TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 286, Number 1, November 1984

ON SKOLEM'S EXPONENTIAL FUNCTIONS BELOW 2^{2^X}

BY
LOU VAN DEN DRIES AND HILBERT LEVITZ

ABSTRACT. A result of Ehrenfeucht implies that the smallest class of number-theoretic functions $f\colon \mathbf{N} \to \mathbf{N}$ containing the constants $0, 1, 2, \ldots$, the identity function X, and closed under addition, multiplication and $f \to f^X$, is well-ordered by the relation of eventual dominance. We show that its order type is ω^{ω^ω}, and that for any two nonzero functions f, g in the class the quotient $f(n)/g(n)$ tends to a limit in $E^+ \cup \{0, \infty\}$ as $n \to \infty$, where E^+ is the smallest set of positive real numbers containing 1 and closed under addition, multiplication and under the operations $x \to x^{-1}$, $x \to e^x$.

Introduction. We define Sk as the least set of functions from $\mathbf{R}^{>0}$ to itself which contains the constant function 1, the identity function X, and such that $f + g$, $f \cdot g$, and f^g belong to Sk for all $f, g \in \mathrm{Sk}$.

It follows from a theorem of G. H. Hardy [**5**] that Sk is *linearly ordered* by the relation $<$ of eventual dominance:

$$f < g \overset{\text{def}}{\Leftrightarrow} \exists y \forall x > y\ (f(x) < g(x)).$$

Skolem [**11**] asked whether $(\mathrm{Sk}, <)$ is well ordered and suggested that its order type should be $\varepsilon_0 = \sup(\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots)$. (He indicated a subset of Sk of order type ε_0.)

Ehrenfeucht [**3**] answered the first part of Skolem's question affirmatively. His proof is as short as any proof could be, but it gives virtually no information on the order type. (The argument applies a combinatorial principle discovered by J. Kruskal; this principle was recently shown to be independent of a 'strong' system of predicative analysis by H. Friedman.)

Several upperbounds have been given for the order type of Sk, the sharpest one, to our knowledge, being the least ordinal α such that $\varepsilon_\alpha = \alpha$ [**7**]. (Note that this is still huge compared to ε_0.) On the other hand, the order types of certain initial segments of Sk have been determined, and this paper goes further in this direction.

To facilitate further discussion we introduce the following *conventions* and *notations*:

It turns out to be convenient to admit also the constant function 0 *as a member of* Sk, *so from now on we let* 0 *be the least element of* Sk. *The letters* i, j, m, n *stand for elements of* $\mathbf{N}$, *and* f, g, h *stand for functions in* Sk. *We put* $\mathrm{Sk}(f) = \{g \mid g < f\}$, *and given* $T \subset \mathrm{Sk}$ *we let* $|T|$ *be the order type* (*an ordinal*) *of* $(T, <)$.

Since $\mathrm{Sk}(X) = \{0, 1, 2, \ldots\} = \mathbf{N}$ we have $|\mathrm{Sk}(X)| = \omega$. Further, $\mathrm{Sk}(2^X) = \mathbf{N}[X]$, or, more precisely,

$$\omega^n \cdot a_n + \omega^{n-1} \cdot a_{n-1} + \cdots + a_0 \mapsto a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \qquad (a_i \in \mathbf{N})$$

Received by the editors December 1, 1983.
1980 *Mathematics Subject Classification*. Primary 03D20, 26A12; Secondary 06F05.

is an isomorphism of ω^ω onto $(\mathrm{Sk}(2^X), <)$; hence, $|\mathrm{Sk}(2^X)| = \omega^\omega$.

Now X and 2^X are just the first two terms in the sequence $X, 2^X, 2^{2^X}, 2^{2^{2^X}}, \dots,$ which is cofinal in Sk; cf. [**7**].

In this article we determine the order type of the initial segment bounded by the next term 2^{2^X}. This again is done in stages: Levitz [**6**] showed that the sequence $2^X, 2^{X^2}, 2^{X^3}, 2^{X^4}, \dots$ is cofinal in 2^{2^X}, and in [**9**] he proved that $|\mathrm{Sk}(2^{X^2})| = \omega^{\omega^3}$. Here we show that, for $n > 0$,

$$\omega^{\omega^{2n-1}} \le |\mathrm{Sk}(2^{X^n})| \le \omega^{\omega^{n^2}}.$$

Hence, $|\mathrm{Sk}(2^{2^X})| = \omega^{\omega^\omega}$.

Let us note that $\mathrm{Sk}(2^{2^X})$ is a rather large class of functions; it is closed under $+, \cdot$, and under $f \mapsto f^p$ for each polynomial $p \in \mathbf{N}[X]$.

The main novelty of our proof compared to previous work on Skolem's problem is the use of *asymptotic expansions*, especially of the behaviour of coefficients in these expansions.

Before we can state our main result we need a bit more notation: $E^+ \stackrel{\text{def}}{=}$ the smallest subset of $\mathbf{R}^{>0}$ containing 1 and closed under $+, \cdot, {}^{-1}$ (reciprocal), and $r \mapsto e^r$; $E \stackrel{\text{def}}{=} E^+ - E^+ = \{a - b | a, b \in E^+\}$. We call $f, g \in \mathrm{Sk} \setminus \{0\}$ of the same *archimedean class*, or of the same *scale*, if $f(x)/g(x)$ tends to a limit in $\mathbf{R}^{>0}$ as $x \to \infty$. (The only other possibilities are that $\lim_{x\to\infty} f(x)/g(x) = 0$ or $= \infty$, by [**5**].)

