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TRIANGULATIONS OF SUBANALYTIC SETS AND LOCALLY SUBANALYTIC MANIFOLDS

BY

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Dedicated to Professor René Thom

ABSTRACT. If two polyhedrons are locally subanalytically homeomorphic (that is, the graph is locally subanalytic), they are PL homeomorphic. A locally subanalytic manifold is one whose coordinate transformations are locally subanalytic. It is proved that a locally subanalytic manifold has a unique PL manifold structure. A semialgebraic manifold also is considered.

1. Introduction. One of the main results is Theorem 4.1, which states that a locally subanalytic homeomorphism of polyhedrons is locally subanalytically isotopic to a PL homeomorphism. Consequently, a subanalytic triangulation of a subanalytic set (whose existence is shown in [**3**, **5**]) is unique up to a PL homeomorphism. These answer questions posed in [**3** and **13**] in weaker forms. The former result also gives a negative answer to a conjecture in [**13**] of whether there exists a semialgebraic homeomorphism from Π^5_7 to $\mathbf{R}P^5$, where Π^5_7 is Siebenmann's example of a PL manifold homeomorphic, but not PL homeomorphic, to $\mathbf{R}P^5$ [**16**].

We sketch its proof. Let $h\colon X \to Y$ be a locally subanalytic homeomorphism of polyhedrons. We find closed neighborhoods X_1, Y_1 of the singular sets of h, h^{-1} in X, Y, respectively, using Whitney stratifications of the singular sets, such that (1) X_1, Y_1 are C^∞ triangulable, and (2) $\overline{X - X_1}$ and $\overline{Y - Y_1}$ are C^∞ diffeomorphic and, hence, PL homeomorphic. For (1) we need a generalization of the C^∞ triangulation theorem of Cairns-Whitehead. The key lemma to (2) is a version (2.14) of Proposition 5.1 of [**14**] (about topological equivalence of subanalytic functions). By the Alexander trick we extend the PL homeomorphism $\overline{X - X_1} \to \overline{Y - Y_1}$ to the global. We once more use the Alexander trick to find a locally subanalytic isotopy of h to a PL homeomorphism.

We show locally subanalytic triangulations of locally subanalytic sets and Proposition 2.14 in §2; and we explain our generalization of the Cairns-Whitehead theorem and the Alexander trick in §3. §4 deals with Theorem 4.1 and its proof.

As the set of all locally subanalytic homeomorphisms of open subsets of $\mathbf{R}^n$ is a pseudo-group, we can define locally subanalytic manifolds as those whose coordinate transformations are locally subanalytic (§5). We show that every locally subanalytic manifold has a unique PL manifold structure (5.3).

In §6 we define semialgebraic manifolds. The relation between semialgebraic manifolds and compact PL manifolds possibly with boundary is similar to that between Nash manifolds and compact C^∞ manifolds possibly with boundary, as

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follows (see [**14**]). By putting boundary on a semialgebraic manifold, we can compactify it; in other words a noncompact semialgebraic manifold is semialgebraically homeomorphic to the interior of some compact semialgebraic manifold with boundary. As a compact semialgebraic manifold possibly with boundary is semialgebraically homeomorphic to a compact PL manifold possibly with boundary, we have a correspondence from the semialgebraic homeomorphism classes of all semialgebraic manifolds to the PL homeomorphism classes of all compact PL manifolds possibly with boundary. We prove that the correspondence is one-to-one and onto. Accordingly there exist two semialgebraic (=finite) PL manifolds PL homeomorphic but semialgebraically distinct.

As a semialgebraic triangulation of semialgebraic sets is always possible [**5**, **6**], it is meaningful to study properties of semialgebraic polyhedrons. We see in §7 that a semialgebraic polyhedron essentially consists of finite cells. A curious property of semialgebraic homeomorphisms is that compact polyhedrons X_1 and X_2 are PL homeomorphic if $X_1 \times \mathbf{R}$ and $X_2 \times \mathbf{R}$ are semialgebraically homeomorphic (7.5). This is derived from finiteness of semialgebraic maps.

In this paper, manifolds do not have boundary and are separable, unless otherwise specified. A topological manifold with boundary with a system of coordinate neighborhoods $\{(U_\alpha, \psi_\alpha)\}$, such that ψ_α are homeomorphic onto open sets of $[0,\infty)^n$ and whose coordinate transformations are of class C^∞, is called a C^∞ *manifold with cornered boundary*.

2. Locally subanalytic sets.

2.1. DEFINITION. Let X be a subset of $\mathbf{R}^n$ and U an open neighborhood of X. We call X *subanalytic* in U if each point $x \in U$ has an open neighborhood U' such that $X \cap U'$ is a finite union of sets of the form $\operatorname{Im} f_1 - \operatorname{Im} f_2$, where f_1 and f_2 are proper analytic maps from analytic manifolds to $\mathbf{R}^n$. If X is subanalytic in some U or in $\mathbf{R}^n$, then X is called *locally subanalytic* or *subanalytic*, respectively.

