

ON SUBALGEBRAS OF SIMPLE LIE ALGEBRAS OF CHARACTERISTIC $p > 0$

BY

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ABSTRACT. The main results of the paper are Theorems I.5.1, II.1.3 and III.2.1. Theorem I.5.1 states that if a maximal subalgebra M of a simple finite-dimensional Lie algebra G has solvable quotients of dimension ≥ 2 , then every nilpotent element of H acts nilpotently on G . Theorem II.1.3 states that if such a simple Lie algebra G contains a maximal subalgebra which is solvable, then G is Zassenhaus-Witt algebra. Theorem III.2.1 states that certain \mathbb{Z} -graded finite-dimensional simple Lie algebras are either classical or the difference between the number of nonzero positive and negative homogeneous components is large.

In the present paper we study subalgebras of simple finite-dimensional Lie algebras in characteristic p . We assume throughout that the ground field is algebraically closed of characteristic $p \geq 7$. In the case $p = 3$ there are counterexamples to some of our results. In characteristic 5 most proofs do not go through, and additional work does not seem to help.

In the study of simple Lie algebras in characteristic p it is important to establish for each such an algebra L that there exists in it a maximal subalgebra L_0 which defines a *long filtration* of L . This means that there is a nilpotent ideal $L_1 \neq 0$ of L_0 such that L_1 acts nilpotently on L .

Our first aim is to establish that if L_0 does not define a long filtration, then L_0 must satisfy certain conditions. Our result in this direction is Theorem I.5.1 which implies, in particular, that if L_0 has solvable quotients of dimension ≥ 2 , then every nilpotent ideal of L_0 acts nilpotently on L . The proof of this result is based on results of H. Zassenhaus and R. Block about representations of Lie algebras and on ideas of V. Kac and I. L. Kantor about realizations of graded Lie algebras.

In trying to apply Theorem I.5.1 one has to consider the case when a maximal subalgebra L_0 has both nilpotent ideals and large solvable quotients. The simplest case which comes to mind where both these conditions hold is that of a solvable maximal subalgebra. Certain special cases of a solvable maximal subalgebra were considered by several people, among them by J. R. Schue [11] and M. I. Kuznetsov [9]. We heavily use results and ideas of the Kuznetsov paper. Because of the existence of a long filtration the question of classifying simple Lie algebras L with a solvable maximal subalgebra L_0 is reduced to the question of classifying certain

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graded simple Lie algebras whose zeroth term is solvable. This turns out to be rather hard. At any rate, the result is not surprising. It says that L is of type A_1 or W_1 . This is the main result, Theorem II.1.3.

Part III contains results about graded Lie algebras which are used in Part II but which seem to have wider applicability. They were also removed to streamline (to an extent) exposition in Part II. We also include in Part III some other results which seem to be in the spirit of that part. In particular, we include Theorem III.2.1 which states that a simple graded Lie algebra $G = \bigoplus_{i=-q}^r G_i$, satisfying certain additional conditions, is either classical (and then $q = r$) or $\max(q, r) \geq ((p-1)/2) \min(q, r)$.

To conclude this account of the results of the present paper we would like to mention that the recent work of R. Block and R. Wilson has resulted in much more final structure results and will in the near future result in a classification of simple Lie p -algebras.

We do not give indications of proofs in this introduction because the paper is split into several parts which are fairly independent and straightforward. More precisely, Part II needs only one result from Part I; however, it uses several auxiliary results placed in Part III. Part III, the Appendix, is completely independent.

Notation and convention. We assume that the ground field k is algebraically closed of characteristic $p \geq 7$. Our terminology is mostly standard. A Lie algebra L over k is classical if it is a direct sum of simple classical Lie algebras.

For subspaces X, Y, Z, \dots of L we denote by $\langle X, Y, Z, \dots \rangle$ the subalgebra of L generated by X, Y, Z, \dots . For subspaces X, Y of L such that $[X, Y] \subseteq Y$ we denote by $\text{ad}_Y X$ the subspace of $\text{End}_k Y$ consisting of linear operators $y \rightarrow [x, y]$, $x \in X$; $\text{ad } X$ refers to the case $Y = L$. A subspace X of L is *nil* if $(\text{ad } X)^{\dim L} = 0$.

We are often led to consider divided power algebras. We denote by $B_m(F)$ the divided power algebra in m variables associated to a flag F (i.e., to a sequence of m integers). In one or two variables we write

$$B_1(n) = k\{x\} = \bigoplus_{0 \leq i < p^n} kx^{(i)}, \quad B_2(q, r) = k\{x, y\} = \bigoplus_{\substack{0 \leq i < p^q \\ 0 \leq j < p^r}} kx^{(i)}y^{(j)}.$$

The Lie algebra of special derivations of $B_m(F)$ is denoted $W_m(F)$. When we say that L is of type W_1 it means that L is isomorphic to $W_1(n)$ for some n . The rings $B_m(F)$ have ring filtration (by total degree, for example, $B_{1,m}(n) = \bigoplus_{m \leq i < p^n} kx^{(i)}$). This induces a filtration on $W_m(F)$ via $W_{m,j}(F) = \{x \in W_m(F) | x(B_{m,r}(n)) \subseteq B_{m,r+j}(n) \text{ for all } r\}$. Automatically $W_m(F) = W_{m,-1}(F)$. We usually use C to denote the maximal ideal of $B_m(F)$. Thus $C = B_{m,1}(F)$. Then $W_{m,0}(F) = \{x \in W_m(F) | x(C) \subseteq C\}$. Both $B_m(F)$ and $W_m(F)$ can be made into graded algebras via $B_m(F) = \bigoplus B_m(F)_{(i)}$, where $B_m(F)_{(i)}$ consists of homogeneous elements of total degree i . Then $W_m(F)_{(i)} = \{x \in W_m(F) | x(B_m(F)_{(j)}) \subseteq B_m(F)_{(i+j)} \text{ for all } j\}$. This grading is called *standard*. If $m = 1$, then $B_1(n)_{(i)} = kx^{(i)}$ and $W_1(n)_{(j)} = ke_j$, where $e_j = x^{(j+1)}\partial$, $j = -1, 0, \dots$, $\partial x^{(j)} = x^{(j-1)}$. In particular, $[e_{-1}, e_j] = e_{j-1}$, $[e_0, e_j] = je_j$.

Finally, we record some terminology and notation, related to \mathbf{Z} -graded and filtered Lie algebras (see [12]). A Lie algebra L with a sequence $\{L_i\}$ of subspaces is

filtered if $L \supset \cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots$ and $[L_i, L_j] \subseteq L_{i+j}$. A Lie algebra $G = \bigoplus_{i=-q}^r G_i$ is graded if $[G_i, G_j] \subseteq G_{i+j}$. We call q the depth and r the height of the graded Lie algebra. If L is filtered with filtration L_i , then the associated graded algebra is $G = \bigoplus G_i$, $G_i = L_i/L_{i+1}$, where, for $x \in L_i$, $y \in L_j$, we define $[x + L_{i+1}, y + L_{j+1}] = [x, y] + L_{i+j+1}$. This gives G a structure of graded Lie algebra. If $G = \bigoplus G_i$ is a \mathbb{Z} -graded Lie algebra, then $M(G)$ is the largest ideal of G contained in $\bigoplus_{i \leq -2} G_i$. It is defined constructively as

$$M^0(G) = 0, \quad M^{i+1}(G) = \left\{ x \in \bigoplus_{i \leq -2} G_i \mid \left[x, \bigoplus_{i > 0} G_i \right] \subset M^i(G) \right\},$$

$$M(G) = \bigcup_i M^i(G).$$

It is easy to see that $M(G)$ inherits a grading from G : $M(G)_i = M(G) \cap G_i$, $M(G) = \bigoplus M(G)_i$. Next, $A(G)$ is the ideal of G generated by $\bigoplus_{i < 0} G_i$. It is also graded, $A(G) = \bigoplus A(G)_i$, $A(G)_i = A(G) \cap G_i$. We have $A(G)_i = G_i$ for $i < 0$, $A(G)_0 = \Sigma[G_i, G_{-i}]$, etc.

PART I. MAXIMAL SUBALGEBRAS OF SIMPLE LIE ALGEBRAS

The purpose of this part is to develop some techniques for working with Lie algebras when there is no long filtration. The main result says that a maximal subalgebra which has large solvable quotients cannot be too bad. Our main tools are representation-theoretic (after H. Zassenhaus and R. Block) and those developed by V. Kac and I. Kantor in their study of graded Lie algebras.

We start by outlining rudiments of the representation theory of Lie algebras in characteristic $p > 0$ after H. Zassenhaus. This permits us to naturally distinguish a class of subalgebras of a Lie algebra which we call "bad". In the case when a bad subalgebra has large solvable quotients we are able to arrive at a continuation using R. Block's theory of differentially irreducible modules together with rudiments of I. Kantor's realization of graded Lie algebras.

Recall that the ground field is algebraically closed of characteristic $p > 5$.

1. Recollections on representations of Lie algebras (after H. Zassenhaus [16]). Let G be a Lie algebra of dimension n over k . Let $U(G)$ be the universal enveloping algebra of G , let Z be its center, and let Z_0 (denoted \mathfrak{o} in [16]) be the subalgebra of Z generated by $k \cdot 1$ and by $Z \cap (G + kG^p + kG^{p^2} + \cdots)$. Set $G^x = \text{Spec } Z_0$. In [16, p. 13 (top)] Zassenhaus establishes that $G^x \simeq \mathbb{A}^n$ (an n -dimensional affine space over k), and at the very end of the paper [16] he states that G^x has a structure of a vector space over \mathbb{F}_p . The additive structure of G^x is described by him as follows. A point $l \in G^x$ is an algebra homomorphism $l: Z_0 \rightarrow k$. For every finite-dimensional Z_0 -module N we have a Fitting decomposition $N = \bigoplus_{l \in G^x} N(l)$, where $N(l) = \{ n \in N \mid (a - l(a))^{\dim N} n = 0 \text{ for all } z \in Z_0 \}$. If $N = N(l)$ we say that N is isotypic and write l_N for the corresponding l . Irreducible modules are automatically isotypic. By [16, Theorem 12 and subsequent discussion] we see that a tensor product of isotypic Z_0 -modules M and N is isotypic and $l_{M \otimes N} = l_M + l_N$. Let us remark that every G -module can be considered as a $U(G)$ -module and, therefore, by restriction as a

Z_0 -module. We call a G -module isotypic if it is isotypic as the Z_0 -module. We record the results of [16] which were discussed above as

1.1. FACTS. (i) In the center Z of $U(G)$ consider the subalgebra Z_0 generated by 1 and $Z \cap (\sum_{i=0}^{\infty} kG^{p^i})$. Then $\text{Spec } Z_0$ has a structure of a vector space over \mathbb{F}_p .

(ii) For two isotypic G -modules M and N we have $l_{M \oplus N} = l_M + l_N$, where the addition is that of G^x .

1.2. LEMMA. Consider G as a G -module via adjoint action. Then G is an isotypic G -module and $l_G = 0$.

PROOF. Let $G = \bigoplus_{l \in G^x} G(l)$ (a direct sum of G -modules). Then each $G(l)$ is a G -module and, therefore, an ideal of G . Evidently, $[G(l_1), G(l_2)] \subseteq G(l_1) \cap G(l_2) = 0$ if $l_1 \neq l_2$. On the other hand, by 1.1(ii) we have $[G(l_1), G(l_2)] \subseteq G(l_1 + l_2)$ whence $[G(l), G(l)] = 0$ if $l \neq 0$. Thus $\tilde{G} = \bigoplus_{l \neq 0} G(l)$ is in the center of G and $G = G(0) \oplus \tilde{G}$ (a direct sum of algebras). Let M be an irreducible nontrivial module for $G(l)$, $l \neq 0$. Make M into a G -module by setting $G(l')M = 0$ if $l \neq l'$. By 1.1(ii) we have $l_{G(l) \otimes M} = l_{G(l)} + l_M = l + l_M$. On the other hand, the action of G on M determines a map $G \otimes M \rightarrow M$ which factors through $G(l) \otimes M$ as M is a trivial $\bigoplus_{l' \neq l} G(l')$ -module. Thus we have $l_{G(l) \otimes M} = l_M$ whence $l = 0$, a contradiction.

1.2.1. REMARKS. (i) M can be a module with trivial action as well.

(ii) A more explicit description of the group structure on G^x would show directly that $l_G = 0$. But we avoid this approach.

1.3. LEMMA. Let $n \in G$ be nilpotent and let m be an integer such that $(\text{ad } n)^{p^m} = 0$. Then $n^{p^m} \in Z_0$.

PROOF. Since $\text{ad } y$ can be written in $U(G)$ as $\text{ad } y = L_y - R_y$, where L_y (resp., R_y) is the operator of left (resp., right) multiplication by y , and since L_y and R_y commute, we have by the binomial formula

$$(\text{ad } y)^q = \sum_{i=0}^q (-1)^i \binom{q}{i} L_y^{q-i} R_y^i = \sum_{i=0}^q (-1)^i \binom{q}{i} L_{y^{q-i}} R_{y^i}.$$

In particular, $0 = (\text{ad } n)^{p^m} = L_{n^{p^m}} - R_{n^{p^m}}$ whence $n^{p^m}x = xn^{p^m}$ for any $x \in U(G)$. Thus $n^{p^m} \in Z$. But then $n^{p^m} \in Z \cap (\sum_{i=0}^{\infty} kG^{p^i}) \subseteq Z_0$, as claimed.

1.4. COROLLARIES. Let M be a simple G -module, $n \in G$, $(\text{ad } n)^{p^m} = 0$.

(i) Then n acts on M as $\lambda \cdot \text{Id}_M + (\text{nilpotent operator})$. Denote this λ by $\chi_M(n)$.

(ii) We have $l_M(n^{p^m}) = (\chi_M(n))^{p^m}$. In particular, if $l_M(n^{p^m}) = 0$, then $\chi_M(n) = 0$, i.e., n acts on M as a nilpotent operator.

PROOF. Since $n^{p^m} \in Z$, it acts as a scalar on M . Let $\lambda_1, \dots, \lambda_t$ be the eigenvalues of n . We have, therefore, $\lambda_1^{p^m} = \lambda_2^{p^m} = \dots = \lambda_t^{p^m}$, whence $\lambda_1 = \lambda_2 = \dots = \lambda_t$ as claimed in (i). By definition of l_M we know that $n^{p^m} - l_M(n^{p^m})$ is nilpotent. Thus $l_M(n^{p^m}) = (\chi_M(n))^{p^m}$. This proves (ii).

1.4.1. REMARKS. (i) Lemma 1.3 and its corollaries enable us to use condition $l_M = 0$: if $l_M = 0$, then every nilpotent element of G acts on M as a nilpotent transformation.

(ii) Actually χ , considered as a function on the set of nilpotents of G suffices for our purposes. It obviously possesses the properties we need: $\chi_{M \oplus N} = \chi_M + \chi_N$ (if M and N are "isotypic") and $\chi_G = 0$. We went to the trouble of looking at Z_0, l_M , etc., only because it seems to be the proper generality.

(iii) In the case when G is a p -algebra, the statements similar to those of the present section are discussed in a paper of V. Kac and the author [13]. However, even for p -algebras l_V may have different meaning here and in [13].

2. Bad maximal subalgebras. Let H be a subalgebra of a Lie algebra G . Let $U(H)$ be the universal enveloping algebra of H , and let Z be the center of $U(H)$. Then we have Fitting decomposition

$$G = \bigoplus_{\lambda \in \text{Spec } Z} G_\lambda,$$

where $G_\lambda = \{y \in G \mid (a - \lambda(a))^{\dim G} y = 0 \text{ for all } a \in Z\}$. Let $\pi: \text{Spec } Z \rightarrow \text{Spec } Z_0 = H^x$ be the canonical projection induced by the embedding $Z_0 \rightarrow Z$. Set $G(l) = \bigoplus_{\pi(\lambda)=l} G_\lambda$.

2.1. DEFINITION. A subalgebra H of a Lie algebra G is called *bad* if $G \neq G(0)$. (By Lemma 1.2, H is automatically a proper subalgebra.)

Our aim in this section is to prove

2.2. THEOREM. *Let G be a Lie algebra and let H be its maximal subalgebra. Suppose that G contains an ideal \hat{G} such that*

- (a) \hat{G} is a simple Lie algebra;
- (b) any minimal ideal of G coincides with \hat{G} ;
- (c) $G = \hat{G} + H$.

Suppose that H is a bad subalgebra of G . Then there are \mathbb{Z}/p -gradings $G = \bigoplus_{i \in \mathbb{Z}/p} G_i$ and $\hat{G} = \bigoplus_{i \in \mathbb{Z}/p} \hat{G}_i$ of G and \hat{G} such that $\hat{G}_i \subseteq G_i$ and

- (i) $G_0 = H, \hat{G}_0 = H \cap \hat{G} \neq 0$;
 - (ii) $G_i = \hat{G}_i, G_i \neq 0$, for all $i \neq 0$;
 - (iii) $G_i, i \neq 0$, is an irreducible and faithful G_0 -module;
 - (iv) $[\hat{G}_i, \hat{G}_j] = \hat{G}_{i+j}$ for all $(i, j) \neq (0, 0)$.
- (Note that if G is simple then $G = \hat{G}$.)

PROOF. By Fact 1.1(ii) we have $[G(l_1), G(l_2)] \subseteq G(l_1 + l_2)$. Thus $G(0)$ is a subalgebra. We have $H \subseteq G(0)$ by Lemma 1.2. Since $G \neq G(0)$ and since H is maximal we have $H = G(0)$. Now take $l \neq 0$ such that $G(l) \neq 0$. Then $G(0)$ and $G(l)$ generate a subalgebra which is contained in $\bigoplus_{i \in \mathbb{Z}/p} G(il)$. Since H is maximal and $\bigoplus G(il) \neq H$ we must have that $G = \bigoplus G(il)$. Set $\hat{G}(il) = G(il) \cap \hat{G}$. Since \hat{G} is an ideal we have $\hat{G} = \bigoplus \hat{G}(il)$. Since $G = \hat{G} + H$ and $H \subseteq G(0)$ we have $\hat{G}(il) = G(il)$ for $i \neq 0$.

