CODIMENSION 1 ORBITS AND SEMI-INVARIANTS FOR THE REPRESENTATIONS OF AN EQUIORIENTED GRAPH OF TYPE D_n

BY

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ABSTRACT. We consider the Dynkin diagram D_n equioriented and the variety $\mathrm{Hom}(V_1, V_3) \times \prod_{i=2}^{n} \mathrm{Hom}(V_i, V_{i+1})$, V_j a vector space over K, on which the group $G = \prod_{i=1}^{n} \mathrm{GL}(V_i)$ acts. We determine the maximal orbit and the codim. 1 orbits of this action, giving their decomposition in terms of the irreducible representations of D_n. We also deduce a set of algebraically independent semi-invariant polynomials which generate the ring of semi-invariants.

0. Introduction. We develop part of a program devoted to study the geometry and the orbit structure for the varieties L_d of representations of an oriented graph of finite type (d a fixed dimension) on which a group G_d (a product of linear groups) acts naturally.

The case of a Dynkin diagram $\mathscr{A}_n$ has been treated in [1–5]. [1, 2] describe the structure of the codim. 1 orbits and determine the corresponding ring of semi-invariants and [3, 4] give a combinatorial description of the closures of the orbits in L_d for any orientation of $\mathscr{A}_n$. In [5] it is proved, when $\mathscr{A}_n$ is equioriented, that the closures of the orbits are algebraic varieties which are normal, Cohen-Macaulay with rational singularities.

We begin the corresponding study for the Dynkin diagram D_n. In particular we are interested in the description of the codim. 1 orbits. Once we have a "combinatorial" method to produce the set $\mathcal{O}_1$ of codim. 1 orbits, it is not difficult to find, using the same method of [1 and 2], the reduced equations for the Zarisky closure $\overline{O}$ of any $O \in \mathcal{O}_1$; in particular we can deduce a set of algebraically independent semi-invariant polynomials which generate the ring of semi-invariants (cf. §6).

We briefly explain the procedure allowing us to describe the set of codim. 1 orbits $\mathcal{O}_1$ when D_n is equioriented. Consider the graph D_n:
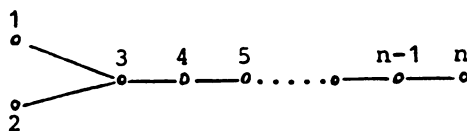
where $\Gamma_0 = \{1, 2, \ldots, n\}$ is the set of its vertices, and Γ_1 the set of its edges, and let Ω be an arbitrary orientation for its edges. For any dimension

$$(*) \qquad\qquad d = \left({}^{d_1}_{d_2}; d_3, \ldots, d_n \right),$$

d_i nonnegative integers, let V_i be a vector space of dimension d_i over a given field K and consider the variety of the representations of the oriented graph (D_n, Ω):

$$L_d(D_n, \Omega) := \prod_{l \in \Gamma_1} \text{Hom}_K\left(V_{i(l)}, V_{f(l)}\right)$$

where $i(l), f(l) \in \Gamma_0$ are, respectively, the initial and the final vertex of l with respect to the orientation Ω. The group $G_d := \prod_{i=1}^{n} \text{GL}(V_i)$ acts naturally on $L_d(D_n, \Omega)$ and the orbits O_B of this action correspond to the isomorphism classes $[B]$ of the representations $B \in L_d(D_n, \Omega)$. Recall now that the indecomposable representations of (D_n, Ω) are in 1-1 correspondence with the set Δ^+ of the positive roots of the Dynkin diagram D_n, independently from the orientation (cf. [6, 8], and §1 for an explicit and graphic description). Let us denote them by E_α, $\alpha \in \Delta^+$ (Δ^+ is finite as D_n is of finite type). Then up to isomorphism we have $B = \bigoplus_{\alpha \in \Delta^+} e_\alpha^B E_\alpha$, where e_α^B is the multiplicity of E_α in $B \in L_d(D_n, \Omega)$. It follows that the sets $e^B = \{ e_\alpha^B, \alpha \in \Delta^+ \}$ parametrize the orbits $O_B \subset L_d(D_n, \Omega)$ provided they satisfy the equalities corresponding to the fact that the dimension d is fixed; but their independence from the orientation makes them unsuitable for studying problems which do depend on the orientation, e.g. the structure of the orbits, in particular the description of the maximal orbit and the codim. 1 orbits.

What we do here is choose a special orientation for D_n, the equiorientation, namely:
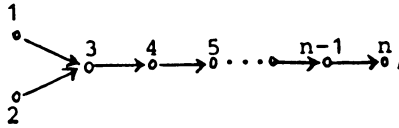
$$L_d := L_d(D_n, \text{eq}) = \text{Hom}_K(V_1, V_3) \times \prod_{i=2}^{n-1} \text{Hom}_K(V_i, V_{i+1})$$

and we reparametrize the orbits with sets of "rank parameters" $N^B = \{ N_\alpha^B, \alpha \in \Delta^+ \}$, where N_α^B is the rank of a suitable map φ_α^B constructed from the representation $B \equiv (B_1, B_2, \ldots, B_{n-1})$:
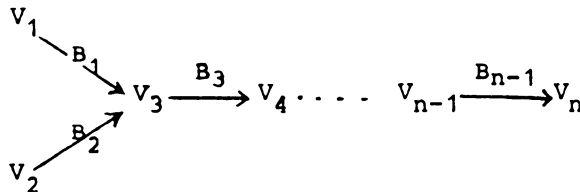
The choice of the maps φ_α^B is related to the geometry of the subspaces given by the images and the kernels of the linear maps B_i and their compositions (cf. §1), therefore they need to be changed if we change the orientation of D_n.

The advantage of this parametrization is the fact that if B, $C \in (D_n, \text{eq})$ and $O_C \subset \overline{O}_B$ ($\overline{O}_B$ the Zariski closure of O_B), then $N_\alpha^C \leqslant N_\alpha^B$ for every $\alpha \in \Delta^+$, since in a degeneration ranks cannot increase. It follows that the unique maximal orbit $O_{\max} \subset L_d(D_n, \text{eq})$ must correspond to a set $N^A = \{ N_\alpha^A, \alpha \in \Delta^+ \}$, $A \in O_{\max}$, where the N_α^A's are "maximum possible". The construction of $O_{\max}$ is suggested now by this property. In §3 we describe explicitly, using a "combinatorial" argument, the indecomposables appearing as factors of $A \in O_{\max}$; equivalently, for any dimension d (see (∗)) we give a combinatorial description of the "canonical decomposition" of the dimension vector d with respect to the chosen orientation of D_n (for this notion cf. [11, 12]).

The classification of the codim. 1 orbits in $L_d(D_n, \text{eq})$ is explicitly given in Theorem 5.1. More precisely, for any d as in (∗) we define a subset I_d of pairs of indices (i, j), $i \geqslant j$ (cf. §4), then we find a bijection $\psi: I_d \to \mathcal{O}_1$, $\mathcal{O}_1$ the set of codim. 1 orbits in $L_d(D_n, \text{eq})$, $\psi(i, j) = O_{B_{ij}}$, where B_{ij} is explicitly constructed in terms of its indecomposable factors. As a consequence of properties (a) and (b) which we establish for ψ in Theorem 5.1, we deduce, for any $O_{B_{ij}} \in \mathcal{O}_1$, the equation $D_{ij} = 0$ for the corresponding algebraic variety $\overline{O}_{B_{ij}}$ (cf. §6). A result due to Sato and Kimura (cf. [13]) allows us to describe the ring of semi-invariants $\mathscr{R}_d(D_n, \text{eq})$, namely if K is an infinite field, then the D_{ij}'s, $(i, j) \in I_d$, are a set of algebraically independent semi-invariant polynomials which generate the ring of semi-invariants $\mathscr{R}_d(D_n, \text{eq})$ (cf. §6).

At this point the proposed problem of classifying the codim. 1 orbits and describing the ring of semi-invariants for any dimension d is solved for (D_n, eq). If we change the orientation, i.e. we consider (D_n, Ω) for any Ω, and want to solve the same type of problems, we must define a suitable set of maps φ_α^B, $\alpha \in \Delta$, for $B \in L_d(D_n, \Omega)$ in such a way that the corresponding ranks parametrize the orbits. Then we can go through the whole procedure developing combinatorial arguments, as in §§3–5, suitable for the orientation Ω. The argument will not be trivial, since it is not trivial for the Dynkin diagram $\mathscr{A}_n$ (cf. [1, 2]), and in any case will use complicated notation related to the sequence of sources and sinks describing the orientation Ω. We do not do it.

On the other hand, the canonical decomposition of d w.r.t. Ω does not strictly need knowledge of the "rank parameters" once it is known for a special orientation. In fact any orientation Ω can be obtained from a given one by performing a suitable number of simple reflections in admissible vertices (cf. [6]), therefore the reflection functors suggest a possible construction of $O_{\max} \subset L_d(D_n, \Omega)$. In any case the formal proof that we have produced the maximal orbit is deducible from the computation of the dimension of its stabilizer (once we have computed $\dim \operatorname{Hom}_K(E_\alpha, E_\beta)$ for any $\alpha, \beta \in \Delta^+$ and for any orientation Ω).

We can also control what happens to the ring of semi-invariants under a change of orientation by using the isomorphisms (2.19) given in [11] (cf. the discussion given in [2, §7]).

1. Rank parameters for the representations of (D_n, eq). Consider an equioriented Dynkin diagram D_n, i.e.
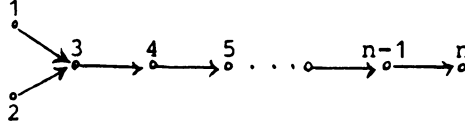
where $\Gamma_0 = \{1, 2, 3, \ldots, n\}$ is the set of its vertices. The orientation induces a partial ordering $\leqslant$ in Γ_0 which is the usual ordering in the set N of natural numbers except for the integers 1 and 2, which are incomparable. We will systematically use this ordering, therefore, for example, the interval $[1, q] \subset N$ does not contain the integer 2.

The subdiagram $\mathcal{A}_{n-1}^{(1)}$ relative to the vertices $\{2, 3, \ldots, n\}$ is an equioriented Dynkin diagram of type $\mathcal{A}_{n-1}$. Similarly we have $\mathcal{A}_{n-1}^{(2)}$ erasing the vertex 1.

Recall now that the set of the indecomposable representations of D_n is in 1-1 correspondence with the set Δ^+ of the positive roots of D_n, independently of the orientation (cf. [6, 8]). We denote by E_{pq}, $p \leqslant q$ (in the ordering specified above), $p, q = 1, 2, \ldots, n$, the indecomposable representations of D_n which are also indecomposable representations either of $\mathcal{A}_{n-1}^{(1)}$ or $\mathcal{A}_{n-1}^{(2)}$, and we denote the remaining ones by E_{pq}, $3 \leqslant q < p \leqslant n + 1$, i.e. the indecomposable representations of D_n which are representations neither of $\mathcal{A}_{n-1}^{(1)}$ nor $\mathcal{A}_{n-1}^{(2)}$. Table 1 lists explicitly the dimensions of the E_{pq}'s and gives a "graphic" description of them which will be useful in the future. We have to read the graphic description of the indecomposable E_{pq} as representations for the given orientation. The dots j or j', j'' always represent a given basis for the corresponding vector space. If E_{pq} is such that $p \leqslant q$, the representation (a canonical form of it) is described by sending the vector j to $j + 1$, $p \leqslant j < q$, and q to zero.

If E_{pq} is such that $p > q$, the representation is described by:

$$1 \to 3', 2 \to 3''; j' \to (j + 1)', j'' \to (j + 1)'' \quad \text{if } 3 \leqslant j < q - 1;$$
$$(q - 1)' \to \tfrac{1}{2}q, (q - 1)'' \to \tfrac{1}{2}q; j \to j + 1 \quad \text{if } q \leqslant j < p - 1;$$
$$p - 1 \to 0.$$

Note that the graphic description of E_{pq} gives rise to an indecomposable ending at q if $p \leqslant q$, and $p - 1$ if $p > q$. From this point of view we will also say that the indecomposable E_{pq} ($p \gtrless q$) contains (or passes through) the vertex j if its graphic description contains the vertex j (or j', j'').

Let A be a representation of (D_n, eq). Then A uniquely decomposes, up to isomorphism, into indecomposable factors

$$[A] = \bigoplus_{p, q} e_{pq}^A E_{pq},$$

where e_{pq}^A denotes the multiplicity of E_{pq}, $p \leqslant q \leqslant n$, $p = 1, 2, \ldots, n$ or $3 \leqslant q < p \leqslant n + 1$.