We remark that every polyhedron contained in a Euclidean space is locally subanalytic, but a polyhedron might not be subanalytic if it is not closed. (This is the reason why we define the concept "locally subanalytic".)

A continuous map of subanalytic sets is called *subanalytic* if the graph is subanalytic, and a *locally subanalytic map* is relatively defined. For example, a PL map of polyhedrons in Euclidean spaces is locally subanalytic. Moreover we define a *locally subanalytic map* of polyhedrons by PL imbedding the polyhedrons in Euclidean spaces. This definition does not depend on the choice of PL imbeddings because the composition of locally subanalytic maps is locally subanalytic. A homotopy f_t: $X \to Y$, $t \in [0,1]$, is called (*locally*) *subanalytic* if $X \times [0,1] \ni (x,t) \to f_t(x) \in Y$ is (locally) subanalytic.

2.2 *Triangulation of locally subanalytic sets.* The following is a locally subanalytic version of Propositions 3.1, 3.1′ and Remark 3.10 in [**15**]. As the proof is the same, we omit it.

Let K be a simplicial complex in $\mathbf{R}^n$ and U an open neighborhood of the underlying polyhedron $|K|$, where $|K|$ is closed. Let $\{X_i\}$ be a family of subsets of $|K|$ locally finite in U and subanalytic in U. Then there exist a subdivision K' of K and a locally subanalytic isotopy τ_t: $|K| \to |K|$, $t \in [0,1]$, of the identity map such that:

(2.2.1) $\tau_t(\sigma) = \sigma$ for any $\sigma \in K$ and $t \in [0,1]$;

(2.2.2) for any $\sigma \in K'$ and $t \in [0,1]$, $\tau_t(\operatorname{Int}\sigma)$ is an analytic submanifold of $\mathbf{R}^n$ and $\tau_t|_{\operatorname{Int}\sigma}\colon \operatorname{Int}\sigma \to \tau_t(\operatorname{Int}\sigma)$ is an analytic diffeomorphism, where $\operatorname{Int}\sigma$, the interior of σ, is called an *open simplex* of K'; and

(2.2.3) each X_i is the image under τ_1 of a union of open simplexes of K'.

Moreover, let L be a subcomplex of K. Assume each $X_i \cap (|K| - |L|)$ is a union of open simplexes of $K - L$. Then

(2.2.4) for any $t \in [0,1]$ and $\sigma \in K$ with $\sigma \cap |L| = \emptyset$, we have $\sigma \in K'$ and $\tau_t = \text{ident}$ on σ. (See Remark 3.8 in [**15**].)

2.3. DEFINITION. A *stratification* of a set $X \subset \mathbf{R}^n$ is a partition of X into C^∞ submanifolds X_i of $\mathbf{R}^n$ such that $\{X_i\}$ is locally finite at X and $\overline{X}_i \cap X_j \neq \emptyset$ implies $\overline{X}_i \supset X_j$.

A *Whitney stratification* is a stratification satisfying the Whitney condition (b) (see [**8**]). An *analytic stratification* means that all strata are of class C^ω. If all strata are subanalytic in an open set $U \subset \mathbf{R}^n$, we call the stratification *subanalytic in* U. From now on we omit U in the case $U = \mathbf{R}^n$.

Let $X \subset \mathbf{R}^n$ be a C^∞ submanifold. A *tube* at X is a triple $T = (|T|, \pi, \rho)$, where $|T|$ is a C^∞ tubular neighborhood of X for some Riemannian metric of $\mathbf{R}^n$, $\pi\colon |T| \to X$ is the projection, and ρ is a nonnegative C^∞ function on $|T|$ such that $\rho^{-1}(0) = X$ and each point $x \in X$ is a nondegenerate critical point of $\rho|_{\pi^{-1}(x)}$ (ρ is called a *distance function*).

It is easily seen (cf. the proof of Lemma 4.11 in [**15**]) that the above definition agrees with [**2**].

Let $\{X_i\}$ be a stratification in $\mathbf{R}^n$. A *controlled tube system* for $\{X_i\}$ consists of one tube $T_i = (|T_i|, \pi_i, \rho_i)$ at each X_i such that

(2.3.1) $\pi_i \circ \pi_j(x) = \pi_i(x)$ and $\rho_i \circ \pi_j(x) = \rho_i(x)$ for $x \in |T_i| \cap |T_j| \cap \pi_j^{-1}(|T_i|)$.

