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STOCHASTIC REPRESENTATION AND SINGULARITIES OF SOLUTIONS OF SECOND ORDER EQUATIONS WITH SEMIDEFINITE CHARACTERISTIC FORM

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ABSTRACT. In the theory of partial differential equations, there is no explicit representation of solutions for *general* degenerate elliptic-parabolic equations. However, Stroock and Varadhan [**15**] have obtained a stochastic representation for such a wider class of equations in L^∞ space. In this paper we establish, by using Stroock and Varadhan's stochastic representation, a method which enables us to construct solutions with singularities of second order equations with semidefinite characteristic form. Our theorems are not probabilistic paraphrases of the results obtained in the theory of partial differential equations. In fact, each assumption of the theorems is much weaker than any assumption of corresponding known results.

Introduction. As is well known, it is one of many interesting problems to characterize the (analytic) singularities of solutions of second order equations with semidefinite characteristic form, and this problem was investigated by many authors. However, most of them gave only sufficient conditions for (analytic-) hypoellipticity. In this paper we shall give sufficient conditions under which a certain class of second order operators with semidefinite characteristic form are not (analytic-) hypoelliptic. The proof depends on Stroock-Varadhan's stochastic representation of solutions of degenerate elliptic-parabolic equations; we solve the Dirichlet problem with singular boundary condition by using probabilistic methods and obtain a solution with (analytic) singularities.

Let G be an open set in $\mathbf{R}^d$ and let us consider a differential operator

$$A = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

with real coefficients belonging to $C^\infty(G)$. Without loss of generality we may assume $a_{ij} = a_{ji}$. Let $X_0, X_1, \ldots, X_d$ be vector fields defined by

$$X_0 = \sum_{i=1}^{d} \left(b_i - \sum_{j=1}^{d} \frac{\partial a_{ij}}{\partial x_j} \right) \frac{\partial}{\partial x_i},$$

$$X_i = \sum_{j=1}^{d} a_{ij} \frac{\partial}{\partial x_j} \qquad (i = 1, 2, \ldots, d).$$

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THEOREM 1. *Assume that*

(1) $$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq 0 \quad \textit{for all } (x,\xi) \in G \times \mathbf{R}^d$$

and assume that there is a submanifold M of G such that

(2) $$\sum_{i,j=1}^{d} |a_{ij}| + \sum_{i=1}^{d} |b_i| \neq 0 \quad \textit{on } M$$

and

(3) $$X_i(x) \in T_x M \quad \textit{for all } x \in M \textit{ and } i = 0, 1, \ldots, d.$$

Then for any point p on M there exist an open neighborhood U of p in G and a function u of the class $L^1_{\mathrm{loc}}(U)$ such that $Au = 0$ in U and

(4) $$(p, \xi^0) \in WF(u) \quad \textit{for some } \xi^0 \in (T^*_p M)^\perp \backslash 0,$$

where $(T^*_{pM})^\perp = \{\xi \in T^*_p G\colon\ \langle \xi, \eta\rangle = 0$ for any $\eta \in T_p M\}$.

We cannot remove condition (2). In fact, the operators

$$|x|^{2k} \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i}\right)^2 - 1 \quad (k = 2, 3, \ldots), \qquad \exp\left(\frac{-1}{|x|}\right) \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i}\right)^2 - 1$$

are hypoelliptic in $\mathbf{R}^d$ (cf. [**8**, **2**]). For any point p in G there passes an integral manifold M through the point p which satisfies condition (3). In fact, let Δ be a distribution (in the sense of differential geometry) spanned by the vector fields $X_0, X_1, \ldots, X_d$, i.e., Δ is a mapping defined in G such that

$$\Delta(x) = \left\{ \sum_{i=0}^{d} \lambda_i X_i(x)\colon\ \lambda_i \in \mathbf{R}\ (i = 0, 1, \ldots, d) \right\}.$$

$\mathfrak{G}$ denotes the group of local C^∞-diffeomorphisms in G generated by $X_0, X_1, \ldots, X_d$. Let D be the smallest $\mathfrak{G}$-invariant distribution in G, i.e., $D(x)$ is the linear hull of all vectors $v \in \mathbf{R}^d$ such that $v \in \Delta(x)$ or $v \in d\phi(\Delta(y))$ for some $y \in G$ and some $\phi \in \mathfrak{G}$ satisfying $x = \phi(y)$. Then, according to Sussmann [**16**], through every point in G there passes a maximal integral manifold of D. Any maximal integral manifold of D satisfies condition (3). Consequently, we obtain the following fact: If the operator A is hypoelliptic in G, then

(5) $$\dim D(x) = d \quad \text{in } G.$$

It is to be noted that $L(x) \subset D(x)$ in G and further, if $X_0, X_1, \ldots, X_d$ are analytic vector fields, then $L(x) = D(x)$ in G, where

$$L(x) = \{X(x)\colon\ X \in \mathrm{Lie}(X_0, X_1, \ldots, X_d)\}.$$

However, (5) is not a sufficient condition for hypoellipticity. In fact, Kusuoka and Stroock [**11**] proved that the operator

$$\left(\frac{\partial}{\partial x}\right)^2 + \exp\left(\frac{-1}{|x|^k}\right)\left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 \qquad (k > 1)$$

is not hypoelliptic in $\mathbf{R}^3$.

In case the coefficients a_{ij}, b_i, c are real analytic, Derridj [**5**] and Oleĭnik and Radkevič [**12**] have proved that on the same assumption of Theorem 1 there exists a solution u of the equation $Au = 0$ with $\operatorname{sing\,supp} u \neq \emptyset$. Amano [**1**] has proved that their theorem remains true when the coefficients are real C^∞-smooth.

THEOREM 2. *Assume that*

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq 0 \quad \textit{for all } (x,\xi) \in G \times \mathbf{R}^d \tag{6}$$

and assume that there is a real valued function ψ belonging to the class $C^\infty(G)$ such that ψ attains 0 at some point in G, $\nabla\psi \neq 0$ in G,

$$\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial\psi}{\partial x_i}(x)\frac{\partial\psi}{\partial x_j}(x) \begin{cases} = 0 & \textit{if } \psi(x) = 0, \quad (7) \\ > 0 & \textit{if } \psi(x) \neq 0 \quad (8) \end{cases}$$

and

$$\langle X_0, \nabla\psi\rangle \neq 0 \quad \textit{on } M, \tag{9}$$

where $M = \{x \in G\colon\ \psi(x) = 0\}$. Then for any point p on M there exist an open neighborhood U of p in G and a function u of the class $L^\infty(U)$ such that $Au = 0$ in U and

$$(p, \xi^0) \in WF_A(u) \quad \textit{for some } \xi^0 \in (T_p^*M)^\perp \backslash 0. \tag{10}$$

Condition (7) means that M is a characteristic hypersurface of the operator A. Condition (8) is not removable. In fact, we have only to consider an operator $A = \partial/\partial x_d$ in $\mathbf{R}^d$ and a hyperplane $M = \{x \in \mathbf{R}^d\colon\ x_d = 0\}$. When (9) is not satisfied, the result of Theorem 2 is not always true. However, if we assume

$$\langle X_0, \nabla\psi\rangle = 0 \quad \text{and} \quad X_0 \neq 0 \quad \text{on } M$$

instead of (9), then, for any point p on M, Theorem 1 ensures the existence of a solution u of the equation $Au = 0$ satisfying (4). It is to be noted that we can apply Theorem 2 to degenerate parabolic operators.

