

INVERSE PRODUCING EXTENSION OF A BANACH ALGEBRA WHICH ELIMINATES THE RESIDUAL SPECTRUM OF ONE ELEMENT

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ABSTRACT. If A is a commutative unital Banach algebra and $G \subset A$ is a collection of nontopological zero divisors, the question arises whether we can find an extension A' of A in which every element of G has an inverse. Shilov [1] proved that this was the case if G consisted of a single element, and Arens [2] conjectures that it might be true for any set G . In [3], Bollobás proved that this is not the case, and gave an example of an uncountable set G for which no extension A' can contain inverses for more than countably many elements of G . Bollobás proved that it was possible to find inverses for any countable G , and gave best possible bounds for the norms of the inverses in [4].

In this paper, it is proved that inverses can always be found if the elements of G differ only by multiples of the unit; that is, we can eliminate the residual spectrum of one element of A . This answers the question posed by Bollobás in [5].

1. Preliminary definitions and statement of the main result. Throughout this paper, a Banach algebra is assumed to be commutative and to possess a unit.

If A is a Banach algebra, $x \in A$, then the *essential spectrum* of x in A is the set

$$\sigma_e(x) = \{\lambda \in \mathbb{C} : \lambda \cdot 1_A - x \text{ is a topological zero divisor}\},$$

and the *residual spectrum* of x in A is the set

$$\sigma_r(x) = \{\lambda \in \mathbb{C} : \lambda \cdot 1_A - x \text{ is not invertible, but is not a topological zero divisor}\}.$$

Thus our main theorem may be stated as follows.

THEOREM 1. *Let A be a commutative Banach algebra, $x \in A$. Then there is an extension A' of A in which the spectrum of x is precisely the essential spectrum of x in A .*

Before proving Theorem 1, we prove the weaker result stated here as Theorem 2.

THEOREM 2. *Let A be a Banach algebra, $x \in A$, and let K be a compact set in the residual spectrum of x in A . Then there is an extension A' of A , such that the spectrum of x in A' does not intersect K .*

The method used to prove Theorem 2 is to take an open neighbourhood U of the essential spectrum of x in A , whose closure does not intersect K . (Such a

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neighbourhood can always be found since the essential spectrum and the set K are two nonintersecting compact sets in C .) We then consider the algebra X of bounded analytic maps $U \rightarrow A$, and give it the supremum norm; this is a Banach algebra in which A is embedded isometrically as the constant functions.

We then let $J \subset X$ denote the closed ideal generated by the function $h \in X$, where

$$\begin{aligned} h: U &\rightarrow A \\ &: \lambda \rightarrow \lambda \cdot 1_A - x \end{aligned}$$

(we shall see that the ideal generated by this function is closed anyway).

We shall find that the algebra A is embedded (up to isomorphism) in the quotient space X/J . But then X/J is an extension of A , and the spectrum of x when embedded in X/J does not intersect K .

To see this, let $\mu \in K$, and let us find an inverse for $\mu \cdot 1_A - x$ when embedded in X/J . Let

$$\begin{aligned} R_\mu: U &\rightarrow A \\ &: \lambda \rightarrow (\mu - \lambda)^{-1} \cdot 1_A. \end{aligned}$$

This is a bounded analytic map $U \rightarrow A$ (since K and U are a finite distance apart). So $R_\mu \in X$. Also, the element $\mu \cdot 1_A - x$ is embedded in X as the constant function

$$\begin{aligned} c_\mu: U &\rightarrow A \\ &: \lambda \rightarrow \mu \cdot 1_A - x. \end{aligned}$$

Let $q: X \rightarrow X/J$ denote the quotient map. Then,

$$\begin{aligned} R_\mu \cdot c_\mu: U &\rightarrow A \\ &: \lambda \rightarrow ((\mu - \lambda)^{-1} \cdot 1_A) \cdot (\mu \cdot 1_A - x) \\ &= (\mu - \lambda)^{-1} \cdot 1_A \cdot ((\mu - \lambda) \cdot 1_A + (\lambda \cdot 1_A - x)) \\ &= 1_A + (\mu - \lambda)^{-1} \cdot (\lambda \cdot 1_A - x). \end{aligned}$$

Thus

$$q(R_\mu) \cdot q(c_\mu) = q(1_x) + q(h),$$

where $h: \lambda \rightarrow (\mu - \lambda)^{-1} \cdot (\lambda \cdot 1_A - x)$ and

$$\begin{aligned} 1_x: U &\rightarrow A \\ &: \lambda \rightarrow 1_A. \end{aligned}$$

But h is in the ideal J , so $q(h) = 0$. Thus $q(R_\mu) \cdot q(c_\mu) = q(1_x)$, so the element $\mu \cdot 1_A - x \in A$ has an inverse $q(R_\mu)$, when embedded as $q(c_\mu) \in X/J$.