Now let V be a simple H -submodule of $G(l)$. Consider the subalgebra \tilde{G} of G generated by V . Clearly, $\tilde{G} \subseteq \hat{G}$ (since $G(l) = \hat{G}(l) \subseteq \hat{G}$). On the other hand, \tilde{G} is spanned by commutators $c(v_1, \dots, v_n) = [v_1[v_2 \cdots [v_{n-1}, v_n] \cdots]]$ with $v_i \in V$ and $n \in \mathbb{Z}, n \geq 1$. We have $c(v_1, \dots, v_p) \in G(0) = H$ for all choices of $v_1, \dots, v_p \in V$. Therefore $c(v, v_1, \dots, v_p) = [v, c(v_1, \dots, v_p)] \in V$ for all $v \in V$ and all $v_i \in V, i = 1, \dots, p$. Therefore \tilde{G} is spanned by the $c(v_1, \dots, v_n)$ with $v_i \in V, p \geq n \geq 1$. Now note that $H = G(0)$ normalizes \tilde{G} and therefore $H + \tilde{G}$ is a subalgebra of G . Since H

is maximal we have $G = H + \tilde{G}$. Therefore \tilde{G} is an ideal of G . By condition (b) we have $\hat{G} \subseteq \tilde{G}$, that is, $\tilde{G} = \hat{G}$. We can write the equality $\tilde{G} = \hat{G}$ as

$$(2.2.i) \quad \hat{G}(il) = [V[V \cdots [V, V] \cdots]] \text{ (i times), } i \leq p,$$

First apply (2.2.1) and get that $\hat{G}(l) = V$ so that $\hat{G}(l)$ is irreducible. Now consider (2.2.p). If $\hat{G}(0) = 0$, then (2.2.p) says that \hat{G} is nilpotent, a contradiction. So $\hat{G}(0) \neq 0$ and now (2.2.p) implies that $G(il) \neq 0$ for all $i \in \mathbb{Z}/p$ and that $[G(-l), C(l)] = \hat{G}(0)$.

Since $G(il) \neq 0$ for all i we can apply the argument above to il , $i \neq 0$, instead of l . This gives us that all $G(il)$, $i \neq 0$, are irreducible and $[G(il), G(-il)] = \hat{G}(0)$ for all $i \neq 0$.

Let us now show that H is faithful on each $G(il)$, $i \neq 0$. Suppose that \tilde{H} is the kernel of the action of H on $G(l)$. Then $[\tilde{H}, G(l)] = 0$ whence by (2.2.i), $p \geq i > 0$, we have that $[\tilde{H}, \hat{G}(il)] = 0$, that is, \tilde{H} centralizes \hat{G} . Since \tilde{H} is an ideal of H and $G = H + \hat{G}$, we conclude that \tilde{H} is an ideal of G . Thus $\tilde{H} \supseteq \hat{G}$, a contradiction (since $[\tilde{H}, \hat{G}] = 0$ and \hat{G} is simple).

Now we want to show that $[\hat{G}(il), \hat{G}(jl)] = \hat{G}((i+j)l)$. First we remark that $[\hat{G}(0), G(il)] = G(il)$ for $i \neq 0$. Indeed, since $G(il)$ is irreducible for $G(0)$, and $\hat{G}(0)$ is an ideal of $G(0)$, we have that $[\hat{G}(0), G(il)] = 0$ or $G(il)$. The first case is impossible since $G(0)$ is faithful on $G(il)$.

Next (2.2.*) applied to jl instead of l , $p > j > 0$, says that $[\hat{G}(jl), \hat{G}(mjl)] = \hat{G}((m+1)jl)$ for all $0 < m \leq p$. Since $j \neq 0$ we can take $m = i/j$, whence $[\hat{G}(jl), \hat{G}(il)] = \hat{G}((i+j)l)$.

To conclude the proof we set $G_i = G(il)$.

2.3. REMARK. It should be remembered that the indexing of G_i depends on the choice of l . We shall use this freedom later.

2.4. PROPOSITION. *Let H be a maximal subalgebra of a Lie algebra G . Suppose that H is bad. Then*

- (i) $G = \bigoplus_{i \in \mathbb{Z}/p} G(il)$ for some $l \in H^x$,
- (ii) $G_0 = H$;
- (iii) $G(il)$, $i \neq 0$, are irreducible for H .

Proof of these statements is contained in the first part of the proof of Theorem 2.2.

2.5. PROPOSITION. *Let H be a maximal subalgebra of a Lie algebra G . Suppose that H is not bad. Then every nilpotent element of H acts nilpotently on G . In particular, if N is a nilpotent ideal of H , then N acts nilpotently on G .*

PROOF. If H is not bad, then $l_G = 0$ (i.e. $G = G(0)$), whence our claim follows in view of Corollary 1.4(ii); see also Remark 1.4.1(i).

2.6. REMARK. The first step of the proof of Theorem 2.2 can be seen as obtaining a statement that a bad maximal subalgebra is the set of fixed points of a subgroup scheme isomorphic to μ_p (p th roots of unity) in the automorphism group scheme of G . For classical Lie algebras G subgroups of $\text{Aut } G$ isomorphic to μ_n were classified

by V. Kac in [7]. Of course, the fixed point subalgebras are not bad in this case, but nevertheless, the conclusions of Theorem 2.2 hold. One can also give a slight “generalization” of Theorem 2.2 as follows.

2.7. THEOREM. *Let B be a simple Lie algebra and let \mathcal{H} be a subgroup scheme of multiplicative type in $\text{Aut } G$. If the subalgebra corresponding to the trivial character of \mathcal{H} is maximal, then $\mathcal{H} = \mu_p$ and the conclusions of Theorem 2.2 hold.*

3. Generalities about graded Lie algebras (after V. Kac and I. Kantor). Our aim is to establish that certain maximal subalgebras of a simple Lie algebra G cannot be bad. To this end we consider a \mathbb{Z}/p -graded simple Lie algebra $G = \bigoplus_{i \in \mathbb{Z}/p} G_i$ which is generated by its subspace $G_{-1} \oplus G_0 \oplus G_1$ (actually, in our case, in view of Theorem 2.2, G is generated by G_{-1} only). Such direct sum of spaces, satisfying natural conditions which automatically hold in our case, was called a *local* Lie algebra by V. Kac [5, 6]. He used local Lie algebras to study \mathbb{Z} -graded Lie algebras. In particular, he considered a \mathbb{Z} -graded Lie algebra $\tilde{G} = \bigoplus \tilde{G}_i$ with $\tilde{G}_i = G_i$ for $i = \pm 1, 0$, which is generated by $G_{-1} \oplus G_0 \oplus \dots$, but does not have graded ideals intersecting its local part. This construction of V. Kac is a variant of the generators and relations description of Lie algebras. I. Kantor in [8] gave a realization of the algebra \tilde{G} . His realization is obtained by actually increasing the algebra G to a certain universal algebra whose realization is quite easy to describe. However, to save space we do not give here the exposition of results of Kac and Kantor but just apply what is immediately useful for our purposes.

3.1. Consider again a \mathbb{Z}/p -graded simple Lie algebra $G = \bigoplus G_i$, satisfying the conclusions of Theorem 2.2. The map $\text{ad } x: G_{-1} \rightarrow G_0$ for $x \in G_1$ defines $\text{ad } x$ as an element of $\text{Hom}_k(G_{-1}, G_0) = G_0 \otimes G_{-1}^*$. Thus we get a map $G_1 \rightarrow G_0 \otimes G_{-1}^*$. This map is injective for: (a) it is not zero because of Theorem 2.2(iv), and (b) its kernel is a G_0 -submodule which must be zero in view of (a) above and Theorem 2.2(iii). The commutator between $G_0 \otimes G_{-1}^*$ and G_{-1} is given by $[g \otimes v^*, v] = v^*(v)g$ for $v \in G_{-1}$, $v^* \in G_{-1}^*$ and $g \in G_0$. We replace for a moment $G_{-1} \oplus G_0 \oplus G_1$ by $G_{-1} \oplus G_0 \oplus (G_0 \otimes G_{-1}^*)$ and we consider the free Lie algebra generated by this subspace modulo relations there; we call it F . We have: F is \mathbb{Z} -graded, $F_{-1} = G_{-1}$, $F_0 = G_0$, $F_1 = G_0 \otimes G_{-1}^*$. On the other hand, $F_{-2} = [F_{-1}, F_{-1}]$ which means that F_{-2} is a quotient of the G_0 -module of the skew-symmetric square of G_{-1} . As $p \neq 2$ we can consider F_{-2} as a quotient module of $G_{-1} \wedge G_{-1}$, where $G_{-1} \wedge G_{-1} = \{A \in G_{-1} \otimes G_{-1} \mid A' = -A\}$, $(a \otimes b)' = b \otimes a$. The commutator of $v_1 \wedge v_2 \in G_{-1} \wedge G_{-1}$ with G_0 is given by $[g, v_1 \wedge v_2] = gv_1 \wedge v_2 + v_1 \wedge gv_2$, and with $G_0 \otimes G_{-1}^*$ it is given by $[g \otimes v^*, v_1 \wedge v_2] = v^*(v_1)gv_2 - v^*(v_2)gv_1$. Indeed,

$$\begin{aligned} [g \otimes v^*, [v_1, v_2]] &= [[g \otimes v^*, v_1]v_2] + [v_1[g \otimes v^*, v_2]] \\ &= [v^*(v_1)g, v_2] + [v_1, v^*(v_2)g] = v^*(v_1)gv_2 - v^*(v_2)gv_1. \end{aligned}$$

Now returning back to our G we identify G_1 with a G_0 -submodule of $G_0 \otimes G_{-1}^*$. Consider the subalgebra \bar{G} of F generated by $G_{-1} \oplus G_0 \oplus G_1$. We have a surjective map $\bar{G} \rightarrow G$ which is an isomorphism on $G_{-1} \oplus G_0 \oplus G_1$. Let $D_{-2} \subseteq F_{-2}$ be the intersection of the kernel of this map with F_{-2} . Since the map is injective on $G_{-1} \oplus G_0 \oplus G_1$ we see that $D_{-2} = \{A \in F_{-2} \mid [G_1, A] = 0\}$.

We sum this up as

3.2. THEOREM. *Let $G = \oplus G_i$ be a graded Lie algebra satisfying the conclusions of Theorem 2.2. Then:*

(i) G_1 can be identified with a G_0 -submodule of $G_0 \otimes G_{-1}^*$ with $[g_1, g_2 \otimes v^*] = [g_1, g_2] \otimes v^* + g_2 \otimes g_1 v^*$ and $[v, g \otimes v^*] = -v^*(v)g$ for $g, g_1, g_2 \in G_0, v^* \in G_{-1}^*$ and $v \in G_{-1}$.

(ii) G_{-2} can be identified with a quotient of $G_{-1} \wedge G_{-1}$ by $D_{-2} = \{A \in G_{-1} \wedge G_{-1} \mid [x, A] = 0 \ \forall x \in G_1\}$, where the commutator of $G_{-1} \wedge G_{-1}$ with $G_0 \otimes G_{-1}^*$ is given by

$$[g \otimes v^*, v_1 \wedge v_2] = v^*(v_1)gv_2 - v^*(v_2)gv_1.$$

4. Generalities on ideals of the G_0 -term of a graded Lie algebra. Consider a local Lie algebra of V. Kac, i.e., a direct sum $G_{-1} \oplus G_0 \oplus G_1$ with G_0 a Lie algebra, $G_{\pm 1}$ its modules, and with a map of G_0 -modules $[\cdot, \cdot]: G_{+1} \otimes G_{-1} \rightarrow G_0$. We assume that $[G_{-i}, x] \neq 0$ for $0 \neq x \in G_i, i = \pm 1$. We make one further

4.1. ASSUMPTION. $[G_1, G_{-1}] = G_0$.

Let K be a proper ideal in G_0 . For a K -module V we denote by V^K (resp., by V_K) the largest submodule (resp., quotient) of V on which K acts trivially. The surjective map $[\cdot, \cdot]: G_1 \otimes G_{-1} \rightarrow G_0$ of G_0 -modules factors through $G_0 \rightarrow G_0/K$ to give a surjective map $G_1 \otimes G_{-1} \rightarrow G_0/K$. As K acts trivially on G_0/K we actually get a surjective map $(G_1 \otimes G_{-1})_K \rightarrow G_0/K$. Dualizing we get an injective map

$$(4.1.1) \quad (G_0/K)^* \rightarrow ((G_1 \otimes G_{-1})^*)^K = \text{Hom}_K(G_1, G_{-1}^*).$$

4.2. COROLLARIES. (i) $\dim(G_0/K) \leq \dim \text{Hom}_K(G_1, G_{-1}^*)$.

(ii) If G_1 and G_{-1} are simple K -modules then $\dim(G/K) = 1$ and $G_1 \simeq G_{-1}^*$ (as G_0 -modules)

(iii) If G_1 and G_{-1} are simple G_0 -modules, then $\dim(G_0/[G_0, G_0]) \leq 1$. Moreover, $G_1 \simeq G_{-1}^*$ if $G_0 \neq [G_0, G_0]$.

PROOF. (i) is a direct consequence of (4.1.1). In case (ii) we have $\dim \text{Hom}_K(G_1, G_{-1}^*) \leq 1$, whence $\dim(G_0/K) \leq 1$. Since $K \neq G_0$ it follows that $\dim(G_0/K) = 1$. But then G_0/K is a trivial G_0 -module. Thus the G_0 -module map $G_1 \otimes G_{-1} \rightarrow G_0/K$ determines a G_0 -invariant pairing between G_1 and G_{-1} . This pairing is nonzero because $[G_1, G_{-1}] = G_0$ and it is nondegenerate because G_1 and G_{-1} are simple G_0 -modules. Finally, (iii) is a partial converse of (ii). Namely, set $K = [G_0, G_0]$. Then G_0/K is a trivial G_0 -module. Thus the image of $(G_0/K)^*$ in $\text{Hom}_K(G_1, G_{-1}^*)$ is actually in $\text{Hom}_{G_0}(G_1, G_{-1}^*)$. Since both G_1 and G_{-1} are simple G_0 -modules we have $\dim \text{Hom}_{G_0}(G_1, G_{-1}^*) \leq 1$. Therefore $\dim(G_0/K) \leq 1$. If $\dim(G_0/K) = 1$, then $\text{Hom}_{G_0}(G_1, G_{-1}^*) \neq 0$, whence $G_1 \simeq G_{-1}^*$.

5. On the derived series of a bad maximal subalgebra. The purpose of this section (and of the first part of this paper as well) is to establish

5.1. THEOREM. *Let G be a simple Lie algebra and let H be a maximal subalgebra of G . If H is bad, then $\dim(H/[H, H]) \leq 1$ and $[[H, H], [H, H]] = [H, H]$. (Thus, solvable quotients of H are at most one-dimensional.)*

We know that G is \mathbb{Z}/p -graded, $G = \bigoplus_{i \in \mathbb{Z}/p} G_i$, with $G_0 = H$. We have certain freedom (cf. Remark 2.3) in choosing G_{-1} . Let us choose G_{-1} so that $\dim G_{-1} \geq \dim G_i$, $i \neq 0$. Our strategy is to show that the assumption that H has solvable quotients of dimension ≥ 2 leads to a contradiction with the assumption $\dim G_{-1} \geq \dim G_i$, $i \neq 0$.

Denote $H' = [H, H]$ and $H'' = [H', H']$.

5.2. LEMMA. $\dim(H/H') \leq 1$.

PROOF. This follows directly from Corollary 4.2(iii) and Theorem 2.2(iii) (Assumption 4.1 is satisfied in view of Theorem 2.2(iv)).

5.3. PROPOSITION. Assume that $H \neq H'$. Then G_{-1} is an irreducible H' -module.

PROOF. Suppose G_{-1} is not irreducible for H' . Then it is differentially irreducible with one derivation (as $\dim(H/H') = 1$). Therefore by R. Block [1, Theorem 2 or 2] we see that $G_{-1} \cong U \otimes B_n$ as an H -module. Here U is an irreducible H' -module, $B_n = k[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$, H' acts trivially on B_n , and B_n has no H -invariant ideals. By R. Ree [10] (explicit in M. I. Kuznetsov [9, Proposition 2.1]) B_n can be identified with an algebra of divided power series in one variable, $B_n \cong B_1(n) = k\{x\}$, and the action of H on B_n respects the structure of divided powers.

Now we look at F_{-2} (in notation of §3.1). Let $N = \dim U$ and let u_1, \dots, u_N be a basis of U . We have $F_{-2} = G_{-1} \wedge G_{-1}$ so that a generic element $b \in G_{-2}$ can be written as

$$b = \sum_{(i, i') < (j, j')} b_{i, i', j, j'} (u_{i'} \otimes x^{(i)}) \wedge (u_j \otimes x^{(j)}),$$

where $b_{i, i', j, j'} \in k$, $0 \leq i, j \leq p^n - 1$, $1 \leq i', j' \leq N$, and $<$ is the lexicographic order. For $b \in F_{-2}$ we set

$$\min(b) = \min\{(i, i') | b_{i, i', j, j'} \neq 0 \text{ for some } j, j'\}.$$

5.3.1. LEMMA. Let D_{-2} be the same as in Theorem 3.2(ii), and let $k\{\overline{x}\}$ be the set of divided powers with zero constant term. Then

$$D_{-2} \cap (U \otimes k\{\overline{x}\} \wedge U \otimes k\{\overline{x}\}) = 0.$$

Before proving this claim let us show how Proposition 5.3 follows from it. Since $G_{-2} = (G_{-1} \wedge G_{-1})/D_{-2}$ it follows that $U \otimes k\{\overline{x}\} \wedge U \otimes k\{\overline{x}\}$ is mapped monomorphically into G_{-2} . Thus

$$\dim G_{-2} \geq \dim(U \otimes k\{\overline{x}\} \wedge U \otimes k\{\overline{x}\}) = \frac{1}{2}N(p^n - 1)(N(p^n - 1) - 1).$$

On the other hand, by our choice of G_{-1} we have $Np^n = \dim G_{-1} \geq \dim G_{-2}$. Thus we have $Np^n \geq \frac{1}{2}N(p^n - 1)(N(p^n - 1) - 1)$. After elementary transformations this inequality becomes $p^{2n} + 1 + 1/N \leq (2 + 3/N)p^n$. This last inequality clearly does not hold if $p^n \geq 5$. Since $p > 5$ is among our general assumptions, we have a contradiction. Thus it remains only to prove Lemma 5.3.1.