THEOREM 3. *Assume $d \geq 2$ and assume that there is a real valued function ψ belonging to the class $C^\infty(G)$ such that ψ attains 0 at some point in G, $\nabla\psi \neq 0$ in G,*

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \begin{cases} \geq 0 & \textit{if } \psi(x) \geq 0 \textit{ and } \xi \in \mathbf{R}^d, \\ \leq 0 & \textit{if } \psi(x) \leq 0 \textit{ and } \xi \in \mathbf{R}^d, \end{cases} \tag{11}$$

and

$$X_0(x) \neq 0 \quad \textit{on } M, \tag{12}$$

where $M = \{x \in G\colon\ \psi(x) = 0\}$.

(i) *If $\langle X_0, \nabla\psi\rangle \equiv 0$ on M and if*

$$\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial\psi}{\partial x_i}(x)\frac{\partial\psi}{\partial x_j}(x)/\psi(x) \to 0 \quad \textit{as } \psi(x) \to 0, \tag{13}$$

then for any point p on M there exist an open neighborhood U of p in G and a function u of the class $L^\infty(U)$ such that $Au = 0$ in U and

$$(p, \xi^0) \in WF(u) \quad \text{for some } \xi^0 \in (T_p^* M)^\perp \backslash 0. \tag{14}$$

(ii) *If $\langle X_0, \nabla\psi \rangle < 0$ on M, then for any point p on M there exist an open neighborhood U of p in G and a function u of the class $L^\infty(U)$ such that $Au = 0$ in U and*

$$p \in \text{sing supp } u. \tag{15}$$

Zuily [**17**] has proved that if $a_{ij}(x)$ are real analytic and if the matrix $(a_{ij}(x))_{d\times d}$ is either positive or negative semidefinite at each point x in G, then for any point p in G there exist an open neighborhood V of p in G and a real analytic function $\psi(x)$ in V such that (11) is valid in V. If $\langle X_0, \nabla\psi\rangle > 0$ on M, then the operator A is hypoelliptic on a moderate assumption (cf. [**17**, **3**]). It is to be noted that Theorem 3 shows that the operators

$$x_d^k \sum_{i=1}^d \left(\frac{\partial}{\partial x_i}\right)^2 - \frac{\partial}{\partial x_d} \qquad (k \text{ is odd}, d \geq 2)$$

are not hypoelliptic in $\mathbf{R}^d$ and on the other hand, Zuily's theorem [**17**] shows that the operators

$$x_d^k \sum_{i=1}^d \left(\frac{\partial}{\partial x_i}\right)^2 + \frac{\partial}{\partial x_d} \qquad (k \text{ is odd}, d \geq 2)$$

are hypoelliptic in $\mathbf{R}^d$. Thus the sign of $\langle X_0, \nabla\psi\rangle$ has a significant influence upon the hypoellipticity in case the characteristic form changes sign.

When $a_{ij}(x)/\psi(x) \in C^\infty(G)$, $\text{rank}(a_{ij}(x)/\psi(x))_{d\times d} > 0$ in G and

$$\sum_{i,j=1}^d a_{ij}(x)\frac{\partial\psi}{\partial x_i}(x)\frac{\partial\psi}{\partial x_j}(x)/\psi(x) \equiv 0 \quad \text{in } G,$$

Beals and Fefferman [**3**] have proved that on the same assumption of Theorem 3 there exists a solution u of the equation $Au = f$ such that sing supp $f \subsetneq$ sing supp u. Helffer and Zuily [**7**] have proved that the operators of Fuchs type are not hypoelliptic. Kannai [**10**] has given a virtually complete characterization of hypoelliptic ordinary differential operators.

THEOREM 4. *Assume that there is a real-valued function ψ belonging to the class $C^\infty(G)$ such that ψ attains 0 at some point in G, $\nabla\psi \neq 0$ in G and*

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \left(\begin{array}{ll} \geq 0 & \text{if } \psi(x) \geq 0 \text{ and } \xi \in \mathbf{R}^d, \\ \leq 0 & \text{if } \psi(x) \leq 0 \text{ and } \xi \in \mathbf{R}^d, \end{array} \right. \tag{16}$$

and assume that

$$\sum_{i,j=1}^d a_{ij}(x)\frac{\partial\psi}{\partial x_i}(x)\frac{\partial\psi}{\partial x_j}(x) \neq 0 \quad \text{in } G\backslash M, \tag{17}$$

where $M = \{x \in G\colon \psi(x) = 0\}$. *If* $\langle X_0, \nabla\psi\rangle < 0$ *on* M, *then for any point* p *on* M *there exist an open neighborhood* U *of* p *in* G *and a function* u *of the class* $L^\infty(U)$ *such that* $Au = 0$ *in* U *and*

$$(p, \xi^0) \in WF_A(u) \quad \text{for some } \xi^0 \in (T_p^* M)^\perp \backslash 0. \tag{18}$$

1. Preliminaries. In this section we assume that the coefficients a_{ij}, b_i, c are real valued functions such that $a_{ij} \in C^2_{\mathrm{bdd}}(\mathbf{R}^d)$, $b_i \in C^1_{\mathrm{bdd}}(\mathbf{R}^d)$ and $c \in C_{\mathrm{bdd}}(\mathbf{R}^d)$, i.e., $\partial^\alpha a_{ij}$, $\partial^\beta b_i$ and c are bounded continuous functions in $\mathbf{R}^d$ for $|\alpha| \le 2$, $|\beta| \le 1$, and assume that

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \ge 0 \quad \text{for all } (x,\xi) \in \mathbf{R}^d \times \mathbf{R}^d.$$

For brevity we set

$$a(x) = \left(a_{ij}(x)\colon {}^{i\downarrow 1\ldots d}_{j\to 1\ldots d}\right), \qquad b(x) = (b_1(x), \ldots, b_d(x))$$

and

$$\sigma(x) = \left(\sigma_{ij}(x)\colon {}^{i\downarrow 1\ldots d}_{j\to 1\ldots d}\right) = (2a(x))^{1/2}.$$

LEMMA 1.1 [**13**]. $\sigma_{ij}(x)$ *are Lipschitz continuous in* $\mathbf{R}^d$.

Let Ω be the space of all $\mathbf{R}^d$-valued continuous functions defined on $[0,\infty)$. $x(t) = x(t,\omega) = \omega(t)$ denotes the value of a function $\omega \in \Omega$ at t. $\mathfrak{F}_t$ is the σ-field generated by the functions $x(s,\cdot)$ for $0 \le s \le t$. $\mathfrak{F}$ is the smallest σ-field containing $\mathfrak{F}_t$ for all $t \ge 0$.