Thus Theorem 2 will be proved.

An important tool in proving Theorem 2 and, later, Theorem 1, is the following

LEMMA 3. *Let B be a Banach algebra, $x \in B$, and let U and V be open sets in C such that*

- (1) U contains the essential spectrum of x in B ,
- (2) V contains U , and
- (3) every component of V intersects U .

Suppose we have $f(\lambda) = (\lambda - x)g(\lambda)$ (all $\lambda \in U$), where f and g are analytic functions $f: V \rightarrow B$ and $g: U \rightarrow B$.

Then there is an analytic extension $g: V \rightarrow B$ of g .

PROOF OF LEMMA 3. We can find an open set $W \subset C$, such that $V = U \cup W$ and $\overline{W} \cap \sigma_e(x) = \emptyset$. Each component of W will intersect U , and our problem is to extend the analytic germs of g from $U \cap W$ to all of W .

Since

$$(*) \quad \overline{W} \cap \sigma_e(x) = \emptyset,$$

we claim that there is an $\varepsilon > 0$ such that $\lambda \in W$, $a \in A$ implies

$$(3.1) \quad \|(\lambda \cdot 1_A - x) \cdot a\| \geq \varepsilon \|a\|.$$

For if not, there are sequences $(\lambda_n)_{n=1}^\infty \subset W$, and $(a_n)_{n=1}^\infty \subset A$, with each $\|a_n\| = 1$, and

$$\|(\lambda_n \cdot 1_A - x) \cdot a_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then $\{|\lambda_n|: n \in \mathbb{N}\}$ must be bounded, so we may assume (taking a subsequence if necessary) that $\lambda_n \rightarrow \lambda \in \overline{W}$. Then

$$\begin{aligned} \|(\lambda \cdot 1_A - x) \cdot a_n\| &\leq \|(\lambda_n \cdot 1_A - x)a_n\| + \|(\lambda - \lambda_n)a_n\| \\ &= \|(\lambda_n \cdot 1_A - x)a_n\| + |\lambda - \lambda_n| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus $\lambda \cdot 1_A - x$ is a topological zero divisor in A , so $\lambda \in \overline{W} \cap \sigma_e(x)$, contradicting our observation $(*)$ that this set is empty. Let us choose an $\varepsilon > 0$ such that condition (3.1) is satisfied.

Now,

$$f(\lambda) = (\lambda \cdot 1_A - x) \cdot g(\lambda)$$

which implies that, for each $n = 1, 2, \dots$,

$$f^{(n)}(\lambda) = (\lambda \cdot 1_A - x) \cdot g^{(n)}(\lambda) + ng^{(n-1)}(\lambda),$$

where $h^{(r)}$ denotes the r th derivative of a function h .

It follows that, if we have an analytic germ of g at some point $\lambda_0 \in W$, then

$$\|g^{(n)}(\lambda_0)\| \leq \sum_{r=0}^n \|f^{(r)}(\lambda_0)\| \cdot \frac{n!}{r!} \cdot \left(\frac{1}{\varepsilon}\right)^{n-r}$$

for all $n \in \mathbb{N}$.

So if the power series for f at λ_0 has radius of convergence $\delta > 0$, then the power series for g at λ_0 has radius of convergence greater than or equal to $\varepsilon \cdot \delta$.

Therefore this radius of convergence is bounded away from zero on any compact set in W ; hence it must be possible to extend g throughout W , as required. Thus Lemma 3 is proved.

COROLLARY 4. Let B be a Banach algebra, $c \in B$, and suppose that

$$c = (\lambda \cdot 1_A - x) \cdot g(\lambda) \quad (\text{all } \lambda \in U),$$

where $g: U \rightarrow B$ is analytic, and U is a neighbourhood of the essential spectrum of c in B . Then $c = 0$.

PROOF. The constant function c can be extended to all of C , hence, by Lemma 3, the function g extends to all of C , and the extension is a bounded entire function,

which must be constant. Thus we have $c = (\lambda \cdot 1_A - x) \cdot g$ (all $\lambda \in \mathbb{C}$) for some $g \in A$; therefore $c = g = 0$.

We now prove Theorem 2.

PROOF OF THEOREM 2. We are given a Banach algebra A , an element $x \in A$, and a compact set K in the residual spectrum of X in A . We wish to exhibit an extension A' of A , such that the spectrum of x in A does not intersect K .