5.3.2. Write $H = kd \oplus H'$. Let $\{g_i\}$ be a basis of H' . By Corollary 4.2(iii) we know that $G_1 = G'_{-1}$. By Theorem 3.2(i), we can consider G_1 as a submodule of

$H \otimes G_{-1}^*$. Then we can write $y \in G_1$ as $y = d \otimes \psi(y) + \sum g_i \otimes \psi_i(y)$, where ψ and the ψ_i are linear maps from G_1 to G_{-1}^* . As H/H' is a trivial H -module, ψ is an H -module map. As $G_1 \cong G_{-1}^*$ is irreducible, ψ is an isomorphism. Set $v^* = \psi(y)$ and $\varphi_i = \psi_i \psi^{-1}$. Then $y = d \otimes v^* + \sum g_i \otimes \varphi_i(v^*)$, where $\varphi_i: G_{-1}^* \rightarrow G_{-1}^*$ are linear maps. Here v^* is the element of G_{-1}^* corresponding to $y \in G_1$ under $G_1 \cong G_{-1}^*$; the coefficient of d is v^* as it is kd that determines the pairing between G_1 and G_{-1} .

As H' acts trivially on $k\{x\}$, $k\{x\}$ has no H -invariant ideals. As H preserves the divided power structure of $k\{x\}$, d acts on $k\{x\}$ as $a_0 \partial + \sum_{i>0} a_i x^{(i)} \partial$, where $\partial x^{(i)} = x^{(i-1)}$ and $a_0 \neq 0$. Therefore d acts on $G_{-1} = U \otimes k\{x\}$ as $\sum_{i \geq 0} A_i \otimes x^{(i)} \partial$, where $A_i \in \text{End } U$, $A_0 \neq 0$, and $(A \otimes P(x) \partial)(u \otimes Q(x)) = Au \otimes (P \cdot \partial Q)(x)$. Consider $V = U \otimes \sum_{i>0} kx^{(i)}$. Then $d^m V \subseteq A_0 U \otimes 1 + V$. Since $V + \sum_{m \geq 0} d^m V$ is H -invariant and G_{-1} is H -irreducible we have $A_0 U = U$, i.e., $\det A_0 \neq 0$.

The following lemma is formulated in a slightly more general form than is immediately necessary. We use notation \tilde{H} for the subalgebra of operators on $U \otimes k\{x\}$ preserving the filtration by degree of x in $U \otimes k\{x\}$.

5.3.3. LEMMA. *Let $y \in H \otimes G_{-1}^* (= F_1)$ be such that $y \equiv A \otimes \partial \otimes \varphi(v^*) \pmod{\tilde{H} \otimes G_{-1}^*}$, where $A \in \text{End } U$, $A \neq 0$, and $\varphi \in \text{End } G_{-1}^*$. Let*

$$b = \sum_{(i,i') < (j,j')} b_{i,i',j,j'} (u_{i'} \otimes x^{(i)} \wedge \tilde{u}_{j'} \otimes x^{(j)}) \in F_{-2}$$

and suppose that all summands in the expression for b are nonzero. Let $(m, q) = \min(b) > 0$. If $m > 0$, then the lowest term (with respect to powers of x) in the expression for $[b, y]$ is

$$\begin{aligned} & - \left[\sum_{i' < j'} b_{m,i',m,j'} \varphi(v^*) (u_{i'} \otimes x^{(m)}) \cdot Au_{j'} \right. \\ & \quad \left. - \varphi(v^*) \left(\sum_{(m,i') < (j,j')} b_{m,i',j,j'} (u_{j'} \otimes x^{(j)}) \right) \cdot Au_{i'} \right] \otimes x^{(m-1)}. \end{aligned}$$

REMARKS. (a) The expression $\varphi(v^*)(\sum u_{j'} \otimes x^{(j)})$ means the value of $\varphi(v^*) \in G_{-1}^*$ on $\sum u_{j'} \otimes x^{(j)} \in G_{-1}$.

(b) We do not rule out the case that the given expression (or even $[b, y]$ itself) is zero.

PROOF. It is clear that the lowest term belongs to $U \otimes x^{(m-1)}$. Therefore this lowest term is contained in $[b, A \otimes \partial \otimes \varphi(v^*)]$, and furthermore it is contained in

$$\begin{aligned} & \left[\sum_{(m,i') < (j,j')} b_{m,i',j,j'} (u_{i'} \otimes x^{(m)} \wedge u_{j'} \otimes x^{(j)}), A \otimes \partial \otimes \varphi(v^*) \right] \\ & = - \sum_{(m,i') < (j,j')} b_{m,i',j,j'} \left([\varphi(v^*)(u_{i'} \otimes x^{(m)}) \cdot A \otimes \partial, u_{j'} \otimes x^{(j)}] \right. \\ & \quad \left. + [u_{i'} \otimes x^{(m)}, \varphi(v^*)(u_{j'} \otimes x^{(j)}) \cdot A \otimes \partial] \right) \\ & = - \sum_{(m,i') < (j,j')} b_{m,i',j,j'} \left(\varphi(v^*)(u_{i'} \otimes x^{(m)}) \cdot Au_{j'} \otimes x^{(j-1)} \right. \\ & \quad \left. - \varphi(v^*)(u_{j'} \otimes x^{(j)}) \cdot Au_{i'} \otimes x^{(m-1)} \right). \end{aligned}$$

Collecting together the terms from $U \otimes x^{(m-1)}$, we readily see that they have the form claimed in Lemma 5.3.3.

5.3.4. Proof of Lemma 5.3.1 concluded. Set $Q = (\sum_{i>q} k \cdot Au_i) \otimes x^{(m-1)} + \sum_{j \geq m} U \otimes x^{(j)}$. Then it follows from Lemma 5.3.3 that

$$[b, y] \equiv \varphi(v^*) \left[\sum_{(m,q) < (j,j')} b_{m,q,j,j'} (u_{j'} \otimes x^{(j)}) \right] \cdot Au_q \otimes x^{(m-1)} \pmod{Q}.$$

Now let $b \in D_{-2}$ be such that $m > 0$ in $(m, q) = \min(b)$ (i.e. $b \in D_{-2} \cap (U \otimes k\{x\} \wedge U \otimes k\{\bar{x}\})$). Then $[b, y] = 0$ since $b \in D_{-2}$, whence $[b, y] \equiv 0 \pmod{Q}$. Since A is nondegenerate and $\varphi = \text{Id} \in \text{End } G_{-1}^*$ we get from the above expression for $[b, y] \pmod{Q}$ that $\sum_{(m,q) < (j,j')} b_{m,q,j,j'} (u_{j'} \otimes x^{(j)}) = 0$, i.e., $b_{m,q,j,j'} = 0$ for $(j, j') > (m, q)$. This contradicts the construction of $(m, q) = \min(b)$. This contradiction proves Lemma 5.3.1.

5.4. PROOF OF THEOREM 5.1. It remains to prove that $H'' = H'$. Suppose that $H'' \neq H'$. Then there is an ideal K of H , $H' \supset K \supseteq H''$ such that $\dim(H/K) = 2$, $H = kd + kh + K$ with $[d, h] = h \pmod{K}$, and $H' = kh + K$.

5.4.1. As in 5.3.2 we write any element $y \in G_1$ in the form $y = d \otimes v^* + h \otimes \tilde{\varphi}(v^*) + \sum \tilde{g}_i \otimes \tilde{\varphi}_i(v^*)$, where $v^* \in G_{-1}^*$ corresponds to $y \in G_1$ under the (fixed) isomorphism $G_1 \simeq G_{-1}^*$, $\tilde{\varphi}, \tilde{\varphi}_i \in \text{End } G_{-1}^*$, and $\{\tilde{g}_i\}$ is a basis of K .

For $x \in H$ we know that $[x, y]$ corresponds to $xv^* \in G_{-1}^*$ under the isomorphism $G_1 \simeq G_{-1}^*$. We use this fact for $x = d$, $x = h$, and $x \in K$. We have

$$[d, y] \equiv d \otimes dv^* + h \otimes \varphi(dv^*) \pmod{K \otimes G_{-1}^*}.$$

On the other hand, a direct computation gives

$$[d, y] \equiv d \otimes dv^* + h \otimes \varphi(v^*) + h \otimes d\varphi(v^*) \pmod{K \otimes G_{-1}^*}.$$

Therefore $\varphi(dv^*) = \varphi(v^*) + d\varphi(v^*)$ or

$$(5.4.1.1) \quad \varphi(dv^*) - d\varphi(v^*) = \varphi(v^*).$$

Similarly, $[h, y] \equiv d \otimes hv^* + h \otimes \varphi(hv^*) \pmod{K \otimes G_{-1}^*}$ or, by direct computation, $[h, y] \equiv -h \otimes v^* + d \otimes hv^* + h \otimes h\varphi(v^*) \pmod{K \otimes G_{-1}^*}$. Thus $\varphi(hv^*) = -v^* + h\varphi(v^*)$, or

$$(5.4.1.2) \quad \varphi(hv^*) - h\varphi(v^*) = -v^*.$$

Finally, $[x, y] \equiv d \otimes xv^* + h \otimes \varphi(xv^*) + K \otimes G_{-1}^*$ for $x \in K$ or by direct computation $[x, y] \equiv d \otimes xv^* + h \otimes x\varphi(v^*) + K \otimes G_{-1}^*$; that is,

$$(5.4.1.3) \quad \varphi(xv^*) = x\varphi(v^*) \quad \text{for } x \in K.$$

The last equality says that $\varphi \in \text{End}_K(G_{-1}^*)$.

5.4.2. By Proposition 5.3 we know that G_{-1} is irreducible for $H' = kh + K$. If G_{-1} were irreducible for K , then $\varphi \in \text{End}_K G_{-1}^*$ would be a scalar. Applying (5.4.1.2) to this case we get a contradiction: $0 = -1$.

Thus G_{-1} is reducible for K .

5.4.3. Since G_{-1} is irreducible for $H' = K + kh$ we see that G_{-1} is differentially irreducible for K with one derivation (namely, h). Therefore we can write (as in Proposition 5.3 and in view of R. Block ([1, Theorem 2] or [2])) that $G_{-1} \simeq U \otimes B_n$

with an irreducible K -module U , and K acting trivially on B_n . Moreover, we have $B_n = \text{End}_K(G_{-1})$. Therefore we have an action of G/K on B_n . Since G_{-1} is irreducible for H' we see that B_n does not have h -invariant ideals. Therefore it does not have G/K -invariant ideals. In view of R. Ree [10] and R. Wilson [15] (explicitly stated in M. I. Kuznetsov [9, Proposition 2.1]) there are two possibilities:

(i) B_n has a structure of a truncated divided power algebra in one variable, i.e., $B_n \simeq B_1(n) = k\{x\}$ and G/K acts on B_n by special derivations of $k\{x\}$. Recall that a derivation D of $k\{x\}$ is special if $Dx^{(s)} = x^{(s-1)} D x$.

(ii) B_n has a structure of a truncated divided power algebra in two variables, i.e., $B_n \simeq B_2(q, r) = k\{\tilde{x}, \tilde{y}\}$ with $q + r = n$, $q \geq r > 0$, G/K acts on B_n by special derivations of $k\{\tilde{x}, \tilde{y}\}$, and there are two linearly independent derivations in G/K which do not preserve filtration. Recall that a derivation D of $k\{\tilde{x}, \tilde{y}\}$ is special if $Dz^{(s)} = z^{(s-1)} D z$ for $z = \tilde{x}$ or \tilde{y} .

5.4.4. Let us look at the first case first. Let ∂ be the derivation of $k\{x\}$ given by $\partial x^{(i)} = x^{(i-1)}$. Then h acts on $\text{End}_K G_{-1}$ as $H\partial$ and d acts as $D\partial$ with $H, D \in k\{x\}$. Also we represent $\varphi \in \text{End}_K G_{-1}^* \simeq \text{End}_K G_{-1} \simeq k\{x\}$ by $F \in k\{x\}$. Write $H = \sum h_i x^{(i)}$, $D = \sum d_i x^{(i)}$ and $F = \sum f_i x^{(i)}$ with $h_i, d_i, f_i \in k$.

Since $\text{End}_K G_{-1}$ has no h -invariant nontrivial ideals, the constant term of H must be nontrivial, $h_0 \neq 0$. Replacing d by $d - (d_0/h_0)h$ we have that $d_0 = 0$ (this change causes also changes in φ and the φ_i).

The relation $[d, h] = h$ (valid for d, h as operators on $\text{End}_K G_{-1}$) becomes $D \cdot \partial H - H \cdot \partial D = H$. Therefore $-h_0 \cdot d_1 = h_0$ whence $d_1 \neq 0$. Now the relations (5.4.1.1), (5.4.1.2) become $[d, \varphi] = -\varphi$, $[h, \varphi] = \text{Id}$ or, by definition of the action of G/K on $\text{End}_K G_{-1}^*$, $d(F) = -F$, $h(F) = 1$. Thus we have $D \cdot \partial F = -F$, $H \cdot \partial F = 1$. The first equality implies that $f_0 = 0$ (because $d_0 = 0$), and the second one that $f_1 \neq 0$ (or, more precisely, $h_0 \cdot f_1 = 1$).

Now, as in 5.3.2 we can write $y \in G_1$ in the form $y \equiv A \otimes \partial \otimes \varphi(v^*) \bmod \tilde{H} \otimes G_{-1}^*$, where \tilde{H} is the algebra of operators preserving the filtration of $U \otimes k\{x\}$; h acts on $G_{-1} = U \otimes k\{x\}$ as $\sum A_i \otimes x^{(i)} \partial$ with $A = A_0$, $\det A \neq 0$ (this is because $d_0 = 0$, so d preserves the filtration). By Lemma 5.3. and the notation of 5.3.4 we see that for $b \in D_{-2}$ with $m > 0$ in $(m, q) = \min(b)$ we still have

$$\varphi(v^*) \left(\sum_{(m, q) < (j, j')} b_{m, q, j, j'} (u_{j'} \otimes x^{(j)}) \right) = 0$$

(we use the fact that A is still nondegenerate). (However, now φ is not an isomorphism!) We have $\varphi(G_{-1}^*) = F \cdot G_{-1}^* = U^* \otimes (F)$, where $(F) = F \cdot k\{x\}$. Since $f_0 = 0$, $f_1 \neq 0$ we have $f_1^{-1} F x^{(i)} = x^{(i+1)} + (\text{higher terms})$ whenever $p \neq i + 1$. Therefore $\dim(F) \geq p^n - p^{n-1}$. Let M be a complement to $(F)^\perp$ in $k\{x\}$ and $V = M \cap k\{x\}$. Then $\dim M \geq p^n - p^{n-1}$, $\dim V \geq p^n - p^{n-1} - 1$. Now for $b \in U \otimes V \wedge U \otimes V$ we have $\sum_{(m, q) < (j, j')} b_{m, q, j, j'} (u_{j'} \otimes x^{(j)}) \in U \otimes V$ and therefore if, in addition, $b \in D_{-2}$, we have $\varphi(G_{-1}^*)(\sum_{(m, q) < (j, j')} b_{m, q, j, j'} (u_{j'} \otimes x^{(j)})) = 0$. By our choice of V this implies that $\sum_{(m, q) < (j, j')} b_{m, q, j, j'} (u_{j'} \otimes x^{(j)}) = 0$, contradicting, as in 5.3.4, the definition of $\min(b)$. Thus $D_{-2} \cap (U \otimes V \wedge U \otimes V) = 0$. Then $U \otimes V \wedge U \otimes V$ is mapped injectively into G_{-2} . Therefore

$$\begin{aligned}\dim G_{-2} &\geq \dim(U \otimes V \wedge U \otimes V) \\ &\geq \frac{1}{2}N(p^n - p^{n-1} - 1) \cdot (N(p^n - p^{n-1} - 1) - 1),\end{aligned}$$

where $N = \dim U$. Since $N \cdot p^n = \dim G_{-1} \geq \dim G_{-2}$ we get the inequality $Np^n \geq \frac{1}{2}N(p^n - p^{n-1} - 1) \cdot (N(p^n - p^{n-1} - 1) - 1)$ which, after multiplying through and collecting terms, becomes

$$2p^{2n-1} + (2 + 3/N)p^n \geq p^{2n} + p^{2n-2} + (2 + 1/N)p^{n-1} + 1 + 1/N.$$

The left-hand side is clearly bounded from above by $7p^{2n-1}$. Therefore for $p \geq 7$ the inequality is violated. This contradiction proves the theorem in case (i).