The following lemma is well known.

LEMMA 1.2. *There exists a probability measure* P *defined on* $(\Omega, \mathfrak{F})$ *such that* $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P)$, $t \ge 0$, *is a Brownian motion.*

For convenience we often rewrite the function $x(t) = x(t,\omega)$ as $w(t) = w(t,\omega)$ when $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P)$, $t \ge 0$, is a Brownian motion.

The following two lemmas are also well known.

LEMMA 1.3. *Let* $(\Omega, \mathfrak{F}, \mathfrak{F}_t, w(t), P)$, $t \ge 0$, *be a Brownian motion. Then for any* $x \in \mathbf{R}^d$ *and any* $T > 0$ *there exists a unique solution, say* $\xi_x(t)$, *of the initial value problem*

$$d\xi = \sigma(\xi)dw(t) + b(\xi)dt, \tag{1.1}$$

$$\xi(0) = x \quad P\text{-a.s.} \tag{1.2}$$

in $M^2_w[0,T]$. *Here* $M^2_w[0,T]$ *is the set of all nonanticipative functions* $f(t)$ *with respect to* $(\Omega, \mathfrak{F}_t)$ *satisfying* $E\left[\int_0^T |f(t)|^2\,dt\right] < +\infty$.

LEMMA 1.4. *By*

$$P_x(F) = P[\{\omega \in \Omega\colon\ \xi_x(\cdot,\omega) \in F\}] \qquad (F \in \mathfrak{F})$$

we define a probability measure P_x *on* $(\Omega, \mathfrak{F})$. *Then* $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P_x)$, $t \ge 0, x \in \mathbf{R}^d$, *is a diffusion process generated by the differential operator* $A - c(x)$.

The proofs of our theorems essentially depend on Itô's formula (Lemma 1.5) and Stroock and Varadhan's stochastic representation (Lemma 1.6).

LEMMA 1.5 (ITÔ'S FORMULA). *If $\xi(t)$ is a solution of the stochastic differential equation* (1.1), *then*

$$d\left[v(\xi(t))\exp\left\{\int_0^t c(\xi(s))\,ds\right\}\right] = \nabla v(\xi(t))\exp\left\{\int_0^t c(\xi(s))\,ds\right\}\sigma(\xi(t))\,dw(t) + Av(\xi(t))\exp\left\{\int_0^t c(\xi(s))\,ds\right\}dt$$

P-a.s. for any $v \in C^2(\mathbf{R}^d)$.

Let U be an open set in $\mathbf{R}^d$ with C^2 boundary ∂U. $\tau \leftrightarrows$ denotes the exit time of $x(t)$ from $\overline{U}$. Γ and Σ are the subsets of ∂U such that

$$\Gamma = \{x \in \partial U\colon\ P_x(\tau > 0) = 0\}$$

and

$$\Sigma = \{x \in \partial U\colon\ \langle a(x)\nu(x), \nu(x)\rangle > 0 \text{ or } \langle X_0(x), \nu(x)\rangle < 0\},$$

where $\nu(x)$ is the inward normal vector at x to ∂U. The sets Γ and Σ are not essentially different; in fact $\Sigma \subset \Gamma \subset \overline{\Sigma}$ (cf. [**15**]). C is a constant defined by

$$C = \sup_{x\in U} c(x) \vee 0.$$

Then we have the following

LEMMA 1.6 [**15**]. *Assume that*

$$\sup_{x\in U} E_x[(1+\tau)e^{C\tau}] < +\infty. \tag{1.3}$$

Then for given $f \in L^\infty(U)$ and $g \in L^\infty(\Gamma) \cap C(\Sigma)$, the function

$$u(x) = E_x\left[g(x(\tau))\exp\left\{\int_0^\tau c(x(s))\,ds\right\} - \int_0^\tau f(x(t))\exp\left\{\int_0^t c(x(s))\,ds\right\}dt\right]$$

is a unique solution of the Dirichlet problem

$$Au = f \quad \text{in } U, \tag{1.4}$$

$$\lim_{\substack{x\to a\\ x\in U}} u(x) = g(a) \qquad (a \in \Sigma) \tag{1.5}$$

in $L^\infty(U)$. Here (1.4) *means*

$$\int uA^*\phi\,dx = \int f\phi\,dx \qquad (\phi \in C_0^\infty(\mathbf{R}^d)).$$

In case $f = 0$, we can replace the assumption (1.3) by

$$\sup_{x\in U} E_x[e^{C\tau}] < +\infty.$$

(1.3) is fulfilled, if the diameter of U is sufficiently small. In fact, we have the following lemma.

LEMMA 1.7 [**1**]. *Let U_ρ be an open neighborhood of a fixed point p in $\mathbf{R}^d$, with diameter $U_\rho = \rho$, and let τ_ρ be the exit time of $x(t)$ from $\overline{U}$. If*

$$\sum_{i,j=1}^{d} |a_{ij}(p)| + \sum_{i=1}^{d} |b_i(p)| \neq 0$$

then

$$\overline{\lim_{\rho \downarrow 0}} \sup_{x \in U} E_x[e^{C\tau_\rho}] < +\infty$$

and

$$\lim_{\rho \downarrow 0} \sup_{x \in U} E_x[(\tau_\rho e^{c\tau_\rho})^k] = 0$$

for any constant C and any $k = 1, 2, \ldots$.

In the proofs of our theorems we use fundamental properties of the probability measures P_x (Lemmas 1.8 and 1.9).

Let us define $S(x)$ to be the set of $\omega \in \Omega$ such that

$$\omega(t) = x + \int_0^t X_0(\omega(s))\, ds + \sum_{i=1}^{d} \int_0^t \psi_i(s) X_i(\omega(s))\, ds \tag{1.6}$$

for some bounded measurable functions $\psi_i\colon [0, \infty) \to \mathbf{R}$ $(i = 1, \ldots, d)$. $\overline{S}(x)$ denotes the closure of $S(x)$ with respect to the topology of the space Ω.

LEMMA 1.8 [**15**]. $\operatorname{supp}(P_x) = \overline{S}(x)$.

LEMMA 1.9 [**6**]. *Let V be an open set in $\mathbf{R}^d$ and let K be a compact set of $\overline{V}$. Assume that there exists a function $w \in C^2(V \backslash K)$ and a constant $\gamma \geq 0$ such that*

$$(A - c)w \leq \gamma w \quad \textit{in } V \backslash K \tag{1.7}$$

and

$$w(x) \to +\infty \quad \textit{if } x \in V \backslash K, d(x, K) \to 0. \tag{1.8}$$

Then

$$P_x[x(t) \in K \ \textit{for some } 0 \leq t < \sigma] = 0 \tag{1.9}$$

for any $x \in V \backslash K$, where σ is the exit time of $x(t)$ from V.

2. Proof of Theorem 1. Throughout this section, we assume (1), and assume that M is a submanifold of G satisfying (2) and (3). Let p be a point on M and let V be an open neighborhood of p in G. Let us take a nonnegative function $\chi \in C_0^\infty(V)$ and consider a diffusion process $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P_x)$ generated by the operator $\chi(A - c(x))$ (cf. Lemma 1.4). Then we have the following two lemmas.