Let us choose a bounded open set $U \subset \mathbb{C}$ such that $U \supset \sigma_e(x)$ and $\bar{U} \cap K = \emptyset$. Let X be the Banach algebra of bounded analytic functions $U \rightarrow A$, with the supremum norm,

$$\|f\|_X = \sup_{\lambda \in U} \|f(\lambda)\|_A.$$

(Note: we do not demand that such a function have a continuous extension to $\bar{U}$; this is important when we come to prove Theorem 1.) Let J be the closed ideal in X generated by the function $h \in X$, where $h(\lambda) = \lambda \cdot 1_A - x$ (all $\lambda \in U$).

Consider the isometric embedding $j: A \rightarrow X$ sending $c \in A$ to the constant function $j(c): \lambda \rightarrow c$ (all $\lambda \in U$).

We wish to show that the morphism $\psi: A \rightarrow X/J$ obtained by composing j and the quotient map $q: X \rightarrow X/J$, is still an isomorphism.

Now it is evident that

$$\|\psi(a)\| \leq \|a\| \quad \text{for all } a \in A;$$

so we need to check that there is no sequence $(c_i)_{i=1}^\infty \subset A$, such that each $\|c_i\| = 1$, but

$$\|\psi(c_i)\|_{X/J} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Now J is the closure of the set of all functions $H \in X$ of form

$$\begin{aligned} H: U &\rightarrow A \\ &: \lambda \rightarrow f(\lambda) \cdot (\lambda \cdot 1_A - x), \end{aligned}$$

where $f \in X$. Thus if $\|\psi(c_i)\|_{X/J} \rightarrow 0$ as $i \rightarrow \infty$, there must be functions $(f_i)_{i=1}^\infty \subset X$ such that

$$\sup_{\lambda \in U} \|f_i(\lambda) \cdot (\lambda \cdot 1_A - x) - c_i\|_A \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

We have to show that such constants and functions cannot exist. We state this as a separate lemma:

LEMMA 2.1. *If A is a Banach algebra, $x \in A$, and $U \subset \mathbb{C}$ is an open set containing the essential spectrum of x in A , then there is an $\varepsilon > 0$ such that for all $c \in A$, and all analytic functions $f: U \rightarrow \mathbb{C}$, we have*

$$\sup_{\lambda \in U} \|f(\lambda) \cdot (\lambda \cdot 1_A - x) - c\|_A \geq \varepsilon \|c\|.$$

PROOF OF LEMMA 2.1. Let B be the Banach algebra of bounded sequences $(a_i)_{i=1}^\infty \subset A$, with pointwise addition and multiplication, and the norm

$$\|(a_i)_{i=1}^\infty\|_B = \sup_{i \in \mathbb{N}} \|a_i\|_A.$$

Let I be the closed ideal of B consisting of those sequences in B which norm converge to zero. Let π be the natural projection $\pi: B \rightarrow B/I$.

Now, for all $(a_i)_{i=1}^\infty \in B$,

$$\|\pi((a_i)_{i=1}^\infty)\|_{B/I} = \limsup_{i \rightarrow \infty} \|a_i\|_A;$$

and B/I is a commutative Banach algebra with unit $1_{B/I} = \pi[(1_A, 1_A, 1_A, \dots)]$ (the equivalence class of the sequence in B consisting entirely of 1's).

A is embedded in B/I by the isometry

$$\begin{aligned} \phi: A &\rightarrow B/I \\ &: a \rightarrow \pi[(a, a, a, \dots)]. \end{aligned}$$

In fact, our element x will have exactly the same spectrum and essential spectrum as $\phi(x) \in B/I$ as it did in A .

Suppose our lemma is false. Let $(c_i)_{i=1}^\infty$ be a sequence of norm 1 elements of A , and $(f_i)_{i=1}^\infty$ a sequence of analytic functions $U \rightarrow A$, such that

$$\sup_{\lambda \in U} \|f_i(\lambda) \cdot (\lambda \cdot 1_A - x) - c_i\| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Now if $W \subset U$ is any set which is bounded away from the essential spectrum of x , then there is an $\eta > 0$ such that

$$\|a \cdot (\lambda \cdot 1_A - x)\| \geq \eta \|a\| \quad \text{for all } a \in A, \lambda \in W.$$

Therefore for all $i \in \mathbb{N}$, $\lambda \in W$ we have

$$\|f_i(\lambda) \cdot (\lambda \cdot 1_A - x) - c_i\| \geq \eta \|f_i(\lambda)\| - \|c_i\| = \eta \|f_i(\lambda)\| - 1.$$

Thus there is a uniform bound on the values $\|f_i(\lambda)\|$ for all i , throughout W .