MAIN THEOREM. *For all $f, g \in \mathrm{Sk}(2^{2^X}) \setminus \{0\}$ of the same scale we have an asymptotic expansion $f(x)/g(x) \sim r_0 + (r_1/x) + (r_2/x^2) + \cdots$ $(x \to \infty)$, where $r_0 \in E^+$, $r_i \in E$ for $i \ge 1$. Moreover, given any $r_0 > 0$, $r_1, \dots, r_k \in \mathbf{R}$ $(k \ge -1)$ and any $g \in \mathrm{Sk}(2^{2^X}) \setminus \{0\}$, the set*

$$\mathcal{R}(g, r_0, \dots, r_k) \stackrel{\text{def}}{=} \{r_{k+1} \in \mathbf{R} | \exists f.\ f \text{ is of the same scale as } g \text{ and}$$
$$f/g \sim r_0 + (r_1/x) + \cdots + (r_k/x^k) + (r_{k+1}/x^{k+1}) + \cdots\}$$

contains only finitely many numbers below any given bound $M \in \mathbf{R}$.

A direct consequence of our method of proof is that the set of scales below 2^{2^X} has order type ω^ω.

Various other results of interest follow from this theorem; e.g., if $0 < f, g < 2^{2^X}$, then $\lim_{x\to\infty} f(x)/g(x) \in E^+ \cup \{0, \infty\}$. (A proof by Richardson [**10**] shows that each number in E^+ actually occurs as such a limit.)

It is by now common experience in solving problems involving repeated exponentiations that the key step is to set up the right induction and to select the right induction hypothesis. This is also the case in the proof of the Main Theorem: one just assumes the theorem holds for functions below 2^{X^n} (induction hypothesis) and uses this to show that it holds for functions below $2^{X^{n+1}}$. A weaker statement of the theorem would not suffice to get the information on order types we are interested in or would not be strong enough to serve as an induction hypothesis.

From the 'logic' point of view it may be of interest that we frequently use Ehrenfeucht's result that Sk is well ordered.

§1 is devoted to the proof of the main theorem; §2 contains the 'ordinal' counting leading up to $|\mathrm{Sk}(2^{2^X})| = \omega^{\omega^\omega}$.

In closing this introduction we think it worth mentioning that Richardson showed that the relation $f = g$ on Sk is decidable [**10**]. He also showed that the relation $<$ on E^+ is Turing reducible to the relation $<$ on $\mathrm{Sk}(2^{2^X})$. Gurevič [**4**], on the other hand, announced that $<$ on $\mathrm{Sk}(2^{2^X})$ is Turing reducible to the relation $<$ on E^+. Van den Dries and Gurevič have shown independently that $<$ is decidable on $\mathrm{Sk}(2^{X^2})$.

0. Preliminaries. Given a real valued function ψ defined on some interval (r, ∞) we say that the series $a_0 + (a_1/x) + (a_2/x^2) + \cdots = \sum_0^\infty a_i/x^i$ $(a_i \in \mathbf{R})$ is an asymptotic expansion of ψ for $x \to \infty$, written

$$\psi(x) \sim \sum_0^\infty a_i/x^i \qquad (x \to \infty),$$

if, for each $k \in \mathbf{N}$,

$$\psi(x) - (a_0 + (a_1/x) + \cdots + (a_k/x^k)) = O(1/x^{k+1}) \qquad (x \to \infty).$$

We shall freely use the basic facts on asymptotic expansions for proofs of which we refer the reader to [**2**, p. 11]. In particular, if ψ has an asymptotic expansion $\sum_0^\infty a_i/x^i$ for $x \to \infty$, it has only one of that form. From now on we shall omit the expression $(x \to \infty)$ in formulas like

$$\phi \sim \psi \quad (x \to \infty), \qquad \phi = O(\psi) \quad (x \to \infty).$$

It is crucial for our purpose that relations like $\psi(x) \sim \sum a_i/x^i$ can be manipulated as if we are dealing with power series in $1/x$ converging to $\psi(x)$ (for large x), although this is usually not the case for the functions we are dealing with. (The power series actually do converge but mostly to other functions.)

We call f a *scale function* if $f \neq 0$ and f is the least element in its scale. (By Ehrenfeucht [**3**] each scale has a least element.) Equivalently, f is a scale function if $f \neq 0$ and $\mathrm{Sk}(f)$ is closed under addition.

We shall repeatedly use the fact that E is a subring of $\mathbf{R}$ which is mapped into E^+ by exp: $\exp(a - b) = \exp(a) \cdot \exp(b)^{-1}$.

1. Proof of the Main Theorem.

(1.1) We start with formulating a convenient induction hypothesis. Fix $n > 0$ and consider the following condition:

(H_n) For each scale function ϕ in $\mathrm{Sk}(2^{X^n})$ and each f of the same scale as ϕ we have an expansion: $f/\phi \sim r_0 + (r_1/x) + (r_2/x^2) + \cdots$ with $r_0 \in E^+$, $r_i \in E$ for $i \geq 1$. Moreover, given any $r_0 > 0$, $r_1, \ldots, r_k \in \mathbf{R}$ $(k \geq -1)$, the set $\mathcal{R}(\phi, r_0, \ldots, r_k)$ contains only finitely many numbers below any given bound $M \in \mathbf{R}$.

(1.2) REMARKS. (1) For $k = -1$ we have $\mathcal{R}(\phi) = \{r \in \mathbf{R} | \exists f.\ f$ is of the same scale as ϕ and $\lim_{x \to \infty} f(x)/\phi(x) = r\}$. Note that 1 is the least element of $\mathcal{R}(\phi)$ (take $f = \phi$) and $\mathcal{R}(\phi)$ is closed under addition. So $\mathcal{R}(\phi)$ is infinite.