LEMMA 2.1. *If the diameter of V is sufficiently small, then*

$$P_x[x(t) \in \overline{M} \ \textit{for some } t \geq 0] = 0 \tag{2.1}$$

for all $x \in \mathbf{R}^d \backslash \overline{M}$.

PROOF. Without loss of generality, we may assume that

$$\overline{M} \cap V = \{x \in V\colon\ x_{r+1} = \cdots = x_d = 0\} \qquad (0 \leq r < d).$$

By (1) and (3), we have

$$a_{ij}(x) = O(|x''|^2), \qquad b_i(x) = O(|x''|) \tag{2.2}$$

for $r+1 \le i,\ j \le d$ when $x \in V \backslash \overline{M}$ and $d(x, \overline{M}) \to 0$, where $x'' = (x_{r+1}, \dots, x_d)$. It is easy to show that $w(x) = 1/|x''|^{\varepsilon}$ ($\varepsilon > 0$) satisfies conditions (1.7) and (1.8) for $K = \overline{M} \cap \overline{V}$. Hence, by Lemma 1.9,

$$P_x[x(t) \in \overline{M} \text{ for some } 0 \le t < \sigma] = 0$$

for all $x \in V \backslash \overline{M}$, where σ is the exit time of $x(t)$ from V. On the other hand, by Lemma 1.8, we have $\sigma = +\infty$ P_x-a.s. if $x \in V$, and

$$P_x[x(t) = x \text{ for all } t \ge 0] = 1 \quad \text{if } x \notin V.$$

Therefore, we obtain (2.1).

LEMMA 2.2. *If the diameter of M is sufficiently small, then*

$$\begin{aligned} d&\left[v(x(t)) \exp\left\{\int_0^t \chi c(x(s))\,ds\right\}\right] \\ &= \nabla v(x(t)) \chi^{1/2} \sigma(x(t)) \exp\left\{\int_0^t \chi c(x(s))\,ds\right\} dw(t) \\ &\quad + \chi A v(x(t)) \exp\left\{\int_0^t \chi c(x(s))\,ds\right\} dt \end{aligned} \tag{2.3}$$

P_x-a.s. for any $v \in C^2(\mathbf{R}^d \backslash \overline{M})$ and any $x \in \mathbf{R}^d \backslash \overline{M}$.

PROOF. For a fixed $v \in C^2(\mathbf{R}^d \backslash \overline{M})$ let us take a function $v_\varepsilon \in C^2(\mathbf{R}^d)$, $\varepsilon > 0$, such that $v_\varepsilon(x) = v(x)$ when $d(x, \overline{M}) > \varepsilon$. For $T > 0$ and $\varepsilon > 0$ we set

$$\begin{aligned} F_{T,\varepsilon} &= \{\omega \in \Omega\colon\ d(x(t), \overline{M}) > \varepsilon \text{ for all } 0 \le t \le T\}, \\ F_T &= \{\omega \in \Omega\colon\ (2.3) \text{ is valid for all } 0 \le t \le T\}. \end{aligned}$$

By Lemma 1.5, we have

$$\begin{aligned} d&\left[v_\varepsilon(x(t)) \exp\left\{\int_0^t \chi c(x(s))\,ds\right\}\right] \\ &= \nabla v_\varepsilon(x(t)) \chi^{1/2} \sigma(x(t)) \exp\left\{\int_0^t \chi c(x(s))\,ds\right\} dw(t) \\ &\quad + \chi A v_\varepsilon(x(t)) \exp\left\{\int_0^t \chi c(x(s))\,ds\right\} dt \end{aligned}$$

P_x-a.s. for any $x \in \mathbf{R}^d$. Hence, for any $x \in \mathbf{R}^d$ there is a set $N_\varepsilon^x \in \mathfrak{F}$ such that $P_x(N_\varepsilon^x) = 0$ and $F_{T,\varepsilon} \subset F_T \cup N_\varepsilon^x$. Since, by Lemma 2.1,

$$P_x\left(\bigcup_{n=1}^{\infty} F_{T,1/n}\right) = 1 \quad \text{for } x \in \mathbf{R}^d \backslash \overline{M},\ T > 0,$$

we have

$$P_x(F_T) = 1 \quad \text{for } x \in \mathbf{R}^d \backslash \overline{M},\ T > 0.$$

This completes the proof of Lemma 2.2.

LEMMA 2.3 [1]. *Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, w(x), P)$ be a Brownian motion and let τ be a stopping time such that $0 \le \tau < \infty$ P-a.s. Then*

$$E\left[\sup_{0\le t\le\tau}\left|\int_0^t f(s)\,dw(s)\right|^2\right] \le 4E\left[\int_0^\tau |f(s)|^2\,ds\right]$$

if $f(t) \in M_w^2[0,T]$ for any $T > 0$.

PROPOSITION 2.4. *Assume that $g \in C^2(G\backslash\overline{M}) \cap L^1_{\mathrm{loc}}(G)$ is a function such that*

$$|(A-c)g(x)| \le C_0, \tag{2.4}$$

$$|\nabla g(x)\sigma(x)| \le C_0 \tag{2.5}$$

in $G\backslash\overline{M}$ for some nonnegative constant C_0 and such that

$$g(x) \to +\infty \quad \text{if } x \in G\backslash\overline{M},\ d(x,\overline{M}) \to 0. \tag{2.6}$$

Then for any point p on M there is an open neighborhood U of p in G and a function u of the class $L^1_{\mathrm{loc}}(U)$ such that

$$Au = 0 \quad \text{in } U \tag{2.7}$$

and

$$C_1 g(x) \le u(x) \le C_2 g(x) \quad \text{in } U\backslash M \tag{2.8}$$

for some positive constants C_1 and C_2.

PROOF. Let U and V be open neighborhoods of a point $p \in M$ in G such that $\overline{U} \subset V \subset G$, ∂U is C^2-smooth and diameter V is sufficiently small. By $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P_x)$ we denote a diffusion process generated by the operator $\chi(A - c(x))$, where $\chi \in C_0^\infty(V)$ is a nonnegative function satisfying $\chi = 1$ in U. We define a function $u(x)$ in $U\backslash M$ by

$$\begin{aligned} u(x) &= E_x\left[\chi g(x(\tau))\exp\left\{\int_0^\tau \chi c(x(s))\,ds\right\}\right] \\ &= E_x\left[g(x(\tau))\exp\left\{\int_0^\tau c(x(s))\,ds\right\}\right], \end{aligned}$$

where τ is the exit time of $x(t)$ from $\overline{U}$.