TABLE 1. List of the indecomposables of D_n

E_{pq}	dimension	graphic description
$3 \leqslant p \leqslant q \leqslant n$	$\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix} \; 0 \cdots 0 \; \underset{\underset{p}{\shortparallel}}{1} \cdots \underset{\underset{q}{\shortparallel}}{1} \; 0 \cdots 0 \right)$	
$1 = p \leqslant q \leqslant n$	$\begin{smallmatrix} p \\ \shortparallel \\ 1 \end{smallmatrix}$ $\left(\begin{smallmatrix} \\ 0 \end{smallmatrix} \; 1 \; 1 \cdots 1 \; \underset{\underset{q}{\shortparallel}}{1} \; 0 \cdots \cdots 0 \right)$	
$2 = p \leqslant q \leqslant n$	$\left(\begin{smallmatrix} 0 \\ \\ 1 \\ \shortparallel \\ p \end{smallmatrix} \; 1 \; 1 \cdots 1 \; \underset{\underset{q}{\shortparallel}}{1} \; 0 \cdots \cdots 0 \right)$	
$n + 1 \geqslant p > q > 3$	$\left(\begin{smallmatrix} 1 \\ \\ 1 \end{smallmatrix} \; 2 \cdots 2 \; \underset{\underset{q}{\shortparallel}}{1} \cdots \underset{\underset{p}{\shortparallel}}{1} \; 0 \cdots 0 \right)$	
$n + 1 \geqslant p > q = 3$ $p \neq 4$	$\left(\begin{smallmatrix} 1 \\ \\ 1 \end{smallmatrix} \; 1 \cdots \cdots 1 \; \underset{\underset{p}{\shortparallel}}{0} \cdots \cdots 0 \right)$	
$p = 4, q = 3$	$\left(\begin{smallmatrix} 1 \\ \\ 1 \end{smallmatrix} \; 1 \; 0 \cdots \cdots \cdots 0 \right)$	

Conversely a set of nonnegative integers $\{e_{pq}\}$ determines a unique isomorphism class $[A]$ of representations of (D_n, eq). Therefore the isomorphism classes of representations of our oriented graph are parametrized by the sets of nonnegative integers $\{e_{pq}\}$, $p \leqslant q \leqslant n$, $p = 1, 2, \ldots, n$, or $3 \leqslant q < p \leqslant n + 1$.

We now fix a dimension d as in $(*)$ for the representations of (D_n, eq) and consider the vector space

$$L_d := L_d(D_n, \mathrm{eq}) = \mathrm{Hom}_K(V_1, V_3) \times \prod_{i=2}^{n-1} \mathrm{Hom}_K(V_i, V_{i+1}),$$

i.e. the variety of all representations of dimension d on which the group $G_d = \prod_{i=1}^n \mathrm{GL}(V_i)$ acts naturally. The orbit $G_d \cdot A = O_A$ of an element $A \in L_d$ coincides with the isomorphism class $[A]$ of A, and the given action of G_d on L_d has a finite number of orbits and a unique maximal orbit (as it follows from the fact that the number of indecomposable representations is finite).

We now want to describe a different parametrization of the orbits in $L_d(D_n, \mathrm{eq})$. Consider first the situation when $n = 4$:

Starting from the three linear maps A_i we can construct the following five linear maps:

$$
\begin{aligned}
V_1 &\to V_4, & v_1 &\rightsquigarrow A_3 A_1 v_1, \\
V_2 &\to V_4, & v_2 &\rightsquigarrow A_3 A_2 v_2, \\
V_1 \oplus V_2 &\to V_3, & (v_1, v_2) &\rightsquigarrow (A_1 v_1 + A_2 v_2), \\
V_1 \oplus V_2 &\to V_4, & (v_1, v_2) &\rightsquigarrow (A_3 A_1 v_1, A_3 A_2 v_2), \\
V_1 \oplus V_2 \oplus V_3 &\to (V_3 \oplus V_3 \oplus V_3)/\Delta \oplus V_4,
\end{aligned}
$$

where Δ is the diagonal in $V_3 \oplus V_3 \oplus V_3$ and the linear map is defined by

$$(v_1, v_2, v_3) \rightsquigarrow \left(\overline{(A_1 v_1, A_2 v_2, v_3)}, A_3 v_3 \right),$$

where $\overline{(A_1 v_1, A_2 v_2, v_3)}$ is the class of $(A_1 v_1, A_2 v_2, v_3)$ modulo the diagonal.

Note that if we include the identity map $V_i \to V_i$, $i = 1, \ldots, 4$, then we have 12 linear maps, as many as the number of indecomposable representations of D_4.

We are interested in computing the rank of these maps (the reason will be clear from the discussion of the general case; cf. Proposition 1.4). In particular, we want to compute them in terms of the dimensions of the subspaces $\mathrm{Im}\, A_1, \mathrm{Im}\, A_2, \ker A_3$ $(\subset V_3)$, $\mathrm{Im}\, A_3 A_1$, $\mathrm{Im}\, A_3 A_2$ $(\subset V_4)$, and their intersections.

We have

$$\mathrm{rk}(V_1 \oplus V_2 \to V_3) = \mathrm{rk}\, A_1 + \mathrm{rk}\, A_2 - \dim(\mathrm{Im}\, A_1 \cap \mathrm{Im}\, A_2)$$

and similarly for $V_1 \oplus V_2 \to V_4$. For the map

$$\theta \colon V_1 \oplus V_2 \oplus V_3 \to (V_3 \oplus V_3 \oplus V_3)/\Delta \oplus V_4$$

we have

$$\mathrm{rk}\, \theta = \dim V_1 + \dim V_2 + \dim V_3 - \dim \ker \theta,$$

$$\ker \theta = \{ (v_1, v_2, v_3) \colon A_1 v_1 = A_2 v_2 = v_3, A_3 v_3 = 0 \}.$$

Let

$$\mathscr{C} = \{ (v_1, v_2) \in V_1 \oplus V_2 \colon A_1 v_1 = A_2 v_2 \in \ker A_3 \};$$

then $\mathscr{C}$ is canonically isomorphic to $\ker \theta$, and we also have the surjective morphism

$$\tau \colon \mathscr{C} \to W = \mathrm{Im}\, A_1 \cap \mathrm{Im}\, A_2 \cap \ker A_3,$$

$$\tau(v_1, v_2) = A_1 v_1 \quad (= A_2 v_2).$$

It follows that

$$\dim \ker \theta = \dim \mathscr{C} = \dim(\mathrm{Im}\, A_1 \cap \mathrm{Im}\, A_2 \cap \ker A_3) + \ker \tau$$

$$= \dim(\mathrm{Im}\, A_1 \cap \mathrm{Im}\, A_2 \cap \ker A_3) + (\dim V_1 - \mathrm{rk}\, A_1)$$

$$+ (\dim V_2 - \mathrm{rk}\, A_2)$$

and

$$\mathrm{rk}\, \theta = \dim V_3 + \mathrm{rk}\, A_1 + \mathrm{rk}\, A_2 - \dim(\mathrm{Im}\, A_1 \cap \mathrm{Im}\, A_2 \cap \ker A_3).$$

We return to the general case. Let $A = (A_1, A_2, \ldots, A_{n-1}) \in L_d(D_n, \mathrm{eq})$:
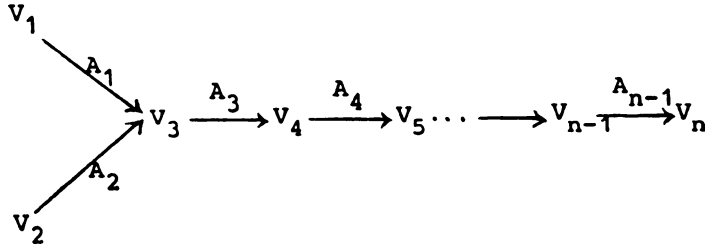
For any pair of indices $w, z \in [3, n]$, $w < z$, we consider the "contracted diagram" D_4 of vertices $1, 2, w, z$,
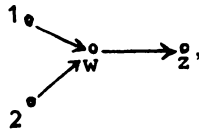
the contracted representation obtained from A via the obvious composition of the A_i's and the linear maps defined above (not all of them are distinct). We deduce the following distinct linear maps associated to A:

DEFINITIONS AND NOTATIONS.
(1)

$$\varphi_{hk}^A = A_{k-1} \cdots A_{h+1} A_h \colon V_h \to V_k, \qquad 2 \leqslant h < k \leqslant n,$$

$$\varphi_{1k}^A = A_{k-1} \cdots A_3 A_1 \colon V_1 \to V_k, \qquad k \leqslant n.$$

(2)

$$\varphi_{n+1,k}^A \colon V_1 \oplus V_2 \to V_k, \qquad 3 \leqslant k \leqslant n,$$

$$\varphi_{n+1,k}^A(v_1, v_2) = \varphi_{1k}^A v_1 + \varphi_{2k}^A v_2.$$

(3)

$$\varphi_{hk}^A \colon V_1 \oplus V_2 \oplus V_k \to (V_k \oplus V_k \oplus V_k)/\Delta \oplus V_h, \qquad 3 \leqslant k < h \leqslant n,$$

$$\varphi_{hk}^A(v_1, v_2, v_k) = \left(\overline{\left(\varphi_{1k}^A v_1, \varphi_{2k}^A v_2, v_k \right)}, \varphi_{kh}^A v_k \right).$$

We now define $N^A = \{ N_{hk}^A \}$, the set of "rank parameters" of the representation A.

DEFINITION.
(1')

$$N_{hk}^A = \mathrm{rk}\, \varphi_{hk}^A, \quad \text{if } h < k, h, k = 1, \ldots, n,$$

$$N_{hk}^A = d_h, \quad \text{if } h = k, h, k = 1, \ldots, n.$$

(2')

$$N_{n+1,k}^A = \mathrm{rk}\, \varphi_{n+1,k}^A$$
$$= \mathrm{rk}\, \varphi_{1k}^A + \mathrm{rk}\, \varphi_{2k}^A - \dim\left(\mathrm{Im}\, \varphi_{1k}^A \cap \mathrm{Im}\, \varphi_{2k}^A\right), \quad \text{if } 3 \leqslant k \leqslant n.$$

(3')

$$N_{hk}^A = \mathrm{rk}\, \varphi_{hk}^A - \dim V_k$$
$$= \mathrm{rk}\, \varphi_{1k}^A + \mathrm{rk}\, \varphi_{2k}^A - \dim\left(\mathrm{Im}\, \varphi_{1k}^A \cap \mathrm{Im}\, \varphi_{2k}^A \cap \ker \varphi_{kh}^A\right) \quad \text{if } 3 \leqslant k < h \leqslant n.$$

The integers N_{hk}^A are all nonnegative. Clearly the same set N^A is associated to all the representations of the isomorphism class $[A] = O_A$; moreover, the number of elements in N^A equals the number of nonisomorphic indecomposable representations of D_n. Note that the number $N_{n+1,k}^A$ defined in (2') can be considered a "special case" of (3') (it is enough to extend D_n by adding a vertex $n+1$ and to assume $d_{n+1} = 0$).

Now consider the decomposition

$$A = \sum e_{pq}^A E_{pq}.$$

Clearly we can express the rank parameters N_{hk}^A ($h \gtreqless k$) in terms of the multiplicities e_{pq}^A, as ranks are additive (and the dimension too), i.e. we have a linear system

(1.1) $$N_{hk}^A = f_{hk}\left(e_{pq}^A\right).$$

We want to show that (1.1) is invertible over the integers. For this we introduce the auxiliary set of parameters $m^A = \{ m_{pq}^A \}$ associated to $[A] = O_A$ and define as follows:

$$(1.2) \quad \begin{cases} m_{pq}^A = \displaystyle\sum_{p \leqslant r \leqslant n+1} e_{rq}^A \quad \text{if } n + 1 \geqslant p > q \geqslant 3; \\[2em] m_{iq}^A = \displaystyle\sum_{q \leqslant k \leqslant n} e_{ik}^A + \sum_{\substack{s > r \\ q+1 \leqslant s \leqslant n+1 \\ 3 \leqslant r \leqslant n}} e_{sr}^A \\[2em] \qquad\qquad\qquad \text{if } i = 1, 2, q = 1, 2, \ldots, n \text{ and } i \leqslant q; \\[2em] m_{pq}^A = \displaystyle\sum_{h \leqslant p \leqslant q \leqslant k} e_{hk}^A + \sum_{\substack{s > r \\ q+1 \leqslant s \leqslant n+1 \\ 3 \leqslant r \leqslant q}} e_{sr}^A + 2 \sum_{n+1 \geqslant s > r > q} e_{sr}^A, \\[2em] \qquad\qquad\qquad \text{if } 3 \leqslant p \leqslant q \leqslant n. \end{cases}$$

The linear system (1.2) can be inverted over the integers, in fact the equations of (1.2) are recursive. It follows that we have the relations

$$(1.2)' \qquad\qquad e_{pq}^A = g_{pq}(m_{uv}^A),$$

which imply that a set $m = \{ m_{uv} \}$ of nonnegative integers determines an orbit in L_d if and only if $g_{pq}(m_{uv}) \geqslant 0$.

