0(\eta)] = \tau_0([\xi,\eta]) - [\tau_0(\xi), \tau_0(\eta)]. \tag{4.6}$$

Using this and our previous expression of $\mathrm{gr}(I', \mathscr{F}')$, we see from Lemma 1.7 that the cocycle conditions of Theorem 2.1 are equivalent to requiring that the restrictions $\tau_0|_M$, $(\pi_j \circ \tau_0)|_{N_j}$ and $(\pi_i \circ \tau_0)|_{N_i''}$ for $1 \leqslant i \leqslant r$ and $r+1 \leqslant j \leqslant q$, be Lie algebra homomorphisms, which we see from (4.4) and (4.5) to be true. Therefore, a morphism ϕ: $L \to L'$ exists such that $I = \phi^{-1}(I')$, and ϕ agrees with ϕ_0 modulo J'. To verify the relationships between the various ideals of L in I and of L' in I' listed in the theorem, we first observe that, in the induced filtrations, we have

$$\mathrm{gr}\big(I_V^{\#}, \mathscr{F}_V^{\#}\big) = V \otimes_K S(W^*), \quad \mathrm{gr}\big(I_0^{\#}, \mathscr{F}_0^{\#}\big) = (V \oplus g_0) \otimes_K S(W^*),$$
$$\mathrm{gr}\big(I_j^{\#}, \mathscr{F}_j^{\#}\big) = (V \otimes_K S(W^*)) \oplus \big(g_j \otimes_{K_{g_j}} S(W_j^*)\big),$$

for $1 \leqslant j \leqslant q$. Let $(\tilde{I}, I^{\#})$ denote one of the pairs $(I_V, I_V^{\#})$, $(I_0, I_0^{\#})$, $(I_j, I_j^{\#})$, $1 \leqslant j \leqslant q$. To verify that $\phi^{-1}(I^{\#}) \subset \tilde{I}$, we first recall that ϕ agrees with ϕ_0 modulo J', and thus, since $\phi_0|_I = \pi|_I$, we have

$$\pi\big(\phi^{-1}(I^{\#})\big) = \mathrm{gr}^0(I^{\#}, \mathscr{F}^{\#}).$$

But for each pair $(\tilde{I}, I^{\#})$, the definition of $\tilde{I}$ was as the largest ideal of L in $\pi^{-1}(\mathrm{gr}^0(I^{\#}, \mathscr{F}^{\#}))$, which proves that $\phi^{-1}(I^{\#})$ lies in $\tilde{I}$. Again because $\phi_0|_I = \pi|_I$, we have $\phi_0(\tilde{I}) \subset I^{\#}$, and viewing the list of submodules above, we see from Lemma 1.7 that the Spencer cohomology groups of $\mathrm{gr}(I^{\#}, \mathscr{F}^{\#})$ vanish in all bidegrees (l, m) with $l > 0$, and, in the Spencer complex of $\mathrm{gr}(I', \mathscr{F}')$, we have

$$\delta^{-1}\left(\mathrm{gr}^0(I^{\#}, \mathscr{F}^{\#}) \otimes_K \Lambda^1 U^*\right) = \mathrm{gr}^1(I^{\#}, \mathscr{F}^{\#}) \subset \mathrm{gr}^1(I', \mathscr{F}').$$

Thus, to verify that the requirement

$$\phi_1(\tilde{I}) \subset I^{\#}/\left(I_2' \cap I^{\#}\right)$$

of Theorem 2.1 holds, it will suffice to show that

$$b_0(\tilde{I}) \subset \mathrm{gr}^0(I^{\#}, \mathscr{F}^{\#}) \otimes_K \Lambda^1 U^*.$$