However, since the essential spectrum is a compact set within U , we may pick a suitable set W so that, by the Maximum Modulus principle, each f_i approaches its supremum norm $\sup_{\lambda \in U} \|f_i(\lambda)\|$ somewhere on W (we must say "approaches" rather than "achieves" since U is an open set so the supremum need not be achieved anywhere).

It follows that the collection $(f_i)_{i=1}^\infty$ is uniformly norm bounded. Similarly, the collection is uniformly differentiable at any point $u \in U$.

Now consider the map

$$\begin{aligned} F: U &\rightarrow B/I \\ &: \lambda \rightarrow \pi[(f_1(\lambda), f_2(\lambda), f_3(\lambda), \dots)]. \end{aligned}$$

Since the f_i 's are uniformly bounded, the sequence $(f_1(\lambda), f_2(\lambda), f_3(\lambda), \dots)$ is always in B ; and since they are uniformly differentiable, we find that F is a uniformly bounded analytic map. It is easy to see that for all $\lambda \in U$,

$$(\lambda \cdot 1_{B/I} - \phi(x)) \cdot F(\lambda) = \pi[(c_1, c_2, c_3, \dots)].$$

Therefore, since U contains the essential spectrum of $\phi(x) \in B/I$, we have by Corollary 4 that $\pi[(c_1, c_2, c_3, \dots)] = 0$. Therefore $\limsup_{i \rightarrow \infty} \|c_i\|_A = 0$, but this is a contradiction since by hypothesis each $\|c_i\| = 1$. Thus Lemma 2.1 is proved.

COROLLARY 2.2. *The map $\psi: A \rightarrow X/J$ is an isomorphism.*

By the result of [6], we can put an equivalent norm on X/J so that A is now embedded isometrically by the map ψ . But this proves the theorem, for X/J contains an inverse to $\mu \cdot 1_A - x$ for every $\mu \in K$, namely the element $q(R_\mu)$, where

$$\begin{aligned} R_\mu: U &\rightarrow A \\ &: \lambda \rightarrow (\mu - \lambda)^{-1} \cdot 1_A. \end{aligned}$$

Having proved Theorem 2, we now make some definitions which lead towards a proof of Theorem 1.

Let Ω denote the collection of all countable ordinals. With each ordinal $\alpha \in \Omega$, we shall associate a Banach algebra $A(\alpha)$; the collection $\{A(\alpha): \alpha \in \Omega\}$ will be directed upwards, in the sense that for all $\alpha, \beta \in \Omega$, $\alpha < \beta$, there will be an isometric embedding

$$\tau_{\alpha, \beta}: A(\alpha) \rightarrow A(\beta).$$

Furthermore, if $\alpha, \beta, \gamma \in \Omega$ ($\alpha < \beta < \gamma$), we will have $\tau_{\beta, \gamma} \circ \tau_{\alpha, \beta} = \tau_{\alpha, \gamma}$.

We now proceed to define our sequence of Banach algebras, using transfinite induction.

(1) We define $A(1)$ to be our original Banach algebra A .

(2) Given $\alpha \in \Omega$ and the Banach algebra $A(\alpha)$, we define $A(\alpha + 1)$ as follows:

Let $B(\alpha)$ be the Banach algebra of sequences $(a_i)_{i=1}^\infty$, $a_i \in A(\alpha)$, with pointwise addition and multiplication and the supremum norm

$$\|(a_i)_{i=1}^\infty\|_{B(\alpha)} = \sup_{i \in \mathbf{N}} \|a_i\|_{A(\alpha)},$$

let $I(\alpha)$ be the closed ideal consisting of all sequences in $B(\alpha)$ which norm converge to zero, and let π_α denote the natural projection

$$\pi_\alpha: B(\alpha) \rightarrow B(\alpha)/I(\alpha).$$

We define $A(\alpha + 1) = B(\alpha)/I(\alpha)$, and the map $\tau_{\alpha, \alpha+1}$ is the isometric embedding

$$\begin{aligned} j_\alpha: A(\alpha) &\rightarrow A(\alpha + 1) \\ &: a \rightarrow \pi_\alpha[(a, a, a, \dots)]. \end{aligned}$$

We must then define $\tau_{\beta, \alpha+1} = j_\alpha \circ \tau_{\beta, \alpha}$ for each ordinal $\beta < \alpha$.