On the other side it is immediately seen, using the definition of the rank parameters N_{pq}^A, that we have

$$(1.3) \qquad \begin{cases} N_{pq}^A = m_{pq}^A \quad \text{if } 1 \leqslant p \leqslant q \leqslant n, \\[1em] N_{pq}^A = m_{1q}^A + m_{2q}^A - \displaystyle\sum_{j-3,\ldots,q} \left(m_{q+1,j}^A - m_{p+1,j}^A \right) \end{cases}$$

which is again a recursive system of linear equations, i.e. it is invertible. (1.1) is obtained by substituting (1.2) in (1.3) and therefore it is invertible over the integers; i.e. we have

$$(1.1)' \qquad\qquad e_{pq}^A = h_{pq}(N_{uv}^A).$$

The consequence is the following

PROPOSITION 1.4. (i) *A set of nonnegative integers* $N = \{ N_{uv}, u \lesseqgtr v \}$ *determines the orbit of a representation of* $L_d(D_n, \text{eq})$ *if and only if* $h_{pq}(N_{uv}) \geqslant 0$.
(ii) *The sets* $N = \{ N_{uv} \}$ *satisfying* (i) *parametrize the orbits in* $L_d(D_n, \Omega)$.

PROPOSITION 1.5. *Let* $A, B \in (D_n, \text{eq})$, $O_B \subset \overline{O}_A$ *and* $O_B \neq O_A$. *Then* $N_{hk}^B \leqslant N_{hk}^A$ *for all pairs* (h, k) *and the strict inequality holds for at least a pair.*

PROOF. In a degeneration the rank of the linear maps φ_{hk} ($h \lesseqgtr k$) cannot increase and the dimension d is constant. Moreover, as a consequence of Proposition 1.4(i), we must have at least a strict inequality.

Assume we know the decomposition $A = \sum_{pq} e^A_{pq} E_{pq}$, i.e. we know the multiplicities e^A_{pq}. Then we can compute N^A_{hk} by computing $N^{E_{pq}}_{hk}$ for every E_{pq} such that $e^A_{pq} > 0$, since ranks are additive. The computation of $N^{E_{pq}}_{hk}$ is immediate for every p, q, h, k by using the graphic description of E_{pq} and definitions (1')–(3').

Before ending we want to justify the choice of the orientation of D_n given at the beginning of this section.

Consider the Dynkin diagram D_n for $n = 4$. We have only two possible orientations,

$$\Omega = \text{eq:} \qquad\qquad\qquad\qquad , \qquad\qquad \Omega':$$
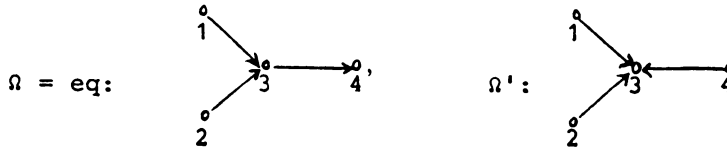
since we can always permute the vertices $(1, 2, 4)$ or simultaneously reverse all the arrows.

The geometry of the representations of (D_4, Ω') immediately suggests how to parametrize the orbits of given dimension using rank parameters.

Nevertheless the orientation $\Omega = \text{eq}$ priviledged in the sense that we are able to deduce for free the "rank parameters" for the representations of (D_n, eq) from that of (D_4, eq).

In the following sections we will systematically use the oriented graph (D_n, eq).

2. Stabilizer formula. Let $A \in L_d(D_n, \text{eq})$ and let O_A be the corresponding orbit. Then

$$(2.1) \qquad\qquad \dim O_A = \dim G_d - \dim \text{St}\, A$$

where $\text{St}\, A$ denotes the subgroup of G_d which fixes A. Moreover, $\dim \text{St}\, A$ equals $\dim \text{Lie}(\text{St}\, A)$; therefore

$$(2.2) \qquad\qquad \dim \text{St}\, A \sum_{p,q,r,s} e^A_{pq} e^A_{rs} \dim \text{Hom}_K \left(E_{pq}, E_{rs} \right).$$

$\text{Hom}_K(E_{pq}, E_{rs})$ depends only on the integers p, q, r, s, as the orientation of D_n has been chosen once for all. We collect the possible values of $\lambda = \dim \text{Hom}_K(E_{pq}, E_{rs})$ in Table 2, which has been constructed case by case using the list of the indecomposables of §1. We explain the more interesting cases. For instance consider $\text{Hom}_K(E_{pq}, E_{rs})$ when $3 \leqslant q < p \leqslant n + 1$ and $3 \leqslant s < r \leqslant n + 1$. We use the graphic description of the indecomposables and the convention on the dots, explicitly writing only the relevant ones for our discussion. First assume $r \leqslant q$. (See Diagram A.) We have two independent elements $\varphi, \psi \in \text{Hom}_K(E_{pq}, E_{rs})$, which generate $\text{Hom}_K(E_{pq}, E_{rs})$ as a vector space, defined as follows:

φ: $1 \mapsto \bar{1}$, $3' \mapsto \bar{3}'$,...,$(s-1)' \mapsto \overline{(s-1)}'$, $s' \mapsto \frac{1}{2}\bar{s}$,...,$(r-1)' \mapsto \frac{1}{2}\overline{(r-1)}$, and all other base vectors in E_{pq} go to zero.

ψ: $2 \mapsto \bar{2}$, $3'' \mapsto \bar{3}''$,...,$(s-1)'' \mapsto \overline{(s-1)}''$, $s'' \mapsto \frac{1}{2}\bar{s}$,...,$(r-1)'' \mapsto \frac{1}{2}\overline{(r-1)}$ and all other base vectors in E_{pq} are sent to zero. (See Diagram B.)

DIAGRAM A

DIAGRAM B

If $s \leqslant q < r \leqslant p$ we have the following morphism ψ which generates $\mathrm{Hom}_K(E_{pq}, E_{rs})$:

$\psi: 1 \mapsto \bar{1}, \ 2 \mapsto \bar{2}, \ 3' \mapsto \bar{3}', \ 3'' \mapsto \bar{3}'', \ldots, (s-1)' \mapsto \overline{(s-1)}', \ (s-1)'' \mapsto \overline{(s-1)}'',$
$\quad s' \mapsto \frac{1}{2}\bar{s}, \quad s'' \mapsto \frac{1}{2}\bar{s}, \ldots, (q-1)' \mapsto \frac{1}{2}\overline{(q-1)}, \quad (q-1)'' \mapsto \frac{1}{2}\overline{(q-1)}, \quad q \mapsto$
$\quad \bar{q}, \ldots, (r-1) \mapsto \overline{(r-1)}$ and all other base vectors are sent to zero.
In all other possible cases it is easy to see that

$$\mathrm{Hom}_K(E_{pq}, E_{rs}) = 0 \qquad (p > q, r > s).$$

DIAGRAM C

Now consider the case for E_{pq} such that $3 \leqslant p \leqslant q \leqslant n$ and E_{rs} such that $3 < s^{pq} < r \leqslant n + 1$. (See Diagram C.) If $p < s \leqslant r - 1 \leqslant q$ we have two independent elements $\varphi, \psi \in \mathrm{Hom}_K(E_{pq}, E_{rs})$, which generate it as a vector space, defined as follows:

φ: $p \to \bar{p}', \ldots, (s - 1) \to \overline{(s - 1)}', \ s \to -\frac{1}{2}\bar{s}, \ldots, (r - 1) \to \frac{1}{2}\overline{(r - 1)}$, and all other base vectors in E_{pq} are sent to zero.

ψ: $p \to \bar{p}'', \ldots, (s - 1) \to \overline{(s - 1)}'', \ s \to \frac{1}{2}\bar{s}, \ldots, (r - 1) \to \frac{1}{2}\overline{(r - 1)}$ and all other base vectors in E_{pq} are sent to zero. (See Diagram D.)
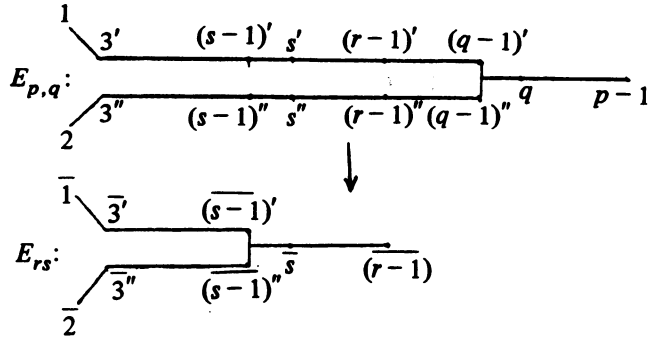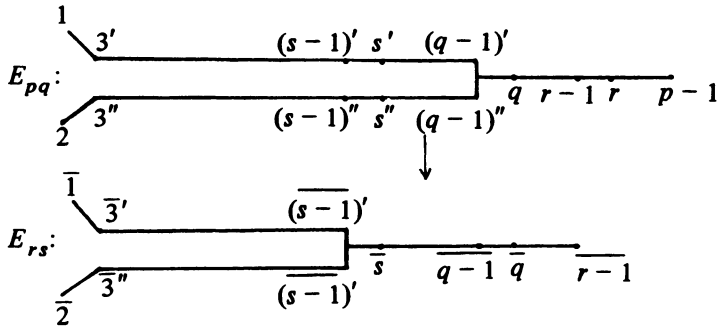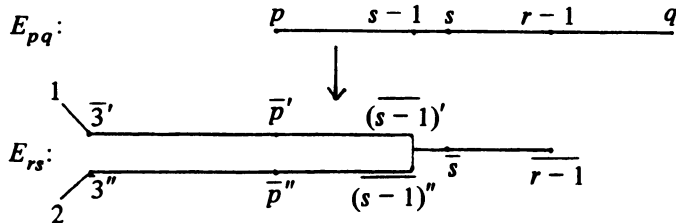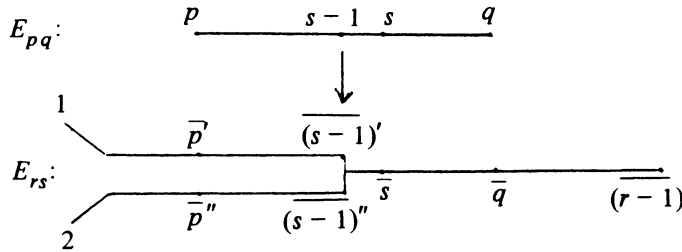

DIAGRAM D

If $p \leqslant s - 1 \leqslant q < r - 1$, the following $\bar{\psi}$ generates $\mathrm{Hom}_K(E_{pq}, E_{rs})$ as a vector space:

$\bar{\psi}$: $p \mapsto \bar{p}' - \bar{p}'', \ldots, (s - 1) \mapsto \overline{(s - 1)}' - \overline{(s - 1)}''$ and all the other base vectors of E_{pq} are sent to zero.

In all other possible cases a generator of $\mathrm{Hom}_K(E_{pq}, E_{rs})$, if any, is forced by the given orientation. In Table 2 we collect all possible values of $\lambda = \dim \mathrm{Hom}_K(E_{pq}, E_{rs})$. The indecomposables for which $\lambda \neq 0$ are described graphically because it is easier to visualize the relative position of the indices p, q, r, s. We do not explicitly consider the case $p = 2$ or $r = 2$. In fact we clearly have $\mathrm{Hom}_K(E_{1q}, E_{2s}) = 0$ for all q, s, and we can deduce the other possibilities permuting the vertices 1 and 2 in the oriented graph (D_n, eq).

From the table we note that if $E_{pq} \neq E_{rs}$ and $\mathrm{Hom}(E_{pq}, E_{rs}) \neq 0$, then $\mathrm{Hom}(E_{rs}, E_{pq}) = 0$; moreover,

$$\lambda = \dim \mathrm{Hom}(E_{pq}, E_{rs}) = 1 \quad \text{if } (p, q) = (r, s).$$

Let us define

$$(2.3) \qquad [E_{pq}, E_{rs}] = \dim \mathrm{Hom}_K(E_{pq}, E_{rs}) + \dim \mathrm{Hom}_K(E_{rs}, E_{pq}).$$

The values of $[E_{pq}, E_{rs}]$ can be read directly from Table 2 using the obvious symmetrization. In particular, $[E_{pq}, E_{pq}] = 2$.

Let A be a representation of (D_n, eq) and assume $A = B \oplus C$. Then from (2.2) we get

$$(2.4) \qquad \dim \mathrm{St}\, A = \dim \mathrm{St}\, B + \dim \mathrm{St}\, C + [B, C]$$

$$= \dim \mathrm{St}\, B + \dim \mathrm{St}\, C + \sum_{p,q,r,s} e_{pq}^B e_{rs}^C [E_{pq}, E_{rs}].$$

TABLE 2. $\lambda = \dim \operatorname{Hom}_K(E_{pq}, E_{rs})$ for (D_n, eq)

3. The canonical construction. We want to describe a construction C which associates to each d (see (*)) a special representation $C(d) \in L_d(D_n, \text{eq})$, whose main property is given by

PROPOSITION 3.1. *Let $A = C(d)$. Then O_A is the maximal orbit of $L_d(D_n, \text{eq})$.*

Construction of $C(d)$. We may always assume $d_2 \leqslant d_1$ (in fact the permutation $(1, 2)$ is a graph automorphism of D_n which preserves the orientation). The construction is given inductively on the vertex i, the first step being, for $i = 3$, the construction of the maximal orbit relative to the dimension $\tilde{d} = (d_1, d_2)$ for the oriented graph of type $\mathscr{A}_n$ ($n = 3$).

$$\overset{1}{\circ} \; \to \; \overset{3}{\circ} \; \leftarrow \; \overset{2}{\circ} \; .$$

The construction was given in [2] for any n and any orientation. We recall the construction since it is very simple.
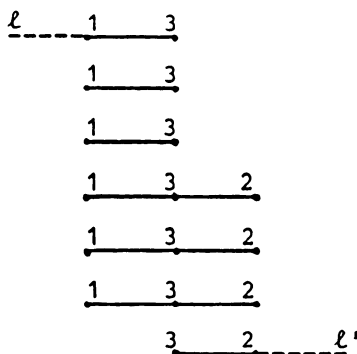
We draw as many dots 1 as d_1 and, on an "orizontal" line l passing through 1, as many dots 3 as d_3 (the first one on the line l). Then we join dots 1 and 3 (if possible) with an orizontal edge. From the last vertex 3 we draw a line l' parallel to l, and we draw as many dots 2 as d_2, (the first one always on l' and in the half-plane containing l), and we proceed as before. The result is a collection of indecomposables for which we use the notations and the graphic description introduced in §1.