By Lemma 2.2 and (2.5), we have

$$u(x) = g(x) + E_x\left[\int_0^\tau Ag(x(t))\exp\left\{\int_0^t c(x(s))\,ds\right\}\,dt\right]$$

and, by (2.4),

$$\begin{aligned} &\left|E_x\left[\int_0^\tau Ag(x(t))\exp\left\{\int_0^t c(x(s))\,ds\right\}\,dt\right]\right| \\ &\qquad \le C_0 E_x[\tau e^{C\tau}] + CE_x\left[e^{C\tau}\int_0^\tau g(x(t))\,dt\right], \end{aligned}$$

where $C = \sup_{x \in U} c(x) \vee 0$. Lemmas 2.2 and 2.3 give

$$\begin{aligned} E_x\left[e^{C\tau}\int_0^\tau g(x(t))\,dt\right] &= E_x\left[e^{C\tau}\int_0^\tau \left\{g(x) + \int_0^t \nabla g(x(s))\sigma(x(s))\,dw(s)\right.\right. \\ &\qquad\qquad \left.\left. + \int_0^t (A-c)g(x(s))\,ds\right\}dt\right] \\ &\le E_x[\tau e^{C\tau}]g(x) + (E_x[(\tau e^{C\tau})^2])^{1/2} \\ &\quad \times \left(E_x\left[\sup_{0\le t\le\tau}\left|\int_0^t \nabla g(x(s))\sigma(x(s))\,dw(s)\right|^2\right]\right)^{1/2} \\ &\quad + C_0 E_x[\tfrac{1}{2}\tau^2 e^{C\tau}] \\ &\le E_x[\tau e^{C\tau}]g(x) + C_0E_x[\tau e^{C\tau}] + 2C_0E_x[(\tau e^{C\tau})^2]. \end{aligned}$$

Consequently, we obtain

$$|u(x) - g(x)| \le CE_x[\tau e^{C\tau}]g(x) + C_0(1+C)E_x[\tau e^{C\tau}] + 2CC_0E_x[(\tau e^{C\tau})^2].$$

This implies, by Lemma 1.7, $u \in L^1_{\mathrm{loc}}(U)$ and (2.8).

Let us define functions $u_n(x)$, $n \in \mathbf{N}$, in U by

$$u_n(x) = E\left[g_n(x(\tau))\exp\left\{\int_0^\tau c(x(s))\,ds\right\}\right],$$

where $g_n(x) = g(x) \wedge n$. Lemmas 1.6 and 1.7 give

$$\int u_n A^*\phi\,dx = 0 \qquad (\phi \in C_0^\infty(U)).$$

By letting $n \to \infty$, we obtain (2.7).

LEMMA 2.5. *Assume that $u \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ is a function such that*

$$\text{(2.9)} \qquad -C_1 \log|x''| \le u(x', x'') \le -C_2\log|x''| \quad \text{in } \mathbf{R}^d_x$$

for some positive constants C_1 and C_2, where $x' = (x_1, \ldots, x_r)$, $x'' = (x_{r+1}, \ldots, x_d)$ and $0 < r < d$. Then

$$(0, \xi^0) \in WF(u) \quad \text{for some } \xi^0 \in \{\xi \in \mathbf{R}^d\colon\ |\xi'| = 0\},$$

where $\xi' = (\xi_1, \ldots, \xi_r)$ and $\xi'' = (\xi_{r+1}, \ldots, \xi_d)$.

PROOF. We assume that $(0, \xi) \notin WF(u)$ when $|\xi'| = 0$. Then there is a nonnegative function $\chi \in C_0^\infty(\mathbf{R}^d)$ such that $\chi = 1$ in a neighborhood of 0 in $\mathbf{R}^d$ and

$$\int_{\mathbf{R}^d_x} e^{-ix\cdot\xi}\chi(x)u(x)\,dx \in \mathcal{S}(\mathbf{R}^{d-r}_{\xi''})$$

for any fixed ξ'. Hence

$$\int_{\mathbf{R}^r_{x'}} \chi(x', x'')u(x', x'')\,dx' \in \mathcal{S}(\mathbf{R}^{d-r}_{x''}).$$

On the other hand we have, by (2.9),

$$\int_{\mathbf{R}^r_{x'}} \chi(x', x'')u(x', x'')\,dx' \to +\infty \quad \text{as } |x''| \to 0.$$

This is a contradiction.

PROOF OF THEOREM 1. Without loss of generality, we may assume that $r = \dim M < d$ and $\overline{M} \cap V = \{x \in V\colon x_{r+1} = \cdots = x_d = 0\}$ for some open neighborhood V of p in G. By (1) and (3) we have

$$a_{ij}(x) = O(|x''|^2), \qquad \sigma_{ij}(x) = O(|x''|), \qquad b_i(x) = O(|x''|) \tag{2.10}$$

for $r+1 \le i,\ j \le d$ when $x \in V \backslash \overline{M}$ and $d(x, \overline{M}) \to 0$, where $x'' = (x_{r+1}, \ldots, x_d)$. Let us consider a function $g(x) = -\log|x''|$ defined in $V\backslash\overline{M}$. It is easy to show that $g(x)$ satisfies (2.4)–(2.6) for $G = V$. Hence, by Proposition 2.4, there is an open neighborhood U of p in V and a function $u \in L^1_{\mathrm{loc}}(U)$ such that (2.7) and (2.8) are valid. By Lemma 2.5, (2.8) implies that $(p, \xi^0) \in WF(u)$ for some $\xi^0 \in (T_p^*M)^\perp$. The proof of Theorem 1 is complete.

3. Proof of Theorem 2.

LEMMA 3.1 [**14**, **4**]. *Assume $u \in \mathcal{D}'(\mathbf{R}^d)$ and $0 \in \operatorname{supp} u \subset \{x \in \mathbf{R}^d\colon x_d \ge 0\}$. Then*

$$(0, \xi^0) \in WF_a(u) \quad \textit{for some } \xi^0 \in \{\xi \in \mathbf{R}^d\colon |\xi'| = 0\},$$

where $\xi' = (\xi_1, \ldots, \xi_{d-1})$.

PROOF OF THEOREM 2. Let U and V be open neighborhoods of a point $p \in M$ in G such that $\overline{U} \subset V \subset G$, ∂U is C^2-smooth and diameter V is sufficiently small. $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P_x)$ denotes a diffusion process generated by the operator $\chi(A - c(x))$, where $\chi \in C_0^\infty(V)$ is a nonnegative function satisfying $\chi = 1$ in U. Without loss of generality, we may assume $\overline{M} \cap V = \{x \in \mathbf{R}^d\colon x_d = 0\}$ and, by (9), $\langle X_0(x), e_d\rangle < 0$ on $\overline{M} \cap V$, where $e_d = (0, \ldots, 0, 1)$. Let us take a function

$$g(x) = \begin{cases} e^{-1/|x_d|} & \text{if } x_d > 0, \\ 0 & \text{if } x_d \le 0 \end{cases}$$

defined in V. By Lemmas 1.6 and 1.7,

$$u(x) = E_x\left[g(x(\tau)) \exp\left\{\int_0^\tau c(x(s))\,ds\right\}\right] \in L^\infty(U)$$

is a solution of the Dirichlet problem

$$Au = 0 \quad \text{in } U, \qquad \lim_{\substack{x \to a \\ x \in U}} u(x) = g(a) \quad (a \in \Sigma),$$

where τ is the exit time of $x(t)$ from $\overline{U}$. Furthermore, by (7)–(9) and Lemma 1.8, we have

$$u(x) \begin{cases} > 0 & \text{if } x_d > 0, \\ = 0 & \text{if } x_d \le 0. \end{cases}$$

Combining this fact with Lemma 3.1, we obtain the desired result.