Applying (4.6) and the definition of b_0, we see that

$$\begin{aligned} b_0(\tilde{I})(U) &\subset \left[\tau_0(\tilde{I}), \tau_0(L)\right] + \tau_0([\tilde{I}, L]) \subset \left[\tau_0(\tilde{I}), \tau_0(L)\right] + \tau_0(\tilde{I}) \\ &\subset \left[\mathrm{gr}^0(I^{\#}, \mathscr{F}^{\#}), \mathrm{Der}(\tilde{M})\right] + \mathrm{gr}^0(I^{\#}, \mathscr{F}^{\#}) = \mathrm{gr}^0(I^{\#}, \mathscr{F}^{\#}). \end{aligned}$$

From Theorem 2.1, we conclude that $\phi(\tilde{I}) \subset I^{\#}$, which finishes our proof of the relation $\phi^{-1}(I^{\#}) = \tilde{I}$ for all of the pairs on our list. To complete the proof, we have only to show that $\phi(I_j) + I_V^{\#} = I_j^{\#}$ for $1 \leqslant j \leqslant q$. We first observe that, for all $l \in \mathbf{Z}$,

$$D_L^l(J_j) = I_j \cap D_L^l(J) + I_V, \tag{4.7}$$

as is proved easily by induction on l. In terms of the filtrations $\mathscr{F} = \mathscr{F}_L(I, J)$ of I and $\mathscr{F}'$ of I', we know that $\phi|_I$ is filtration-preserving and that $\mathrm{gr}(\phi|_I)$ has zeroth graded term equal to $\pi|_I$, by Theorem 2.1. Write $\tilde{\mathscr{F}}_j$ for the filtration $\mathscr{F}_L(I_j, J_j)$ of I_j; then, comparing the filtrations $\mathscr{F} \cap I_j$ and $\tilde{\mathscr{F}}_j$ by means of (4.7), we see that

$$\mathrm{gr}(I_j, \tilde{\mathscr{F}}_j) = \mathrm{gr}(I_j, \mathscr{F} \cap I_j)/\mathrm{gr}(I_V, \mathscr{F} \cap I_V).$$

Because $\phi^{-1}(I_V^{\#}) = I_V$, it now follows that $\mathrm{gr}(\phi|_{I_j})$ induces a morphism of graded Lie algebras and $S(U)$-modules

$$\nu\colon \mathrm{gr}(I_j, \tilde{\mathscr{F}}_j) \to \left(\mathrm{gr}\left(I_j^{\#}, \mathscr{F}_j^{\#}\right)/\mathrm{gr}\left(I_V^{\#}, \mathscr{F}_V^{\#}\right)\right) = g_j \otimes_{K_{g_j}} S(W_j^*)$$

whose zeroth graded term ν_0 is the isomorphism $I_j/J_j \to g_j$ induced by $\pi_V|_{I_j}$. From Proposition 3.1 and Lemma 3.1, we deduce that ν is an isomorphism, and thus, that

$$\mathrm{gr}(\phi|_I)\left(\mathrm{gr}(I_j, \mathscr{F} \cap I_j)\right) + \mathrm{gr}\left(I_V^{\#}, \mathscr{F}_V^{\#}\right) = \mathrm{gr}\left(I_j^{\#}, \mathscr{F}_j^{\#}\right).$$

Because $\phi(I_j) + I_V^{\#}$ is closed in $I_j^{\#}$, the equality $\phi(I_j) + I_V^{\#} = I_j^{\#}$ now follows at once from Chevalley's theorem. The proof of Theorem 4.5 is thus complete.

We pause to examine the realization which Theorem 4.5 gives to the closed abelian ideal I_V of L. Assume that $I_V \neq \{0\}$, and for each index j with $1 \leqslant j \leqslant p$ for which I_j/I_V is not reduced to $\{0\}$, consider the associative subalgebra $\mathscr{E}_j$ of $\mathrm{End}_K(V \hat{\otimes}_K G)$ generated by the operators

$$\mathrm{ad}(\phi(\xi))|_{V \hat{\otimes}_K G} \quad \text{for } \xi \in I_j.$$

According to Theorem 4.5, we have

$$(V \hat{\otimes}_K G) + \phi(I_j) = (V \hat{\otimes}_K G) + \left(g_j \hat{\otimes}_{K_{g_j}} G_j\right);$$