EXAMPLES.
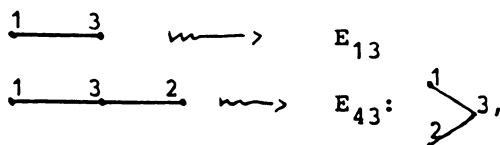
We denote this construction by C_3.

Assume we have performed the construction C_i up to the vertex i (and for the dimension

$$\tilde{d}_i = \left(\genfrac{}{}{0pt}{}{d_1}{d_2}; d_3, d_4, \ldots, d_i\right),$$

C_i is the maximal orbit, $i \geqslant 3$, for the subdiagram of vertices $1, \ldots, i$). Then we want to add the vertex $i + 1$ and describe the construction C_{i+1}. This construction is suggested from the fact that the sets of rank parameters parametrize the orbits (cf. Propositions 1.4, and 1.5), therefore the maximal orbit must correspond to a set with maximum possible rank parameters. It follows that all the indecomposables of C_i which do not end at the vertex i are also factors in C_{i+1} (and their multiplicity is unchanged). The construction C_{i+1} will "extend" to $i + 1$ some (or all) of the indecomposables of C_i which end at the vertex i (the remaining ones being unchanged in C_{i+1}), according to the following rule.

Assume that in C_i we have factors $E_{i+1,t}$ with $i + 1 > t$ and let us denote by:

f_0 the multiplicity of the factor E_{i+1,i_0} (in C_i);

f_1 the multiplicity of the factor E_{i+1,i_1} (in C_i);

$\vdots$

f_r the multiplicity of the factor E_{i+1,i_r} (in C_i),

where $3 \leqslant i_0 < i_1 < \cdots < i_r \leqslant i$ and in C_i there is no factor $E_{i+1,q}$ such that $3 \leqslant q < i_0$ or $i_h < q < i_{h+1}$, $h = 0, \ldots, r - 1$, or $q > i_r$. $\{f_0, f_1, \ldots, f_r\}$ is a sequence of positive integers (empty if in C_i there is no factor $E_{i+1,t}$, $i + 1 > t$). Moreover consider the following nonnegative integers;

k the number of pairs $E_{1i} \oplus E_{2i}$ (in C_i),

l the number of the remaining E_{1i} (in C_i),

m_3 the multiplicity of E_{3i} (in C_i),

$\vdots$

m_{i-1} the multiplicity of $E_{i-1,i}$ (in C_i),

m_i the multiplicity of E_{ii} (in C_i).

Let $m = \sum_{p=3}^{i} m_p$. We have automatically ordered the factors of C_i and ending at i as follows (the other ones do not appear as they play no role in the construction C_{i+1}) (see Diagram E). Clearly we have

$$d_i = \sum_{u=0}^{r} f_u + 2k + l + m,$$

and we have several possibilities for the dimension d_{i+1}:

(a) If $d_{i+1} \leqslant f_0$, d_{i+1} of the factors E_{i+1,i_0} of C_i become E_{i+2,i_0} in C_{i+1}; the remaining $f_0 - d_{i+1}$ are unchanged.
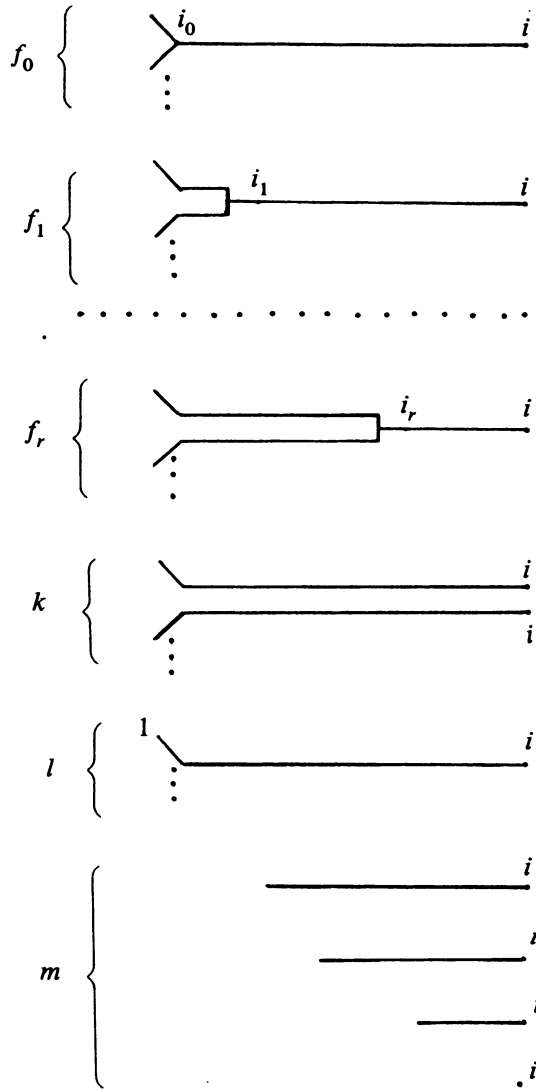
DIAGRAM E

(b) If $d_{i+1} = \sum_{u=0}^{h} f_u + t$, $0 \leqslant h \leqslant r-1$, $0 < t \leqslant f_{h+1}$, all f_u factors E_{i+1,i_u}, $u = 0,\ldots,h < r$, of C_i become E_{i+2,i_u}; t of the f_{h+1} factors $E_{i+1,i_{h+1}}$ become $E_{i+2,i_{h+1}}$; the remaining $f_{h+1} - t$ are unchanged.

(c) If $d_{i+1} = \sum_{u=0}^{r} f_u + t$, $0 < t \leqslant k$, all f_u factors E_{i+1,i_u}, $u = 0,\ldots,r$, become E_{i+2,i_u} and t of the pairs $E_{1i} \oplus E_{2i}$ give rise to t factors $E_{i+2,i+1}$; the remaining $k - t$ $E_{1i} \oplus E_{2i}$ are unchanged.

(d) If $d_{i+1} = \sum_{u=0}^{r} f_u + k + t$, $0 < t \leqslant l$, all f_u factors E_{i+1,i_u}, $u = 0,\ldots,r$, become E_{i+2,i_u}; all k pairs $E_{1i} \oplus E_{2i}$ give rise to k factors $E_{i+2,i+1}$, and t of the l factors E_{1i} become $E_{1,i+1}$; the remaining $l - t$ E_{1i} are unchanged.

(e) If $d_i = \sum_{u=0}^{r} f_u + k + l + t$, $0 < t \leqslant k$, all f_u factors E_{i+1,i_u} become E_{i+2,i_u}; $k - t$ of the pairs $E_{1i} \oplus E_{2i}$ become $E_{i+2,i+1}$; the remaining t pairs $E_{1i} \oplus E_{2i}$ become $E_{1,i+1} \oplus E_{2,i+1}$; all the l E_{1i} become $E_{1,i+1}$.

(f) If $d_{i+1} = \sum_{u=0}^{r} f_u + 2k + l + t$, $0 < t \leqslant m$, all indecomposables containing the vertices 1 or 2 (or both) and end at i are extended to $i + 1$, t of the remaining factors ending at i are extended to $i + 1$, the priority being given to the E_{pi} ($p \leqslant i$) with p small; i.e. if $t = \sum_{p=3}^{s} m_p + \tau$, $s < i$, $0 < \tau \leqslant m_{s+1} \neq 0$, then all the E_{pi} of C_i, $p = 0, \ldots, s$, become $E_{p,i+1}$ in C_{i+1}, τ of the $m_{s+1}E_{s+1,i}$ become $E_{s+1,i+1}$ and the remaining are unchanged.

(g) If $d_{i+1} > d_i$ all factors of C_i ending at i in C_{i+1} are extended to $i + 1$, and in C_{i+1} there are also $d_{i+1} - d_i$ factors $E_{i+1,i+1}$.

Finally we define $C(d) := C_n$. The graphic description of the construction C_{i+1} is very easy once we order the factors of C_i ending at i as indicated at the beginning of the discussion. In fact in (a), (b), (f), (g) we drown d_{i+1} vertices $i + 1$, starting from the top, each one next to a vertex i, if possible, and we join with an orizontal edge i and $i + 1$.

EXAMPLE. (D_6, eq), $d = (\tfrac{3}{2}; 4, 5, 4, 1)$.

C_3	C_4	C_5	$C_6 = C(d)$

In (c), (e), (d) we distribute the first $\sum_{u=0}^{r} f_u$ of the d_{i+1} vertices $i + 1$ as before; with the remaining ones we try to extend all factors E_{1i} and E_{2i} simultaneously to $i + 1$, if possible, but not E_{pi} where $p \geqslant 3$. This implies some of the remaining vertices $i + 1$ need to be drawn between a pair $E_{1i} \oplus E_{2i}$, creating a factor $E_{i+2,i+1}$.

EXAMPLE. (D_6, eq), $d = (\tfrac{4}{3}; 8, 9, 6, 3)$.

C_3	C_4	C_5	$C_6 = C(d)$

Note that once we have ordered the indecomposables of the construction C_i as indicated, the indecomposables of C_{i+1} ending at $i + 1$ are automatically ordered to perform C_{i+2}. This fact makes the whole construction $C(d) = C_n$ very easy to be performed graphically.

LEMMA 3.2. *Let* $A = C(d)$.

(i) *Assume in* A *we have a factor* E_{zt}, $n + 1 \geqslant z > t \geqslant 3$. *Then no factor* E_{pq} *can appear in* A *such that*:

(a) $p > q > t, p > z$;
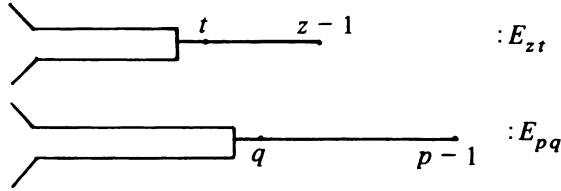
(b) $p \leqslant q$ *and either* $3 \leqslant p \leqslant t$, $q \geqslant t$; *or* $p = z$, $q \geqslant z$; *or* $p = 1, 2, q < t - 1$; *or* $p = 1, 2, q > z - 1$;
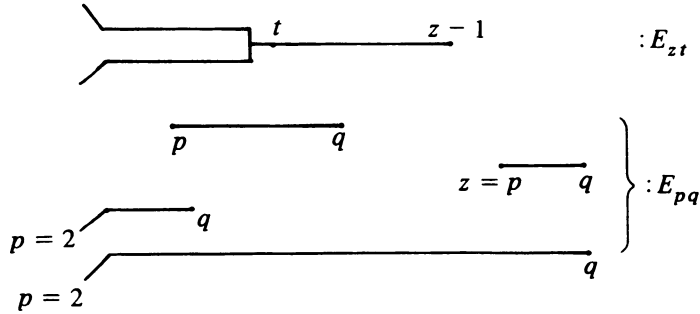
(ii) *Assume in* A *we have a factor* E_{tz}, $t \leqslant z$, $t, z = 1, \ldots, n$. *Then no factor* $E_{p,q}$ *can appear in* A *with* $p \leqslant q$ *and* $p < t, t - 1 \leqslant q < z$.
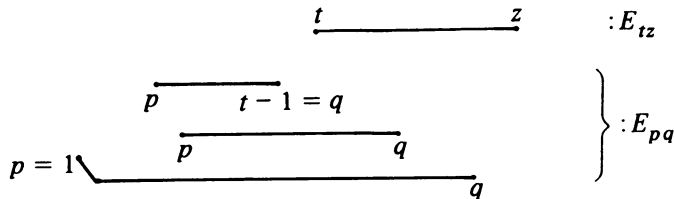
(iii) *Assume there is a factor* E_{1z} *in* A. *Then no factor* E_{2q} *can appear in* A *with* $q \neq z$.
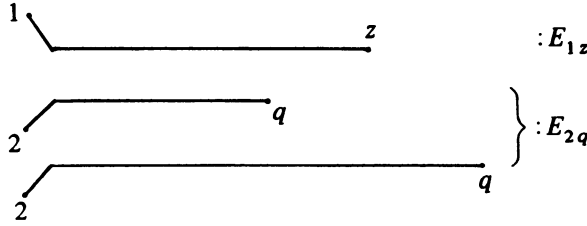
PROOF. It is a direct consequence of the construction $C(d)$. If in (i) we assume E_{zt} and E_{pq} in A, E_{pq} as in (a), then we have violated the construction $C(d)$ at the vertex z, i.e. C_z is not correct. The proof is similar in all the other cases.