5.4.5. Let us look at the second case. So $\text{End}_K G_{-1} \cong B_2(q, r) = k\{\tilde{x}, \tilde{y}\}$ with $q + r = n$, $q \geq r > 0$. Let ∂_1, ∂_2 be the derivations of $k\{\tilde{x}, \tilde{y}\}$ defined by $\partial_1 \tilde{x}^{(i)} = \tilde{x}^{(i-1)}$, $\partial_1 \tilde{y} = 0$, $\partial_2 \tilde{x} = 0$, $\partial_2 \tilde{y}^{(j)} = \tilde{y}^{(j-1)}$. Since $\text{End}_K G_{-1}$ has no h -invariant ideals we see that h acts on $k\{\tilde{x}, \tilde{y}\}$ as $a\partial_1 + b\partial_2 + (\text{higher terms})$ with $a\partial_1 + b\partial_2 \neq 0$. If $a \neq 0$ we can replace \tilde{y} by $b\tilde{x} - a\tilde{y}$ and thereby achieve that h acts on $k\{\tilde{x}, \tilde{y}\}$ as $\partial_1 + (\text{higher terms})$. Otherwise, h acts as $b\partial_2 + (\text{higher terms})$, $b \neq 0$. To simplify our notation let us set $x = \tilde{x}$, $y = \tilde{y}$ in the first case and $x = \tilde{y}$, $y = \tilde{x}$ in the second one. Then $\text{End}_K G_{-1} = k\{x, y\}$ and h acts on it as $c\partial_x + (\text{higher terms})$ with $c \neq 0$. We can write $d = D_1\partial_x \oplus D_2\partial_y$ and $h = H_1\partial_x + H_2\partial_y$, where $D_i = \sum d_{ijk}x^{(j)}y^{(k)}$ and $H_i = \sum h_{ijk}x^{(j)}y^{(k)}$. We have by the above that $h_{100} \neq 0$, $h_{200} = 0$. Replacing d by $d - (d_{100}/h_{100})h$ we have that $d_{100} = 0$. By the remark at the end of §2 of [9] we have $d_{200} \neq 0$. To $\varphi \in \text{End}_K G_{-1}^*$ we put into correspondence $F \in k\{x, y\}$, $F = \sum f_{ij}x^{(i)}y^{(j)}$.

The conditions (5.4.1.1) and (5.4.1.2) become $d(F) = -F$, $h(F) = 1$, or explicitly $D_1\partial_x F + D_2\partial_y F = -F$, $H_1\partial_x F + H_2\partial_y F = 1$. Looking at the constant term of F in these expressions we get $d_{200}f_{01} = -f_{00}$ and $h_{100}f_{10} = 1$. Thus either $f_{00} \neq 0$ (and then φ is invertible) or $f_{00} = f_{01} = 0$, $f_{10} \neq 0$. Write $d \equiv A \otimes \partial_y \pmod{\tilde{H}}$ and $h \equiv B \otimes \partial_x \pmod{\tilde{H}}$ for d and h acting on $G_{-1} = U \otimes k\{x, y\}$. Here \tilde{H} is the algebra of operators preserving filtration of $U \otimes k\{x, y\}$ and $A, B \in \text{End } U$. Since G_{-1} is irreducible for $H' = kh + K$ the same argument as in §5.3.2 (with d replaced by h) gives that $\det B \neq 0$.

Now we can write any element $z \in G_1$ in the form $z = A \otimes \partial_y \otimes v^* + B \otimes \partial_x \otimes \varphi(v^*) \pmod{\tilde{H} \otimes G_{-1}^*}$, $\det B \neq 0$, $v^* \in G_{-1}^*$. Then any element from G_{-1} has the form $\sum u_{ij}x^{(i)}y^{(j)}$ and an element $b \in F_{-2} = G_{-1} \wedge G_{-1}$ has the form

$$b = \sum_{(i,j,l) < (i',j',l')} b_{i,j,l,i',j',l'} u_i \otimes x^{(i)}y^{(j)} \wedge u_{l'} \otimes x^{(i')}y^{(j')},$$

where u_i , $i = 1, \dots, N = \dim U$, is a basis of U and the triples of integers are ordered lexicographically. Let

$$\min(b) = \min\{(i, j, l) | b_{i,j,l,i',j',l'} \neq 0 \text{ for some } i', j', l'\}.$$

Write $\min(b) = (s, t, r)$. Then, as in Lemma 5.3.3, if $s > 0$ then the lowest term

(with respect to powers of x and y) in $[b, z]$ is

$$\left[\varphi(v^*) \left(\sum_{(s,t,l) < (i',j',l')} b_{s,t,l,i',j',l'} (u_{l'} \otimes x^{(i')} y^{(j')}) \right) Bu_{l'} \right. \\ \left. - \sum_{l < l'} b_{s,t,l,s,t,l} \varphi(v^*) (u_l \otimes x^{(s)} y^{(t)}) Bu_{l'} \right] \otimes x^{(s-1)} y^{(t)}.$$

Setting $Q = \sum_{i>r} k \cdot Bu_i \otimes x^{(s-1)} y^{(t)} + \sum_{(i,j) \geq (s,t)} U \otimes x^{(i)} y^{(j)}$ we see as in 5.3.4 that

$$[b, z] \equiv \varphi(v^*) \left(\sum_{(s,t,r) < (i,j,l)} b_{s,t,r,i,j,l} (u_l \otimes x^{(i)} y^{(j)}) \right) Bu_r \otimes x^{(s-1)} y^{(t)} \pmod{Q}.$$

Thus if $b \in D_{-2}$, $s > 0$ in $(s, t, r) = \min(b)$, then $\lambda(b, v^*) Bu_r = 0$ for all $v^* \in G_{-1}^*$, where

$$\lambda(b, v^*) = \varphi(v^*) \left(\sum_{(s,t,r) < (i,j,l)} b_{s,t,r,i,j,l} (u_l \otimes x^{(i)} y^{(j)}) \right).$$

Since $\det B \neq 0$ it follows that $\lambda(b, v^*) = 0$ for all $v^* \in G_{-1}^*$. Now, as in 5.4.4, $\varphi(G_{-1}^*) = U^* \otimes (F)$, $F \in k\{x, y\}$. If F is invertible, then $(F) = k\{x, y\}$ and $\varphi(G_{-1}^*) = G_{-1}^*$. Therefore in this case, $\lambda(b, v^*) = 0$ for all $v^* \in G_{-1}^*$ is equivalent to $\sum_{(s,t,r) < (i,j,l)} b_{s,t,r,i,j,l} (u_l \otimes x^{(i)} y^{(j)}) = 0$ in contradiction with the assumption $s > 0$. Thus in this case $D_{-2} \cap (U \otimes K\{\overline{x, y}\} \wedge U \otimes k\{x, y\}) = 0$, where $k\{x, y\} = \{P \in K\{x, y\} \mid P = \sum_{i \geq 1} p_{ij} x^{(i)} y^{(j)}\}$.

If F is not invertible, then $f_{00} = f_{01} = 0$ and $f_{10} \neq 0$. As in 5.4.4 this implies that $\dim(F) \geq p^n - p^{n-1}$. Again as in 5.4.4 we take M to be a complement of $(F)^\perp$ in $k\{x, y\}$ and set $V = M \cap k\{\overline{x, y}\}$. For $b \in U \otimes V \wedge U \otimes V$ we have $\sum_{(s,t,r) < (i,j,l)} b_{s,t,r,i,j,l} (u_l \otimes x^{(i)} y^{(j)}) \in U \otimes V$ and if, in addition, $b \in D_{-2}$, then $\lambda(b, v^*) = 0$. By our choice of V this implies a contradiction with a definition of $\min(b)$.

We have $\dim k\{\overline{x, y}\} = \dim k\{x, y\} - \dim k\{y\} \geq p^n - p^{n-1}$ and therefore $\dim V \geq p^n - 2p^{n-1}$. Since $D_{-2} \cap (U \otimes V \wedge U \otimes V) = 0$ (we take $V = k\{\overline{x, y}\}$ if F is invertible), we see that

$$\dim G_{-2} \geq \dim(U \otimes V \wedge U \otimes V) \geq \frac{1}{2} N(p^n - 2p^{n-1}) \cdot [N(p^n - 2p^{n-1}) - 1].$$

As before, we have (by our choice of G_{-1}) the inequality $\dim G_{-1} \geq \dim G_{-2}$ which becomes $2Np^n \geq N^2(p^{2n} - 4p^{2n-1} + 4p^{2n-2}) - N(p^n - 2p^{n-1})$. We divide it by N^2p^{n-1} and collect:

$$4p^n + \frac{3}{N}p \geq p^{n+1} + 4p^{n-1} + \frac{2}{N}.$$

As $n \geq 2$ and $p \geq 7$ the right-hand side is $\geq 7p^n + 4p$, which is larger than the left-hand side. This contradiction concludes the proof of Theorem 5.1.

5.5. Remarks and comments on the proof.

5.5.1. We derived a contradiction from $\dim G_{-2} > \dim G_{-1}$. Probably a more general fact takes place: If $p > 5$ and if $G = \bigoplus_{i \in \mathbb{Z}} G_i$ is such that G_0 has solvable

quotients of dimension ≥ 2 and $[G_1, G_{-1}] = G_0$ with G_1 and G_{-1} irreducible, then G is infinite-dimensional (maybe even of exponential growth).

What we did was probably just verifying the first step.

5.5.2. In characteristic 3 there exist graded algebras of the type we rejected; see Frank [4].

PART II. SOLVABLE MAXIMAL SUBALGEBRAS

In this part we consider simple Lie algebras which contain a solvable maximal subalgebra. Using the results of the preceding part we conclude that such a subalgebra gives rise to a long filtration. We pass then to the graded Lie algebra and work hard to prove that certain graded Lie algebras with solvable zeroth term are of type A_1 or W_1 . Our main tools are papers [5, 9, 12].

The steps of the proof are the following: We consider a minimal counterexample $S = \bigoplus_{i=-q}^r S_i$. We show by constructing different smaller subalgebras that S is generated by S_{-1} and S_1 and then that S_1 is irreducible. This latter step required very detailed study of the structure of our algebra. Then, once we established that S_1 is irreducible (and it is automatically faithful), we are able to invert the grading of S , that is, to replace S_i by S_{-i} . Then we can assume that $q \leq r$. Using interplay between $\bigoplus_{i \geq -1} S_{qi}$ and $\bigoplus_{i \geq -1} S_{(q-1)i}$ we are able to arrive at a contradiction.

The ground field is still assumed to be algebraically closed of characteristic $p > 5$.

1. Setup and statements of results. A solvable maximal subalgebra is a maximal subalgebra which is solvable.

1.1. Suppose that L is a simple Lie algebra which contains a solvable maximal subalgebra H . Suppose that L_0 is bad. By Theorem I.5.1 this means, in particular, that $\dim H = 1$. Then by Theorem I.2.2, $L = \bigoplus_{i \in \mathbb{Z}/p} L_i$ and L_i , $i \neq 0$, are irreducible and faithful for L_0 . But then $L_1 \oplus L_0$ is a proper subalgebra (and solvable at that). Thus L_0 is never bad. Let A be a commutative ideal of L_0 . Then A acts nilpotently on L_0 . By Corollary I.1.4(ii), A acts nilpotently on G . This shows that L_0 gives rise to a long filtration of L .

1.2. More generally, consider a Lie algebra L with a maximal subalgebra L_0 which contains no ideals of L . Pick a simple L_0 -submodule F in L/L_0 and let L_{-1} be its preimage in L . Suppose that $\text{Ker } L_0|_F \neq 0$ (which is certainly the case if L_0 has ideals acting on L nilpotently). Then set $L_{i-1} = [L_i, L_i] + L_i$ for $i < 0$ and $L_{i+1} = \{x \in L_i \mid [x, L_{-1}] \subseteq L_i\}$ for $i \geq 0$. In particular, $L_1 = \text{Ker } L_0|_F$. Then $\{L_i\}$ is a Lie algebra filtration in L (see [12]).

Let $G = \bigoplus_{i \in \mathbb{Z}} G_i$, $G_i = L_i/L_{i+1}$, be the associated graded Lie algebra. The main result of Part II is

1.3. THEOREM. *Suppose that L_0 is a solvable maximal subalgebra of L and that L_0 defines a long filtration of L . Then there exist a simple Lie algebra S of type A_1 or W_1 with standard grading $S = \bigoplus S_i$, a number m and a solvable subalgebra T of derivations of degree 0 of $S \otimes B_m$ such that $G \cong S \otimes B_m + T$, $G_i = S_i \otimes B_m$ for $i \neq 0$, and $G_0 = S_0 \otimes B_m + T$, B_m has no T -invariant ideals. In particular:*

- (i) $L = L_{-1}$ (i.e., L_0 is irreducible on L/L_0);
- (ii) $\dim(L_i/L_{i+1}) = m$ if $i \neq 0$ and $L_i/L_{i+1} \neq 0$.

In view of this theorem and M. I. Kuznetsov [9, Theorem 6.1], we have

1.4. COROLLARY. *Suppose that L is simple and contains a solvable maximal subalgebra. Then L is of type A_1 or W_1 .*

1.4.1. REMARK. In characteristic 3, as was already remarked, there are counter-examples to the above statements, see, e.g., M. Frank [4].

1.5. To start proving Theorem 1.3 we denote by $M(G)$ (see [12, §1.5]) the largest ideal of G contained in $\bigoplus_{i < 0} G_i$. We set $\bar{G} = G/M(G)$, $\bar{G} = \bigoplus \bar{G}_i$. Let $A(\bar{G}) = \bigoplus A(\bar{G})_i$ (see [12, §1.6]) be the smallest nontrivial ideal of \bar{G} . According to [12] there are two cases: degenerate case (when $A(\bar{G})_1 = 0$) and nondegenerate case (when $A(\bar{G})_1 \neq 0$).

1.6. LEMMA. *We are in a nondegenerate case. More precisely, if L_0 is a solvable maximal subalgebra of L , then there exists a simple graded Lie algebra $S = \bigoplus S_i$, a natural number m and a solvable subalgebra T_0 of derivations of degree zero of $S \otimes B_m$ such that $A(\bar{G}) = S \otimes B_m$, $A(G)_i = S_i \otimes B_m$, $\bar{G}_i = S_i \otimes B_m$ for $i < 0$ and $G_0 = \bar{G}_0 = S_0 \otimes B_m + T_0$.*

PROOF. Suppose that \bar{G} is degenerate. Then \bar{G}_0 contains an ideal of the form $S \otimes B_u$ with S simple. This is impossible since G_0 is a quotient of L_0 , and L_0 is solvable.

2. The case when S is of the type A_1 or W_1 . Suppose that $L = \{L_i\}$ is a filtered Lie algebra and L_0 a solvable maximal subalgebra in L and suppose that L_0 contains no ideals of L . We take notation from the preceding section. As a first step and to simplify reference, we record some properties of \bar{G} in the case when S is type A_1 or W_1 .

2.1. LEMMA. *Let \bar{G} be as in §1.5. Suppose that S is of type A_1 or W_1 with standard grading. Write $S_i = ke_i$. We have:*

- (i) $\bar{G}_i = 0$ for $i \leq -2$; $\bar{G}_i = G_i$ for $i \geq -1$;
- (ii) $G_i = A(\bar{G})_i$ for $i \neq 0$, $i \geq -1$;
- (iii) $G_i = e_i \otimes B_m$ for $i \neq 0$, $i \geq -1$;
- (iv) $e_0 \otimes B_m \subseteq G_0$ is a commutative ideal of G_0 ;
- (v) G_0 acts on $e_0 \otimes B_m$ by derivations of B_m ; B_m has no G_0 -invariant ideals;
- (vi) $e_0 \otimes 1$ is in the center of G_0 and it acts as (multiplication by) i on G_i ;
- (vii) the G_i , $p \nmid i$, $i \geq -1$, are irreducible and faithful G_0 -modules;
- (viii) $e_0 \otimes B_m$ is the kernel of the action of G_0 on G_{p^i} , $i \neq 0$.

PROOF. We have $S_i = 0$ for $i < -1$, whence (i). Now (ii), (iii), (iv) and (v) follow from M. I. Kuznetsov [9, Theorem 5.1 and Proposition 2.1]. (They can also be derived with little work from results of [12]). Now it is clear from (v) that $e_0 \otimes 1$ is in the center of G_0 and that $G_0 = e_0 \otimes B_m + T_0$ with $T_0 \subseteq 1_{S_0} \otimes \text{Der } B_m$ acting on $e_0 \otimes B_m$ in such a way that $e_0 \otimes I$ is not a T_0 -invariant subspace for any ideal I of B_m .

Now $e_0 \otimes 1$ acts as 0 on G_0 and it acts as i on $G_i = A(\bar{G})_i$, $i \geq -1$, since $[e_0 \otimes 1, e_i \otimes b] = [e_0, e_i] \otimes b = ie_i \otimes b$ for $b \in B_m$. Since $G_{i-1} = [G_i, G_{-1}]$ for $i < -1$ we have that $e_0 \otimes 1$ acts as i also on G_i , $i < -1$. This establishes (vi). To prove (vii) and (viii) note that $e_0 \otimes b$, $b \in B_m$, acts on $G_i = e_i \otimes B_m$, $i \geq -1$, $i \neq 0$, as multiplication by ib ; in particular, it acts trivially on G_i , $p|i$. So $\text{Ker } G_0|G_i \supseteq e_0 \otimes B_m$ if $i \neq 0$, $p|i$, and the only invariant subspaces for $e_0 \otimes B_m$ on G_i , $p \nmid i$, are of the form $e_i \otimes I$, where I is an ideal of B_m . As T_0 acts faithfully on B_m we see that $\text{Ker } G_0|G_i = e_0 \otimes B_m$ if $p|i$, $i \neq 0$, and $\text{Ker } G_0|G_i = 0$ if $p \nmid i$. Since B_m has no T_0 -invariant ideals it follows that G_0 acts irreducibly on G_i , $p \nmid i$. This concludes the proof of (vii) and (viii).