Let $\{T_i\}$ be a controlled tube system for a stratification $\{X_i\}$ of $X \subset \mathbf{R}^n$. A *vector field* ξ on $\{X_i\}$ consists of one C^∞ vector field ξ_i on each X_i. We call it *semicontrolled* if, for each i, j,

(2.3.2) $$d(\rho_i|_{X_j})\xi_{jx} = 0 \quad \text{for } x \in X_j \cap U_i,$$

where $U_i \subset |T_i|$ is some neighborhood of X_i in $\mathbf{R}^n$. If, furthermore,

(2.3.3) $$d(\pi_i|_{X_j})\xi_{jx} = \xi_{i\pi_i(x)} \quad \text{for } x \in X_j \cap U_i,$$

ξ is called *controlled*. If ξ is continuous as a map from X to $\mathbf{R}^n$, then we call it *continuous*.

2.4. *Existence of controlled tube system* [**2**, *Corollary* 2.7, *Chapter* II]. Every Whitney stratification admits a controlled tube system. Moreover, we have the following (see the proof of the corollary quoted above). Let $\{X_i\}$ be a Whitney stratification in $\mathbf{R}^n$ such that $X_1 \in \{X_i\}$ and $\dim X_1 \leq \dim X_i$ for any i, and let $\{T_i = (|T_i|, \pi_i, \rho_i)\}$ be a tube system for $\{X_i\}$ not necessarily controlled. Then there exists a controlled tube system $\{T_i' = (|T_i'|, \pi_i', \rho_i')\}$ for $\{X_i\}$ such that for each i,

$$|T_i'| \subset |T_i|, \qquad \rho_i = \rho_i' \text{ on } |T_i'| \qquad \text{and} \qquad \pi_1 = \pi_1' \text{ on } |T_1'|.$$

Hence we can choose a controlled tube system so that ρ_i is the square of the distance function from X_i in the usual metric of $\mathbf{R}^n$. We call such a tube system *canonical.*

2.5. *Lift of vector fields* [**15**, Lemma 4.11]. Let $\{T_i\}$ be a controlled tube system for a Whitney stratification $\{X_i\}$ in $\mathbf{R}^n$. Assume $X_1 \in \{X_i\}$. Let ξ_1' be a C^∞

vector field on X_1. Then there exists a continuous controlled vector field $\xi = \{\xi_i\}$ on $\{(X_i \cap U,\ T_i|_{X_i \cap U})\}$, U an open neighborhood of X_1, such that $\xi_1 = \xi_1'$.

2.6. *Flows of vector fields.* Let $\{X_i\}$ be a stratification of a locally closed set $X \subset \mathbf{R}^n$, $\{T_i\}$ a controlled tube system for $\{X_i\}$, and $\xi = \{\xi_i\}$ a vector field on $\{X_i\}$. For each i, let θ_i: $D_i \to X_i$, $D_i \subset X_i \times \mathbf{R}$, be the maximal C^∞ flow defined by ξ_i. Put $D = \bigcup D_i$ and define θ: $D \to X$ by $\theta|_{D_i} = \theta_i$ for each i. We call θ the *flow* of ξ.

If ξ is semicontrolled, the flow of ξ clearly has the following property. Let C be a compact subset of one X_i. Then there is a positive constant ε such that for any $x \in \pi_i^{-1}(C) \cap X$ with $\rho_i(x) < \varepsilon$, ρ_i is constant on $\theta(D \cap x \times \mathbf{R}) \cap \pi_i^{-1}(C)$.

Corollary 4.7 in [**2**, Chapter II] shows, moreover, that if ξ is controlled then D is open in $X \times \mathbf{R}$ and θ is continuous.

2.7. *Properties of subanalytic sets* [**4**, **15**]. Let $X, Y \subset \mathbf{R}^n$ be subanalytic sets.

(2.7.1) $\overline{X}, X \cap Y$ and $X - Y$ are subanalytic.

(2.7.2) The image of X under a proper subanalytic map from X to a Euclidean space is subanalytic.

(2.7.3) There exists a subanalytic subset X' of X with $\dim X' < \dim X$ such that $X - X'$ is an analytic manifold.

(2.7.4) Assume, moreover, X is a connected analytic manifold. Let f: $\mathbf{R}^n \to \mathbf{R}^m$ be an analytic map. Then there exists a subanalytic closed subset X' of X such that $\dim X' < \dim X$ and the differential $d(f|_X)$ has constant rank on $X - X'$.

(2.7.5) Assume X and Y are analytic manifolds, $X \cap Y = \emptyset$ and $\overline{X} \supset Y$. Then there exists a closed subanalytic subset Y' of Y such that $\dim Y' < \dim Y$ and $(X, Y - Y')$ satisfies the Whitney condition (b).

2.8. LEMMA (SEE [**15**, (5.3.6), (5.3.7)]). *Let $\{X_1, X_2\}$ be a subanalytic analytic stratification of a set $X \subset \mathbf{R}^n$, and f_1, f_2 subanalytic functions on X. Assume $f_1^{-1}(0) = f_2^{-1}(0) = X_1$, $\overline{X}_2 \supset X_1$ and the restrictions to X_2 of f_1 and f_2 are positive analytic functions. Then the set*

$$S = \{x \in X_2 | a_1 \text{grad } f_{1x} + a_2 \text{grad } f_{2x} = 0 \textit{ for some } a_1, a_2 \geq 0 \textit{ with } a_1 + a_2 > 0\}$$

is empty in a neighborhood of X_1.