(2) Another way of formulating the second part of (H_n) is: $\mathcal{R}(\phi, r_0, \ldots, r_k)$ is either finite, or of order type ω and cofinal in $\mathbf{R}$.

(1.3) Let us verify (H_1): the scale functions below 2^X are the powers X^m, $m \geq 0$, and if f is of the same scale as X^m, then $f = a_m X^m + a_{m-1} X^{m-1} + \cdots + a_0$, $a_i \in$

$\mathbf{N}$, $a_m > 0$, so $f/X^m \sim a_m + (a_{m-1}/x) + \cdots + (a_0/x^m) + (0/x^{m+1}) + \cdots$. It is clear that (H_1) holds: $\mathcal{R}(X^m) = \mathbf{N}^{>0}$, etc.

Until further notice we assume (H_n).

(1.4) LEMMA. *Let $0 < g < 2^{X^n}$. Then for each f which is of the same scale as g we have an expansion $f/g \sim c_0 + (c_1/x) + (c_2/x^2) + \cdots$ with $c_0 \in E^+$, $c_i \in E$ for $i \geq 1$. Moreover, given any $c_0 > 0, c_1, \ldots, c_k \in \mathbf{R}$ $(k \geq -1)$, the set $\mathcal{R}(g, c_0, \ldots, c_k)$ contains only finitely many numbers below any given bound M in $\mathbf{R}$.*

PROOF. Take the unique scale function ϕ which is of the same scale as g and write

$$g/\phi \sim b_0 + (b_1/x) + b_2/x^2) + \cdots,$$

according to (H_n). Given any f of the same scale as g we can write

$$f/\phi \sim a_0 + (a_1/x) + (a_2/x^2) + \cdots.$$

So $f/g = (f/\phi)/(g/\phi) \sim c_0 + (c_1/x) + (c_2/x^2) + \cdots$, where the power series $\sum c_i/x^i$ is obtained by dividing $\sum a_i/x^i$ by $\sum b_i/x^i$ in the ring of formal power series $\mathbf{R}[[1/x]]$. In other words, the c's are determined by the equations $b_0c_0 = a_0$, $b_0c_{k+1} + b_1c_k + \cdots + b_{k+1}c_0 = a_{k+1}$ or, more explicitly,

$$c_0 = a_0/b_0, \qquad c_{k+1} = a_{k+1}/b_0 - (b_1c_k + \cdots + b_{k+1}c_0)/b_0.$$

From these equations it is apparent how $(a_0, \ldots, a_k)$ determines $(c_0, \ldots, c_k)$ (and conversely), and one can simply see that the statements of the lemma follow from the assumption (H_n). □

(1.5) Let $(\phi_\lambda)_{\lambda \in \Lambda}$ be the family of all scale functions $< 2^{X^n}$, where the index set Λ is (well) ordered in such a way that $\lambda < \mu \Rightarrow \phi_\lambda < \phi_\mu$. For each λ we note that $\mathcal{R}(\phi_\lambda)$, as a subset of $\mathbf{R}^{\geq 1}$, has order type ω.

(1.6) DEFINITION (a) For each $\lambda \in \Lambda$ and $r \in \mathcal{R}(\phi_\lambda)$ we let $\phi_{\lambda,r}$ be the least f such that $\lim_{x\to\infty} f(x)/\phi_\lambda(x) = r$.

(b) The ordered set Λ^* is the set of all triples (λ, r, d) with $\lambda \in \Lambda$, $r \in \mathcal{R}(\phi_\lambda)$ and $d \in \mathbf{N}$, ordered lexicographically. For each $\lambda^* = (\lambda, r, d) \in \Lambda^*$ we put $\phi_{\lambda^*} = \phi_{\lambda,r}^X \cdot X^d$.

(1.7) EXAMPLE. For $n = 1$ we take $\Lambda = (\mathbf{N}, <)$, $\phi_m = X^m$, and we have $\mathcal{R}(\phi_m) = \mathbf{N}^{>0}$, $\phi_{m,k} = kX^m$, so $\phi_{(m,k,d)} = (kX^m)^X \cdot X^d$ $(m \in \mathbf{N},\ k \in \mathbf{N}^{>0},\ d \in \mathbf{N})$.

(1.8) PROPOSITION. (a) *The map $\lambda^* \mapsto \phi_{\lambda^*}$ is an embedding of the ordered set Λ^* into* $(\mathrm{Sk}(2^{X^{n+1}}), <)$.

(b) *Each $F \in \mathrm{Sk}(2^{X^{n+1}})$ of the same scale as ϕ_{λ^*} is $\geq \phi_{\lambda^*}$, and there is an expansion*

$$F/\phi_{\lambda^*} \sim r_0 + (r_1/x) + (r_2/x^2) + \cdots$$

with $r_0 \in E^+$, $r_i \in E$ for $i \geq 1$.

(c) *The ϕ_{λ^*}, $\lambda^* \in \Lambda^*$, are exactly the scale functions which are $< 2^{X^{n+1}}$.*

(1.9) Before we start the proof we need a result which is basic in our inductive set up.

LEMMA. (a) *For each* f *we have* $f < 2^{X^{n+1}}$ *if and only if* $f = f_1^X \cdot X^{d_1} + \cdots + f_m^X \cdot X^{d_m}$ *for certain* $f_i < 2^{X^n}$, $d_i \in \mathbf{N}$.