4. Proof of Theorem 3. Throughout this section, we assume that $\psi \in C^\infty(G)$ is a real valued function such that $\psi = 0$ at some point in G, $\nabla\psi \ne 0$ in G and (11) is valid, and furthermore, we assume (12). Without loss of generality, we may assume that $\psi(x) = x_d$. We set $G_+\{x \in G\colon x_d > 0\}$, $G_- = \{x \in G\colon x_d < 0\}$ and $M = \{x \in G\colon x_d = 0\}$.

LEMMA 4.1. *Given $u \in L^\infty(G)$ and $f \in L^\infty(G)$, if*

$$|a_{dd}(x)u(x)| = o(|x_d|), \tag{4.1}$$

$$\left|\left(b_d(x) - \frac{\partial a_{dd}}{\partial x_d}(x)\right) u(x)\right| = o(1) \tag{4.2}$$

uniformly in $x' = (x_1, \ldots, x_{d-1})$ as $x_d \to 0$ and if

$$Au = f \quad \text{in } G_+ \cup G_-, \tag{4.3}$$

then $Au = f$ in G.

PROOF. Let us take a function $\chi \in C_0^\infty(\mathbf{R})$ such that $0 \le \chi \le 1$, $\chi(t) = 1$ when $|t| \le 1/2$ and $\chi(t) = 0$ when $|t| \ge 1$. For $\varepsilon > 0$ we define a function $\chi_\varepsilon \in C^\infty(\mathbf{R}^d)$ by $\chi_\varepsilon(x) = \chi(x_d/\varepsilon)$. By (4.3), we easily have

$$\int u(x) A^* \phi \, dx = \int f(1 - \chi_\varepsilon)\phi \, dx + \int u A^* \chi_\varepsilon \phi \, dx \tag{4.4}$$

for any $\varepsilon > 0$ and any $\phi \in C_0^\infty(G)$. Direct computation gives

$$\begin{aligned} \int u A^* \chi_\varepsilon \phi \, dx = {} & \int_{|x_d| \le \varepsilon} u a_{dd} \frac{\partial^2 \chi_\varepsilon}{\partial x_d^2} \phi \, dx + 2 \sum_{i=1}^d \int_{|x_d| \le \varepsilon} u a_{id} \frac{\partial \chi_\varepsilon}{\partial x_d} \frac{\partial \phi}{\partial x_i} \, dx \\ & - \int_{|x_d| \le \varepsilon} u \left(b_d - \sum_{j=1}^d \frac{\partial a_{dj}}{\partial x_j} \right) \frac{\partial \chi_\varepsilon}{\partial x_d} \phi \, dx + \int_{|x_d| \le \varepsilon} u \chi_\varepsilon A^* \phi \, dx. \end{aligned}$$

Since

$$\frac{\partial \chi_\varepsilon}{\partial x_d}(x) = \frac{1}{\varepsilon} \chi'(x_d/\varepsilon) \quad \text{and} \quad \frac{\partial^2 \chi_\varepsilon}{\partial x_d^2}(x) = \frac{1}{\varepsilon^2} \chi''(x_d/\varepsilon),$$

we obtain, by (4.1), (4.2) and (11),

$$\lim_{\varepsilon \to 0} \int u A^* \chi_\varepsilon \phi \, dx = 0.$$

Combining this fact with (4.4), we have $\int u A^* \phi \, dx = \int f \phi \, dx$ for any $\phi \in C_0^\infty(G)$.

Modifying the proof of Lemma 2.5, we easily have the following

LEMMA 4.2. *Assume that $u \in L^\infty(\mathbf{R}^d)$ is a function such that*

$$u(x', x_d) \left(\begin{array}{ll} \ge C_1 & \text{if } x_d > 0, \\ \le C_2 & \text{if } x_d < 0 \end{array} \right. \tag{4.5}$$

for some constants $C_1 > C_2$, where $x' = (x_1, \ldots, x_{d-1})$. Then

$$(0, \xi^0) \in WF(u) \quad \text{for some } \xi^0 \in \{\xi \in \mathbf{R}^d \colon\ |\xi'| = 0\},$$

where $\xi' = (\xi_1, \ldots, \xi_{d-1})$.

PROPOSITION 4.3. *Assume that*

$$\langle X_0, e_d \rangle = 0 \quad \text{on } M, \tag{4.6}$$

where $e_d = (0, \ldots, 0, 1) \in \mathbf{R}^d$, and assume that

$$a_{dd}(x)/x_d \to 0 \quad \text{as } x_d \to 0 \tag{4.7}$$

uniformly in $x' = (x_1, \ldots, x_{d-1})$. *Then for any point* $p \in M$ *there is an open neighborhood* U *of* p *in* G *and a function* $u \in L^\infty(U)$ *such that* $Au = 0$ *in* U *and*

$$u(x) \begin{pmatrix} \geq 1 & \text{in } U \cap G_+, \\ = 0 & \text{in } U \cap G_-. \end{pmatrix} \tag{4.8}$$

PROOF. Let U and V be open balls with center $p \in M$ such that $\overline{U} \subset V \subset G$ and diameter V is sufficiently small. By (11), (12), (4.6) and (4.7), we may assume that

$$b_1(x) \neq 0 \quad \text{in } V. \tag{4.9}$$

(11) and (4.9) show that for any constant C there are constants γ_1 and γ_2 such that

$$a_{11}(x)\gamma_1^2 + b_1(x)\gamma_1 + C \geq 0 \quad \text{in } V \tag{4.10}$$

and

$$a_{11}(x)\gamma_2^2 + b_1(x)\gamma_2 + C \leq 0 \quad \text{in } V, \tag{4.11}$$

if diameter V is sufficiently small. Here we note that $Au = 0$ means

$$Av + 2\gamma_i \sum_{j=1}^{d} a_{1j}(x) \frac{\partial v}{\partial x_j} + (a_{11}(x)\gamma_i^2 + b_1(x)\gamma_i)v = 0 \tag{4.12}$$

when $u = e^{\gamma_i x_1} v$.