PROOF. Considering the graph of $f_1 \times f_2$, we can assume that f_1, f_2 are the restrictions to X of analytic functions F_1, F_2 on $\mathbf{R}^n$ respectively. If we prove that S is subanalytic, the lemma follows from (5.3.7) in [**15**]. Lemma 1.6 of [**17**] says that the subset of $\mathbf{R}^n \times G_{n,m}$ consisting of (x, TX_{2x}), $x \in X_2$, is subanalytic, where $m = \dim X_2$, TX_2 is the tangent space, and $G_{n,m}$ is the Grassmann manifold with naturally given affine algebraic structure. It is easy to see that the sets $\{(H, H') \in G_{n,m} \times G_{n,n-m} \mid H \perp H'\}$ and $\{(H, x) \in G_{n,m} \times \mathbf{R}^n \mid x \in H\}$ are algebraic in $G_{n,m} \times G_{n,n-m}$ and $G_{n,m} \times \mathbf{R}^n$ respectively. Hence

$$\{(x, TX_{2x}, (TX_{2x})^\perp, u, v) \mid x \in X_2,\ u \in TX_{2x},\ v \perp TX_{2x}\}$$

is a subanalytic subset of $\mathbf{R}^n \times G_{n,m} \times G_{n,n-m} \times \mathbf{R}^n \times \mathbf{R}^n$. Take its image X_2' under the projection $\mathbf{R}^n \times G_{n,m} \times G_{n,n-m} \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$. Then

$$X_2' = \{(x, u, v) \in X_2 \times \mathbf{R}^n \times \mathbf{R}^n | u \in TX_{2x},\ v \perp TX_{2x}\}.$$

(2.7.2) says that X_2' is subanalytic, and so is

$$X_2'' = \{(x, u, v, w) \in X_2' \times \mathbf{R}^n \mid w = u + v\}.$$

Clearly the subset of $\mathbf{R}^n \times \mathbf{R}^n$ consisting of points $(x, \operatorname{grad} F_{1x})$ is analytic. Hence

$$\tilde{X}_2 = \{(x, u, v, w) \in X_2'' \mid w = \operatorname{grad} F_{1x}\}$$

is subanalytic. Now $\{(x, \operatorname{grad} f_{1x}) \mid x \in X_2\}$ is the projection image of $\tilde{X}_2$ onto the first two factors. Therefore, by (2.7.2), $\{(x, \operatorname{grad} f_{1x})\}$ is subanalytic and, moreover, so is $\{(x, \operatorname{grad} f_{1x}, \operatorname{grad} f_{2x}) \mid x \in X_2\}$. After considering the subanalytic set

$$\{(x, \operatorname{grad} f_{1x}, \operatorname{grad} f_{2x}, a_1, a_2) \mid x \in X_2,\ a_1 a_2 \geq 0, \\ a_1 + a_2 = 1,\ a_1 \operatorname{grad} f_{1x} + a_2 \operatorname{grad} f_{2x} = 0\},$$

we see in the same way as above that S is subanalytic.

2.9. COROLLARY. *Under the same notations and assumptions as* 2.8, *there exist a C^∞ vector field v on X_2 and a neighborhood U of X_1 in X such that vf_1 and vf_2 are positive on $U - X_1$.*

PROOF. Define v on $U - X_1$ for small U by $|\operatorname{grad} f_2| \operatorname{grad} f_1 + |\operatorname{grad} f_1| \operatorname{grad} f_2$, and extend v to X_2.

2.10. LEMMA. *Let $X \subset \mathbf{R}^n$ be a subanalytic set, $\{X_i\}$ its subanalytic analytic Whitney stratification, and $\{T_i\}$ a controlled tube system for $\{X_i\}$. Assume $X_1 \in \{X_i\}$ and $X_1 \subset \overline{X}_i$ for any i. Let f_1, f_2 be the restrictions to X of analytic functions on $\mathbf{R}^n$ such that $f_1^{-1}(0) = f_2^{-1}(0) = X_1$ and $f_1, f_2 > 0$ on $X - X_1$. Then there exist a neighborhood U of X_1 in $\mathbf{R}^n$ and a controlled vector field $\xi = \{\xi_i\}$ on $\{X_i\}_{i \neq 1}$ such that for each i, $\xi_i f_1$ and $\xi_i f_2$ are positive on $U \cap X_i$.*

PROOF. Apply 2.9 to f_1, f_2 and $\{X_1, X_i\}$ for each $i \neq 1$. Then there exists a vector field $\xi' = \{\xi_i'\}$ on $\{X_i\}_{i \neq 1}$ such that $\xi_i' f_1$ and $\xi_i' f_2$ are positive on $U \cap X_i$ for some neighborhood U. Hence the lemma follows from [**15**, Lemma 4.14]. But we repeat the proof because we later use the same idea of proof.