(3) If $\alpha_i \in \Omega$, $i = 1, 2, \dots$, then we require that $A(\bigcup_i \alpha_i)$ be the completion of the direct limit of the collection $\{A(\alpha_i), i = 1, 2, 3, \dots\}$ of Banach algebras, which is directed by the maps $\tau_{\alpha_i, \alpha_j}$ ($\alpha_i < \alpha_j$). For $\beta < \bigcup_i \alpha_i$, the map

$$\tau_{\beta, \bigcup_i \alpha_i}: A(\beta) \rightarrow A\left(\bigcup_i \alpha_i\right)$$

is the direct limit of the maps τ_{β, α_i} : $\beta < \alpha_i$, followed by the map which sends the direct limit of the algebras $A(\alpha_i)$ to its completion, which is $A(\bigcup_i \alpha_i)$.

Now these three conditions define the collection $\{A(\alpha), \alpha \in \Omega\}$ uniquely, together with the linking maps $\tau_{\beta, \alpha}$: $\beta < \alpha$.

LEMMA 5. If $\alpha, \beta \in \Omega$ ($\alpha \leq \beta$) and $a \in A(\alpha)$, then

$$(*) \quad \inf_{y \in A(\beta)} \left(\frac{\|\tau_{\alpha, \beta}(a) \cdot y\|}{\|y\|} \right) = \inf_{y_0 \in A(\alpha)} \left(\frac{\|ay_0\|}{\|y_0\|} \right).$$

PROOF. The left-hand side of this identity is an infimum similar to that on the right-hand side, but evaluated with the element $a \in A(\alpha)$ embedded in a larger space $A(\beta)$. It is therefore less than or equal to the right-hand side.

The opposite inequality is proved by transfinite induction on β ; the result is trivially true if $\beta = \alpha$.

If the result is true for $\beta = \beta_0$, then given $y \in A(\beta_0 + 1)$ let us say

$$y = \pi_\alpha[(y_i)_{i=1}^\infty] \quad (y_i \in A(\beta_0));$$

then

$$\begin{aligned} \|y \cdot \tau_{\alpha, \beta_0+1}(a)\|_{A(\beta_0+1)} &= \limsup_{i \rightarrow \infty} (\|y_i \cdot \tau_{\alpha, \beta_0}(a)\|_{A(\beta_0)}) \\ &\geq \limsup_{i \rightarrow \infty} \|y_i\| \cdot \inf_{y' \in A(\beta_0)} \left(\frac{\|y' \cdot \tau_{\alpha, \beta_0}(a)\|}{\|y'\|} \right) \\ &\geq \limsup_{i \rightarrow \infty} \|y_i\| \cdot \inf_{y_0 \in A(\alpha)} \left(\frac{\|y_0 a\|}{\|y_0\|} \right) \\ &\quad \text{(by induction hypothesis)} \\ &= \|y\|_{A(\beta_0+1)} \cdot \inf_{y_0 \in A(\alpha)} \left(\frac{\|y_0 a\|}{\|y_0\|} \right). \end{aligned}$$

This is the result for $\beta_0 + 1$. But it is clear that an equality such as $(*)$ is preserved under direct limits and completions. So the result is true for all $\beta \in \Omega$.

We now prove Theorem 1.

PROOF OF THEOREM 1. Given a Banach algebra A and $x \in A$, let us choose a sequence $(U_i)_{i=1}^\infty$ of bounded open sets in $\mathbb{C}$, such that:

- (1) For each i , $U_i \supset U_{i+1}$.
- (2) For each i , every component of U_i intersects U_{i+1} .
- (3) $\bigcap_{i=1}^\infty U_i = \sigma_\epsilon(x)$.

DEFINITION. A sequence $(\epsilon_i)_{i=1}^\infty$ of strictly positive real numbers is said to be "admissible" for $\alpha \in \Omega$ if, whenever there are bounded analytic functions

$$g_i: U_i \rightarrow A(\alpha) \quad (i = 1, 2, \dots, n),$$

an analytic function $f: U_n \rightarrow A(\alpha)$ and a constant $c \in A(\alpha)$ such that, for all $\lambda \in U_n$,

$$c = \sum_{i=1}^n g_i(\lambda) + \tau_{1, \alpha}(\lambda \cdot 1_A - x) \cdot f(\lambda),$$

then

$$\|c\|_{A(\alpha)} \leq \sum_{i=1}^n \epsilon_i^{-1} \cdot \sup_{\lambda \in U_i} \|g_i(\lambda)\|_{A(\alpha)}.$$