(b) $2^{X^{n+1}}$ *is a scale function.*

The conjunction of (a) and (b) of the Lemma is proven by induction on n; see [**6**].

(1.10) PROOF OF (1.8). The Lemma just stated shows that $\phi_{\lambda^*} < 2^{X^{n+1}}$ for $\lambda^* \in \Lambda^*$. It follows from $X^d < r^X$ ($d \in \mathbf{N}$, $r > 1$), that if $\lambda^* < \mu^*$, then $\phi_{\lambda^*} < \phi_{\mu^*}$. (In fact, ϕ_{λ^*} and ϕ_{μ^*} are even in different scales.) This concludes the proof of (a).

To prove (b) and (c), consider any nonzero $F \in \mathrm{Sk}(2^{X^{n+1}})$. According to the lemma in (1.9) we can write

$$F = f_1^X \cdot X^{d_1} + \cdots + f_p^X \cdot X^{d_p} + \cdots + f_{p+q}^X \cdot X^{d_{p+q}}, \tag{1}$$

where $0 < f_i < 2^{X^n}$, $d_i \in \mathbf{N}$. Here we have arranged the terms such that, if we assign to each f_i the pair (λ, r) with f_i of the same scale as ϕ_λ and $\lim_{x\to\infty} f_i(x)/\phi_\lambda(x) = r$, then $f_1, \ldots, f_p$ are assigned the highest pair, say (λ, r), among the $p+q$ pairs, while $f_{p+1}, \ldots, f_{p+q}$ are assigned lower pairs. Further we arrange the first p terms so that in the sequence $(d_1, \ldots, d_p)$ the first m terms are maximal, say $d = d_1 = \cdots = d_m$, while $d_{m+i} < d$ for $m < m+i \le p$. With these definitions of λ, r, d in mind we claim

$$F \ge \phi_{\lambda^*}, \quad \text{where } \lambda^* = (\lambda, r, d), \tag{2}$$

$$F/\phi_{\lambda^*} \sim r_0 + (r_1/x) + (r_2/x^2) + \cdots \tag{3}$$

with $r_0 \in E^+$, $r_i \in E$ for $i \ge 1$.

Inequality (2) follows easily: $f_1 \ge \phi_{\lambda,r}$, by definition of λ, r and $\phi_{\lambda,r}$, so $F \ge f_1^X X^{d_1} = f_1^X X^d \ge \phi_{\lambda,r}^X \cdot X^d = \phi_{\lambda^*}$.

For (3), consider an $f = f_i$ with $1 \le i \le p$. Lemma (1.4) enables us to write

$$f/\phi_{\lambda,r} \sim 1 + (a_1/x) + (a_2/x^2) + \cdots. \tag{4}$$

Now

$$f^X/\phi_{\lambda,r}^X = e^{X \log(f/\phi_{\lambda,r})} \tag{5}$$

and (4) gives

$$\log(f/\phi_{\lambda,r}) \sim (b_1/x) + (b_2/x^2) + \cdots, \tag{6}$$

where $\sum_1^\infty b_i/x^i$ is obtained by substituting $\sum_1^\infty a_i/x^i$ for the variable y into the power series

$$\log(1+y) = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \cdots;$$

note that $b_1 = a_1$. Together with (6) this gives

$$X \log(f/\phi_{\lambda,r}) \sim a_1 + (b_2/x) + (b_3/x^2) + \cdots$$

and in combination with (5) we get

$$f^X/\phi_{\lambda,r}^X \sim e^{a_1} + (e^{a_1}c_1/x) + (e^{a_1}c_2/x^2) + \cdots, \tag{7}$$

where the series $1 + \sum_1^\infty c_i/x^i$ is obtained by substituting $\sum_1^\infty b_{i+1}/x^i$ for the variable y into the power series $e^y = 1 + y + (y^2/2) + (y^3/6) + \cdots$.

It is not difficult to keep track of the coefficients b_i and c_i; the b's are in E since the a's are, and the c's are in E since the b's are. So all coefficients in (7) are in E, and the constant term e^{a_1} is in E^+.

Now, since $\phi_{\lambda^*} = \phi_{\lambda,r}^X \cdot X^d$ we derive from (7), for $1 \leq i \leq p$,

$$f_i^X X^{d_i}/\phi_{\lambda^*} \sim r_{i0} + (r_{i1}/x) + (r_{i2}/x^2) + \cdots, \tag{8}$$

where $r_{i0} \in E^+$ if $1 \leq i \leq m$, $r_{i0} = 0$ if $m < i \leq p$, and $r_{ij} \in E$ for $1 \leq i \leq p$, $j \geq 1$.

Further, it follows easily from the definitions that, for $p < i \leq p+q$,

$$f_i^X X^{d_i}/\phi_{\lambda^*} \sim 0 + (0/x) + (0/x^2) + (0/x^3) + \cdots. \tag{9}$$

Adding the expansions in (8) and (9) and using (1) we get the desired expansion stated in (3). Moreover, since $r_0 > 0$ in (3), we see that F is of the same scale as ϕ_{λ^*} and by the parenthetical observation made in the proof of (a) we know that $\lambda^* \in \Lambda^*$ is unique with this property. This concludes the proof of statements (b) and (c). $\square$

(1.11) REMARKS. (1) Applying the proposition to the case $n = 1$ we recover a result by Levitz [**9**]: *the functions* $\phi_{(m,k,d)}$ *from* (1.7),

$$\phi_{(m,k,d)} = (kX^m)^X \cdot X^d \qquad (m \in \mathbf{N},\ k \in \mathbf{N}^{>0},\ d \in \mathbf{N}),$$

are exactly the scale functions $< 2^{X^2}$.