We assume for simplicity $U = \mathbf{R}^n$, the index set$= \{1, 2, \ldots\}$ and $\dim X_i < \dim X_{i+1}$ for any i. We modify ξ' so as to be controlled by induction as follows. Let $k \geq 2$. Assume $\xi'|_{V_{k-1}}$ is controlled for some neighborhood V_{k-1} of $\bigcup_{i=2}^{k-1} X_i$. We will modify $\{\xi_i'\}_{i \geq k}$ so that $\xi'|_{V_k}$ is controlled for some V_k. By 2.5 we can lift ξ_k', namely there exists a continuous controlled vector field $\{\xi_{jk}\}$ on $\{(X_j \cap W_k, T_j|_{X_j \cap W_k})\}_{j \geq k}$ such that $\xi_{kk} = \xi_k'$ for some neighborhood W_k of X_k. By the continuity of $\{\xi_{jk}\}$, we can suppose that $\xi_{jk} f_1$ and $\xi_{jk} f_2$ are positive on $X_j \cap W_k$. Let V_{k-1}' and W_k' be neighborhoods of $\bigcup_{i=2}^{k-1} X_i$ and X_k, respectively, such that

$$\overline{V}_{k-1}' - (\overline{X} - X) - X_1 \subset V_{k-1}, \qquad \overline{W}_k' - (\overline{X}_k - X_k) \subset W_k.$$

Let φ be a C^∞ function on $\mathbf{R}^n - (\overline{X} - X) - X_1$ such that

$$0 \leq \varphi \leq 1, \quad \varphi^{-1}(1) \supset \overline{(W_k' - V_{k-1})}, \quad \operatorname{supp} \varphi \subset W_k - \overline{V}_{k-1}',$$

and for each $x \in X_k$, φ is constant on $\pi_k^{-1}(x) \cap W_k'$. This is possible when we choose sufficiently small V_{k-1}'. Put

$$\xi_j'' = \begin{cases} \varphi \xi_{jk} + (1 - \varphi) \xi_j' & \text{for } j > k, \\ \xi_j' & \text{for } j \leq k. \end{cases}$$

Then $\xi_j'' f_1$ and $\xi_j'' f_2$ are positive on each X_j, $j \neq 1$, and it is easy to check also that $\{\xi_j''\}|_{V_k}$, $V_k = V_{k-1}' \cup W_k'$, is controlled. Hence the lemma follows.

We require the vector field in 2.10 to satisfy the following additional property in exchange for (2.2.3).

2.11. LEMMA. *Let* $X, \{X_i\}, \{T_i\}, f_1$ *and* f_2 *be the same as* 2.10. *Assume* X *is locally closed in* $\mathbf{R}^n$ *and*

$$X_1 = \mathbf{R}^m \times 0 = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_{m+1} = \cdots = x_n = 0\}.$$

Then there exist an open neighborhood U *of* X_1 *and a semicontrolled vector field* $\xi = \{\xi_i\}$ *on* $\{X_i\}_{i\neq 1}$ *such that for each* i, $\xi_i f_1$ *and* $\xi_i f_2$ *are positive on* $U \cap X_i$ *and for each* i *and* $1 \leq l \leq m$, $\xi_i|_{\{x_l=0\}\cap X_i \cap U}$ *is tangent to* $\{x_l = 0\} \cap X_i \cap U$.

To prove this we need the following, which is inspired by the proof of Thom's First Isotopy Lemma (see [**2**, Theorem 5.2, Chapter II]).

2.12. LEMMA. *Let* $\{X_i\}$ *be a Whitney stratification of a locally closed set* $X \subset \mathbf{R}^n$ *such that* $X_1 = \mathbf{R}^m \times 0 \in \{X_i\}$, $X_1 \subset \overline{X}_i$ *for any* i. *Let* $\{T_i\}$ *be a controlled tube system for* $\{X_i\}$. *Given* $1 \leq m' \leq m$. *Then we have open neighborhoods* V, W *and* O *of* 0 *in* $\mathbf{R}^{m'} \times 0$ $(\subset \mathbf{R}^{m'} \times \mathbf{R}^{n-m'})$, $0 \times \mathbf{R}^{n-m'}$ $(\subset \mathbf{R}^{m'} \times \mathbf{R}^{n-m'})$ *and* $\mathbf{R}^n$, *respectively, and a homeomorphism* α: $V \times W \to O$ *such that: for each* i, $\alpha|_{V\times X_{i0}}$ *is a diffeomorphism onto* $X_i \cap O$, *where* $X_{i0} = W \cap X_i$; *for each* i,

$$\rho_i \circ \alpha(x, y) = \rho_i(y), \qquad x \in V,\ y \in W \text{ near } X_{i0}; \tag{2.12.1}$$

and for each $1 \leq l \leq m$,

$$O \cap \{x_l = 0\} = \begin{cases} \alpha((V \cap \{x_l = 0\}) \times W) & \text{if } l \leq m', \\ \alpha(V \times (W \cap \{x_l = 0\})) & \text{if } l > m'. \end{cases} \tag{2.12.2}$$