The most important step in proving Theorem 1 is the following

LEMMA 6. *There is a sequence $(\varepsilon_i)_{i=1}^\infty$ such that, for each n , the sequence $(\varepsilon_i)_{i=1}^n$ is admissible for every $\alpha \in \Omega$.*

PROOF OF LEMMA 6. It is enough to show the following:

- (1) There is an ε_1 , which is admissible, as a sequence of length 1, for every $\alpha \in \Omega$.
- (2) If the sequence $(\varepsilon_i)_{i=1}^n$ is admissible for every $\alpha \in \Omega$ and $\delta > 0$, then there is an $\varepsilon_{n+1} > 0$ such that the sequence

$$(1 + \delta)^{-1} \cdot \varepsilon_1, (1 + \delta)^{-1} \cdot \varepsilon_2, \dots, (1 + \delta)^{-1} \cdot \varepsilon_n, \varepsilon_{n+1}$$

is also admissible for every $\alpha \in \Omega$.

Let us apply Lemma 2.1 with A replaced by $A(\alpha)$ for some ordinal α , x replaced by $\tau_{1,\alpha}(x)$ and U replaced by U_1 . We find that there is an $\varepsilon > 0$ such that, for all $c \in A(\alpha)$ and all analytic functions $f: U_1 \rightarrow A(\alpha)$, we have

$$\sup_{\lambda \in U_1} \|f(\lambda) \cdot (\lambda \cdot 1_A - x) - c\|_{A(\alpha)} \geq \varepsilon \|c\|.$$

Writing $g_1(\lambda) = c - f(\lambda) \cdot (\lambda \cdot 1_A - x)$ ($\lambda \in U_1$) we see the value ε , a “sequence” of length 1, is admissible for α .

Denote by $\varepsilon_1(\alpha)$ the supremum of all admissible values of ε for a given α . This decreases with increasing α , since the Banach algebras involved are always getting larger.

But $\varepsilon_1(\alpha)$ must be bounded away from zero. For if we could find a sequence $(\alpha_i)_{i=1}^\infty$ such that $\alpha_i \in \Omega$ and $\varepsilon_1(\alpha_i) \rightarrow 0$ as $i \rightarrow \infty$, then we must have $\varepsilon_1(\bigcup_{i=1}^\infty \alpha_i) = 0$, which is impossible.

Thus there is an ε_1 which is admissible for every $\alpha \in \Omega$. This proves assertion (1).

Now suppose that $(\varepsilon_i)_{i=1}^n$ is admissible for every $\alpha \in \Omega$. Choose a particular $\alpha \in \Omega$, and, given $\delta > 0$, suppose that we cannot find an ε_{n+1} such that

$$(1 + \delta)^{-1} \varepsilon_1, (1 + \delta)^{-1} \varepsilon_2, \dots, (1 + \delta)^{-1} \varepsilon_n, \varepsilon_{n+1}$$

is admissible for α .

Then we must be able to find constants $(c_j)_{j=1}^\infty$ in $A(\alpha)$, and analytic functions

$$g_i^{(j)}: U_i \rightarrow A(\alpha) \quad (i = 1, 2, \dots, n+1; j \in \mathbf{N})$$

and

$$f^{(j)}: U_{n+1} \rightarrow A(\alpha) \quad (j \in \mathbf{N}),$$

such that, for each j , $\|c_j\| = 1$, and for all $\lambda \in U_{n+1}$,

$$c_j = \sum_{i=1}^{n+1} g_i^{(j)}(\lambda) + \tau_{1,\alpha}(\lambda \cdot 1_A - x) \cdot f^{(j)}(\lambda),$$

but

$$1 > (1 + \delta) \sum_{i=1}^{n+1} \sup_{\lambda \in U_i} \|g_i^{(j)}(\lambda)\| + 2^j \cdot \sup_{\lambda \in U_{n+1}} \|g_{n+1}^{(j)}(\lambda)\|.$$

Extracting a subsequence from $(c_k)_{k=1}^\infty$, we may assume that

$$\sup_{\lambda \in U_i} \|g_i^{(j)}(\lambda)\|_{A(\alpha)} \rightarrow \eta_i \quad \text{as } j \rightarrow \infty, \quad 1 \leq i \leq n,$$

where

$$(*) \quad (1 + \delta) \cdot \sum_{i=1}^n \varepsilon_i^{-1} \cdot \eta_i \leq 1.$$

But now, for each $i \leq n$ and $\lambda \in U_i$, define

$$G_i(\lambda) = \pi_\alpha[(g_i^{(1)}(\lambda), g_i^{(2)}(\lambda), \dots, g_i^{(j)}(\lambda), \dots)].$$

Define also, for $\lambda \in U_{n+1}$,

$$F(\lambda) = \pi_\alpha[(f^{(1)}(\lambda), f^{(2)}(\lambda), \dots, f^{(j)}(\lambda), \dots)] \in A(\alpha + 1).$$

These functions are analytic by arguments similar to those used in the proof of Theorem 2.