(2) If Λ has order type ω^{2n-1} ("induction hypothesis"), then Λ^* has order type $\omega^{2n+1} = \omega^{2(n+1)-1}$. So we can keep track of the order type of the set of scale function $< 2^{X^n}$ when we replace n by $n+1$. Note that for $n = 1$ the hypothesis that Λ has order type $\omega^{2n-1} = \omega$ is satisfied; see (1.7).

(1.12) By refining the arguments in (1.10) we shall arrive at the following result.

PROPOSITION. *Condition* (H_{n+1}) *holds.*

PROOF. Let $M \in \mathbf{R}^{>0}$ and $\lambda^* = (\lambda, r, d) \in \Lambda^*$. Given $r_0 \in E^+$, $r_1, \ldots, r_k \in E$ we have to prove that $\mathcal{R}(\phi_{\lambda^*}, r_0, \ldots, r_k)$ contains only finitely many numbers below M. The proof is unavoidably a bit messy, and it is helpful to first do the special case $k = -1$; that is, we first prove that $\mathcal{R}(\phi_{\lambda^*})$ contains only finitely many numbers below M. So we consider functions $F \in \mathrm{Sk}(2^{X^{n+1}})$ of the same scale as ϕ_{λ^*} and write them as in (1). (We use the notations and numbering from (1.10).) From (4), (7), (8) and (9) it follows that the number $r_0 = r_{10} + \cdots + r_{m0}$ from (3) equals $e^{a_{11}} + \cdots + e^{a_{m1}}$, where the a_{i1} are in $\mathcal{R}(\phi_{\lambda,r}, 1)$. More precisely, $\mathcal{R}(\phi_{\lambda^*})$ is the set of all $e^{t_1} + \cdots + e^{t_m}$ with $m \in \mathbf{N}^{>0}$ and $t_i \in \mathcal{R}(\phi_{\lambda,r}, 1)$. Since such t_i are ≥ 0 (use the minimality property of $\phi_{\lambda,r}$), we have $e^{t_i} \geq 1$, and by the hypothesis that $\mathcal{R}(\phi_{\lambda,r}, 1)$ contains only finitely many numbers below any given bound, we conclude that for $r_0 = r_{10} + \cdots + r_{m0} \leq M$ we must have $m \leq M$, and there are only finitely many possibilities for r_0. This not only proves the desired result for $\mathcal{R}(\phi_{\lambda^*})$ but also gives the bound $m \leq M$.

Consider now the case $k \geq 0$. It turns out that the number c_{k+1} in (7) only depends on the initial coefficients $a_1, a_2, \ldots, a_{k+2}$ in (4). Let us study this dependence in more detail.

We can write

$$\begin{aligned}\log(1+(a_1/x)+(a_2/x^2)+\cdots) &= ((a_1/x)+(a_2/x^2)+\cdots)\\ &\quad -\tfrac{1}{2}((a_1/x)+(a_2/x^2)+\cdots)^2\\ &\quad \vdots\\ &\quad +((-1)^{k+1}/(k+2))((a_1/x)+(a_2/x^2)+\cdots)^{k+2}\\ &\quad \vdots\\ &= (b_1/x)+(b_2/x^2)+\cdots+(b_{k+2}/x^{k+2})+\cdots;\end{aligned}$$

so

$$\begin{aligned}b_{k+2} &= a_{k+2}+\left(-\tfrac{1}{2}\right)\sum_{i+j=k+2} a_i a_j+\cdots+\frac{(-1)^{k+1}}{k+2}a_1^{k+2}\\ &= a_{k+2}+P_{k+1}(a_1,\ldots,a_{k+1}),\end{aligned}$$

where P_{k+1} is a certain polynomial over $\mathbf{Q}$ in $k+1$ variables.

Similarly,

$$\begin{aligned}\exp((b_2/x)+(b_3/x^2)+\cdots) &= 1+((b_2/x)+(b_3/x^2)+\cdots)\\ &\quad +\tfrac{1}{2}((b_2/x)+(b_3/x^2)+\cdots)^2\\ &\quad \vdots\\ &\quad +(1/(k+1)!)((b_2/x)+(b_3/x^2)+\cdots)^{k+1}\\ &\quad \vdots\\ &= 1+(c_1/x)+(c_2/x^2)+\cdots+(c_{k+1}/x^{k+1})+\cdots;\end{aligned}$$

so $c_{k+1}=b_{k+2}+B_k(b_2,\ldots,b_{k+1})$, where B_k is a certain polynomial over $\mathbf{Q}$ in k variables.

In combination with the previous formula for b_{k+2}, this gives

$$(*)\qquad c_{k+1}=a_{k+2}+A_{k+1}(a_1,\ldots,a_{k+1}),$$

where A_{k+1} is a certain polynomial over $\mathbf{Q}$ in $k+1$ variables.

Now fix $r_0\in E^+, r_1,\ldots,r_k\in E$ and $M\in\mathbf{R}^{>0}$. Our task is to show that $\mathcal{R}(\phi_{\lambda^*},r_0,\ldots,r_k)$ contains only finitely many numbers r_{k+1} below M. The essential tools here are formula $(*)$ and the (inductive) assumption (H_n). A detailed proof would be messy and not very enlightening to most readers and therefore we prefer to treat only the case $k=0$ since this conveys the basic idea. (The (nontypical) case $k=-1$ was treated before.)