PROOF. We consider the case $m' = 1$. By 2.4, we have a tube system $\{T'_i = (|T'_i|, \pi'_i, \rho'_i)\}$ for $\{X_i\}$ such that $|T'_i| \subset |T_i|, \rho_i = \rho'_i$ on $|T'_i|$ and π'_1 is the orthogonal projection. Hence, we may suppose that π_1 is the orthogonal projection. Since $\{X_i, \mathbf{R}^n - \overline{X}\}$ is a Whitney stratification and X is locally closed, we also assume X is a neighborhood of X_1. Apply 2.5 to $\partial/\partial x_1|_{X_1}$. Then we have a controlled vector fied $\xi = \{\xi_i\}$ on $\{(X_i \cap U, T_i|_{X_i\cap U})\}$ such that $\xi_1 = \partial/\partial x_1$ for some neighborhood U of X_1. It follows that

$$d(\rho_i|_{X_j})\xi_j = 0 \quad \text{on } X_j \cap U_i \tag{2.12.3}$$

for each i, j and some neighborhood U_i of $X_i \cap U$, and

$$d(\pi_1|_{X_i})\xi_i = \partial/\partial x_1 \quad \text{on } X_i \cap U' \tag{2.12.4}$$

for each i and some neighborhood U' of X_1. In particular,

$$\text{each } \xi_i \text{ is transversal to } \{x_1 = 0\} \cap U'. \tag{2.12.5}$$

We remark that $\{X_i \cap \{x_1 = 0\} \cap U''\}$ is a Whitney stratification for some neighborhood U'' of X_1. Let θ be the flow of ξ, and let V_1 and W_1 be open connected small neighborhoods of 0 in $\mathbf{R} \times 0$ $(\subset \mathbf{R} \times \mathbf{R}^{n-1})$ and $0 \times \mathbf{R}^{n-1}$ $(\subset \mathbf{R} \times \mathbf{R}^{n-1})$, respectively. Define α_1: $V_1 \times W_1 \to \mathbf{R}^n$ by

$$\alpha_1(x_1, 0, \ldots, 0, x_2, \ldots, x_n) = \theta(0, x_2, \ldots, x_n, x_1).$$

Put $\alpha_1(V_1 \times W_1) = O_1$. Then α_1 is a homeomorphism onto O_1 by 2.6, and $\alpha_1|_{V_1 \times X_{i1}}$ is a diffeomorphism onto $X_i \cap O_1$ for each i by (2.12.5), where $X_{i1} = W_1 \cap X_i$. It also follows from (2.12.3) that, for each i,

$$\rho_i \circ \alpha_1(x, y) = \rho_i(y), \qquad x \in V_1,\ y \in W_1 \text{ near } X_{i1},$$

and, from (2.12.4), that for each $2 \le l \le m$,

$$\alpha_1(V_1 \times (W_1 \cap \{x_l = 0\})) = O_1 \cap \{x_l = 0\}.$$

It is trivial that $\alpha_1(0 \times W_1) = O_1 \cap \{x_1 = 0\}$. Hence the lemma for $m' = 1$ is proved.

If $m' = 2$ argue in the same way as above about $\{X_i \cap \{x_1 = 0\} \cap U''\}$. Then we have neighborhoods V_2', W_2', O_2' of 0 in $0 \times \mathbf{R} \times 0$, $0 \times 0 \times \mathbf{R}^{n-2}$, $0 \times \mathbf{R}^{n-1}$, respectively, and a homeomorphism α_2': $V_2' \times W_2' \to O_2'$ such that the properties corresponding to (2.12.1) and (2.12.2) are satisfied. We choose V_2' and W_2' so small that $O_2' \subset W_1$. Put

$$V_2 = \{(x_1, x_2, 0, \dots, 0) \mid (x_1, 0, \dots, 0) \in V_1, (0, x_2, 0, \dots, 0) \in V_2'\},$$
$$W_2 = W_2', \qquad O_2 = \alpha_1(V_1 \times O_2'),$$

and define α_2: $V_2 \times W_2 \to O_2$ by

$$\alpha_2(x_1, x_2, 0, \dots, 0, x_3, \dots, x_n) = \alpha_1(x_1, 0, \dots, 0, \alpha_2'(0, x_2, 0, \dots, 0, x_3, \dots, x_n)).$$

Then the lemma for $m' = 2$ follows. For general m', repeating the above argument, we obtain the lemma.