PROOF OF PROPOSITION 3.1. Let $A := C(d) = C_n$. We have to verify that

$$\dim \operatorname{St} A = \dim G_d - \dim L_d$$

$$= d_1^2 + \cdots + d_n^2 - (d_1 d_3 + d_2 d_3 + d_3 d_4 + \cdots + d_{n-1} d_n)$$

$$= d_1(d_1 - d_3) + d_2(d_2 - d_3) + \cdots + d_{n-1}(d_{n-1} - d_n) + d_n^2.$$

We proceed by induction and assume

$$A' := C_{n-1} \in O_{\max}^{d'} \subset L_{d'}, \qquad d' = \left(\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}; d_3, \ldots, d_{n-1}\right),$$

i.e.,

$$\dim \operatorname{St} A' = d_1(d_1 - d_3) + d_2(d_2 - d_3) + \cdots + d_{n-2}(d_{n-2} - d_{n-1}) + d_{n-1}^2.$$

Then we need only prove

$$\Delta := \dim \operatorname{St} A - \dim \operatorname{St} A' = d_n(d_n - d_{n-1}).$$

We compute Δ using the stabilizer formula (2.2) for both A and A'. Therefore we need only compare the contribution given to $\dim \operatorname{St} A'$ by the factors in C_{n-1} ending at $n - 1$ and the contribution given to $\dim \operatorname{St} A$ by the factors in C_n ending at $n - 1$ and n. We read the integers $f_0, \ldots, f_r, k, l, m$ relative to the construction C_{n-1} and we examine the various possibilities for $C(d) = C_n$. If $d_n = d_{n-1}$, all the indecomposables ending at $n - 1$ in C_{n-1} are extended to n. Then from (2.2) we have $\dim \operatorname{St} A' = \dim \operatorname{St} A$ and the claim is proved.

If $d_n < d_{n-1}$, i.e. in cases (a) to (f) of the construction, using Table 2 we see that $\dim \operatorname{St} A < \dim \operatorname{St} A'$, i.e. $\Delta < 0$. Moreover, in (a), (b), (f) the decrease comes from the contribution given in (2.2) from the d_n factors ending at n, once paired with the $d_n - d_{n-1}$ factors which remain unchanged in passing from C_{n-1} to C_n. Therefore $\Delta = d_n(d_n - d_{n-1})$. In (c) the decrease comes from the d_n factors ending at n once paired with those which remain unchanged, and with the t factors $E_{n+1,n}$, i.e.

$$-\Delta = \left[m + l + 2(k - t)\right]d_n + td_n = (m + l + 2k - t)d_n$$

$$= (d_{n-1} - d_n)d_n.$$

In (d) the decrease comes from the d_n factors ending at n once paired with those which remain unchanged, and with the k factors $E_{n+1,n}$, i.e.

$$-\Delta = d_n[m + l - t] + d_n \cdot k = d_n[m + l + k - t] = d_n(d_{n-1} - d_n).$$

In (e) the decrease comes from the d_n factors ending at n once paired with m factors which remain unchanged, and with the $k - t$ factors $E_{n+1,n}$, i.e.

$$-\Delta = d_n \cdot m + d_n(k - t) = d_n(m - k - t) = d_n(d_{n-1} - d_n).$$

If $d_n > d_{n-1}$ then $\dim \operatorname{St} A > \dim \operatorname{St} A'$ and the increase Δ comes from the $(d_n - d_{n-1})$ factors $E_{n,n}$ once paired with the d_n factors ending at n and coming from C_{n-1}, i.e. $\Delta = d_n(d_n - d_{n-1})$. The proposition is now completely proved.

4. The set $I_d(D_n, \mathrm{eq})$. Let d as in (∗) be a given dimension for the representations of (D_n, eq) such that $d_h > 0$ for all $h = 1, \ldots, n$ and assume $d_{n+1} = 0$. We define

$$(4.1) \qquad\qquad\qquad\qquad I_d = I'_d \cup I''_d,$$

where

$$(4.1)' \quad \begin{aligned} I'_d := \{ (i, j) &: i, j \in [3, n + 1], i > j \text{ and} \\ & d_1 < d_j < d_t, d_2 < d_j < d_t \text{ for all } t \in [3, j - 1], \\ & d_1 + d_2 = d_j + d_i, d_t > d_i \text{ for all } t \in [j, i - 1] \}, \end{aligned}$$

$$(4.1)'' \quad \begin{aligned} I''_d := \{ (i, j) &: i, j = 1, 2, \ldots, n, i < j \text{ and} \\ & d_i = d_j < d_t \text{ for all } t \in [i + 1, j - 1] \}. \end{aligned}$$

REMARK 4.2. (a) If in $(4.1)'$ we assume $i = n + 1$, i.e. $(n + 1, j) \in I'_d$, then the conditions on the d_h are

$$j \in [3, n], \qquad d_1 + d_2 = d_j < d_t \quad \text{for all } t \in [3, j - 1].$$

(b) The conditions defining $(i, j) \in I''_d$ for $i \neq 1$ (resp. $i \neq 2$) are the same as those in [1] for the equioriented graph $\mathscr{A}^{(1)}_{n-1}$ (resp. $\mathscr{A}^{(2)}_{n-1}$) and the dimension $d^{(1)} = (d_2, d_3, \ldots, d_n)$ (resp. $(d_1, d_3, \ldots, d_n)$).

I_d is entirely defined in terms of d. Its main property here will be to have the same cardinality of the codimension 1 orbits in L_d (cf. 4.6 and 4.7).

The following properties are immediate:

LEMMA 4.3. (α) *If* $(i, j) \in I'_d$ *then for every* $(u, v) \neq (i, j)$, $u \in [i, n + 1]$, $v \in [j, i]$, $(u, v) \notin I'_d$.

(β) *If* $(i, j) \in I''_d$ *then for every* $(u, v) \neq (i, j)$, $u \in [i, j - 1]$, $v \in [j, n]$, $(u, v) \notin I''_d$.

(γ) *If* $(i, j) \in I''_d$ *for every* $u \in [j + 1, n + 1]$, $(u, j) \notin I'_d$.

PROOF. It follows from the fact that if we assume (i, j) and (u, v) are simultaneously in I_d, the inequalities imposed on d are incompatible. For instance, in (α) if we assume $(i, j), (u, v) \in I'_d$ and $u \in [i, n + 1]$, $v \in [j, i]$, we must have $d_u + d_v = d_1 + d_2 = d_i + d_j$; moreover, either $d_j \geqslant d_v$, or $d_i \geqslant d_u$ must be a strict inequality since $(u, v) \neq (i, j)$, a contradiction.

Next we want to analyze the condition $(i, j) \in I_d$ in terms of the indecomposables appearing as factors in $A = C(d)$.

Interpretation 4.4. Case $(i, j) \in I'_d$. The discussion is related to the following sets of indices:

$$\Theta_1 := \Theta_1^{ij} = \left\{ s: s > i > j, e_{sj}^A > 0 \right\},$$
$$\Theta_2 := \Theta_2^{ij} = \left\{ t: i > t > j, e_{it}^A > 0 \right\}.$$

If $i \neq n + 1$ the conditions $d_1 < d_j < d_t$, $d_2 < d_j < d_t$, $t \in [3, j - 1]$ imply that in A every factor containing the vertex 1 or 2 also contains the vertex j. The condition $d_1 + d_2 > d_j$ implies that in A there is at least a factor E_{sr}, $s > j \geqslant r$. The condition $d_{j-1} > d_j$ implies there is at least a factor E_{sj}, $s > j$. The conditions $d_t > d_i > 0$, $t \in [j, i - 1]$, and $d_1 + d_2 = d_j + d_i$ imply all the factors E_{sr}, $s > r \leqslant j$, are such that $s > i > j$ and in A there is at least a factor E_{sj}, $s > i$. All the factors E_{wt}, $w > t > j$, are such that $w \leqslant i$, and in A there is either a factor E_{it}, $i > t > j$, or a pair $E_{1,i-1} \oplus E_{2,i-1}$. It follows for $(i, j) \in I'_d$ and $i \neq n + 1$:

$$(4.4)_1 \quad \begin{cases} \Theta_1 = \left\{ s: s > i > j, e_{sj}^A > 0 \right\} \neq \varnothing, & \text{and either} \\ \Theta_2 = \left\{ t: i > t > j, e_{it}^A \right\} \neq \varnothing, \text{or} \\ E_{1,i-1} \oplus E_{2,i-1} \text{ is a pair of factors in } A. \end{cases}$$

If $i = n + 1$ the condition $d_1 + d_2 = d_j < d_t$, $t \in [3, j - 1]$, again implies all factors of A containing the vertex 1 or 2 also contain the vertex j, but in A there is no factor E_{sr} with $s > r \geqslant j$. Then the same argument holds for Θ_2 with $i = n + 1$. Moreover if $j > 3$ and therefore $d_1 + d_2 < d_3$, the condition $d_t > d_j$, $t \in [3, j - 1]$, implies in A there is at least a factor $E_{3,j-1}$; if $j = 3$ we will conventionally consider $E_{3,j-1}$ as the zero representation.

It follows for $(n + 1, j) \in I'_d$:

$$(4.4)_2 \quad \begin{cases} \Theta_1 = \varnothing, E_{3,j-1} \text{ is a factor in } A, \text{ and either} \\ \Theta_2 = \left\{ t: t > j, e_{n+1,t}^A > 0 \right\} \neq \varnothing, \text{or} \\ E_{1n} \oplus E_{2n} \text{ is a pair of factors in } A. \end{cases}$$

$(4.4)_1$ and $(4.4)_2$ will classify the possible different cases treated in Table I of §5.

COROLLARY 4.4. *If $(i, j) \in I'_d$ then φ_{ij}^A is an isomorphism.*

PROOF. The condition $d_1 + d_2 = d_i + d_j$ if $i \neq n + 1$, or $d_1 + d_2 = d_j$ if $i = n + 1$ says φ_{ij}^A is a linear map between vector spaces of the same dimension. But we also deduce that ker $\varphi_{ij}^A = 0$, since, according to the Interpretation 4.4, no indecomposable contributing to ker φ_{ij}^A is a factor in A.

Interpretation 4.5. Case $(i, j) \in I''_d$. The discussion is now related to the following sets:

$$\Lambda_0 := \Lambda_0^{ij} = \left\{ (r, s): 3 \leqslant r \leqslant i < j \leqslant s \leqslant n, e_{rs}^A > 0 \right\},$$
$$\Lambda_1 := \Lambda_1^{ij} = \left\{ (s, r): s > r > j, e_{sr}^A > 0 \right\},$$
$$\Lambda_2 := \Lambda_2^{ij} = \left\{ k: k \geqslant j, e_{1k}^A > 0, e_{2k}^A > 0 \right\},$$
$$\Lambda_3 := \Lambda_3^{ij} = \left\{ k: k \geqslant j, e_{1k}^A > 0, e_{2k}^A = 0 \right\},$$

$$\Lambda_4 := \Lambda_4^{ij} = \left\{ (s,r) : s > j > i \geqslant r, e_{sr}^A > 0 \right\},$$

$$\Lambda_5 := \Lambda_5^{ij} = \left\{ (s,r) : s > j \geqslant r \geqslant 3, e_{sr}^A > 0 \right\}.$$

Consider $i \neq 1, 2$. Then we can assume $d_1 \geqslant d_2$ (cf. construction $C(d)$ in §3). The condition $d_i = d_j < d_t, t \in [i+1, j-1]$, implies that in A there is at least a factor $E_{i+1, j-1}$ (we read it as the zero representation if $j = i + 1$). Every factor which contains the vertex i also contains j and conversely; moreover the assumption $d_h > 0$ for all h implies that in A there are factors passing through i and j. Therefore we have the following possibilities: either $\Lambda_0^{ij} \neq \varnothing$ or all the factors of A containing i and j also contain at least one of the vertices $1, 2$. Moreover if E_{sr}, $s > r$, is such a factor then either $s > j > i \geqslant r$ or $s > r > j$. The previous discussion implies that if $i \neq 1, 2$ then

$$(4.5)_1 \qquad \begin{cases} \text{either } \Lambda_0^{ij} \neq \varnothing \text{ or} \\ \Lambda_\tau^{ij} \neq \varnothing \text{ for at least a } \tau, \tau = 1, 2, 3, 4. \end{cases}$$

Consider $i = 1, 2$. For the symmetry of (D_n, eq) we may assume $i = 1$, i.e. $(1, j) \in I_d''$, $j \in [3, n]$, but then we have to consider $d_1 \gtrless d_2$. The condition $d_1 = d_j < d_t$, $t \in [3, j-1]$, implies every factor of A containing 1 also contains j and conversely, and such factors exist in A (as $d_h > 0$ for all h). If $d_2 \geqslant d_1$ each factor containing 1 also contains 2 and j. If $d_2 < d_1$ in A there is no factor E_{2h}.