So we consider those $F\in \mathrm{Sk}(2^{X^{n+1}})$ for which $F/\phi_{\lambda^*}\sim r_0+(r_1/x)+\cdots$, $\lambda^*=(\lambda,r,d)$ and r_0 is given. Note that

$$(10)\qquad r_1=r_{11}+\cdots+r_{m1}+r_{m+11}+\cdots+r_{m+l1}\qquad (l\geq 0),$$

where the terms in (1) are arranged such that $d_1=\cdots=d_m=d$, $d_{m+1}=\cdots=d_{m+l}=d-1$, while $d_i<d-1$ if $m+l<i\leq p$. Note that (7) and $(*)$ imply

$$(11)\qquad r_{i1}=e^{a_{i1}}\cdot(a_{i2}+A_1(a_{i1}))\quad \text{for } 1\leq i\leq m,$$

$$(12)\qquad r_{i1}=e^{a_{i1}}\qquad \text{for } m+1\leq i\leq m+l.$$

Now we combine (10), (11) and (12) with the following 4 facts: $m \leq r_0$ (as we saw in the proof of the case $k = -1$); there are only finitely many possible values of a_{i1} for $1 \leq i \leq m$ (again, consult the proof of the case $k = -1$); for each such value of a_{i1} there are only finitely many possible values of a_{i2} below any given bound, $1 \leq i \leq m$ (because of (H_n), or rather its consequence (1.4)); finally $a_{i1} \geq 0$ for $m + 1 \leq i \leq m + l$.

Because r_0 and M are given and we require $r_1 \leq M$ we draw the conclusion that there are only finitely many possibilities for the sequence $(m, l, a_{11}, \ldots, a_{m+l1}, a_{12}, \ldots, a_{m2})$. This implies, in particular, that there are only finitely many possible values for r_1, finishing the proof of the case $k = 0$.

To handle the case $k = 1$ we proceed essentially as above, using the previous conclusion about the possible sequences $(m, l, a_{11}, \ldots, a_{m2})$ as an inductive assumption.

By now the pattern should be clear. □

(1.13) Since we have proved that (H_n) implies (H_{n+1}), we have established the Main Theorem, of course taking into account (1.3) and Lemma (1.4). Besides that we have established the following:

(1.14) *The order type of the set of scale functions below* 2^{X^n} is ω^{2n-1}. (By (1.11)(2).)

(1.15) *If f, g are of the same scale and below* 2^{2^X}, *then* $\lim_{x\to\infty} f(x)/g(x) \in E^+$.

(1.16) *If f is a scale function below 2^{2^X} then $f \cdot X$ is the next scale function.*

(Say $f < 2^{X^{n+1}}$; then $f = \phi_{\lambda^*}$ for some $\lambda^* = (\lambda, r, d)$. The successor of (λ, r, d) in the set Λ^* is $(\lambda, r, d+1)$ and $\phi_{(\lambda,r,d+1)} = \phi_{(\lambda,r,d)} \cdot X$.)

(1.17) If f, g are below 2^{2^X} and $\lim_{x\to\infty} f(x)/g(x) = 1$, then

$$0 < \lim_{x\to\infty} (f(x)/g(x))^x < \infty.$$

(This is easily established if we recall from the argument (1.10) that for any f, the scale function of f^X is $(\phi_{\lambda,r})^X$, where ϕ_λ is the scale function of f and $\lim_{x\to\infty} f(x)/\phi_\lambda(x) = r$. So if $\lim_{x\to\infty} f(x)/g(x) = 1$, then f^X and g^X have the same scale function.)

2. Bounds on the order type of $\mathrm{Sk}(2^{X^n})$.

(2.1) In this section we show, as promised in the introduction, that, for $n > 0$,

$$\omega^{\omega^{2n-1}} \leq |\,\mathrm{Sk}(2^{X^n})| \leq \omega^{\omega^{n^2}}. \tag{13}$$

(2.2) On the basis of (1.14) we have that the scale functions in $\mathrm{Sk}(2^{2^X})$ can be enumerated in order as a transfinite sequence $\{\phi_\alpha\}_{\alpha<\omega^\omega}$ and that for $n > 0$, $\phi_{\omega^{2n-1}} = 2^{X^n}$. This enumeration is continuous; that is, $\sup_{\beta<\alpha} \phi_\beta = \phi_\alpha$ whenever α is a limit ordinal. Using the fact that the scale functions determine initial segments closed under addition, it is easy to show that for all α, $\phi_{\alpha+1} = \sup_n(\phi_\alpha \cdot n)$. (The suprema are taken in the well-ordered set Sk.)

(2.3) We establish the left-hand inequality of (13) by showing an order preserving mapping from the set of ordinals less than $\omega^{\omega^{2n-1}}$ into $\mathrm{Sk}(2^{X^n})$. To each $\beta < \omega^{\omega^{2n-1}}$ write β in Cantor normal form as $\beta = \sum_i \omega^{\beta_i} \cdot n_i$, where $\beta_1 > \beta_2 > \cdots$ and $n_i < \omega$. The desired mapping is the one which sends β to $\sum_i \phi_{\beta_i} \cdot n_i$. That the mapping is order preserving can be seen by first showing that the ordering on the image functions is lexicographical.

(2.4) We now need to introduce further notation. # will be used for the Hessenberg natural sum of ordinals, and $\otimes$ will be used for the Hessenberg product [**1**]. We also use the notation

$$\sum_{i=1}^{k}{}^{\#} \alpha_i \quad \text{and} \quad \prod_{i=1}^{k}{}^{\otimes} \alpha_i$$

for these sums and products. If A is a subset of Skolem's family, we let $\sum^k A = \{\sum_{i=1}^k f_i | f_i \in A\}$. We use $\sum A$ to denote the set of all finite sums of members of A.