2.2. REMARK. We assumed that S has the standard grading. The only other possibility is that S has inverse to standard grading; that is, $S_i = Ke_{-i}$. But then $[S_{-1}, S_{-1}] = 0$; that is, S is type A_1 and for type A_1 there is no difference between standard and inverse to standard grading.

2.3. PROPOSITION. *Suppose that S is of type A_1 or W_1 with standard grading. Then $M(G) = 0$.*

PROOF. Since the grading of S is standard we have $S = \bigoplus_{i \geq -1} S_i$. Thus $\bar{G}_i = 0$ for $i \leq -2$ and therefore $M(G) = \bigoplus_{i < -1} G_i$. Since $G_{-2} \subseteq M(G)$ we have $[G_{-2}, G_1] = 0 = [G_{-2}, G_2]$. Therefore $[L_{-2}, L_1] \subseteq L_0$, $[L_{-2}, L_2] \subseteq L_1$.

Set $M = L$, $M_0 = L_0$, $M_{-1} = L_{-2}$, $M_{i-1} = [M_i, M_{-1}] + M_i$ for $i \leq -1$, and $M_{i+1} = \{x \in M_i | [x, M_{-1}] \subseteq M_i\}$ for $i \geq 0$. Then M_i is a Lie algebra filtration in M (see proof of Proposition 1.2 in [12]). We know that $[M_{-1}, L_1] = [L_{-2}, L_1] \subseteq L_0 = M_0$. Thus $L_1 \subseteq M_1$. On the other hand, $M_1 = \text{Ker } M_0|_{M_{-1}/M_0} \subseteq \text{Ker } M_0|_{L_{-1}/L_0} = L_1$. Thus $M_1 = L_1$.

Similarly, $[M_{-1}, L_2] = [L_{-2}, L_2] \subseteq L_1 = M_1$ implies that $L_2 \subseteq M_2$. Now M_2/L_3 is a submodule of $G_1 = L_1/L_2$. But G_1 is irreducible (by Lemma 2.1(vii)). Since $M_2 \neq L_1$ we must have $M_2 = L_2$.

Now let $H = \bigoplus H_i$, $H_i = M_i/M_{i+1}$, be the graded algebra associated to the filtered algebra $\{M_i\}$. We have $H_0 = M_0/M_1 = L_0/L_1 = G_0$, $H_1 = M_1/M_2 = L_1/L_2 = G_1$. In particular, $c = e_0 \otimes 1 \in G_0$ (cf. Lemma 2.1) acts as 1 on H_1 and as 0 on H_0 . Therefore, for $x \in H_1$ and $y \in H_{-1}$ we have $[x, [c, y]] = [[x, c]y] + [c[x, y]] = -[x, y]$, whence $[H_1, [c, y] + y] = 0$. Let $[c, y] + y = v + M_0$. Then $[M_1, v] \subseteq M_1$, i.e., $[L_1, v] \subseteq L_1$. Since L_1 is not an ideal of L and since L_0 is maximal in L the inclusion $[L_1, v] \subseteq L_1$ implies that $v \in L_0 = M_0$, whence $[c, y] + y = 0$, i.e., c acts as -1 on H_{-1} . However, H_{-1} involves G_{-2} and $c = e_0 \otimes 1 \in G_0$ acts as -1 on G_{-1} and as -2 on G_{-2} . Thus it cannot act as -1 on H_{-1} . This contradiction concludes the proof of Proposition 2.3.

2.4. COROLLARY (M. I. KUZNETSOV [9, THEOREM 6.1]). *If L is simple and S is of type A_1 or W_1 , then L is of the same type.*

3. Initial properties of a minimal counterexample. Our aim is to show that S appearing in Lemma 1.6 is of type A_1 or W_1 . To show this we pick S of minimal dimension which can appear in Lemma 1.6 and which is not of type A_1 or W_1 .

3.1. LEMMA. Let S be as above, $S = \bigoplus_{i=-q}^r S_i$. Let $D_0 = (\text{Der } S)_0$. Then:

- (i) S is simple;
- (ii) S_0 is solvable;
- (iii) S is not of type A_1 or W_1 (i.e., $\dim S_0 > 1$);
- (iv) S_{-1} is a faithful D_0 -module;
- (v) S_{-1} is an irreducible D_0 -module;
- (vi) $S_{i-1} = [S_i, S_{-1}]$ for $i \leq -1$;
- (vii) for any $x \in S_i$, $x \neq 0$, $i > 0$, we have $[x, S_{-1}] \neq 0$;
- (viii) S_{-q} is irreducible for S_0 ;
- (ix) D_0 acts faithfully on every D_0 -submodule of S_1 .

PROOF. First, (i) and (ii) hold by construction and (iii) is our assumption. Next, (v), (vi) and (vii) are contained in [12, Theorem 4.1(iii)] and (viii) in [12, Lemma 4.5.1(ii)]. Let us establish (iv). Let $x \in (\text{Der } S)_0$, $[x, S_{-1}] = 0$. Then $[x, S_i] = 0$ for $i < 0$ in view of (vi). Assume by induction that $[s, S_i] = 0$ for $i \leq m$, $m \geq -1$. Then for $y \in S_{m+1}$ we have $[S_{-1}[x, y]] = [x[S_{-1}, y]] \subseteq [x, S_m] = 0$. By (vii) it means that $[x, y] = 0$; that is, $[x, S_{m+1}] = 0$. Thus x acts as 0 on s , i.e., $x = 0$. This proves (iv). Now (ix) follows from [12, Proposition 3.2.1(ii)] and the fact that S_0 is solvable.

3.2. DEFINITION. A graded Lie algebra $S = \bigoplus_{i=-q}^r S_i$ of minimal dimension among the Lie algebras satisfying the conclusions of Lemma 3.1 is called a *minimal counterexample*.

3.3. LEMMA. Let $S = \bigoplus_{i=-q}^r S_i$ be a minimal counterexample. Let $V \neq 0$ be a D_0 -submodule of S_1 and let $\tilde{S} = \langle S_{-1}, V \rangle + D_0$. If $\tilde{S} \neq S$, then $\tilde{S}/M(\tilde{S}) = R \otimes B_m + D_0$, where $R = ke_{-1} \oplus ke_0 \oplus ke_1$ is a simple graded Lie algebra of type A_1 . In particular:

- (i) D_0 has a commutative ideal $e_0 \otimes B_m = [S_{-1}, V]$;
- (ii) D_0 has a nontrivial center, $\text{Center}(D_0) \supset e_0 \otimes 1$.

PROOF. First of all, $\tilde{S}/M(\tilde{S})$ is nondegenerate in view of Lemma 3.1(ix). Thus $A(\tilde{S}/M(\tilde{S})) = R \otimes B_m$ (by [12, Theorem 4.1]) with R simple and satisfying conclusions of Lemma 3.1(iv)–(ix). Properties (i) and (ii) of Lemma 3.1 are also satisfied. Since S was a minimal counterexample we have that R is of type A_1 or W_1 . As $R = \langle R_{-1}, R_1 \rangle$ we have that R is of type A_1 . The rest is easily checked.

3.4. PROPOSITION. Let S be a minimal counterexample. Then $S = \langle S_{-1}, S_1 \rangle$ (i.e., S is generated by S_1 and S_{-1}).

PROOF. Suppose not. Let $\tilde{S} = \langle S_{-1}, S_1 \rangle + D_0$. Then by Lemma 3.3 we have $\tilde{S}/M(\tilde{S}) = R \otimes B_m + D_0$, where R is simple of type A_1 , $R = \bigoplus_{i=-1}^1 R_i$, $\dim R_i = 1$. We have $S_{-1} = R_{-1} \otimes B_m$, $S_1 = R_1 \otimes B_m$ and, since $[S_1, S_{-1}] = S_0$ (by Lemma III.1.1), we also have $S_0 = R_0 \otimes B_m$.

Next, since $R_2 = 0$, we have $[S_1, S_1] = 0$. Both S_1 and S_{-1} are irreducible for D_0 . If $m = 0$ then $[S_{-1}, S_{-1}] = 0$, i.e., $S_{-2} = 0$, i.e., $q = 1$. But $q > 1$ by (an independently proved) Proposition 3.6 below. So $m \geq 1$.

Let C be the maximal ideal of B_m . Then $R_0 \otimes C$ acts nilpotently on S_{-1} . Therefore (since $S_{j-1} = [S_j, S_{-1}]$ for -1) $R_0 \otimes C$ acts nilpotently on all S_i , $i < 0$. By Lemma 3.1(viii) it acts trivially on S_{-q} and therefore $\dim S_{-q} = 1$. By Proposition III.1.12(iv), S_0 acts nontrivially on S_{-q} . Therefore D_0 acting on B_m by derivations must preserve C . But then it also preserves $R_{-1} \otimes C$, i.e., D_0 is not irreducible on S_{-1} , a contradiction.

3.5. PROPOSITION. *Let S be a minimal counterexample. Then S_1 has at most one proper D_0 -submodule. In particular, S_1 is indecomposable.*

PROOF. Suppose the contrary. Let \tilde{V}, \bar{V} be two proper submodules, $\tilde{V} \neq \bar{V}$. Consider $\tilde{S} = \langle S_{-1}, \tilde{V} \rangle + D_0$ and $\bar{S} = \langle S_{-1}, \bar{V} \rangle + D_0$. By Lemma 3.3 we have $\tilde{S}/M(\tilde{S}) = \tilde{R} \otimes B_m + D_0$, $\bar{S}/M(\bar{S}) = \bar{R} \otimes B_m + D_0$ with \tilde{R}, \bar{R} of type A_1 . In particular, both $\tilde{V} = \tilde{R}_1 \otimes B_m$ and $\bar{V} = \bar{R}_1 \otimes B_m$ are irreducible for D_0 .

We claim that $\tilde{V} + \bar{V} = S_1$. If not, then $\tilde{V} + \bar{V}$ is a proper submodule of S_1 , and we can assume that it is one of our submodules \tilde{V} or \bar{V} , say \bar{V} . Then $\bar{V} \supseteq \tilde{V}$. Since both \tilde{V} and \bar{V} are irreducible we must have $\tilde{V} = \bar{V}$, a contradiction.

Thus $\tilde{V} + \bar{V} = S_1$. Now we know that $S_{-2} \subseteq M(\tilde{S})$ and $S_{-2} \subseteq M(\bar{S})$. Therefore $[S_{-2}, \tilde{V}] = 0$, $[S_{-2}, \bar{V}] = 0$, whence $[S_{-2}, S_1] = 0$. Since $\bigoplus_{i>0} S_i = \langle S_1 \rangle$ we have that $S_{-2} \subseteq M(S)$, and since S is simple, $S_{-2} = 0$. Thus $q = 1$, which is a contradiction in view of the following

3.6. PROPOSITION. *Let S be a minimal counterexample. Then $q \geq 2$.*

PROOF. Suppose $q = 1$. As S is simple it follows that S_{-1} is irreducible for S_0 (Proposition III.1.12(iii) with $q = 1$). Now our claim follows by M. I. Kuznetsov [9, Theorem 5.1].

4. Proof that S_1 is irreducible for D_0 .

4.1. PROPOSITION. *Let $S = \bigoplus_{i=-q}^r S_i$ be a minimal counterexample. Then S_1 is irreducible for D_0 .*

To start the proof we suppose the contrary. Let V be a proper submodule of S_1 . Set $\tilde{S}_i = S_i$ for $i < 0$, $\tilde{S}_0 = D_0$ and $\tilde{S}_1 = V$, $\tilde{S}_{i+1} = \{x \in S_{i+1} \mid [S_{-1}, x] \subseteq \tilde{S}_i\}$ for $i \geq 1$. Set $\tilde{S} = \bigoplus \tilde{S}_i$. It is a subalgebra by Lemma III.1.8.

- 4.2. LEMMA.** (i) S_1/V is irreducible;
(ii) \tilde{S} is a maximal subalgebra of $S + D_0$;
(iii) $\tilde{S} \cap S_1 = V$;
(iv) S_1/V is contained in every nontrivial \tilde{S} -submodule of $(S + D_0)/\tilde{S}$;
(v) $\tilde{S}/M(\tilde{S}) = R \otimes B_m + D_0$, where R is a simple Lie algebra of type A_1 or W_1 with standard grading.

PROOF. First, (i) follows from Proposition 3.5. Then (ii) and (iv) follow from Proposition III.1.9. Next, (iii) is just the definition of \tilde{S}_1 .

To prove (v) note that $\tilde{S}/M(\tilde{S})$ is nondegenerate in view of Lemma 3.1(ix). Thus $A(\tilde{S}/M(\tilde{S})) = R \otimes B_m$ (by [12, Theorem 4.1]) with R simple and satisfying conclusions of Lemma 3.1(iv)–(ix). Properties (i) and (ii) of Lemma 3.1 are also satisfied. Since S is a minimal counterexample we have that R is of type A_1 or W_1 as required.

4.3. COROLLARY. (i) $M(\tilde{S}) = \bigoplus_{i \leq -2} S_i \neq 0$.

(ii) Write $R = \bigoplus_{i \geq -1} ke_i$. Then $e_0 \otimes 1$ belongs to the center of D_0 and acts as multiplication by i on each S_i .

PROOF. By Lemma 4.2(v) we know that $(\tilde{S}/M(\tilde{S}))_{-2} = 0$. So $S_{-2} \subseteq M(\tilde{S})$. Therefore $M(\tilde{S}) = \bigoplus_{i \leq -2} S_i$. Since $q > 1$ we have $M(\tilde{S}) \neq 0$. This proves (i).

We have $[e_0 \otimes 1, e_{-1} \otimes b] = [e_0, e_{-1}] \otimes b = -e_{-1} \otimes b$ for $b \in B_m$. Therefore $e_0 \otimes 1$ acts as -1 on $S_{-1} = \tilde{S}_{-1}$. Since D_0 is faithful on S_{-1} it follows that $e_0 \otimes 1$ is in the center of D_0 . Since $S_{-1} = [S_i, S_{-1}]$ for $i < 0$, it follows that $e_0 \otimes 1$ acts as i on S_i , $i < 0$. Suppose that $e_0 \otimes 1$ acts as i on all S_i , $i \leq t$. Then for $x \in S_{-1}$, $y \in S_{t+1}$ we have

$$\begin{aligned} [x[e_0 \otimes 1, y]] &= [[x, e_0 \otimes 1], y] + [e_0 \otimes 1, [x, y]] \\ &= [x, y] + t[x, y] = (t+1)[x, y]. \end{aligned}$$

In view of Lemma 3.1(vii) this implies that $e_0 \otimes 1$ acts as $(t+1)$ on S_{t+1} . This proves (ii).

4.4. Now set $L = S + D_0$, $L_0 = \tilde{S}$, and let L_{-1} be a minimal subspace of L such that $L_{-1} \supset L_0$, $L_{-1} \neq L_0$ and $[L_{-1}, L_0] \subseteq L_{-1}$. Let $\{L_i\}$ be the resulting filtration (cf. §1.2).

LEMMA (i) L_{-1} is the L_0 -submodule of L generated by $S_1 + L_0$;

(ii) $L_1 = M(\tilde{S}) \neq 0$; in particular, the filtration is long.

PROOF. The first claim directly follows from Lemma 4.2(iv). To prove the second claim, note that $M(\tilde{S})$ is a nilpotent ideal of L_0 , and $M(\tilde{S})$ acts nilpotently on L . Therefore, $M(\tilde{S})$ acts trivially on the irreducible L_0 -module L_{-1}/L_0 . Thus $M(\tilde{S}) \subseteq L_1$. Next, L_1 is a nilpotent ideal of L_0 , and the algebra $\tilde{S}/M(\tilde{S})$ does not have nilpotent ideals (by Lemma 4.2(v)). So $L_1 = M(\tilde{S})$.

4.5. LEMMA. Let $F = \bigoplus F_i$ be the graded Lie algebra associated to the filtered Lie algebra $\{L_i\}$:

(i) $F/M(F)$ is nondegenerate;

(ii) $F_0 = R \otimes B_m + D_0$;

(iii) $[F_1, F_{-1}] = R \otimes B_m$;

(iv) $F_{-1} \supset S_1/V$;

(v) $A(F/M(F))_1 \supseteq S_{-2}$.

(Inclusions and equalities here and below have the natural interpretation.)

PROOF. We clearly have (iv) by Lemma 4.4(i), and (ii) by Lemma 4.4(ii). In particular, $S_{-1} \subseteq F_0$. By construction of the graded Lie algebra we see that $[S_{-1}, S_1/V] = 0$ in F . Since F_{-1} is generated by S_1/V as an F_0 -module by Lemma

4.4(i), to show that $[F_1, F_{-1}] \subseteq R \otimes B_m$ it is sufficient to check that $[F_1, S_1/V] \subseteq R \otimes B_m$. But $[\sum_{i \leq -2} S_i, S_1/V] \subseteq \sum_{i \leq -1} S_i$. So $[F_1, S_1/V] \cap F_0 = S_{-1} \subseteq R \otimes B_m$. So $[F_1, F_{-1}] \subseteq R \otimes B_m$ and $[F_1, F_{-1}] \supseteq [S_{-2}, S_1/V] = S_{-1}$. Since $[F_1, F_{-1}]$ is an ideal of F_0 we must have $[F_1, F_{-1}] = R \otimes B_m$. This proves (iii).

To prove (v) we have to establish first that $F_1 \supseteq S_{-2}$, i.e. that $S_{-2} \cap L_2 = 0$. For $x \in S_{-2} \cap L_2$ we have $[x, L_{-1}] \subset L_1 = M(\tilde{S})$. Since, $S_1 \subset L_{-1}$ this implies that $[x, S_1] = 0$ whence, by Proposition 3.4, $x \in M(S) = 0$. Now by (iii) (and in view of Corollary 4.3(ii)) we have that $c = e_0 \otimes 1 \in A(F/M(F))_0$. Since c acts as -2 on S_{-2} it follows that $S_{-2} = [c, S_{-2}] \subseteq A(F/M(F))$. This proves (v). Since it is now established that $A(F/M(F))_1 \neq 0$ we also have (i).