For $n \in \mathbf{N}$ we set $U_n = U \cap \{x \in G\colon x_d > 1/n\}$ and $V_n = V \cap \{x \in G\colon x_d > 1/2n\}$. $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P_x^n)$ denotes a diffusion process generated by the operator $\chi_n(A - c(x))$ (cf. Lemma 1.4), where $\chi_n \in C_0^\infty(V_n)$ is a nonnegative function satisfying $\chi_n = 1$ in U_n. τ_n is the exit time of $x(t)$ from $\overline{U}_n$. Applying Lemma 1.5 to functions $w_i(x) = e^{\gamma_i x_1}$ $(i = 1, 2)$, we have, by (4.10)–(4.12),

$$E_x^n[w_1(x(\tau_n))e^{C\tau_n}] \geq w_1(x), \qquad E_x^n[w_2(x(\tau_n))e^{C\tau_n}] \leq w_2(x)$$

in U_n for any $n \in \mathbf{N}$. Hence, there are positive constants C_1 and C_2 independent of n such that

$$C_1 \leq E_x^n[e^{C\tau_n}] \leq C_2 \quad \text{in } U_n. \tag{4.13}$$

Lemma 1.6 and (4.13) show that, for each $n \in \mathbf{N}$,

$$u_n(x) = \begin{pmatrix} E_x\left[\exp\{\int_0^{\tau_n} c(x(s))\,ds\}\right] & \text{if } x \in U_n, \\ 0 & \text{if } x \in U \backslash U_n \end{pmatrix} \tag{4.14}$$

is a solution of the equation $Au = 0$ in $U_n \cup (U \cap G_-)$ such that

$$C_1' \leq u_n(x) \leq C_2' \quad \text{in } U_n \tag{4.15}$$

for some positive constants C_1' and C_2' independent of n. Since, by (4.14) and (4.15), $\{u_n\colon n \in \mathbf{N}\}$ is a bounded subset of the Hilbert space $L^2(U)$, there is a subsequence $\{u_{n_i}\}_{i=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and a function $u \in L^2(U)$ such that

$$\lim_{i \to \infty} u_{n_i} = u \quad \text{weakly in } L^2(U). \tag{4.16}$$

$Au = 0$ in $U_n \cup (U \cap G_-)$ means

$$\int u_{n_i} A^* \phi \, dx = 0 \quad \text{for } \phi \in C_0^\infty(U_{n_i}) \cup C_0^\infty(U \cap G_-). \tag{4.17}$$

Combining (4.17) with (4.16), we have $Au = 0$ in $U \cap (G_+ \cup G_-)$. Hence, by (11), (4.6), (4.7) and Lemma 4.1, $Au = 0$ in U. (4.15) implies that

$$\int C_1' \phi \, dx \leq \int u_{n_i} \phi \, dx \leq \int C_2' \phi \, dx \tag{4.18}$$

for any nonnegative function $\phi \in L^2(U_{n_i})$. Letting $i \to \infty$ in (4.18), we obtain

$$C_1' \leq u(x) \leq C_2' \quad \text{a.e. in } U \cap G_+. \tag{4.19}$$

(4.14) and (4.16) easily give

$$u(x) = 0 \quad \text{a.e. in } U \cap G_-. \tag{4.20}$$

Therefore, $u(x)/C_1'$ is the desired solution.

PROPOSITION 4.4. *Assume that*

$$\langle X_0, e_d \rangle < 0 \quad \text{on } M, \tag{4.21}$$

where $e_d = (0, \ldots, 0, 1) \in \mathbf{R}^d$. *Then for any function* $g \in C^2(M)$ *and any point* $p \in M$ *there is an open neighborhood* U *of* p *in* G *and a function* $u \in L^\infty(U)$ *such that* $Au = 0$ *in* U *and*

$$\lim_{\substack{x \to a \\ x \in U \setminus M}} u(x) = g(a) \quad \text{for } a \in M \cap U. \tag{4.22}$$

PROOF. Let U and V be open balls with center $p \in M$ such that $\overline{U} \subset V \subset G$ and diameter V is sufficiently small. Let us take an auxiliary function $w(x) = x_d^\alpha$, where α is a sufficiently small positive constant. By (11) and (4.21),

$$(A - c(x))w = \left\{ \alpha \left(b_d - \frac{a_{dd}}{x_d} \right) + \alpha^2 \frac{a_{dd}}{x_d} \right\} x_d^{\alpha - 1} \to -\infty$$

as $x_d \downarrow 0$. Hence, we may assume that

$$Aw \leq -1 \quad \text{in } V \tag{4.23}$$

and

$$(A - c(x) + C)(w + 1) \leq 0 \quad \text{in } V, \tag{4.24}$$

where $C = \sup_{x \in V} c(x) \vee 0$.

For $n \in \mathbf{N}$ we set $U_n = U \cap \{x \in G: x_d > 1/n\}$ and $V_n = V \cap \{x \in G: x_d > 1/2n\}$. $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P_x^n)$ denotes a diffusion process generated by the operator $\chi_n(A - c(x))$, where $\chi_n \in C_0^\infty(V_n)$ is a nonnegative function satisfying $\chi_n = 1$ in U_n. τ_n is the exit time of $x(t)$ from $\overline{U}_n$. Let $h(x)$ be a C^2 function defined in V such that $h(x) = g(x)$ on $M \cap U$. Lemma 1.5 and (4.24) give

$$E_x^n[(w(x(\tau_n)) + 1)e^{C\tau_n}] \leq w(x) + 1 \quad \text{in } U_n;$$

this implies

$$E_x^n[e^{C\tau_n}] \leq w(x) + 1 \quad \text{in } U_n. \tag{4.25}$$

By (4.25) and Lemma 1.6, for each $n \in \mathbf{N}$,

$$u_n(x) = \begin{cases} E_x^n \left[h(x(\tau_n)) \exp\left\{\int_0^{\tau_n} c(x(s))\,ds\right\}\right] & \text{in } U_n, \\ 0 \quad \text{in } (U \cap G_+)\backslash U_n \end{cases}$$

is a solution such that

$$Au_n = 0 \quad \text{in } U_n \tag{4.26}$$

and

$$|u_n| \leq \sup_{x \in U \cap G_+} |h(x)|(w+1). \tag{4.27}$$

Since, by (4.27), $\{u_n\colon n \in \mathbf{N}\}$ is a bounded subset of the Hilbert space $L^2(U \cap G_+)$, there is a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ and a function $u_+ \in L^2(U \cap G_+)$ such that

$$\lim_{i \to \infty} u_{n_i} = u_+ \quad \text{weakly in } L^2(U \cap G_+). \tag{4.28}$$

(4.26)–(4.28) imply that $u_+ \in L^\infty(U \cap G_+)$ is a solution of the equation $Au = 0$ in $U \cap G_+$. Lemma 1.5 and (4.23) give

$$\begin{aligned} u_n(x) - h(x) + C_0 w(x) \geq & E_x^n \left[\int_0^{\tau_n} Ah(x(t)) \exp\left\{\int_0^t c(x(s))\,ds\right\} dt \right. \\ & \left. + \int_0^{\tau_n} C_0 \exp\left\{\int_0^t c(x(s))\,ds\right\} dt\right] \\ \geq & 0 \quad \text{in } U_n \end{aligned}$$

and

$$\begin{aligned} u_n(x) - h(x) - C_0 w(x) \leq & E_x^n \left[\int_0^{\tau_n} Ah(x(t)) \exp\left\{\int_0^t c(x(s))\,ds\right\} dt \right. \\ & \left. - \int_0^{\tau_n} C_0 \exp\left\{\int_0^t c(x(s))\,ds\right\} dt\right] \\ \leq & \, 0 \quad \text{in } U_n, \end{aligned}$$

where $C_0 = \sup_{x \in U} |Ah(x)|$. Hence, we have

$$-\int C_0 w \phi\,dx \leq \int (u_n - h)\phi\,dx \leq \int C_0 w \phi\,dx \tag{4.29}$$

for any nonnegative function $\phi \in L^2(U_n)$. Combining (4.29) with (4.28), we obtain

$$|u_+ - h| \leq C_0 w \quad \text{in } U \cap G_+;$$

this implies

$$\lim_{\substack{x \to a \\ x \in U \cap G_+}} u_+(x) = g(a) \quad \text{for } a \in M \cap U. \tag{4.30}$$