To each scale function ϕ_α we use $[\phi_\alpha, \phi_{\alpha+1})$ as customary to denote an interval. The notation $[\phi_\alpha, \phi_{\alpha+1})'$ will stand for the set of all those $f \in [\phi_\alpha, \phi_{\alpha+1})$ which can be written in the form $f = g^X \cdot X^d$ for some $g \in \mathrm{Sk}$ and some $d \in \mathbf{N}$.

If W is a well ordered set and $b \in W$, then I_b will denote the initial segment of W determined by b. If W is not otherwise specified then it should be presumed to be Sk.

(2.5) It is a simple consequence of the results in [**8**] that if the B_i are subsets of Skolem's family, then

$$\text{(a)} \quad \left|\bigcup_{i=1}^{k} B_i\right| \leq \sum_{i=1}^{k}{}^{\#} |B_i|,$$

$$\text{(b)} \quad \left|\sum_{i=1}^{k} B_i\right| \leq \prod_{i=1}^{k}{}^{\otimes} |B_i|.$$

(2.6) LEMMA. *Suppose* $W = \bigcup_{k=1}^\infty A_k$ *is a well-ordered set and*

(a) $|A_k| < \omega^\gamma$, *all* $k = 1, 2, \ldots$;

(b) *the sequence* $\{a_k\}_{k=1}^\infty$, *where* a_k *is the least element of* A_k, *is unbounded in* W.

Then $|W| \leq \omega^\gamma$.

PROOF. We shall show that whenever $b \in W$, then $|I_b| < \omega^\gamma$. Let such a b be given. Choose n_0 such that $b < a_{n_0+1}$. Then $I_b \subseteq \bigcup_{k=1}^{n_0} A_k$. So using (2.5)(a) together with the fact that ordinals of the form ω^γ determine initial segments of the ordinals closed under addition, we get

$$|I_b| \leq \left|\bigcup_{k=1}^{n_0} A_k\right| \leq \sum_{k=1}^{n_0}{}^{\#} |A_k| < \omega^\gamma. \quad \square$$

(2.7) To show that the right-hand inequality of (13) holds for all $n > 0$, we proceed by induction on n. Recall from the Introduction that $|\mathrm{Sk}(2^X)| = \omega^\omega$; so this disposes of the cases $n = 1$. From now on $n \geq 2$ is fixed, and using the right-hand inequality of (13) as an induction hypothesis we show that it also holds when n is replaced by $n + 1$.

(2.8) We first show that

$$\left|\sum [\phi_\alpha, \phi_{\alpha+1})'\right| \leq \omega^{\omega^{n^2}} \quad \text{for all } \phi_\alpha < 2^{X^{n+1}}. \tag{14}$$

Let such a $\phi_\alpha = f^X \cdot X^d$ be given. Using (1.16) and (1.17) we can see that there exists $\phi_\beta < 2^{X^n}$ such that

$$[\phi_\alpha, \phi_{\alpha+1})' \subset \{g^X \cdot X^d | g < \phi_{\beta+1}\}$$

and from this follows

$$|[\phi_\alpha, \phi_{\alpha+1})'| \leq |I_{\phi_{\beta+1}}|. \tag{15}$$

Note also that $\phi_{\beta+1} < 2^{X^n}$. Using this and the induction hypothesis we get

$$|I_{\phi_{\beta+1}}| < |\operatorname{Sk}(2^{X^n})| \leq \omega^{\omega^{n^2}};$$

so

$$|I_{\phi_{\beta+1}}| < \omega^{\omega^{n^2-1}\cdot m} \quad \text{for some } m \in \mathbf{N}. \tag{16}$$

Now using (2.5)(b), (15) and (16) we see that, for any $k \geq 1$,

$$\begin{aligned}\left|\sum^k [\phi_\alpha, \phi_{\alpha+1})'\right| &\leq \prod_{i=1}^{k}{}^{\otimes} |[\phi_\alpha, \phi_{\alpha+1})'| \leq \prod_{i=1}^{k}{}^{\otimes} |I_{\phi_{\beta+1}}| \\ &< \prod_{i=1}^{k}{}^{\otimes} \omega^{\omega^{n^2-1}\cdot m} = \omega^{\omega^{n^2-1}\cdot m\cdot k} < \omega^{\omega^{n^2}}.\end{aligned}$$

We now wish to use the inequality just obtained along with Lemma (2.6). In applying the lemma let $A_k = \sum^k [\phi_\alpha, \phi_{\alpha+1})'$ so that $a_k = \phi_\alpha \cdot k$; let $\gamma = \omega^{n^2}$. Applying the lemma we get

$$\left|\sum [\phi_\alpha, \phi_{\alpha+1})'\right| = \left|\bigcup_{k=1}^{\infty} \sum^k [\phi_\alpha, \phi_{\alpha+1})'\right| \leq \omega^{\omega^{n^2}}.$$

Thus the desired (14) has been established.