We have the following possibilities:

$(4.5)_2$ $d_2 > d_1$, $\Lambda_5^{ij} \neq \varnothing$ and in A there is at least a factor $E_{2, j-1}$ if $j > 3$ or E_{22} if $j = 3$.

$(4.5)_3$ $d_1 = d_2$, then $(s, j) \in \Lambda_5^{ij}$ for some s, and $\Lambda_3^{ij} = \varnothing$.

$(4.5)_4$ $d_2 < d_1$, $\Lambda_3^{ij} \neq \varnothing$, $(s, j) \in \Lambda_5^{ij}$ for some s and $s > k$ for all $k \in \Lambda_5^{ij}$.

We will use $(4.5)_1$–$(4.5)_4$ to classify all possible cases listed in Table II of §5.

COROLLARY 4.5. *If $(i, j) \in I_d''$ then φ_{ij}^a is an isomorphism.*

PROPOSITION 4.6. *The cardinality of I_d is $|I_d| = n - \tau$, where n is the number of vertices of the Dynkin diagram D_n and τ is the number of nonisomorphic indecomposable factors of $A = C(d)$.*

PROOF. As usual, we assume $d_2 \leqslant d_1$. Let

$$d' = \left(\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}; d_3, \ldots, d_{n-1} \right)$$

and consider it as a dimension for the subdiagram of D_n obtained by erasing the vertex n (and with the induced orientation). By induction assume $|I_{d'}| = n - 1 - \tau'$, where τ' is the number of nonisomorphic factors in $A' = C(d') = C_{n-1}(d)$.

The case $d_2 < d_1$. We may have either $|I_d| = |I_{d'}|$ or $|I_d| = |I_{d'}| + 1$, since by adding the vertex n and passing from d' to d we can create at most a new pair $(i, j) \in I_d$ with i or j equal to n (cf. Lemma 4.3). We now use the same notation as in §3 and in the proof of Proposition 3.1, i.e. the multiplicities $f_0, \ldots, f_r, k, l, m = \sum_{i=3}^{n-1} m_i$ are referred to the construction $C_{n-1}(d)$. If $k \neq 0$ in $A = C(d)$ there is no factor $E_{1,t}$ or E_{2t} with $t < n - 1$ (cf. Lemma 3.2(iii)), and the condition $d_2 < d_1$ implies $l \neq 0$. It follows that in any case we can only have $\tau = \tau'$ or $\tau = \tau' + 1$. Using the inductive assumption we need only prove $\tau = \tau'$ if and only if $|I_d| = |I_{d'}| + 1$.

The definition of $C(d)$ implies we have $\tau = \tau'$ if and only if $0 < d_n \leqslant d_{n-1}$ and d_n satisfies one of the following equalities:

(1)
$$d_n = \sum_{u=0}^{h} f_u, \quad h \leqslant r;$$

(2)
$$d_n = \sum_{u=0}^{r} f_u + k;$$

(2')
$$d_n = \sum_{u=0}^{r} f_u + k + l;$$

(3)
$$d_n = \sum_{u=0}^{r} f_u + 2k + l;$$

(4)
$$d_n = \sum_{u=0}^{r} f_u + 2k + l + \sum_{t=3}^{h} m_t, \quad h \leqslant n - 1, m_h \neq 0.$$

On the other hand if $d_n > d_{n-1}$, no pair $(i, j) \in I_d$ is such that either i or j equals n. If $d_n = d_{n-1}$ then $\tau = \tau'$ (special case of (4) when $h = n - 1$), and obviously we have $(n - 1, n) \in I_d$ and conversely. Therefore we are only left with the case $d_n < d_{n-1}$. Note that (1) disappears if the sequence $\{ f_0, \ldots, f_r \}$ is empty. Moreover (1)–(4) need not be distinct; in fact, for example, if $k = 0$ and $h = r$, (1) and (2) coincide; if $k = l = 0$ and $m \neq 0$ then (2), (2') and (3) coincide. If this happens we only consider the case corresponding to the equality of higher index.

Assume $\tau = \tau'$. Then $(I_d) = |I_{d'}| + 1$. In fact:
if (1) holds, $(n, i_h) \in I'_d$;
if (2) holds, $(2, n) \in I''_d$;
if (2') holds, $(1, n) \in I''_d$;
if (3) holds $(i, n) \in I''_d$, if $i + 1 > 3$, or $(n + 1, n) \in I'_d$, if $i + 1 = 3$ and $i + 1$ is always the minimum index such that $m_{i+1} > 0$;
if (4) holds, $(i, n) \in I''_d$, with $h < i + 1 \leqslant n + 1$ and $i + 1$ is the minimum index such that $m_{i+1} > 0$, which follows from the fact that

$$d_{n-1} = \sum_{u=0}^{r} f_u + 2k + l + m > d_n$$

and Lemma 3.2. The converse is also true, i.e. if we have a pair $(i, j) \in I_d$ and either i or j are equal to n, then $\tau = \tau'$ as an easy consequence of Interpretations 4.4 and 4.5 of $(i, j) \in I_d$ in terms of the canonical construction.

The case $d_1 = d_2$. Adding the vertex n we may only have $|I_d| = |I_{d'}| + \rho, \rho = 0, 1, 2$, and the equality holds with $\rho = 2$ if and only if $d_n = d_1 = d_2$ and $d_t > d_n$, $t \in [3, n - 1]$. On the other hand we also have $\tau = \tau' + \delta, \delta = 0, \pm 1$, and we have $\delta = -1$ if and only if $d_n < d_{n-1}, k \neq 0$, and $d_n = \sum_{u=0}^{r} f_u + k$, but $d_1 = d_2$ now implies $d_n = d_1 = d_2$ and $d_t > d_n, t \in [3, n - 1]$. Therefore also in this case, using the inductive assumption, we need only prove $\tau = \tau'$ if and only if $|I_d| = |I_{d'}| + 1$. The discussion now is the same as in the case $d_2 < d_1$, we need only erase (2), (2') and the corresponding discussion if $k \neq 0$.

The importance of this proposition comes from the following results of Happel [9] and Sato and Kimura [13] (cf. also §6).

PROPOSITION 4.7. (a) $n - \tau$ *equals* dim $L_d/\Pi \, SL(V_i)$,

(b) $n - \tau$ *equals the number of codim.* 1 *orbits in* L_d.

5. The set $\mathcal{O}_1$ of codim. 1 orbits in $L_d(D_n, \text{eq})$.

THEOREM 5.1. *There exists a unique bijection* $\psi: I_d \to \mathcal{O}_1, \psi(i, j) = O_{ij}, B_{ij} \in O_{ij}$ *a representative such that*

(a) $N_{ij}^{B_{ij}} = N_{ij}^{A} - 1$,

(b) $N_{uv}^{B_{ij}} = N_{uv}^{A}$ *for all* $(u, v) \in I_d, (u, v) \neq (i, j)$.

PROOF. We need only construct an injective map ψ satisfying (a) and (b). Then ψ is automatically bijective as a consequence of Propositions 4.6 and 4.7.

The definition of ψ is contained in Table I if $(i, j) \in I'_d$ and Table II if $(i, j) \in I''_d$, and is inspired by the fact that the rank parameters parametrize the orbits and by the following consideration: $(i, j) \in I_d$ implies φ_{ij}^A is an isomorphism (cf. Corollaries 4.4, 4.5). Then if B_{ij} must be a representative for a codim. 1 orbit satisfying (a), we have to produce a "damage" to φ_{ij}^A and make it as small as possible. Now the rank parameters for a representation satisfy the inequalities of Proposition 1.4, therefore from the "damage" produced to φ_{ij}^A it may follow a "damage" to φ_{uv}^A for other pairs (u, v) and in any case we have to try to minimize their number. We explain how to read the tables. In column 1 are listed the various cases which can occur when $(i, j) \in I_d$ according to Interpretations 4.4, 4.5. In column 3 we define (graphically, for convenience) a representation $A'_{ij} \neq 0$ for a suitable choice of the indices listed in column 2. A'_{ij} is the maximal orbit corresponding to its dimension and consists of at most three factors. Clearly we have the decomposition

$$(5.2) \qquad\qquad\qquad A = A'_{ij} \oplus \overline{A}.$$

In column 4 we define a representation B'_{ij} such that $B'_{ij} \neq A'_{ij}$ and dim $B'_{ij} = $ dim A'_{ij}. B'_{ij} is constructed from A'_{ij} via a suitable degeneration (a violation to the canonical construction either at vertex i or vertex j).

We define the map ψ on the pair $(i, j) \in I_d$ via the following representative:

$$(5.3) \qquad\qquad B_{ij} = B'_{ij} \oplus \overline{A}, \qquad \psi(i, j) = O_{ij} \ni B_{ij}$$

and we need to check that:

$$(5.4) \qquad\qquad\qquad \text{codim}_{L_d} \overline{O}_{B_{ij}} = 1.$$

The pair $(i, j) \in I_d$ determines the decomposition (5.2) of A, i.e. the representation A'_{ij}. This claim is trivial in Table I; in Table II the ambiguity comes from the fact that in some cases we have to deal in column 2 with sets of pairs of indices, but the claim follows from Lemma 3.2 applied to the factors of A'_{ij}. Consider, for example, case 1 of Table II and assume, by contradiction, $(r, s) \in \Lambda_0, (r', s') \in \Lambda_0$, $(r, s) \neq (r', s')$ and $s - r = s' - r'$ the least possible. Then in A there are simultaneously two factors $E_{rs}, E_{r's'}$, with $r' < r < s' < s$, a contradiction to Lemma 3.2(ii). For all other cases we always use Lemma 3.2(i).

TABLE I

$(i,j) \in I'_d$ $A = A'_{ij} \oplus \bar{A}, B_{ij} = B'_{ij} \oplus \bar{A}$ $\Theta_1 = (s: s > i > j, c'_{ij} > 0)$, $\Theta_2 = (t: i > t > j, c^A_{it} > 0)$

Column 1		Column 2	Column 3	Column 4	Column 5		Column 6
Case		Choice of the indices	A'_{ij}	B'_{ij}	dim St A'_{ij}	dim St B'_{ij}	$N^{By}_{uv} < N^A_{uv}$
1	(i,j) $i \neq n+1$	$\Theta_1 \neq \emptyset$ $s \in \Theta_1$, s min.	*(diagram: j, i, i, $s-1$)*	*(diagram: j, i, $i-1$, $s-1$)*	2	3	$u \in [i, s-1]$
		$\Theta_2 \neq \emptyset$ $t \in \Theta_2$, t max.	*(diagram: t, j, $i-1$)*	*(diagram: t, j, $s-1$, i)*		4	$v \in [j, t-1]$
2		$\Theta_1 \neq \emptyset$ $s \in \Theta_1$, s min.	*(diagram: j, i, i, $s-1$)*	*(diagram: j, i, $i-1$, $s-1$)*	3	3	$u = n+1$
		$\Theta_2 = \emptyset$	*(diagram: 3, $j-1$, $i-1$)*	*(diagram: 3, $j-1$, $i-1$)*		4	$v \in [j, t-1]$
3	$(n+1, j)$	$\Theta_1 = \emptyset$ $t \in \Theta_2$, t min.	*(diagram: $3 < j$, t, $i-1=n$)*	*(diagram: $3 < j$, t, $i-1=n$)*	2	3	$u = n+1$
4			*(diagram: $j=3$, 3, t, $i-1=n$)*	*(diagram: $j=3$, 3, $t-1$, $i-1=n$)*	1	2	$v \in [j, t-1]$
5		$\Theta_1 = \emptyset$	*(diagram: $3 < j$, $i-1=n$)*	*(diagram: $3 < j$, $i-1=n$)*	3	4	$u = n+1$
6		$\Theta_2 = \emptyset$	*(diagram: $j=3$, 3, $i-1=n$)*	*(diagram: $j=3$, 3, n, $i-1=n$)*	2	3	$v \in [j, n]$

TABLE II

$(i,j) \in \Gamma_d$ $\qquad A = A'_{ij} \oplus \bar{A}$ $\qquad \Lambda_0 = ((r,s): 3 < r < i \le j < s \le n, c^A_{rs} > 0)$ $\qquad \Lambda_1 = ((s,r): s > r > j, c^A_{sr} > 0)$ $\qquad \Lambda_2 = (k: k > j, c^A_{1k} > 0, c^A_{2k} > 0)$

$\qquad B_{ij} = B'_{ij} \oplus \bar{A}$ $\qquad \Lambda_3 = (k: k > j, c^A_{jk} > 0, c^A_{2k} = 0)$ $\qquad \Lambda_4 = ((s,r): s > j > i > r, c^A_{sr} > 0)$ $\qquad \Lambda_5 = ((s,r): 3 < r < s, c^A_{sr} > 0)$