4.6. LEMMA. $A(F/M(F)) = P \otimes B_m$, where P is a simple graded Lie algebra, $P = \bigoplus_{i=-s}^t P_i$.

PROOF. We have $A(F/M(F)) = P \otimes B_d$ with P simple graded, $P = \bigoplus_{i=-s}^t P_i$ in view of Lemma 4.5(i) and [12, Theorem 4.1]. By Lemma 4.5(iii) we have $P_0 = R \otimes B_{m-d}$. By [12, Lemma 4.5.1(ii)] we have that P_{-s} is irreducible for P_0 , and by Proposition III.1.12(iii) we see that $[P_0, P_{-s}] \neq 0$. Let C be the maximal ideal of B_{m-d} . Then the preimage of $R \otimes CB_m$ in L acts nilpotently on S . Therefore $R \otimes CB_m$ acts nilpotently on $\bigoplus_{i \geq 0} F_i$, i.e., $R \otimes C$ acts nilpotently on $\bigoplus_{i \geq 0} P_i$. It follows that $R \otimes C$ acts nilpotently on P (if $n \in R \otimes C$ has an eigenvalue $\lambda \neq 0$ on some P_j , $j < 0$, then using commutation with $\bigoplus_{i > 0} P_i$ we shall see that n has the same eigenvalue on some P_r , $r \geq 0$). Since P_{-s} is irreducible, $R \otimes c$ acts trivially on P_{-s} . Since $[P_0, P_{-s}] \neq 0$ this means that D_0 preserves the ideal $R \otimes CB_m$ of $R \otimes B_m$. But then D_0 does not act irreducibly on S_{-1} (and neither does F_0 act irreducibly on F_{-1}). Thus $m - d = 0$, i.e., $d = m$. This proves our claim.

4.7. Let Q be the maximal ideal of $A(F/M(F))$. Then $P = A(F/M(F))/Q$. Let \mathcal{T} be the one-dimensional subtorus of $\text{Aut } S$ defining the grading $S = \bigoplus S_i$ (see [12, §14]). Since $L_0 = \tilde{S}$, $L_1 = M(\tilde{S})$, and L_{-1} (by Lemma 4.4(i)) are invariant under \mathcal{T} , \mathcal{T} goes over to F . Next, Q is also invariant under \mathcal{T} . So we consider \mathcal{T} as acting also on P and preserving each P_i . If χ is the character of \mathcal{T} on S_1 we can write $P_{ij} = \{x \in P_i \mid t \cdot x = j\chi(t) \cdot x\}$. This gives us a bigrading of P .

- LEMMA. (i) $[P_{ij}, P_{kl}] = P_{i+k, j+l}$;
(ii) $P_0 = \bigoplus_{i \geq -1} P_{0,i}$, and $\dim P_{0,i} = 1$ if $P_{0,i} \neq 0$;
(iii) $P_{-1} = \bigoplus_{i \geq 1} P_{-1,i}$;
(iv) $P_1 = \bigoplus_{i \leq -2} P_{1,i}$.

PROOF. We have: (i) is true since P_{ij} is a bigrading; (ii) is true by Corollary 4.3(ii); (iii) is true since F_{-1} is generated by S_1/V and $[S_1/V, S_{-1}] = 0$; (iv) is true because $L_1 \subseteq \bigoplus_{i \leq -2} S_i$.

4.8. LEMMA. (i) There exist $e_{0,0} \in P_{0,0}$, $e_{0,1} \in P_{0,1}$ and $e_{0,-1} \in P_{0,-1}$ such that $[e_{0,1}, e_{0,-1}] = -e_{0,0}$ and $e_{0,0}$ acts as (a multiplication by) j on every P_{ij} ;

- (ii) there exist $e_{-1,1} \in P_{-1,1}$ and $e_{1,-2} \in P_{1,-2}$ such that $[e_{-1,1}, e_{1,-2}] = e_{0,-1}$.

PROOF. We have $P_0 = R$ so $P_0 = \bigoplus_{i \geq -1} ke_i$. Set $e_{0,0} = e_0$, $e_{0,1} = e_1$ and $e_{0,-1} = e_{-1}$. Then (i) holds in view of Corollary 4.3(ii).

To prove (ii) recall that $[S_{-2}, S_1/V] = S_{-1}$, where $S_{-2} \subseteq F_1$, $S_1/V \subseteq F_{-1}$ and $S_{-1} \subseteq [F_1, F_{-1}]$. This gives us (ii).

4.9. Set $e_{1,-i-1} = (\text{ad } e_{0,-1})^{i-1} e_{1,-2} \in P_{1,-i-1}$ for $i \geq 1$ and $e_{-1,i+1} = (\text{ad } e_{0,1})^i e_{-1,1}$. We are going to compute certain commutators. First of all, in view of Lemma 4.8(i), (ii) we have

$$(4.9.1) \quad [e_{0,1}, e_{0,-1}] = -e_{0,0},$$

$$(4.9.2) \quad [e_{0,0}, e_{ij}] = je_{i,j},$$

$$(4.9.3) \quad [e_{-1,1}, e_{1,-2}] = e_{0,-1}.$$

Next, we have by Lemma 4.7(iii), (iv)

$$(4.9.4) \quad [e_{0,1}, e_{1,-2}] = 0,$$

$$(4.9.5) \quad [e_{0,-1}, e_{-1,1}] = 0$$

Now we claim that

$$(4.9.6) \quad [e_{0,1}, e_{1,-i}] = (i(i-1)/2 - 1)e_{1,-i+1} \quad \text{for } i \geq 2.$$

We prove it by induction. It holds if $i = 2$ in view of (4.9.4). Suppose it holds for $i \leq m$. Then

$$\begin{aligned} [e_{0,1}, e_{1,-m-1}] &= [e_{0,1}[e_{0,-1}, e_{1,-m}]] = [[e_{0,1}, e_{0,-1}]e_{1,-m}] + [e_{0,-1}[e_{0,1}, e_{1,-m}]] \\ &= [-e_{0,0}, e_{1,-m}] + [e_{0,-1}, (m(m-1)/2 - 1)e_{1,-m+1}] \\ &= me_{1,-m} + (m(m-1)/2 - 1)e_{1,-m} = ((m+1)m/2 - 1)e_{1,-m}, \end{aligned}$$

as required.

Next, since $[e_{-1,1}, e_{1,-i}], [e_{-1,2}, e_{1,-i-1}] \in P_{0,-i+1}$ and $P_{0,-i+1} = 0$ for $i > 2$ we have

$$(4.9.7) \quad [e_{-1,1}, e_{1,-i}] = 0 \quad \text{for } i > 2,$$

$$(4.9.8) \quad [e_{-1,2}, e_{1,-i}] = 0 \quad \text{for } i > 3.$$

Next we have $[e_{-1,2}, e_{1,-2}] = [[e_{0,1}, e_{-1,1}]e_{1,-2}] = [e_{0,1}[e_{-1,1}, e_{1,-2}]] = [e_{0,1}, e_{0,-1}] = -e_{0,0}$; that is,

$$(4.9.9) \quad [e_{-1,2}, e_{1,-2}] = -e_{0,0}.$$

Now $[[e_{0,-1}, e_{0,1}]e_{-1,1}] = [e_{0,0}, e_{-1,1}] = e_{-1,1}$; that is, by (4.9.5)

$$(4.9.10) \quad [e_{0,-1}, e_{-1,2}] = e_{-1,1}.$$

Next $[e_{-1,2}, e_{1,-3}] = [[e_{0,1}, e_{-1,1}]e_{1,-3}] = [[e_{0,1}, e_{1,-3}]e_{-1,1}] = [2e_{1,-2}, e_{-1,1}] = -2e_{0,-1}$ (we used the definition of $e_{-1,2}$, (4.9.7), (4.9.6) with $i = 3$, and (4.9.3)). Thus

$$(4.9.11) \quad [e_{-1,2}, e_{1,-3}] = -2e_{0,-1}.$$

Now

$$\begin{aligned} [e_{1,-3}, e_{-1,3}] &= [e_{1,-3}, [e_{0,1}, e_{-1,2}]] = [[e_{1,-3}, e_{0,1}]e_{-1,2}] + [e_{0,1}[e_{1,-3}, e_{-1,2}]] \\ &= [-2e_{1,-2}, e_{-1,2}] + [e_{0,1}, 2e_{0,-1}] = -2e_{0,0} - 2e_{0,0} = -4e_{0,0}. \end{aligned}$$

We used the definition of $e_{-1,3}$, the Jacobi identity (4.9.6) with $i = 3$, (4.9.11), (4.9.9) and (4.9.1). Thus

$$(4.9.12) \quad [e_{1,-3}, e_{-1,3}] = -4e_{0,0}.$$

Next, $[e_{1,-2}, e_{-1,4}] = [e_{1,-2}[e_{0,1}, e_{-1,3}]] = [e_{0,1}[e_{1,-2}, e_{-1,3}]] = [e_{0,1}, \lambda e_{0,1}] = 0$. We used the definition of $e_{-1,4}$, (4.9.4) and Lemma 4.7(i), (ii). Thus

$$(4.9.13) \quad [e_{1,-2}, e_{-1,4}] = 0.$$

Now $[e_{1,-4}, e_{-1,3}] = [e_{1,-4}, [e_{0,1}, e_{-1,2}]] = [[e_{1,-4}, e_{0,1}]e_{-1,2}] = [-5e_{1,-3}, e_{-1,2}] = -10e_{0,-1}$. We used the definition of $e_{-1,3}$, (4.9.8) with $i = 4$, (4.9.6) with $i = 4$ and, finally, (4.9.11). We got

$$(4.9.14) \quad [e_{1,-4}, e_{-1,3}] = -10e_{0,-1}.$$

Finally,

$$\begin{aligned} [e_{1,-4}, e_{-1,4}] &= [e_{1,-4}[e_{0,1}, e_{-1,3}]] = [[e_{1,-4}, e_{0,1}]e_{-1,3}] + [e_{0,1}[e_{1,-4}, e_{-1,3}]] \\ &= [15e_{1,-3}, e_{-1,3}] + [e_{0,1}, -10e_{0,-1}] = 20e_{0,0} + 10e_{0,0} = 30e_{0,0}. \end{aligned}$$

We used the definition of $e_{-1,4}$, the Jacobi identity, (4.9.6) with $i = 4$, (4.9.14), (4.9.12) and (4.9.1). Our result is

$$(4.9.15) \quad [e_{1,-4}, e_{-1,4}] = 30e_{0,0}.$$

4.10. Now look at the subalgebra of P generated by

$$E_1 = e_{-1,2}, \quad E_2 = e_{-1,4}, \quad H = e_{0,0}, \quad F_1 = e_{1,-2}, \quad F_2 = e_{1,-4}.$$

We have $[E_i, F_j] = \lambda_i \delta_{ij} H$, $\lambda_1 = -1$, $\lambda_2 = -30$, and $[H, E_i] = a_i E_i$, $a_i \neq 0$, which makes our algebra into a contragredient algebra. As $p > 5$ we have $\lambda_1 \neq 0$, $\lambda_2 \neq 0$. By V. Kac [5, Lemma 2.1 or 2.3] it follows that our subalgebra and, therefore, P , and, therefore, S are infinite-dimensional. This contradiction establishes that S_1 must be irreducible.

5. Proof of Theorem 1.3 concluded. We have established in the previous section that S_1 is irreducible. It is faithful by Lemma 3.1(ix). This makes it possible to invert the grading of $S = \bigoplus_{i=-q}^r$ without violating conclusions of Lemma 3.1. So we can (and shall) assume that $q \leq r$. For easy reference we record

- 5.1. LEMMA. (i) $r \geq ((p-1)/2)q$; $q > 1$;
(ii) S_1 is irreducible;
(iii) S is generated by S_1 and S_{-1} ;
(iv) $S_0 \subseteq [S_{-q}, S_q] + [S_{-q+1}, V]$ for any D_0 -submodule $V \neq 0$, of S_{q-1} .

PROOF. First, (i) follows from Theorem III.2.1 (applicable in view of Remark III.2.1.1) and Proposition 3.6. Now (ii) and (iii) are Propositions 4.1 and 3.4, respectively. Finally, (iv) is Proposition III.1.7(ii).

5.2. LEMMA. Let $V \neq 0$ be a D_0 -submodule of S_i .

- (i) If $p \nmid i$, then S_0 acts faithfully on V .
(ii) If $p \mid i$, then every commutative ideal of D_0 (in particular, the last nonzero term of the derived series of S_0) acts nilpotently on V (and therefore trivially if V is irreducible).

PROOF. Let I be a commutative ideal of D_0 . Then I is a nilpotent ideal, and (ii) follows from Corollary III.1.11(ii). If R is the kernel of S_0 on V , then R is an ideal of D_0 . Let J be the last nonzero term of the derived series of R . Then J is a nilpotent ideal of D_0 and (i) follows from Corollary III.1.11(i).

5.3. PROPOSITION. *Suppose $p \nmid q - 1$, $q > 2$. Then $\langle S_{-q+1}, S_{q-1} \rangle$ is isomorphic (as a graded Lie algebra) to $R \otimes B_n$, where \tilde{R} is graded of type A_1 . In particular, S_{-q+1} and S_{q-1} are irreducible.*

PROOF. Let V be a D_0 -irreducible submodule of S_{-q+1} . Consider $\tilde{S} = \langle V, S_{q-1} \rangle + D_0$. By Proposition III.1.5(iv) (applicable in view of Proposition III.1.4 and the Remark following it) we have $[V, x] \neq 0$ for any $x \in S_{q-1}$, $x \neq 0$. By Lemma 5.2, S_0 is faithful on V . Consider $\bar{D}_0 = \{x \in D_0 \mid [x, V] = 0\}$. Then \bar{D}_0 is an ideal of D_0 and $[\bar{D}_0, S_0] = 0$ as both are ideals and S_0 is faithful on V . Then $[\bar{D}_0, S_{q-1}] = 0$ as well (for if $[\bar{D}_0, x] \neq 0$ for $x \in S_{q-1}$, then $0 \neq [V[\bar{D}_0, x]] = [\bar{D}_0[V, x]] \subseteq [\bar{D}_0, S_0] = 0$, a contradiction). Thus \bar{D}_0 is an ideal of \tilde{S} . Replacing \tilde{S} by $\bar{S} = \tilde{S}/\bar{D}_0$ we see that \bar{S} satisfies properties (i), (ii), (iv), (v), (vi), (viii) and (ix) of Lemma 3.1. Consider the ideal $N(\bar{S})$ constructed in the proof of [12, Lemma 4.5.1]: $N^0(\bar{S}) = 0$, $N^{i+1}(\bar{S}) = x \in \bigoplus_{j \geq 0} \bar{S}_{j(q-1)} \mid [S_{-q+1}, x] \subseteq N^i(\bar{S})$, $N(\bar{S}) = \bigcup N^i(\bar{S})$. Then $N(\bar{S}) \cap S_{q-1} = 0$ (since $[V, x] \neq 0$ for $x \neq 0$, $x \in S_{q-1}$). Then $\bar{S}/N(\bar{S})$ satisfies all conclusions of Lemma 3.1, except (iii). Therefore, since S is a minimal counterexample, $\bar{S} = \bar{R} \otimes B_m + D_0/\bar{D}_0$ with \bar{R} of type A_1 or W_1 . Since \bar{R} is generated by S_{-q+1} and S_{q-1} we see that \bar{R} must be of type A_1 . In particular, S_{q-1} is irreducible.

Now consider $H = \langle S_{-q-1}, S_{q-1} \rangle$. Again S_0 is faithful on S_{q-1} by the preceding part and $[x, S_{q-1}] \neq 0$ for $x \neq 0$, $x \in S_{-q+1}$ (by Propositions III.1.7(i) and III.1.12(vi)). Then $H/M(H)$ satisfies all conclusions of Lemma 3.1 except possibly (iii). As S is a minimal counterexample, and $q > 2$ implies that $\dim H < \dim S$, we see that $H/M(H) = \tilde{R} \otimes B_m + D_0$ with \tilde{R} of type A_1 or W_1 . Again, since \tilde{R} is generated by S_{-q+1} and S_{q-1} we must have that \tilde{R} is of type A_1 . And then S_{-q+1} must be irreducible. This concludes the proof.

5.4. COROLLARY. $p \nmid q$.

PROOF. If $p \mid q - 1$, then there is nothing to prove. So assume that $p \nmid q - 1$, but $p \mid q$. Then $[S_{-q+1}, S_{q-1}]$ is (by Proposition 5.3) a nilpotent ideal of D_0 . Since S_{-q} is irreducible it follows from Lemma 5.2(ii) that $[S_{-q+1}, S_{q-1}]$ acts trivially on S_{-q} . But this contradicts Proposition III.1.5(ii).

5.5. Since $p \nmid q$ we have that S_0 is faithful and irreducible on S_{-q} (by Lemma 5.2(i)). By Proposition III.1.12 the algebra $\bigoplus_{i \geq -1} S_{iq}$ satisfies the conditions of M. I. Kuznetsov [9, Theorem 5.1]. Therefore, $\bigoplus_{i \geq -1} S_{iq} = R \otimes B_m + S_0$ with R simple graded of type A_1 or W_1 with standard grading (magnified q times). We write $R_{iq} = ke_{iq}$ so that $S_{iq} = e_{iq} \otimes B_m$. We write e_{iq} for $e_{iq} \otimes 1$.

PROPOSITION. (i) $S_{iq} = e_{iq} \otimes B_m$, $i \neq 0$, are irreducible for S_0 ;

(ii) $e_0 \otimes B_m$ is a commutative ideal of D_0 ;

(iii) e_0 is in the center of D_0 ;

(iv) e_0 acts as i/q on S_i .