2.13. PROOF OF 2.11. For each subset $I = \{i_1, \dots, i_{m'}\}$ of $\{1, \dots, m\}$, put

$$R^I = \{(x_1, \dots, x_n) \in \mathbf{R}^n | x_{i_1} = \dots = x_{i_{m'}} = 0\},$$
$$X^I = X \cap R^I, \qquad \{X_i^I\} = \{X_i \cap R^I\},$$
$$f_j^I = f_j|_{X^I} \quad \text{and} \quad \rho_i^{I\prime} = \rho_i|_{|T_i| \cap R^I}.$$

Then $\{X_i^I\}$ is a subanalytic analytic Whitney stratification of X^I in a neighborhood U^I of X_1^I in R^I, and each $\rho_i^{I\prime}$ satisfies the conditions of the distance function (2.3). Hence, by 2.4, we have a controlled tube system $\{T_i^I = (|T_i^I|, \pi_i^I, \rho_i^I)\}$ for $\{X_i^I \cap U^I\}$ such that

$$|T_i^I| \subset |T_i| \cap R^I \quad \text{and} \quad \rho_i^I = \rho_i \text{ on } |T_i^I|. \tag{2.13.1}$$

Apply 2.10 to $\{X_i^I \cap U^I\}, \{T_i^I\}, f_1^I$ and f_2^I, and let $\xi^I = \{\xi_i^I\}_{i \ne 1}$ be a resultant controlled vector field. If $I = \{1, \dots, m'\}$, consider the vector field $(0, \xi^I)$ on $\mathbf{R}^{m'} \times 0 \times U^I$ $(\subset \mathbf{R}^{m'} \times \mathbf{R}^{n-m'} \times \mathbf{R}^{m'} \times \mathbf{R}^{n-m'})$ and an extension of $\xi^I d\alpha(0, \xi^I)$ on $X \cap O^I - X_1$, where α is given in 2.12 and O^I is a neighborhood of 0 in $\mathbf{R}^n$. In another case of I, by changing coordinates we reduce to the case above. We keep the notation ξ^I for the extension. Then we have the following. $\{\xi_i^I\}$ is semicontrolled by (2.12.1) and (2.13.1). For each i and $l \in I$, $\xi_i^I|_{\{x_l=0\} \cap X_i \cap O^I}$ is tangent to $\{x_l = 0\} \cap X_i \cap O^I$ by (2.12.2). For each i, $\xi_i^I f_1$ and $\xi_i^I f_2$ are positive on $V_i^I \cap X_i$ for some small neighborhood V_i^I of $X_i \cap R^I \cap O^I/k$ in $\mathbf{R}^n$ by 2.10, where k is a large integer, since f_1 and f_2 are restrictions of analytic functions and the extension of ξ_i^I is of class C^∞. Hence, we have to restrict ξ_i^I to $V_i^I \cap X_i$. Moreover, to avoid the difficulty that $\xi_i^I|_{\{x_l=0\} \cap X_i \cap O^I}$ may be not tangent to $\{x_l = 0\} \cap X_i \cap O^I$ for some $l \in I^c = \{1, \ldots, m\} - I$, we restrict ξ_i^I to $V_i^I \cap X_i - \{\prod_{i \in I^c} x_i = 0\}$. Now we remark that for each $i \neq 1$, at least one of ξ_i^I is defined at each point of X_i near 0 even after the above restriction. Hence, using a C^∞ partition of unity in the same way as in the proof of 2.10, we can contruct a vector field ξ_i on each X_i, $i \neq 1$, in a neighborhood of 0 so that $\{\xi_i\}$ satisfies the requirement of the lemma. It is easy to define ξ globally once more by a C^∞ partition of unity.

2.14. PROPOSITION. *Let $\{X_i\}_{i=1,\ldots,m}$ be a subanalytic analytic Whitney stratification of a locally closed set $X \subset \mathbf{R}^n$ such that $\dim X_i < \dim X_{i+1}$ for any i. Let $\{T_i\}$ be a controlled tube system for $\{X_i\}$. Let f_1, f_2 be nonnegative analytic functions on a neighborhood U of X_1 whose zero sets in X are X_1. Let $\varphi_1, \ldots, \varphi_k$ be continuous functions on a neighborhood of X, and $Y_1, \ldots, Y_k$ the respective zero sets. For each i, set*

$$X^+ = X \cap \bigcap_j \{\varphi_j \geq 0\}, \qquad X_i^+ = X_i \cap X^+.$$

Assume that: each φ_j is analytic and C^∞ regular near Y_j; each i and each subset $\{j_1, \ldots, j_{k'}\}$ of $\{1, \ldots, k\}$, Y_{j_1} is transversal to $X_i \cap Y_{j_2} \cap \cdots \cap Y_{j_k}$; and