Column 1		Column 2	Column 3	Column 4	Column 5				Column 6	
Case		Choice of the indices	A'_{ij}	B'_{ij}	dim St A'_{ij}		B'_{ij}			
1		$\Lambda_0 \ne \varnothing$	$(r,s) \in \Lambda_0$ $s - r$ min.	[diagram]	[diagram]	$j \ne i+1$	2	3	$u \in [r, i]$ $v \in [j, s]$	
						$j = i+1$	1	2		
2		$\Lambda_0 = \varnothing$ $\Lambda_1 \ne \varnothing$	$(s,r) \in \Lambda_1$ $s - r$ max.	[diagram]	[diagram]	$j \ne i+1$	2	3	$u \in [3, i]$ $v \in [j, k]$	$u \in [s, n+1]$ $v \in [j, r-1]$
						$j = i+1$	1	2		
3	(i,j)	$\Lambda_1 = \varnothing$ $\tau = 0, 1$ $\Lambda_2 \ne \varnothing$	$k \in \Lambda_2$ k min.	[diagram]	[diagram]	$j \ne i+1$	3	4	$u \in [3, i]$ $v \in [j, k]$	$u \in [k+1, n+1]$ $v \in [j, k]$
						$j = i+1$	2	3		
4	$i \ne 1,2$ $d_1 > d_2$	$\Lambda_\tau = \varnothing$ $\tau = 0,1,2$ $\Lambda_3 \ne \varnothing$	$k \in \Lambda_3$ k min.	[diagram]	[diagram]	$j \ne i+1$	2	3	$u \in [1, i]$ $v \in [j, k]$	$u \in [j+1, n+1]$ $v \in [j, k]$
						$j = i+1$	1	2		
5		$\Lambda_\tau = \varnothing$ $\tau = 0,1,2,3$ $\Lambda_4 \ne \varnothing$	$(s,r) \in \Lambda_4$ $s - r$ min.	[diagram]	[diagram]	$j \ne i+1$	2	3	$u \in (1,2,3,\ldots,t)$ $v \in [j, s-1]$	$u \in [j, s-1]$ $v \in [r, j-1]$ or $u \in [j+1, n+1]$ $v \in [j, s-1]$
						$j = i+1$	1	2	$v \in [j, s-1]$	
6		$\Lambda_5 \ne \varnothing$ $(d_2 > d_1)$	$(s,r) \in \Lambda_5$ $s - r$ min.	[diagram]	[diagram]	$r \ne j$	2	3	$u = 1$ $v \in [j, s-1]$	$u \in [j, s-1]$, $v \in [r, s-1]$
7	$(1,j)$	$\Lambda_5 \ne \varnothing$ $\Lambda'_3 = \varnothing$ $(d_2 = d_1)$	$(s,j) \in \Lambda_5$ s min.	[diagram]	[diagram]	$r = j$	3	4	$u = 1$ $v \in [j, s-1]$	$u \in [j+1, s-1]$ $v \in [j, s-1]$
8		$\Lambda_5 \ne \varnothing$ $\Lambda'_3 \ne \varnothing$ $(d_2 < d_1)$	$(s,j) \in \Lambda_5$ s min. $k \in \Lambda_3$ k min.	[diagram]	[diagram]	$k < s-1$	2	3	$u = 1$ $v \in [j, k]$	$u \in [j+1, s-1]$ $v \in [j, k]$
						$k = s-1$	3	4		

In Table I we have collected all possible cases which can occur when $(i, j) \in I'_d$. If $(i, j) \in I''_d$ and $i \neq 1, 2$ in Table II, we have not explicitly drawn, for economy, the "special case" $j = i + 1$; if so we mean that the factor $E_{i+1, j-1}$ in A'_{ij} is the zero representation (i.e. we erase this factor in column 3) and in column 4 the representation B'_{ij} is formally unchanged (but $i + 1 = j!$). Moreover, we have assumed $d_1 \geq d_2$, as we can read the case $d_1 < d_2$ using the symmetry of (D_n, eq) with respect to the permutation of the vertices 1 and 2. The only representation A'_{ij} which is not invariant under such a permutation is the one listed in case 4; therefore if $d_1 < d_2$ we must replace the set Λ_3 with $\Lambda'_3 = \{ k: k \geq j, e^A_{2k} > 0, e^A_{1k} = 0 \}$, the factor E_{1k} with E_{2k} in column 3 and $E_{1, j-1}$ with $E_{2, j-1}$ in column 4. In Table II we have only considered the case $(1, j) \in I''_d$, but we have assumed $d_1 \gtrless d_2$; therefore the situation for $(2, j) \in I_d$ can be read from the previous one by permuting the indices 1 and 2. If $(1, j) \in I''_d$ and $d_2 > d_1$, we have the "special case" $j = r$ in column 3. Then in column 4 we replace the factor E_{jr} with the pair $E_{1, j-1} \oplus E_{2, j-1}$ if $j \neq 3$, or with $E_{11} \oplus E_{22}$ if $j = 3$. If $(1, 3) \in I''_d$ and $d_1 = d_2$ in column 4, we replace $E_{1, j-1}$ with E_{11}. If $(1, j) \in I''_d$ and $d_2 < d_1$ we may have $k = s - 1$; in this case in column 4 we replace the factor $E_{s, k+1}$ with the pair $E_{1, s-1} \oplus E_{2, s-1}$.

In order to have $\psi: I_d \to \mathcal{O}_1$ well defined we have to prove (5.4) on the codimension. We use the following procedure. We have $\operatorname{codim}_{L_d} \overline{O}_{ij} = \dim \operatorname{St} B_{ij} - \dim \operatorname{St} A$. We use the stabilizer formula (2.4) and the decompositions (5.2) and (5.3), respectively, for A and B_{ij}; we get

$$\operatorname{codim}_{L_d} \overline{O}_{ij} = \dim \operatorname{St} B'_{ij} - \dim \operatorname{St} A'_{ij} + \left[B'_{ij}, \overline{A} \right] - \left[A'_{ij}, \overline{A} \right]$$

$$= \dim \operatorname{St} B'_{ij} - \dim \operatorname{St} A'_{ij} + \sum_{p, q} e^{\overline{A}}_{pq} \{ \left[B'_{ij}, E_{pq} \right] - \left[A'_{ij}, E_{pq} \right] \} .$$

In column 5 of Tables I or II we have listed the values of $\dim \operatorname{St} A'_{ij}$ and $\dim \operatorname{St} B'_{ij}$ (this computation is immediate from Table 2 and (2.3), since we know explicitly the factors in A'_{ij} and B'_{ij}). From column 5 we read

$$\dim \operatorname{St} B'_{ij} - \dim \operatorname{St} A'_{ij} = 1.$$

It follows that

$$\operatorname{codim}_{L_d} \overline{O}_{ij} = 1 + \sum_{p, q} e^{\overline{A}}_{pq} \{ \left[B'_{ij}, E_{pq} \right] - \left[A'_{ij}, E_{pq} \right] \} ,$$

and we need only verify that

$$(5.5) \qquad \sum_{p, q} e^{\overline{A}}_{pq} \{ \left[B'_{ij}, E_{pq} \right] - \left[A'_{ij}, E_{pq} \right] \} = 0.$$

(5.5) has been checked case by case using the following argument. In (5.5) we have no contribution whenever E_{pq} is such that $e^A_{pq} = 0$. In view of this we take the list of the indecomposables $E_{p, q}$, $p \gtrless q$, of D_n given in §1 and erase the ones which are incompatible with the interpretation of $(i, j) \in I_d$ in 4.4 or 4.5, the definition of A'_{ij} (i.e. the indices in column 2) and Lemma 3.2 applied to the factors of A'_{ij} (in this case in fact we know a priori that $e^A_{pq} = 0$); then we draw the list of the remaining ones (specifying the mutual position of the indices p, q with respect to the given i, j) and check directly, using Table 2, that $[B'_{ij}, E_{pq}] - [A'_{ij}, E_{pq}] = 0$.

The proof of (5.5) is therefore just routine and is very boring due to the large number of cases. We explicitly give just one case, and we choose case 1 of Table II because the orbit O_{ij} is constructed in the same way as for an equioriented Dynkin diagram of type $\mathscr{A}_m$ (cf. Remark 4.2(b)) and [1]; nevertheless we must prove (5.4) and, therefore, (5.5) as in D_n there are indecomposable representations which are not representations for any subdiagram $\mathscr{A}_m$ of D_n. Case 1. Table II: the list of the E_{pq} for which we do not know a priori that $e_{pq}^A = 0$ is the following diagram (Diagram F). We immediately read, using Table 2, that

$$\left[E_{r,j-1} \oplus E_{i+1,s}, E_{pq} \right] - \left[E_{rs} \oplus E_{i+1,j-1}, E_{pq} \right] = 0.$$

DIAGRAM F

To end the description of the tables we explain column 6. For B_{ij} given by (5.3) we have $N_{uv}^{B_{ij}} \leqslant N_{uv}^A$ and strict inequality holds for at least a pair (u, v), since $B_{ij} \in L_d$ and A is a representative of the maximal orbit (cf. Proposition 1.5). Column 6 lists all pairs for which strict inequality holds. These pairs are easily computed once we observe that ranks are additive; therefore we need only compare rank parameters for A_{ij}' and B_{ij}', which are explicit, using definitions (1'), (2'), (3'). We always get $N_{uv}^{B_{ij}} = N_{uv}^A - 1$, except for $u > v$ and $u \in [j + 1, s - 1], v \in [j, s - 1]$ of case 5, Table II, where $N_{uv}^{B_{ij}} = N_{uv}^A - 2$.

Note that if we assume A_{ij}' and B_{ij}' as in columns 3 and 4, but with an arbitrary choice of the indices in the sets Λ_r or Θ_p, column 6 remains unchanged; suitable

choice of the indices of column 2 minimizes the intervals listed in column 6, i.e. the number of rank parameters which have decreased their value passing from A to B_{ij}.

Injectivity of ψ. We must show that (i, j), $(i', j') \in I_d$, $(i, j) \neq (i', j')$ imply $\psi(i, j) \neq \psi(i', j')$ (i.e. $B_{ij} \neq B_{i'j'}$). Consider simultaneously the decompositions (5.2) for the pairs (i, j) and (i', j'):

$$A = A'_{ij} \oplus \bar{A}, \qquad A = A'_{i'j'} \oplus \tilde{A},$$

and the corresponding degenerations:

$$B_{ij} = B'_{ij} \oplus \bar{A}, \qquad B_{i'j'} = B'_{i'j'} \oplus \tilde{A}.$$

If $A'_{ij} \neq A'_{i'j'}$ in A'_{ij} there is at least a factor E_{zw} which is not a factor of $A'_{i'j'}$ and therefore is a factor in $\tilde{A}$ and $B_{i'j'}$. But E_{zw} has been destroyed by the construction $\psi(i, j)$, i.e. E_{zw} is not a factor in B'_{ij}. If $A'_{ij} = A'_{i'j'}$ (it can happen even if $(i, j) \neq (i', j')$, cf. case 7 in Table II), then $B'_{ij} \neq B'_{i'j'}$, and $\bar{A} = \tilde{A}$. In any case $B_{ij} \neq B_{i'j'}$ and the injectivity of ψ is proved.

PROOF OF (a), (b). (a) is trivial since the pair (i, j) is in the list of column 6 and $N_{ij}^{B_{ij}} = N_{ij}^A - 1$. To prove (b) assume $(i, j) \in I_d$, $N_{uv}^{B_{ij}} < N_{uv}^A$ for $(u, v) \neq (i, j)$ and $(u, v) \in I_d$. Then $\psi(i, j)$ is defined via the representative $B_{ij} = B'_{ij} \oplus \bar{A}$, for B'_{ij} listed in one of the cases in Tables I, II(column 4), and (u, v) is a pair listed in column 6 for the same case.

But we claim these conditions are incompatible. For Table I the claim follows directly from Lemma 4.3(α). The discussion for Table II is more delicate, and from Lemma 4.3(β), (γ) the claim follows only if u or v is one of the indices i, j, and for $u < v$ in cases 6, 7, 8.

For the remaining cases of Table II we proceed as follows: we consider the decomposition $A = A'_{uv} \oplus \tilde{A}$ coming from the construction of the map ψ on the pair $(u, v) \in I_d$ and we show that at least a factor in A'_{uv} contradicts the construction of ψ on the pair (i, j).

Case 1. We have $u \in [r, i]$, $v \in [j, s]$, $u \neq i$, $v \neq j \geqslant 3$; then $(u, v) \in I''_d$, $3 \leqslant r \leqslant u < i < j < v \leqslant s$. It follows from the construction ψ on (u, v) (cf. Table II, cases 1–5 and read (u, v) instead of (i, j)) that $A = A'_{uv} \oplus \tilde{A} = E_{u+1,v-1} \oplus \cdots$ and $r < u + 1 \leqslant i < j \leqslant v - 1 < s$, a contradiction to the choice of the indices $(r, s) \in \Lambda'_0$ in column 2.