(2.9) For any $\phi_\alpha < 2^{X^{n+1}}$ the following set identity is a simple consequence of the fact that I_{ϕ_α} is closed under addition:

$$I_{\phi_{\alpha+1}} = I_{\phi_\alpha} \cup \left(I_{\phi_\alpha} + \sum [\phi_\alpha, \phi_{\alpha+1})'\right). \tag{17}$$

From this we get, using (2.5)(a) and (b),

$$|I_{\phi_{\alpha+1}}| \leq |I_{\phi_\alpha}| \# \left(|I_{\phi_\alpha}| \otimes \left|\sum [\phi_\alpha, \phi_{\alpha+1})'\right|\right).$$

Using this with (14) we get

$$\begin{aligned}|I_{\phi_{\alpha+1}}| &\leq |I_{\phi_\alpha}| \# (|I_{\phi_\alpha}| \otimes \omega^{\omega^{n^2}}) \leq (|I_{\phi_\alpha}| \otimes \omega^{\omega^{n^2}}) \# (|I_{\phi_\alpha}| \otimes \omega^{\omega^{n^2}}) \\ &= |I_{\phi_\alpha}| \otimes \omega^{\omega^{n^2}\cdot 2} \leq |I_{\phi_\alpha}| \otimes \omega^{\omega^{n^2}\cdot 2}.\end{aligned} \tag{18}$$

(2.10) Using (18) we show by an inner transfinite induction on γ that

$$|\phi_\gamma| \leq (\omega^{\omega^{n^2}\cdot 2})^\gamma \quad \text{whenever } \phi_\gamma < 2^{X^{n+1}}. \tag{19}$$

For brevity we write κ for the ordinal $\omega^{n^2} \cdot 2$.

Case 1. $\gamma = 0$. Trivial.

Case 2. $\gamma = \delta + 1$. Then, using (18) and the transfinite induction hypothesis,

$$\begin{aligned}|\phi_\gamma| &= |\phi_{\delta+1}| = |I_{\phi_{\delta+1}}| \leq |I_{\phi_\delta}| \otimes \omega^\kappa = |\phi_\delta| \otimes \omega^\kappa \\ &\leq (\omega^\kappa)^\delta \otimes \omega^\kappa = \omega^{\kappa\cdot\delta\#\kappa} = \omega^{\omega^{n^2}\cdot 2\cdot\delta\#\omega^{n^2}\cdot 2} \\ &= \omega^{\omega^{n^2}\cdot 2\cdot\delta+\omega^{n^2}\cdot 2} = (\omega^\kappa)^{\delta+1} = (\omega^\kappa)^\gamma.\end{aligned}$$

Case 3. γ is a limit ordinal. Then, using the transfinite induction hypothesis,

$$|\phi_\gamma| = \sup_{\delta<\gamma} |\phi_\delta| \le \sup_{\delta<\gamma}(\omega^\kappa)^\delta = (\omega^\kappa)^\gamma.$$

Thus (19) has been established.

(2.11) Finally we are in position to finish our proof. As noted in (2.2), $2^{X^{n+1}} = \phi_\mu$, where $\mu = \omega^{2(n+1)-1} = \omega^{2n+1}$. Using this and (19) with κ denoting $\omega^{n^2 \cdot 2}$ we have

$$|2^{X^{n+1}}| = |\phi_\mu| = \left|\sup_{\gamma<\mu} \phi_\gamma\right| = \sup_{\gamma<\mu} |\phi_\gamma| \le \sup_{\gamma<\mu}(\omega^\kappa)^\gamma = (\omega^\kappa)^\mu$$
$$= (\omega^{\omega^{n^2}\cdot 2})^{\omega^{2n+1}} = \omega^{\omega^{(n+1)^2}}.$$

Thus our induction proof of the right-hand equality of (13) is concluded.

POSTSCRIPT. In a recent article *The limit behaviour of exponential terms* (to appear in Fund. Math.), B. Dahn constructs an ordered abelian group G and an order preserving embedding of a field of (germs at ∞ of) exponential functions into the formal power series field $\mathbf{R}((t^G))$, $t = X^{-1}$. According to Dahn (private communication), Hardy's field LE of logarithmic-exponential functions can also be represented by a field of formal power series over $\mathbf{R}$ (with exponents in a group $\supset G$). Since Sk $\subset$ LE, this offers hope that the techniques from our paper can be extended to all of Sk. For example, one can use the details of Dahn's embedding to prove by induction on the 'complexity' of f and g in Sk, with $g \neq 0$, that

$$\lim_{x\to\infty} f(x)/g(x) \in E^+ \cup \{0, \infty\}.$$

REFERENCES

1. H. Bachman, *Transfinite Zahlen*, 2nd ed., Springer-Verlag, Berlin, Heidelberg and New York, 1967.
2. N. G. de Bruijn, *Asymptotic methods in analysis*, North-Holland, Amsterdam, 1958.
3. A. Ehrenfeucht, *Polynomial functions with exponentiation are well ordered*, Algebra Universalis **3** (1973), 261–262.
4. R. Gurevič, *Transcendent numbers and eventual dominance of exponential functions*, Abstracts Amer. Math. Soc. **4** (1983), 310.
5. G. H. Hardy, *Orders of infinity: The 'infinitärcalcül' of Paul du Bois-Reymond*, 2nd ed., Cambridge Univ. Press, Cambridge, 1924.
6. H. Levitz, *An initial segment of the set of polynomial functions with exponentiation*, Algebra Universalis **7** (1977), 133–136.
7. ——, *An ordinal bound for the set of polynomial functions with exponentiation*, Algebra Universalis **8** (1978), 233–243.
8. ——, *The Cartesian product of sets and the Hessenberg natural product of ordinals*, Czechoslovak Math. J. **29** (1974), 353–357.
9. ——, *Calculation of an order type: An application of non-standard methods*, Z. Math. Logik Grundlag. Math. **28** (1982), 219–228.
10. D. Richardson, *Solution of the identity problem for integral exponential functions*, Z. Math. Logik Grundlag. Math. **15** (1969), 333–340.
11. T. Skolem, *An ordered set of arithmetic functions representing the least ε-number*, Norske Vid. Selsk. Forh. (Trondheim) **29** (1956), 54–59.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306