The statements (i)–(iii) are, essentially, proved in the proof of Lemma 2.1. To prove (iv) note that e_0 acts as -1 on S_{-q} . Since e_0 is in the center of D_0 it acts as a

scalar, say λ , on S_{-1} . Therefore it acts as λi on S_i . In particular, it acts as $-\lambda q = -1$ on S_{-q} . Thus $\lambda = 1/q$, whence (iv).

- 5.6. PROPOSITION.** (i) $\text{ad } e_{-q}: S_{iq+j} \rightarrow S_{iq+j-q}$ is injective for $i \geq 0, q > j > 0$;
(ii) $\text{ad } e_{-q} \circ \text{ad } e_q = -(q-j)q^{-1} \text{Id}_{S_{-q-j}}$ for $0 < j < q$.
(iii) $\text{ad } e_q: S_{-q+j} \rightarrow S_j$ and $\text{ad } e_{-q}: S_j \rightarrow S_{-q+j}, 0 < j < q$, are bijective if $p \nmid (q-j)$; $\text{ad } e_q: S_{-q+j} \rightarrow S_j$ is trivial if $p \mid (q-j)$.
(iv) $\text{ad } e_q \circ \text{ad } e_{-q} = (q-j)q^{-1} \text{Id}_{S_j}$ if $0 < j < q$.

PROOF. Set $\tilde{S}_0 = [S_{-q}, S_q]$ and $\tilde{S}_i = S_i$ for $i \neq 0$. We know that $\text{ad } e_{-q}: \tilde{S}_{iq} \rightarrow \tilde{S}_{(i-1)q}$ is an isomorphism of vector spaces for $i \geq 0$ (as $[e_{-q}, e_{qi} \otimes b] = [e_{-q}, e_{qi}] \otimes b = e_{q(i-1)} \otimes b$). Now take $x \in \tilde{S}_{iq+j}, i \geq 0, q > j > 0$, and assume $[e_{-q}, x] = 0$. Then $0 = (\text{ad } S_{-1})^j [e_{-q}, x] = [e_{-q}, (\text{ad } S_{-1})^j x]$. But $(\text{ad } S_{-1})^j x \neq 0$ if $x \neq 0$ and $\text{ad } e_{-q}$ is injective on $\tilde{S}_{iq}, i \geq 0$. Therefore $x = 0$. This proves (i).

Now for $x \in S_{-q+j}, 0 < j < q$, we have $[e_{-q}, [e_q, x]] = [[e_{-q}, e_q]x] = [c_0, x] = -(q-j)q^{-1}x$, whence (ii). If $p \nmid (q-j)$, then since $\text{ad } e_{-q} \circ \text{ad } e_q(x) = -(q-j)q^{-1}x$ with $(q-j)q^{-1} \neq 0$ we must have that $\text{ad } e_q(x) \neq 0$; that is, $\text{ad } e_q$ is injective. Thus $\dim S_j \geq \dim S_{-q+j}$. But $\text{ad } e_{-q}: S_j \rightarrow S_{-q+j}$ is injective as well (by (i)), so $\dim S_j = \dim S_{-q+j}$ and both $\text{ad } e_q: S_{-q+j} \rightarrow S_j$ and $\text{ad } e_{-q}: S_j \rightarrow S_{-q+j}$ are bijective. If $p \mid q-j$ and $\text{ad } e_q(x) \neq 0$ for $x \in S_{q+j}$, then also $\text{ad } e_{-q}(\text{ad } e_q(x)) \neq 0$ (by (i)). But $\text{ad } e_{-q} \circ \text{ad } e_q = 0_{S_{-q+j}}$ if $p \mid (q-j)$. This concludes the proof of (iii).

To prove (iv) first assume that $p \nmid q-j$. Then $\text{ad } e_q: S_{-q+j} \rightarrow S_j$ is trivial so (iv) holds. So $p \mid q-j$. Take a basis in S_j and a basis in S_{-q+j} . Then $\text{ad } e_q: S_{-q+j} \rightarrow S_j$ is given by a matrix, say A , and $\text{ad } e_{-q}: S_j \rightarrow S_{-q+j}$ is given by a matrix, say B . By (ii) we know that $BA = -(q-j) \cdot q^{-1} \cdot \text{Id}$. By (iii) both B and A are square matrices. Therefore they are invertible and $BA = AB = -(q-1)q^{-1} \text{Id}$, which proves (iv).

5.7. PROPOSITION. $q > 2$.

PROOF. Suppose $q = 2$. Take $x \in S_1$ and apply Proposition 5.6(iv) with $j = 1$. Then $q-j = 1$ so

$$\begin{aligned} -\frac{1}{2}x &= [e_2[e_{-2}, x]] = [[e_2, e_{-2}]x] + [e_{-2}[e_2, x]] \\ &= -[e_0, x] + [e_{-2}[e_2, x]] = -\frac{1}{2}x + [e_{-2}[e_2, x]]. \end{aligned}$$

So $[e_{-2}[e_2, x]] = 0$. Now apply Proposition 5.6(i) with $i = 1, j = 1$. This gives $[e_2, x] = 0$. Therefore $[e_2, S_1] = 0$. Now take $e_2 \otimes b \in S_2$ and write $e_2 \otimes b = [e_0 \otimes b, e_2]$. Then $[e_2 \otimes b, S_1] = [[e_0 \otimes b, e_2]S_1] = [[e_0 \otimes b, S_1]e_2] \subseteq [S_1, e_2] = 0$. Thus $[S_2, S_1] = 0$.

Now $S = \langle S_1, S_{-1} \rangle$ by Lemma 5.1(iii). So $[S_2, S_1] = 0$ means that $S_3 = 0$, that is $S = \bigoplus_{i=-2}^2 S_i$. Now Theorem III.2.1 implies that S is classical, whence a contradiction, since there are no classical graded Lie algebras with $q = 2$ and S_0 solvable.

5.7.1. REMARK. It is curious that an algebra with similar properties has appeared in [14, Theorem 2.11].

5.8. PROPOSITION. $p \mid q-1$.

PROOF. Suppose $p \nmid q - 1$. Then by Proposition 5.3 (which is applicable because of Proposition 5.7) we have that $[S_{-q+1}, S_{q-1}] = \bar{R} \otimes B_n$ is a commutative ideal I_{q-1} in S_0 . In all cases, by Proposition 5.5(ii) we see that $[S_{-q}, S_q] = R_0 \otimes B_m$ is a commutative ideal I_q of S_0 . By Lemma 5.1(iv) we see that $S_0 = I_q + I_{q-1}$; in particular, S_0 is at most two-step nilpotent.

As S_0 is irreducible on S_{-q} the central ideal $I = I_q \cap I_{q-1}$ acts as a scalar on S_{-q} . Since S_0 is faithful on S_{-q} we must have $\dim I \leq 1$. As both I_q and I_{q-1} contain a central element of D_0 we see that $\dim I = 1$.

Now we see that $[I_q, I_{q-1}] \subseteq I_q \cap I_{q-1} = I$. Therefore $[\cdot, \cdot]: (I_q/I) \times (I_{q-1}/I) \rightarrow I = k$ defines a pairing. This pairing is nondegenerate. Indeed, if say, $J \subseteq I_q$, $[J, I_{q-1}] = 0$, then J is a central ideal of S_0 and therefore acts by scalars on S_{-q} . Therefore $J \subseteq I$. Therefore $n = m$, $\dim S_0 = 2p^n - 1$, and S_0 is a Heisenberg Lie algebra.

It is known (say by [13, Theorem 1]) that all faithful irreducible representations of a Heisenberg algebra of dimension $2d + 1$ are of dimension p^d . Therefore we must have $\dim S_{-q} = p^{p^n-1}$. But $\dim S_{-q} = p^n$. So we have the equation $p^{p^n-1} = p^n$, i.e., $p^n - 1 = n$, which is impossible.

5.9 COROLLARY. (i) $q \geq 8$;

(ii) $[S_q, S_{-q+1}] = 0$;

(iii) $[[S_q, S_{-q}], S_{-q+1}] = 0$;

(iv) $[[S_q, S_{-q}], V] = 0$ for every irreducible D_0 -submodule V of S_{q-1} .

PROOF. Since $p|q - 1$ and $p \geq 7$ we have (i). By Proposition 5.6(iii) (case $j = 1$, $p|q - j$) we have $[e_q, S_{-q+1}] = 0$. Thus $0 \neq \{x \in S_q | [x, S_{-q+1}] = 0\}$. Since S_q is irreducible (by Proposition 5.5(i)), $\{x \in S_q | [x, S_{-q+1}] = 0\} = S_q$, whence (ii). Now (iii) is evident since $[S_{-q+1}, S_{-q}] = [S_{-q+1}, S_q] = 0$. Finally, (iv) holds in view of Lemma 5.2(ii) since $[S_q, S_{-q}]$ is a commutative ideal of D_0 .

5.10. Arriving at a contradiction: end of proof.

Write $\tilde{S}_0 = [S_q, S_{-q}]$. By Corollary 5.9(iii), (iv) and Proposition 5.5(iii) we have $[\tilde{S}_0, \tilde{S}_0] = [\tilde{S}_0, S_{-q+1}] = [\tilde{S}_0, V] = 0$ (where V is any irreducible D_0 -submodule of S_{q-1}). By Lemma 5.1(iv), $S_0 = \tilde{S}_0 + [S_{-q+1}, V]$. The preceding equalities now give $[S_0, \tilde{S}_0] = 0$. Thus \tilde{S}_0 is a central ideal of S_0 . As S_0 acts irreducibly on S_{-q} it follows that \tilde{S}_0 must act by scalars whence $m = 1$, i.e., $\dim S_{-q} = 1$ (by Proposition 5.5(i)). Since S_0 is faithful on S_{-q} (by Lemma 5.2(i) and Corollary 5.4) we have $\dim S_0 = 1$.

Now $[\cdot, \cdot]: S_{-1} \times S_1 \rightarrow S_0$ defines a pairing of S_{-1} and S_1 . This pairing is nondegenerate since both S_{-1} and S_1 are \bar{D}_0 -irreducible, where $\bar{D}_0 = \text{Ker } D_0|_{S_{-q}}$ (since $D_0 = S_0 \oplus \bar{D}_0$). Suppose that $\dim S_{-1} = n > 1$. Let E_1, \dots, E_n be a basis of S_{-1} , let F_1, \dots, F_n be the dual basis of S_1 , and let $S_0 = kH$. Then we have $[E_i, F_j] = \delta_{ij}H$ and $[H, E_i] = E_i$, $[H, F_j] = -F_j$. Therefore we can apply V. Kac [5, Lemma 2.1 or 2.3]. It says that if $n > 1$, S is infinite-dimensional. Thus $n = 1$. But then $S_{-2} = S_2 = 0$, i.e., S is of type A_1 , a contradiction with the assumption that S is a minimal counterexample.

PART III. APPENDIX: SOME PROPERTIES OF \mathbb{Z} -GRADED LIE ALGEBRAS

In this part we collect some results which were used in Part II and which seem to have potential for further applicability. We also include some results which seem to

go naturally with those we needed in Part II. Finally, we include a result on “asymmetry” between depth and height of a graded Lie algebra. This simple result is used only tangentially, but seems to be applicable and goes well with other results of this part.

We continue to assume that k is algebraically closed of characteristic $p > 5$.

1. General graded Lie algebras. We start by recording for future reference a list of different statements (compare [12]). Let $G = \bigoplus_{i=-q}^r G_i$ be a graded Lie algebra.

g1. $[G_i, G_j] \subseteq G_{i+j}$.

g2. $[G_i, G_{-1}] = G_{i-1}$ for $i < 0$.

g3. For any $x \in G_i$, $x \neq 0$, $i > 0$, one has $[x, G_{-1}] \neq 0$.

g4. G_{-1} is an irreducible G_0 -module.

g5. G_{-1} is a faithful G_0 -module.

g6. For any $x \in G_i$, $x \neq 0$, $i < 0$, one has $[x, \bigoplus_{j>0} G_j] \neq 0$ (i.e., $M(G) = 0$).

We set $\tilde{G} = \bigoplus_{i \neq 0} G_i + \sum_{i \neq 0} [G_i, G_{-i}]$.

1.1. LEMMA. *If g1 and g2 hold in G , then $[G_i, G_{-1}] \subseteq [G_1, G_{-1}]$ for any $i \neq 0$.*

PROOF. We can assume that $i > 0$. Our statement is true for $i = 1$. Suppose it is true for $i \leq m$. Then we have

$$\begin{aligned} [G_{m+1}, G_{-m-1}] &= [G_{m+1}[G_{-m}, G_{-1}]] \subseteq [G_{-m}[G_{m+1}, G_{-1}]] + [[G_{m+1}, G_{-m}]G_{-1}] \\ &\subseteq [G_{-m}, G_{-m}] + [G_1, G_{-1}] \subseteq [G_1, G_{-1}]. \end{aligned}$$

(We used g2, the Jacobi identity, g1, and inductive assumptions.)

1.2. PROPOSITION. *Suppose that g1, g2 and g3 hold in G . For $V \subseteq G_i$, $i < 0$, such that $[G_0, V] \subseteq V$, $[G_{-1}, V] = 0$, set $H'_i = V$, $H'_{i+t} = \sum_{0 < j \leq t} (\text{ad } G_j)H'_{i+t-j}$ and $H' = \bigoplus H'_i$. Then H' is an ideal of G .*

PROOF. The proof of [12, Lemma 4.5.1] goes through in our case with only notational differences and gives that H' is an ideal of G .

1.3. COROLLARY. *Suppose that g1, g2, g3, g4, g5 and g6 hold in G . Then $H'_i = G_i$ for $i < 0$. In particular:*

- (i) *for any $x \in G_i$, $x \neq 0$, $i > -q$, one has $[x, G_{-1}] \neq 0$;*
- (ii) $G_{-q+j} = \sum_{\alpha_i > 0, \sum \alpha_i = j} (\text{ad } G_{\alpha_1}) \cdots (\text{ad } G_{\alpha_s}) G_{-q}$, $0 < j < q$;
- (iii) $G_{-q} = [G_{-q}, [G_1, G_{-1}]]$;
- (iv) G_{-q} *is irreducible for G_0 .*

PROOF. Let $x \in H'_i$. Applying $\text{ad}(\bigoplus_{j>0} G_j)$ to x repeatedly we can, by g6, obtain $y \in H'_j$, $y \neq 0$, $j \geq 0$. Repeated application of $\text{ad } G_{-1}$ to y produces, by g3 and g5, a $z \in G_{-1}$, $z \neq 0$. By Proposition 1.2, $z \in H'$. Thus $H'_{-1} \neq 0$. By Proposition 1.2, $[G_0, H'_{-1}] \subseteq H'_{-1}$. Then g4 gives $H'_{-1} = G_{-1}$, whence, by g2, $H'_i = G_i$ for $i < 0$.

Now (i) holds for $i \geq 0$ by g3. Suppose it does not hold for some $i < 0$. Then we can construct H' from $V = \{x \in G_i | [x, G_{-1}] = 0\}$ and $H'_i = V$. Therefore $V = G_i$. But then $G_{i-1} = [V, G_{-1}] = 0$. Therefore $i = -q$ in view of g2. This proves (i).

The right-hand side of (ii) is exactly the definition of H'_{-q+j} . So (ii) just says that $H'_{-q+j} = G_{-q+j}$ which is already established. Next we have $[G_{-q}[G_{-1}, G_1]] = [G_{-1}[G_{-q}, G_1]]$. By (ii), with $j = 1$, we know that $[G_{-q}, G_1] = G_{-q+1}$. Thus $[G_{-q}[G_1, G_{-1}]] = [G_{-1}, G_{-q+1}] = G_{-q}$ as required.

Suppose that G_{-q} is not irreducible. Take $V \subset G_{-q}$, $V \neq G_{-q}$, $[V, G_0] \subseteq V$. Then $[V, G_{-1}] = 0$ and so V satisfies the assumptions of Proposition 1.2. Then $H'_{-q} = V$, a contradiction with the fact that $H'_{-q} = G_{-q}$.

1.4. PROPOSITION. *Suppose that g1, g2, g3 and g6 hold in G . Assume, in addition, that G_1 is irreducible and $[G_{-q}, G_1] \neq 0$. Then for any $x \in G_i$, $i > 0$, one has $[x, G_{-q}] \neq 0$.*

REMARK. The second condition, $[G_{-q}, G_i] \neq 0$, follows from g4 and g5 in view of Corollary 1.3(ii) or (iii).

PROOF. Set $V_i = \{x \in G_i \mid [x, G_{-q}] = 0\}$. Then V_i is a G_0 -submodule of G_i . Since G_1 is irreducible and $[G_{-q}, G_1] \neq 0$ by assumptions, we have $V_1 = 0$. Now assume by induction that $V_i = 0$ for $i \leq m$, $m > 0$. Let $x \in V_{m+1}$. Then $0 = [G_{-1}[G_{-q}, x]] = [G_{-q}[G_{-1}, x]]$. Thus $[G_{-1}, x] \subset V_m$. Since $V_m = 0$, we have $[G_{-1}, x] = 0$, whence, by g3, $x = 0$. Thus $V_{m+1} = 0$, as claimed.

1.5. PROPOSITION. *Assume that g1, g2, g3, g4, g5 and g6 hold in G . Assume, in addition, that $[G_{-q}, x] \neq 0$ for any $x \in G_{q-1}$, $x \neq 0$. Let $V \subseteq G_{-q+1}$, $V \neq 0$, and $[V, G_0] \subseteq V$. Then:*

- (i) $[V, G_{-1}] = G_{-q}$;
- (ii) for any $y \in G_{-1}$, $y \neq 0$, one has $[V, y] \neq 0$;
- (iii) if $q > 1$, then for any $x \in G_{q-1}$, $x \neq 0$, one has $[G_{-q}[V, x]] \neq 0$;
- (iv) if $q > 1$, then for any $x \in G_{q-1}$, $x \neq 0$, one has $[V, x] \neq 0$.