Case 2. If $u \in [3, i]$, $v \in [j, r - 1]$ we repeat the argument of Case 1 and get $A = A'_{uv} \oplus \bar{A} = E_{u+1,v-1} \oplus \cdots$, $3 < u + 1 \leqslant i < j < v$, a contradiction to the assumption $\Lambda^{ij}_0 = \varnothing$. If $u \in [s, n + 1]$, $v \in [j, r - 1]$, $v \neq j$, then $(u, v) \in I'_d$ and $v > j > 3$ and the construction ψ on the pair (u, v) is listed in Table I, cases 1–5. Therefore either we have

$$A = A'_{uv} \oplus \bar{A} = E_{wv} \oplus \cdots, \qquad v < r, \quad w - 1 \geqslant u \geqslant s,$$

i.e. a contradiction to the choice of the indices $(s, r) \in \Lambda^{ij}_1$ in column 2, or

$$A = A'_{uv} \oplus \tilde{A} = E_{3,v-1} \oplus \cdots, \qquad v - 1 \geqslant j > 3,$$

i.e. a contradiction to the assumption $\Lambda^{ij}_0 = \varnothing$.

Case 3. If $u \in [3, i]$, $v \in [j, k]$, $u \neq i$, $v > j > 3$ we have

$$A = A'_{uv} \oplus \bar{A} = E_{u+1,v-1} \oplus \cdots, \qquad 3 < u + 1 \leqslant i < j \leqslant v - 1,$$

i.e. a contradiction to the assumption $\Lambda_0^{ij} = \varnothing$. If $u = n + 1$, $v \in [j, k]$, $v \neq j > 3$, we have

$$A = A'_{uv} \oplus \tilde{A} = E_{3,v-1} \oplus \cdots, \qquad v - 1 \geqslant j > i \geqslant 3,$$

i.e. a contradiction to the assumption $\Lambda_0^{ij} = \varnothing$. If $u \in [k + 1, n]$, $v \in [j + 1, n]$, either we have

$$A = A'_{uv} \oplus \tilde{A} = E_{wv} \oplus \cdots, \qquad w > v \geqslant j + 1, \quad w > v,$$
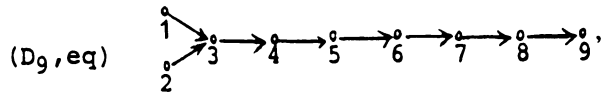
a contradiction to the assumption $\Lambda_1^{ij} = \varnothing$, or

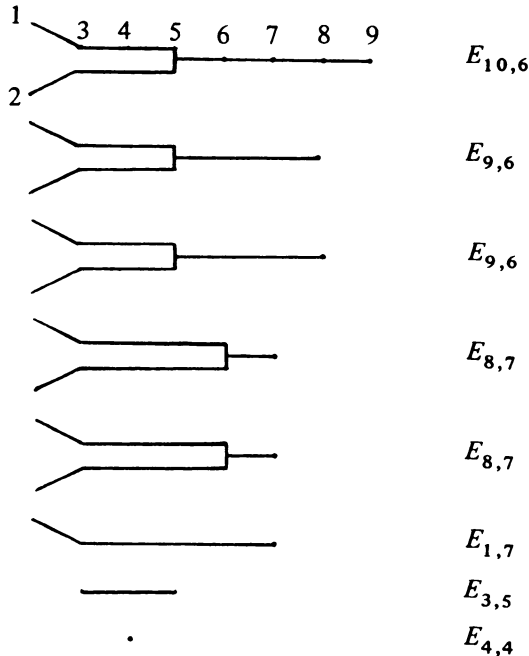$$A = A'_{uv} \oplus \tilde{A} = E_{1,v-1} \oplus E_{2v-1} \oplus \cdots, \qquad v - 1 < k,$$

a contradiction to the choice of the index k in Λ_2^{ij}.

We leave the other cases to the reader. Note that the map ψ gives for any dimension d the codim. 1 orbits through the factors of their decomposition, as we know the decomposition of $A = C(d) \in O_{\max}$.

EXAMPLE.

$$(D_9, \text{eq}) \quad \begin{matrix} 1 \\ 2 \end{matrix} \!\!\searrow\!\! 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \,,$$

$$d = (\begin{matrix} 6 \\ 5 \end{matrix} \; ; 12 \, , \, 13 \, , \, 12 \, , \, 8 \, , \, 6 \, , \, 3 \, , \, 1),$$

$$E_{10,6}$$

$$E_{9,6}$$

$$E_{9,6}$$

$$E_{8,7}$$

$$E_{8,7}$$

$$E_{1,7}$$

$$E_{3,5}$$

$$E_{4,4}$$

$$A = E_{10,6} \oplus 2E_{9,6} \oplus 2E_{8,7} \oplus E_{1,7} \oplus E_{3,5} \oplus E_{4,4},$$

$$I_d' = \{(8,6)\}, \quad I_d'' = \{(3,5),(1,7)\}, \quad |I_d| = 3,$$

$$\tau = \# \text{ distinct ind. in } A = 6, \quad n = 9.$$

Construction ψ for the codim. 1 orbits:

$$(8,6) \in I_d'\text{: Case 1, Table I, } A = E_{9,6} \oplus E_{8,7} \oplus \bar{A},$$

$$B_{8,6} = E_{8,6} \oplus E_{9,7} \oplus \bar{A};$$

$$(3,5) \in I_d''\text{: case 1, Table II, } A = E_{3,5} \oplus E_{4,4} \oplus \bar{A},$$

$$B_{3,5} = E_{3,4} \oplus E_{4,5} \oplus \bar{A};$$

$$(1,7) \in I_d''\text{: case 8, Table II (special case),}$$

$$A = E_{8,7} \oplus E_{1,7} \oplus \bar{A}, \quad B_{1,7} = E_{1,7} \oplus E_{2,7} \oplus E_{1,6} \oplus \bar{A}.$$

6. The ring of semi-invariant polynomials $\mathscr{R}_d(D_n, \text{eq})$. We first recall some known facts. Let K be an algebraically closed field and ρ a rational representation of a connected algebraic group G on a finite-dimensional vector space over K. Then G acts on the polynomial ring $K[V]$ of V via the action

$$f^g(v) = f(\rho(g^{-1})v), \quad g \in G, v \in V.$$

A polynomial $f \in K[V]$ is said to be semi-invariant (or relative invariant) if $f^g = \chi(g)f$ (χ a character of G). The ring of semi-invariants of the representation (G, ρ, V) is by definition the ring generated by the semi-invariant polynomials. (G, ρ, V) is called prehomogeneous if there is an open dense orbit $O_{\max} \subset V$. In this case it has been proved by M. Sato and T. Kimura (cf. [13]) that the reduced equations $f_i = 0$ of the codim. 1 components in $V \setminus O_{\max}$ are such that the f_i's are an algebraically independent system of semi-invariant polynomials which generate the ring of semi-invariants.

Let us now consider the Dynkin diagram D_n equioriented. For a given dimension d (see (*)) we have the variety

$$L_d = L_d(D_n, \text{eq}) = \text{Hom}_K(V_1, V_3) \times \prod_{t=2}^{n-1} \text{Hom}_K(V_t, V_{t+1})$$

(V_t vector spaces over K of dimension d_t) on which the group $G_d = \prod_{t=1}^{n} \text{GL}(V_t)$ acts. We denote by $\mathscr{R}_d(D_n, \text{eq})$ the corresponding ring of semi-invariants. The representation of G_d on L_d is prehomogeneous and the codim. 1 components in $L_d \setminus O_{\max}$ are exactly the closure of the codim. 1 orbits in L_d (which have been classified in §5). In view of the Sato-Kimura theorem we want to compute, for every codim. 1 orbit, the equation of its closure. If we fix bases in the V_t's, a representation $X \in L_d$ is given by a $(n-1)$-tuple of matrices $(X_1, X_2, \ldots, X_{n-1})$, X_1 a $d_3 \times d_1$ matrix, X_t a $d_{t+1} \times d_t$ matrix, $2 \leqslant t \leqslant n - 1$. Then the map φ_{ij}^X, $i \neq j$, defined in §1 is expressed by a matrix Y_{ij} which, with a suitable choice of the bases for the spaces involved, can be written in terms of the X_t's and their compositions. For any

$h < k \leqslant n$ let us define the $d_k \times d_h$ matrix

$$X_{hk} = X_{k-1} \cdot X_{k-2} \cdots X_h, \qquad h \neq 1, h < k,$$
$$X_{1k} = X_{k-1} \cdot X_{k-2} \cdots X_3 \cdot X_1, \qquad h = 1.$$

Then we have

(6.1) $$\varphi_{ij}^x: V_i \to V_j, \quad Y_{ij} = X_{ij}, \qquad i < j,$$

(6.2) $$\varphi_{n+1,j}^x: V_1 \oplus V_2 \to V_j, \quad Y_{ij} = \begin{bmatrix} X_{1j} X_{2j} \end{bmatrix}, \qquad i = n + 1$$

(and the choice of the bases is obvious). If $i > j$ and $i \neq n + 1$ we have the map

$$\varphi_{ij}^x: V_1 \oplus V_2 \oplus V_j \to (V_j \oplus V_j' \oplus V_j'')/\Delta \oplus V_i$$

where V_j' and V_j'' are copies of V_j. Let $w_1, \ldots, w_{d_j}$ be the given basis in V_j and w_h', w_h'', $h = 1, \ldots, d_j$ the same base in the copies V_j', V_j''. Then a basis in $(V_j \oplus V_j' \oplus V_j'')/\Delta$ is given by $\overline{w}_1, \ldots, \overline{w}_{d_j}, \overline{w'}_1, \ldots, \overline{w'}_{d_j}$ where $\overline{w}_h$ (resp. $\overline{w'}_h$) is the class of w_h (resp. w_h') in the quotient space. Then the matrix corresponding to φ_{ij}^x is

(6.3) $$Y_{ij} = \begin{bmatrix} X_{1j} & 0 & -I \\ 0 & X_{2j} & -I \\ 0 & 0 & X_{ji} \end{bmatrix}, \qquad i > j$$

(I the identity matrix of size d_j).

We are interested in the matrices Y_{ij} only if $(i, j) \in I_d$. In this case Y_{ij} is a square matrix and we define

(6.4) $$D_{ij} = \det Y_{ij}, \qquad (i, j) \in I_d.$$

THEOREM 6.5. *If K is an infinite field then*:

(1) *The D_{ij}'s, $(i, j) \in I_d$, are a set of algebraically independent semi-invariant polynomials.*

(2) *Any semi-invariant polynomial is a product of the D_{ij}'s and $\mathscr{R}_d(D_n, \text{eq}) = K[D_{ii}]$.*

PROOF. Let $(i, j) \in I_d$ and consider the corresponding polynomial D_{ij}, which is not identically zero on L_d since it does not vanish on the open orbit $O_{\max}$ (in fact φ_{ij}^A is an isomorphism for $A \in O_{\max}$, cf. Corollaries 4.4, and 4.5). $D_{ij} = 0$ on $\overline{O}_{ij} = \overline{\psi(i, j)}$ as a consequence of Theorem 5.1(a) and D_{ij} does not vanish on any other codim. 1 orbit of L_d as a consequence of Theorem 5.1(b). At this point we first prove the theorem if K is algebraically closed, showing that $D_{ij} = 0$ is the reduced equation of $\overline{O}_{ij}$, and then we extend the result to the case of an infinite field K, passing through its algebraic closure $\overline{K}$. The argument is omitted since it is completely similar to the one given in [2, Theorem 0.1].

REFERENCES

1. S. Abeasis, *On the ring of semi-invariants for the representations of an equioriented quiver of type A_n*, Boll. Un. Mat. Ital. (6) 1-A (1982), 233–240.

2. _____, *Codimension 1 orbits and semi-invariants for the representations of an oriented graph of type $\mathscr{A}_n$*, Trans. Amer. Math. Soc. **282** (1984), 463–485.

3. S. Abeasis and A. Del Fra, *Degenerations for the representations of an equioriented quiver of type A_m*, Boll. Un. Mat. Ital. Suppl. Algebra e Geometria 2 (1980), 158–171.

4. _____, *Degenerations for the representations of a quiver of type A_m*, J. Algebra (to appear).

5. S. Abeasis, A. Del Fra and H. Kraft, *The geometry of representations of A_m*, Math. Ann. **256** (1981), 401–418.

6. I. N. Bernstein, I. M. Gelfand and V. A. Ponomarev, *Coxeter functors and Gabriel's theorem*, Uspekhi Mat. Nauk **28** (1973), 19–33; English transl., Russian Math. Surveys **28** (1973), 17–32.

7. V. Dlab and G. M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. No. 173 (1976).

8. P. Gabriel, *Représentations indécomposables*, Sém. Bourbaki, 26e année (1973/1974), Exp. 444, Lecture Notes in Math., Vol. 431, Springer, Berlin, 1975, pp. 143–169.

9. D. Happel, *Relative invariants and subgeneric orbits of quivers of finite and tame type*, Lecture Notes in Math., vol. 903, Springer, Berlin and New York, 1981, pp. 116–124.

10. _____ , *Relative invariants and subgeneric orbits of quivers of finite and tame type*, J. Algebra **78** (1982), 445–459.

11. V. Kac, *Infinite root systems, representations of graphs and invariant theory*, Invent. Math. **56** (1980), 57–92.

12. _____ , *Infinite root systems, representations of graphs and invariant theory. II*, J. Algebra **77** (1982), 141–162.

13. M. Sato and T. Kimura, *A classification of irreducible pre-homogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–555.

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