thus, because $V \hat{\otimes}_K G$ is abelian, the algebra $\mathscr{E}_j$ is the same as the subalgebra of $\mathrm{End}_K(V \hat{\otimes}_K G)$ generated by the transformations $\mathrm{ad}(\eta)|_{V\hat{\otimes}_K G}$ with $\eta \in g_j \hat{\otimes}_{K_{g_j}} G_j$. The reductive Lie algebra $\tilde{L}^0 = g_0 \oplus g_1 \oplus \cdots \oplus g_p$ is faithfully and irreducibly represented on V. Therefore **[1]**, we can write V as the tensor product

$$V = W_1 \otimes_\Gamma W, \quad \text{with } K \subseteq \Gamma \subseteq \mathrm{End}_{g_j}(W),$$

of a nontrivial g_j-module W which is irreducible over K, and an irreducible module W_1 over the remaining summands $g_0 \oplus \cdots \oplus g_{j-1} \oplus g_{j+1} \oplus \cdots \oplus g_p$ of $\tilde{L}^0$; this module W_1 is defined with scalars lying in Γ, which is a division subring, containing K, of the division ring $\mathrm{End}_{g_j}(W)$. When $K = \mathbf{C}$, this endomorphism ring $\mathrm{End}_{g_j}(W)$ is just our ground field $\mathbf{C}$, but when $K = \mathbf{R}$, the division ring $\mathrm{End}_{g_j}(W)$ can be equal to $\mathbf{R}$ or $\mathbf{C}$ or the quaternions $\mathbf{H}$. Write $\mathscr{G}_j$ for the associative subalgebra of $\mathrm{End}_K(W)$ generated by the action of g_j; then, by the Jacobson-Bourbaki density theorem **[1]**, any transformation of W linear over $\mathrm{End}_{g_j}(W)$ lies in $\mathscr{G}_j$. When $K_{g_j} = K$, it follows at once that the algebra $\mathscr{E}_j$ contains all of the operators which correspond to multiplication on the right in $V \hat{\otimes}_K G$ by elements of the subring G_j of G. Since I_V is an ideal of L and $\phi(I_V)$ lies in $V \hat{\otimes}_K G$, we conclude that $\phi(I_V)$ is invariant under $\mathscr{E}_j$, and thus the image $\phi(I_V)$ is stable under multiplication on the right by G_j. Phrased analytically, we have shown that the partial differential equations which define I_V, in the realization given by ϕ, do not depend on those variables which generate the subring G_j in G.

A similar result applies to the case $K_{g_j} \neq K$, that is, when our ground field is $\mathbf{R}$ and g_j is a complex simple Lie algebra viewed as a Lie algebra over $\mathbf{R}$. In the course of Theorem 4.5, we have fixed on I_j/I_V, and thus on g_j, one of its two possible complex structures. The real irreducible g_j-module W can be described, then, in one of the following ways (see, for example, **[16]**):

(a) Assume that there exists on W a structure of complex vector space with respect to which g_j acts by $\mathbf{C}$-linear transformations. Then with respect to the complex structure fixed on g_j, the complex vector space W is either:

(i) a complex simple g_j-module $\mathscr{M}$; or

(ii) the conjugate $\overline{\mathscr{M}}$ of a complex simple g_j-module $\mathscr{M}$; or

(iii) a tensor product $\mathscr{M} \otimes_{\mathbf{C}} \overline{\mathscr{N}}$, where $\mathscr{M}$ and $\mathscr{N}$ are nonisomorphic simple complex g_j-modules.

(b) If W has no complex structure compatible with the action of g_j, then the complexification of W is a tensor product $W_{\mathbf{C}} = \mathscr{M} \otimes_{\mathbf{C}} \overline{\mathscr{M}}$, where $\mathscr{M}$ is a complex simple module over the complex Lie algebra g_j. Conjugation in $W_{\mathbf{C}}$ is expressed, on decomposable elements, as $u \otimes \bar{v} \mapsto v \otimes \bar{u}$, for all $u, v \in \mathscr{M}$, and W is identified with the invariant real subspace of $\mathscr{M} \otimes_{\mathbf{C}} \overline{\mathscr{M}}$ consisting of those elements which remain fixed under conjugation.