PROOF. First, (i) follows from Corollary 1.3(i), (iv). To prove (ii) note that $U = \{y \in G_1 \mid [y, V] = 0\}$ is a G_0 -submodule of G_{-1} . Corollary 1.3(i) implies that $U \neq G_{-1}$ and then g4 implies that $U = 0$, whence (ii). To prove (iii) observe that $[G_{-q}[V, x]] = [V[G_{-q}, x]]$. Now $[G_{-q}, x] \neq 0$ by assumption. Since $[G_{-q}, x] \subseteq G_{-1}$ we get (iii) from (ii). Finally, (iv) follows directly from (iii).

1.6. LEMMA. *Suppose that g1, g2, g3, g4, g5 and g6 hold in G . Assume, in addition, that $q > 1$ and $[G_{-q}, x] \neq 0$ for any $x \in G_{q-1}$, $x \neq 0$. Let $U \neq 0$ be a G_0 -submodule of G_{q-1} . Then $[y, U] \neq 0$ for any $y \in G_{-q+1}$, $y \neq 0$.*

PROOF. We have $[G_{-q}, U] = G_{-1}$ in view of g4 and since $[G_{-q}, U] \neq 0$ by assumption. Therefore $[G_{-q}[y, U]] = [y, [G_{-q}, U]] = [y, G_{-1}]$ and the latter is not zero by Corollary 1.3(i).

1.7. PROPOSITION. *Suppose that g1, g2, g3, g4, g5 and g6 hold in G . Assume, in addition, that $q > 1$ and $[G_{-q}, x] \neq 0$ for any $x \in G_{q-1}$, $x \neq 0$. Let $V \neq 0$ be a G_0 -submodule of G_{q-1} . Then:*

- (i) $[y, V] \neq 0$ for any $y \in G_{-q+1}$, $y \neq 0$;
- (ii) $[G_1, G_{-1}] \subseteq [G_{-q}, G_q] + [G_{-q+1}, V]$.

PROOF. Note first that $[G_{-q}, V] \neq 0$ by assumption. Therefore $[G_{-q}, V] = G_{-1}$ by g4. Therefore $[G_{-q}[y, V]] = [y[G_{-q}, V]] = [y, G_{-1}]$ and now (i) follows from Corollary 1.3(i). Next,

$$\begin{aligned} [G_1, G_{-1}] &= [G_1[G_{-q}, V]] \subseteq [G_{-q}[G_1, V]] + [[G_1, G_{-q}]V] \\ &\subseteq [G_{-q}, G_q] + [G_{-q+1}, V] \end{aligned}$$

as asserted.

1.8. LEMMA. Suppose that g1 and g2 hold in G . Let $V \subseteq G_1$ and $[V, G_0] \subseteq V$. Set $H_i = G_i$ for $i \leq 0$, $H_1 = V$ and $H_{i+1} = \{x \in G_{i+1} \mid [G_{-1}, x] \subseteq H_i\}$, $i > 0$. Then $H = \bigoplus H_i$ is a subalgebra of G .

PROOF. We have $[H_i, H_j] \subseteq H_{i+j}$ for $i, j \leq 0$ by g1. By construction, $[H_i, H_{-1}] = [H_i, G_{-1}] \subseteq H_{i-1}$ for any i . Therefore, $[H_i, H_j] \subseteq H_{i+j}$ for any i and any $j < 0$. It is easy to check that $[H_0, H_i] \subseteq H_i$. Suppose by induction that $[H_i, H_j] \subseteq H_{i+j}$ if $i + j \leq m$. Then

$$[H_{-1}[H_i, H_{m+1-i}]] \subseteq [H_{i-1}, H_{m+1-i}] + [H_i, H_{m-i}] \subseteq H_m + H_m \subseteq H_m,$$

whence $[H_i, H_{m+1-i}] \subseteq H_{m+1}$ by definition of H_i , $i > 0$.

1.9. PROPOSITION. Suppose that g1, g2 and g3 hold in G . Assume, in addition, that $G_{i+1} = [G_1, G_i]$ for $i > 0$. Let V be a maximal G_0 -submodule of G_1 , and let H be the algebra constructed from V as in Lemma 1.8. Then:

- (i) H is a maximal subalgebra of G ;
- (ii) G_1/V is contained in every nontrivial H -submodule of G/H .

PROOF. Let us first prove (ii). Let $\bar{W} \neq 0$ be an H -submodule of G/H , and let W be its preimage in G . So $W \supseteq H$, $W \neq H$ and $[W, H] \subseteq W$. Take $x \in W$ and let m be the largest integer such that $(\text{ad } G_{-1})^m x \notin H$. Take $y \in (\text{ad } G_{-1})^m x$, $y \notin H$. Write $y = \bigoplus y_i$. Then $[G_{-1}, y_i] \subseteq H_{i-1}$ for all i . For $i \leq 0$ we have $H_i = G_i$, whence $y_i \in H_i$. For $i > 1$ the condition $[G_{-1}, y_i] \subseteq H_{i-1}$ means that $y_i \in H_i$ (by construction of H_i). Thus $y \equiv y_1 \pmod{H}$. Since $y \notin H$ it means that $y_1 \notin V$. Since G_1/V is irreducible it means that $G_1 \subseteq W$; that is, $G_1/V \subseteq \bar{W}$. This proves (ii).

To prove (i) take a subalgebra $M \supset H$, $M \neq H$. Then $[M, H] \subseteq M$, whence, by (ii), $M \supseteq G_1$. Since $G_{i+1} = [G_i, G_1]$ for $i > 0$, it means that $M = G$. This proves (i).

1.10. LEMMA. Assume that g1, g2, g3 and g4 hold in G . Let x be a nilpotent element of G_0 (i.e., $(\text{ad } x)^{\dim G_0} = 0$). Then x acts as $i\lambda \cdot \text{Id} + (\text{nilpotent operator})$ on G_i .

PROOF. Our statement is true by assumption for $i = 0$. Let (see Lemma 1.1.3) m be such that $n = x^{p^m}$ is in the center of the universal enveloping algebra U of G_0 . We extend action of G_0 on G to the action of U . Then n acts as a scalar, say μ , on G_{-1} by g4. Therefore, by g2 it acts as $i\mu$ on G_i , $i < 0$. Of course, n acts as 0 on G_0 . Using g3 we establish successively that n acts as $i\mu$ on all G_i . Set $\lambda = \sqrt[p^m]{\mu}$. Then x acts as $i\lambda \cdot \text{Id} + (\text{nilpotent operator})$ on G_i , as asserted.

1.11. COROLLARY. *Assume that g1, g2, g3, g4 and g5 hold in G . Let I be a nilpotent ideal of G_0 . Then:*

- (i) *if $p \nmid i$, I acts faithfully on G_i and $[I, V] = V$ for every G_0 -submodule of G_i ;*
- (ii) *I acts nilpotently on G_{pi} and it acts trivially on every simple G_0 -submodule V of G_{pi} .*

PROOF. Since I is a nilpotent ideal of G_0 , every element of I is a nilpotent element of G_0 . By Lemma 1.10 there exists a function $\lambda: I \rightarrow k$ such that $x \in I$ acts as $i\lambda(x)\text{Id} + (\text{nilpotent operator})$ on G_i . This directly implies (ii). Let J be the kernel of the action of I on a G_0 -submodule $V \subseteq G_i$. Then J is a nilpotent ideal. If $p \nmid i$ we see that $\lambda(J) = 0$, i.e. J acts nilpotently on G_{-1} . Since G_{-1} is irreducible it means that $[J, G_{-1}] = 0$ in contradiction with g5. This proves (i).

1.12. PROPOSITION. *Suppose that \tilde{G} is simple and that g1 and g2 hold in G . Then:*

- (i) *g3, g5 and g6 hold in G ;*
- (ii) *$\tilde{G}_0 = [G_1, G_{-1}]$;*
- (iii) *G_{-q} is irreducible for $[G_1, G_{-1}]$;*
- (iv) *$[G_{-q}[G_1, G_{-1}]] = G_{-q}$;*
- (v) *$G_{-q+i} = \sum_{\alpha_i > 0, \sum \alpha_j = i} (\text{ad } G_{\alpha_1}) \cdots (\text{ad } G_{\alpha_i}) G_{-q}$ for $i > 0$;*
- (vi) *if G_1 is irreducible, then $x \in G_i, i > 0, [x, G_{-q}] = 0$ implies that $x = 0$.*

PROOF. We have (ii) by Lemma 1.1. Now (iii) and (v) are contained in [12, Lemma 4.5.1(ii) and (iii)]. Again by [12, Lemma 4.5.1(i)], g3 and g5 hold in G . If g6 does not hold, then $M(G) \neq 0$ (see [12, §1.5]). But $M(G) = 0$ since \tilde{G} is simple. Thus g6 holds. This proves (i). Now Proposition 1.4 is applicable and gives (vi). Finally, we have $[G_{-1}, G_{-q+1}] = G_{-q}$ (by g2) and $[G_1, G_{-q}] = G_{-q+1}$ (by (v)). Therefore $[G_{-q}[G_1, G_{-1}]] = [[G_{-q}, G_1]G_{-1}] = [G_{-q+1}, G_{-1}] = G_{-q}$, whence (iv).

2. On asymmetry of graded Lie algebras. The purpose of this section is to establish

2.1. THEOREM. *Let $G = \bigoplus_{i=-q}^r G_i$. Suppose that G satisfies g4 and g5, and that:*

- (a) *\tilde{G} is simple;*
- (b) *for any $x \in G_i, i > 0, [x, G_{-q}] = 0$ implies $x = 0$;*
- (c) *for any $x \in G_i, i < 0, [x, G_r] = 0$ implies $x = 0$;*
- (d) *\tilde{G} is generated by G_1 and G_{-1} .*

Then either \tilde{G} is a classical Lie algebra or $\max(q, r) \geq ((p-1)/2) \min(q, r)$.

2.1.1. REMARK. The conditions of Theorem 2.1 are satisfied if \tilde{G} is simple, G_{-1} and G_1 are irreducible and faithful G_0 -modules, and \tilde{G} is generated by G_1 and G_{-1} . Indeed, in this case Proposition 1.12(vi) applied both to G and to G with inverted grading gives the desired properties.

2.1.2. REMARK. The condition $\max(q, r) < ((p-1)/2) \max(q, r)$ is equivalent to two inequalities: $2r < pq - q$ and $2q < pr - r$.

Indeed, these inequalities can be written as $2(r+q) < pq + q, 2(r+q) < pr + r$, or $2(r+q) < (p+1) \min(q, r)$. But $r+q = \max(q, r) + \min(q, r)$. That is,

$$2 \max(q, r) + 2 \min(q, r) < (p+1) \min(q, r),$$

or

$$\max(q, r) < ((p - 1)/2) \min(q, r),$$

as required.

2.1.3. REMARK. This part of the paper draws heavily on [14] although more in spirit than in results.

2.2. LEMMA. (i) If $2r < pm - q$, $m > 0$, then $\exp \operatorname{ad} G_{-j}$, $j \geq m$, exists and consists of automorphisms of G .

(ii) If $2q < pn - r$, $n > 0$, then $\exp \operatorname{ad} G_j$, $j \geq n$, exists and consists of automorphisms of G .

PROOF. We prove only (i), as (ii) is proved in the same way. For $\exp \operatorname{ad} G_{-j}$ to exist one must have $(\operatorname{ad} G_{-j})^p = 0$. But $(\operatorname{ad} G_{-j})^p G \subseteq \bigoplus_i G_{i-pj}$. Since $j \geq m$, we have $2r < pj - q$, i.e., $r - pj < -r - q < -q$, whence $\bigoplus_i G_{i-pj} = 0$.

Next, in view of [14, Lemma 1.2] we know that $\exp \operatorname{ad} x$ is an automorphism if $h_{\alpha\beta} = [(\operatorname{ad} x)^\alpha y, (\operatorname{ad} x)^\beta z] = 0$ for all $y, z \in G$, $\alpha + \beta \geq p$. Take $x \in G_{-j}$, $y \in G_s$, $z \in G_t$, $\alpha + \beta \geq p$. Then $h_{\alpha\beta} \in G_{-j(\alpha+\beta)+s+t}$. But $-j(\alpha + \beta) + s + t \leq -mp + 2r < -q$, whence $h_{\alpha\beta} = 0$. This concludes the proof of (i).

2.3. PROPOSITION. Suppose that \tilde{G} has no center and $2r < pm - q$, $2q < pn - r$ with $m > 0$, $n > 0$. Then the subalgebra H of G generated by $H_1 = \bigoplus_{i \leq -m} G_i$ and $H_2 = \bigoplus_{i \geq n} G_i$ is, modulo its radical, a direct sum of simple classical Lie algebras (we say simply: classical). Moreover, if $\operatorname{Nilrad} H$ is the maximal ideal of H acting nilpotently on G then $H/\operatorname{Nilrad} H$ has only central radical and $H/\operatorname{Nilrad} H$ coincides with its derived algebra.

PROOF. Since \tilde{G} has no center we have an imbedding $\operatorname{ad}: \tilde{G} \rightarrow \operatorname{Der} \tilde{G}$. We identify subsets of \tilde{G} with their images under ad . Consider the subgroup \mathcal{H} of $\operatorname{Aut} \tilde{G}$ generated by $\exp \operatorname{ad} H_1$ and $\exp \operatorname{ad} H_2$ (meaningful by Lemma 2.2). For the Lie algebra of \mathcal{H} , $\operatorname{Lie} \mathcal{H}$, we have $\operatorname{Lie} \mathcal{H} \subseteq \operatorname{Der} G$ and $\operatorname{Lie} \mathcal{H} \supseteq H_1 \oplus H_2$. Therefore $\operatorname{Lie} \mathcal{H} \supseteq H$. Clearly, H is stable under \mathcal{H} and therefore H is an ideal of $\operatorname{Lie} \mathcal{H}$. Let \mathcal{N} be the unipotent radical of \mathcal{H} . Then \mathcal{H}/\mathcal{N} is reductive and generated by unipotent subgroups $\exp \operatorname{ad} H_i \pmod{\mathcal{N}}$, $i = 1, 2$. Therefore \mathcal{H}/\mathcal{N} is semisimple. Write \mathcal{H}/\mathcal{N} as an almost direct product of factors \mathcal{S}_i such that all simple factors of \mathcal{S}_i are of the same type. Write $S_i = \operatorname{Lie} \mathcal{S}_i$. If simple factors of an \mathcal{S}_i are not of type A_{p_t-1} , then S_i is a direct sum of simple Lie algebras. Otherwise S_i may have center and may be distinct from their derived algebras. However, since $S_i \cap H/(H \cap \operatorname{Lie} \mathcal{N})$ is generated by nilpotent elements, we always have that $S_i \cap H/(H \cap \operatorname{Lie} \mathcal{N})$ coincides with its derived algebra. Since $H \cap \operatorname{Lie} \mathcal{N}$ is an ideal of H which acts nilpotently on G , our proposition is proven.

2.4. COROLLARY. Take assumptions and notation of Proposition 2.3 and assume, in addition, that H is not nil. Then $q \geq n$, $r \geq m$.

PROOF. Let \mathcal{T} be the one-dimensional subtorus of $\operatorname{Aut} \tilde{G}$ defining the grading of \tilde{G} (see [12, §1.4]). Then \mathcal{T} normalizes H and \mathcal{H} . Since \mathcal{T} preserves $\operatorname{Nilrad} H$, it acts on

$H/\text{Nilrad } H$. By condition $H/\text{Nilrad } H \neq 0$, it is classical. But in a classical Lie algebra any \mathbf{Z} -grading is symmetric (i.e., it has height equal to depth). Since $H/\text{Nilrad } H$ is generated by images of $H_1 = \bigoplus_{i=-q}^{-m} G_i$, $H_2 = \bigoplus_{i=n}^r G_i$, this implies that $q \geq n$, $r \geq m$.

2.5. LEMMA. *Take assumptions of Theorem 2.1. Then the subalgebra H of G generated by G_{-q} and G_r is not nil.*

PROOF. Suppose it is. Let \mathcal{T} be the one-dimensional torus defining the grading of G . Since H is invariant under \mathcal{T} so is its center C . So $C = \bigoplus C_i$. By conditions (b) and (c) of Theorem 2.1 we have that $C = C_0$, i.e., $C \subseteq G_0$. But then C is an ideal of G_0 . Since the action of C on \tilde{G} is nilpotent and in view of g4 and g5 we have $C = 0$, a contradiction, since a nilpotent algebra must have a nontrivial center.

2.6. PROOF OF THEOREM 2.1. Suppose $\max(q, r) < ((p-1)/2)\min(q, r)$. By Remark 2.1.2 this is equivalent to $2r < pq - q$, $2q < pr - r$. By Lemma 2.5 we know that $H = \langle G_{-q}, G_r \rangle$ is not nil. Therefore we can apply Corollary 2.4 with $m = q$, $n = r$. It gives us $q \geq r$, $r \geq q$; that is, $q = r$.

If $q = 1$ then we are done by Proposition 2.3 (since $H = \tilde{G}$ is simple). If $q \geq 2$, we consider $H_1 = G_{-q} \oplus G_{-q+1}$ and $H_2 = G_q \oplus G_{q-1}$. We have $p(q-1) - q = (p-1)q - p > 2q$ if $p \geq 7$ (and also if $q > 2$, $p \geq 5$). Therefore we can apply Proposition 2.3 (with $m = n = q-1$). In this case $H = \langle H_1, H_2 \rangle$ contains $[G_{-q}, G_{q-1}] = G_{-1}$, $[G_q, G_{-q+1}] = G_1$ and since \tilde{G} is generated by G_1 and G_{-1} we have $H = \tilde{G}$. So Proposition 2.3 says that \tilde{G} is classical.

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