Then, for every $\lambda \in U_{n+1}$,

$$\sum_{i=1}^n G_i(\lambda) + (\lambda \cdot 1_{A(\alpha+1)} - \tau_{1, \alpha+1}(x)) \cdot F(\lambda) = \pi_\alpha[(c_1, c_2, c_3, \dots)].$$

Therefore, by Lemma 3, we can extend F to U_n (this being the domain of definition of G_n), and so, since $(\varepsilon_i)_{i=1}^n$ is admissible for $\alpha + 1$, we must have

$$1 = \|\pi_\alpha[(c_1, c_2, c_3, \dots)]\| \leq \sum_{i=1}^n \sup_{\lambda \in U_i} \|G_i(\lambda)\| \cdot \varepsilon_i^{-1}.$$

But $\sup_{\lambda \in U_i} \|G_i(\lambda)\| \leq \eta_i$ so, by (*), we have a contradiction.

Thus, for each $\alpha \in \Omega$, there is a suitable $\varepsilon_{n+1} > 0$ so that

$$(1 + \delta)^{-1} \varepsilon_1, (1 + \delta)^{-1} \varepsilon_2, \dots, (1 + \delta)^{-1} \varepsilon_n, \varepsilon_{n+1},$$

is an admissible sequence for α . Let $\varepsilon_{n+1}(\alpha)$ denote the supremum of possible values of ε_{n+1} for a given α .

By the transfinite induction argument of part (1) of this lemma, $\inf_{\alpha \in \Omega} \varepsilon_{n+1}(\alpha) > 0$.

Thus there is an ε_{n+1} such that

$$(1 + \delta)^{-1} \varepsilon_1 (1 + \delta)^{-1} \varepsilon_2, (1 + \delta)^{-1} \varepsilon_3, \dots, (1 + \delta)^{-1} \varepsilon_n, \varepsilon_{n+1}$$

is admissible for every $\alpha \in \Omega$. Thus Lemma 6 is proven.

Let $(\varepsilon_i)_{i=1}^\infty$ be a sequence such that $(\varepsilon_i)_{i=1}^n$ is admissible for each n and $\alpha \in \Omega$. Assume each $\varepsilon_i < 1$. Let Z be the algebra of all analytic functions taking values in A , which are defined on a neighbourhood of $\sigma_e(x)$.

For each $g \in Z$, define

$$\|g\|^{(1)} = \inf \left\{ \|c\|_A + \sum_{i=1}^N \varepsilon_i^{-1} \cdot \sup_{\lambda \in U_i} \|g_i(\lambda)\|_A : \right. \\ \left. (\lambda \cdot 1_A - x)f(\lambda) + g(\lambda) = c + \sum_{i=1}^N g_i(\lambda) \text{ (all } \lambda \in U_N) \right\}$$

(this is a seminorm on Z); and

$$\|g\|^{(2)} = \sup_{h \in Z, \|h\|^{(1)} \neq 0} (\|gh\|^{(1)} / \|h\|^{(1)}).$$

We shall see that $\|\cdot\|^{(2)}$ is also a seminorm on Z ; it is always finite, and has the algebra norm property that $\|g_1 g_2\|^{(2)} \leq \|g_1\|^{(2)} \cdot \|g_2\|^{(2)}$ for all $g_1, g_2 \in Z$.

First, we claim that A is isometrically embedded in $(Z, \|\cdot\|^{(2)})$. For, by the definition of $\|\cdot\|^{(1)}$, we see that for all $c \in A$, $h \in Z$,

$$\|c \cdot h\|^{(1)} \leq \|c\|_A \cdot \|h\|^{(1)};$$

thus $\|c\|^{(2)} \leq \|c\|_A$. However,

$$\begin{aligned} \|c\|^{(1)} &= \inf \left\{ \|d\|_A + \sum_{i=1}^N \varepsilon_i^{-1} \cdot \sup_{\lambda \in U_i} \|g_i(\lambda)\|_A : \right. \\ &\quad \left. c = (\lambda \cdot 1_A - x)f(\lambda) + d + \sum_{i=1}^N g_i(\lambda) \text{ (all } \lambda \in U_N) \right\} \\ &\geq \inf \{ \|d\|_A + \|c - d\|_A \} = \|c\|_A, \end{aligned}$$