To obtain from this a statement concerning $\mathscr{E}_j$, we assume first that W is a g_j-module of one of the types described in (a). Fix on $V = W_1 \otimes_\Gamma W$ the complex structure it inherits from W, and identify $V \hat{\otimes}_{\mathbf{R}} G$ with $V \hat{\otimes}_{\mathbf{C}} G_{\mathbf{C}}$, via this complex structure on V. As in the last paragraph, we see from the Jacobson-Bourbaki density theorem that if W is of type (i), then $\mathscr{E}_j$ contains all of the transformations of

$V \hat{\otimes}_{\mathbf{C}} G_{\mathbf{C}}$ given by right multiplication by elements of the subring G_j of $G_{\mathbf{C}}$, while if W is of type (ii), the algebra $\mathscr{E}_j$ contains right multiplication by all elements of the complex conjugate $\overline{G}_j$ of G_j. If W is of type (iii), we can apply the density theorem to conclude that $\mathscr{E}_j$ contains all multiplications on the right by elements of G_j, as well as right multiplication by all elements of $\overline{G}_j$. An argument which is essentially the same as above shows that when W is of the type described in (b), then $\mathscr{E}_j$ contains all right multiplications in $V \hat{\otimes}_{\mathbf{R}} G$ by elements of the subring $\operatorname{Re}(G_j)$ of G made up of all real (and imaginary) parts of elements of $G_j \subset G_{\mathbf{C}}$. As before, we observe that in all cases $\phi(I_V)$ forms a subspace of $V \hat{\otimes}_K G$ invariant under $\mathscr{E}_j$. Thus, the image $\phi(I_V)$ is stable under multiplication by the appropriate subring of G or $G_{\mathbf{C}}$, as determined above.

We return to our list of realizations to give the final case.

THEOREM 4.6. *Adopt the notation and hypotheses of case* (v) *of Theorem* 4.1 *and assume that* $I \neq \{0\}$. *Let* N *denote the normalizer* $N_L(J)$, *which is an open subalgebra of* L *containing* M. *Set* $U = L/M$, $F = F\{U^*\}$ *and choose a transitive representation* $\lambda\colon (L, M) \to \operatorname{Der}(F)$. *Write* G *for the* λ-*invariant unitary subring of* F *which has isotropy subalgebra* N (*cf. Theorem* 1.4). *Let* E *denote the unique two-dimensional nonabelian Lie algebra over* K; *then* E *has a basis* $\{e_0, e_{-1}\}$ *such that* $[e_0, e_{-1}] = -e_{-1}$, *and thus* $E_0 = \operatorname{span}\{e_0\}$ *is a fundamental subalgebra for* E, *and* $E_{-1} = [E, E] = \operatorname{span}\{e_{-1}\}$ *is the unique nontrivial ideal of* E. *The stabilizer of* G *in* $\operatorname{Der}(F)$ *is a transitive subalgebra* (A_G, A_G^0) *of* $\operatorname{Der}(F)$, *and the semidirect product* $S = (E \hat{\otimes}_K G) \oplus \operatorname{Der}(F)$ *is a transitive Lie algebra, with fundamental subalgebra*

$$S^0 = \left(E_0 \hat{\otimes}_K G + E \hat{\otimes}_K G^0\right) \oplus A_G.$$

Let $\rho\colon S \to A_G$ *be the natural projection, and continue to denote by* λ *the induced mapping* $L \to A_G$.

Then, there exists an embedding of transitive Lie algebras $\phi\colon (L, L^0) \to (S, S^0)$ *such that* $\rho \circ \phi = \lambda$, *and*

$$I = \phi^{-1}(E_{-1} \hat{\otimes}_K G) = \phi^{-1}(E \hat{\otimes}_K G).$$

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