For $n \in \mathbf{N}$ we see $U_{-n} = U \cap \{x \in G\colon x_d < -1/n\}$ and $V_{-n} = V \cap \{x \in G\colon x_d < -1/2n\}$. Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P_x^{-n})$ be a diffusion process generated by the operator $-\chi_{-n}(A - c(x))$, where $\chi_{-n} \in C_0^\infty(V_{-n})$ is a nonnegative function satisfying $\chi_{-n} = 1$ in U_{-n}. Then, by modifying the above argument, we can prove that there is a function $u_- \in L^\infty(U \cap G_-)$ such that $Au_- = 0$ in $U \cap G_-$ and

$$\lim_{\substack{x \to a \\ x \in U \cap G_-}} u_-(x) = g(a) \quad \text{for } a \in M \cap U. \tag{4.31}$$

Since

$$u(x) = \begin{cases} u_+(x) & \text{if } x \in U \cap G_+, \\ u_-(x) & \text{if } x \in U \cap G_- \end{cases}$$

is a solution of the equation $A(u - h) = -Ah$ in $U \cap (G_+ \cup G_-)$, we have, by (11), (4.30), (4.31) and Lemma 4.1, $A(u - h) = -Ah$ in U, i.e., $Au = 0$ in U. (4.30) and (4.31) easily give (4.22). The proof of Proposition 4.4 is complete.

PROOF OF THEOREM 3. Theorem 3(i) follows immediately from Proposition 4.3 and Lemma 4.2. In Proposition 4.4, if we take a function $g \in C^2(M)$ such that $p \in \operatorname{sing\,supp} g$, then we have, by (4.22), $p \in \operatorname{sing\,supp} u$. Thus Theorem 3(ii) follows from Proposition 4.4.

5. Proof of Theorem 4. Without loss of generality, we may assume that $\psi(x) = x_d$. We set $G_+ = \{x \in G\colon\ x_d > 0\}$, $G_- = \{x \in G\colon\ x_d < 0\}$ and $M = \{x \in G\colon\ x_d = 0\}$. Let U and V be open balls with center $p \in M$ such that $\overline{U} \subset V \subset G$ and diameter V is sufficiently small. For $n \in \mathbf{N}$ we set $U_n = \{x \in U\colon\ x_d > 1/n\}$ and $V_n = \{x \in V\colon\ x_d > 1/2n\}$. $(\Omega, \mathfrak{F}, \mathfrak{F}_t, x(t), P_x^n)$ denotes a diffusion process generated by the operator $\chi_n(A - c(x))$, where $\chi_n \in C_0^\infty(V)$ is a nonnegative function satisfying $\chi_n = 1$ in U_n. τ_n is the exit time of $x(t)$ from $\overline{U}_n$.

Let us take a function

$$h(x) = \begin{cases} e^{-1/|x_d|} & \text{if } x_d > 0, \\ 0 & \text{if } x_d \le 0. \end{cases}$$

Modifying the proof of Proposition 4.4, we can prove that

$$u_n(x) = \begin{cases} E_x^n\left[h(x(\tau_n)) \exp\left\{\int_0^{\tau_n} c(x(s))\,ds\right\}\right] & \text{in } U_n, \\ 0 \quad \text{in } U \backslash U_n \end{cases} \tag{5.1}$$

are functions $\in L^\infty(U)$ such that $Au_n = 0$ in U_n and $\{u_n\}_{n=1}^\infty$ is uniformly bounded in U. Furthermore, we can show that there is a subsequence $\{u_{n_i}\}_{i=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and a function $u \in L^\infty(U)$ such that $Au = 0$ in U and

$$\lim_{i \to \infty} u_{n_i} = u \quad \text{weakly in } L^2(U). \tag{5.2}$$

(5.1) and (5.2) easily give $\operatorname{supp}\ u \subset \overline{U \cap G_+}$.

Since $Au = 0$ means

$$Av + 2\gamma \sum_{i,j=1}^d a_{ij} \frac{\partial \psi}{\partial x_i} \partial\psi\partial x_j$$
$$+ \gamma \left\{ \sum_{i,j=1}^d a_{ij} \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j} + \frac{\partial \psi}{\partial \psi_i} \frac{\partial \psi}{\partial x_j} \right) + \sum_{i=1}^d b_i \frac{\partial \psi}{\partial x_i} \right\} v = 0$$

when $u = e^{\gamma\psi} v$, we may assume, by (16) and (17), that

$$c(x) \ge 0 \quad \text{in } U_N \tag{5.3}$$

for any fixed N. We set

$$F_n = \{\omega \in \Omega\colon\ x(t,\omega) \in \overline{U}_N \text{ for all } t \leq \tau_n\}$$

for $n \geq N+1$. It is easy to show that $\tau_n = \tau_N$ on F_{N+1} and $F_n = F_{N+1}$ for any $n \geq N+1$. Since $\chi_n = 1$ in $\overline{U}_N$ for $n \geq N+1$, we have

$$\begin{aligned} &E_x^n\left[\exp\left\{\int_0^{\tau_n} c(x(s))\,ds\right\}\chi_{F_n}\right] \\ &\qquad = E_x^{N+1}\left[\exp\left\{\int_0^{\tau_{N+1}} c(x(s))\,ds\right\}\chi_{F_{N+1}}\right]\end{aligned}$$

for $n \geq N+1$ and $x \in \overline{U}_N$. By (17) and Lemma 1.8,

$$E_x^{N+1}\left[\exp\left\{\int_0^{\tau_{N+1}} c(x(s))\,ds\right\}\chi_{F_{N+1}}\right] > 0$$

for $x \in \overline{U}_N$. Then we have

$$\begin{aligned} u_n(x) &\geq E_x^n\left[h(x(\tau_n))\exp\left\{\int_0^{\tau_n} c(x(s))\,ds\right\}\chi_{F_n}\right] \\ &\geq \left(\inf_{y\in U_N} h(y)\right) E_x^{N+1}\left[\exp\left\{\int_0^{\tau_{N+1}} c(x(s))\,ds\right\}\chi_{F_{N+1}}\right] \\ &> 0\end{aligned}$$

for $x \in U_N$ and $n \geq N+1$; this implies that $u > 0$ in U_N. Therefore, we have $\operatorname{supp} u = \overline{U \cap G_+}$. By Lemma 3.1, u is the desired solution.

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