since $(\varepsilon_i)_{i=1}^N$ is admissible for every N . So

$$\|c\|^{(2)} \geq \|c \cdot 1_A\|^{(1)} / \|1_A\|^{(1)} = \|c\|_A;$$

therefore A is indeed embedded isometrically. Next, we show that there is an inverse to $\mu \cdot 1_A - x$ in $(Z, \|\cdot\|^{(2)})$ for all $\mu \notin \sigma_e(x)$; and to do this we must show that the function $R_\mu: \lambda \rightarrow (\mu - \lambda)^{-1} \cdot 1_A$ has finite norm $\|R_\mu\|^{(2)}$. But this is true of any function $g \in Z$. For if g is defined on a neighbourhood of $\sigma_e(x)$, then for some n it will be defined and bounded on U_n . Then, if $h \in Z$, let us say $\delta > 0$ and

$$h(\lambda) = (\lambda - x)k(\lambda) + c + \sum_{i=1}^M \varepsilon_i^{-1} \sup_{\lambda \in U_i} \|g_i(\lambda)\|_A - \delta,$$

with

$$\|h\|^{(1)} \geq \|c\|_A + \sum_{i=1}^M \varepsilon_i^{-1} \sup_{\lambda \in U_i} \|g_i(\lambda)\| - \delta.$$

Then

$$gh(\lambda) = (\lambda - x)gk(\lambda) + \sum_{i=1}^M gf_i(\lambda) + cg(\lambda) \quad (\lambda \in U_m \cap U_n)$$

and

$$\begin{aligned} \|gh\|^{(1)} &\leq \left(\sup_{\lambda \in U_n} \|g(\lambda)\| \right) \cdot \left[\left(\|c\| + \sum_{i=1}^n \sup_{\lambda \in U_i} \|g_i(\lambda)\| \right) \cdot \varepsilon_n^{-1} \right. \\ &\quad \left. + \sum_{i=n+1}^M \varepsilon_i^{-1} \sup_{\lambda \in U_i} \|g_i(\lambda)\| \right] \\ &\leq \varepsilon_n^{-1} \sup_{\lambda \in U_n} \|g(\lambda)\| (\|h\|^{(1)} + \delta) \quad (\text{for each } \varepsilon_i \leq 1). \end{aligned}$$

Hence

$$\|g\|^{(2)} \leq \varepsilon_n^{-1} \cdot \sup_{\lambda \in U_n} \|g(\lambda)\|.$$

The third remark we make about $\|\cdot\|^{(2)}$ is that it has the algebra norm property that

$$\|g_1 g_2\|^{(2)} \leq \|g_1\|^{(2)} \cdot \|g_2\|^{(2)}$$

for all $g_1, g_2 \in Z$.

For

$$\begin{aligned}
 \|g_1, g_2\|^{(2)} &= \sup_{\substack{h \in Z \\ \|h\|^{(1)} \neq 0}} \left(\frac{\|g_1 g_2 h\|^{(1)}}{\|h\|^{(1)}} \right) \\
 (*) \qquad &= \sup_{\substack{h \in Z \\ \|h\|^{(1)} \neq 0 \\ \|g_2 h\|^{(1)} \neq 0}} \left(\frac{\|g_1 g_2 h\|^{(1)}}{\|g_2 h\|^{(1)}} \cdot \frac{\|g_2 h\|^{(1)}}{\|h\|^{(1)}} \right).
 \end{aligned}$$

It is legitimate to restrict our attention to functions h such that $\|g_2 h\|^{(1)} \neq 0$ since if $\|g_2 h\|^{(1)} = 0$, then the expression $(*)$ is certainly zero for this function h ; thus $\|g_1 g_2\|^{(2)} \leq \|g_1\|^{(2)} \cdot \|g_2\|^{(2)}$.

Now, let I be the ideal in Z consisting of all functions g whose seminorm $\|g\|^{(2)}$ is zero. Let A' denote the completion of the quotient space Z/I . Then A' is a Banach algebra in which A is embedded isometrically, just as it is embedded isometrically in $(Z, \|\cdot\|^{(2)})$. But for all $\mu \notin \sigma_e(x)$, there is an inverse for the element $\mu \cdot 1_A - x$ in Z , hence also in A' . Thus A' is an extension of A in which the spectrum of x is precisely the essential spectrum of x in A .

This concludes the proof of Theorem 1.

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