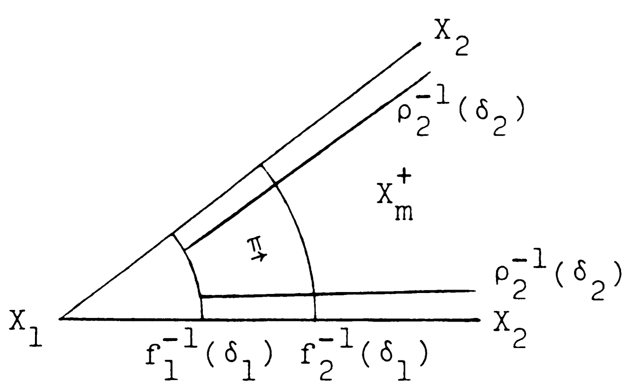
(2.14.1) *$X^+ \cap \{f_l \leq \varepsilon\}$ is compact for some $\varepsilon > 0$, $l = 1$ or 2.*

Then we have a positive number and positive functions $\varepsilon_1, \varepsilon_2(t_1)$ on $\mathbf{R}, \ldots, \varepsilon_{m-1}(t_1, \ldots, t_{m-2})$ on $\mathbf{R}^{m-2}$ such that for any sequence of positive numbers $\delta_1 \leq \varepsilon_1$, $\delta_2 \leq \varepsilon_2(\delta_1), \ldots, \delta_{m-1} \leq \varepsilon_{m-1}(\delta_1, \ldots, \delta_{m-2})$ there exists a C^∞ diffeomorphism π of X_m such that:

(2.14.2) *$\{\pi(x) \neq x\}$ is contained in $U \cap U'$, where U' is a given small neighborhood of X^+;*

$$(2.14.3) \quad \begin{aligned} &\pi(X_m^+ - \{f_1 \leq \delta_1\} - \{\rho_2 \leq \delta_2\} - \cdots - \{\rho_{m-1} \leq \delta_{m-1}\}) \\ &\quad = X_m^+ - \{f_2 \leq \delta_1\} - \{\rho_2 \leq \delta_2\} - \cdots - \{\rho_{m-1} \leq \delta_{m-1}\}; \end{aligned}$$

$$(2.14.4) \qquad \pi(X_m \cap Y_j) = X_m \cap Y_j \quad \textit{for any } j.$$



PROOF. By 2.11 there exist open neighborhoods U_i of X_i, $i = 1, \ldots, m-1$, and a C^∞ vector field ξ_m on X_m such that $U_1 \subset U$,

$$(2.14.5) \qquad \xi_m f_1, \xi_m f_2 > 0 \quad \text{on } X_m \cap U_1,$$

(2.14.6) $$d(\rho_i|_{X_m})\xi_m = 0 \quad \text{on } X_m \cap U_i \text{ for each } i > 1,$$

(2.14.7) $$d(\varphi_j|_{X_m})\xi_m = 0 \quad \text{on } X_m \cap Y_j \text{ for each } j.$$

Let $\theta_m\colon D_m \to X_m$ be the flow of ξ_m. Shrinking U_1, we can assume by (2.14.1), (2.14.5) and (2.14.7)

$$X_m^+ \cap U_1 = X_m^+ \cap \{f_1 < \varepsilon'\} \quad \text{for some } 0 < \varepsilon' < \varepsilon,$$

$$\theta_m(D_m \cap ((X_m^+ \cap U_1) \times \mathbf{R}^-)) = X_m^+ \cap U_1,$$

where $\mathbf{R}^- = (-\infty, 0]$, and for each $x_0 \in X_m^+ \cap \overline{U}_1$, $f_1 \circ \theta_m(x_0, t)$ and $f_2 \circ \theta_m(x_0, t)$ are C^∞ regular and increasing on $x_0 \times (t^-, 0] = D_m \cap (x_0 \times \mathbf{R}^-)$ and tend to 0 as $t \to t^-$. Hence, we may regard $X_m^+ \cap U_1$ as $(0, \varepsilon') \times (f_1^{-1}(\varepsilon') \cap X_m^+)$ and $f_1|_{X_m^+ \cap U_1}$ as the projection onto the first factor.

Choose a small $\varepsilon_1 > 0$ and so that $\varepsilon_1 < \varepsilon'$ and $X_m^+ \cap \{f_2 \le \varepsilon_1\} \subset U_1$. Let $0 \le \delta_1 \le \varepsilon_1$. Then for any point $x \in X_m^+$ with $f_1(x) = \delta_1$ there exists uniquely $y \in X_m^+$ such that $y \in \theta_m((x \times \mathbf{R}) \cap D_m)$ and $f_2(y) = \delta_1$. Put $\pi(x) = y$. Then π is a diffeomorphism from $f_1^{-1}(\delta_1) \cap X_m^+$ to $f_2^{-1}(\delta_1) \cap X_m^+$ such that

$$\pi(f_1^{-1}(\delta_1) \cap X_m^+ \cap Y_j) = f_2^{-1}(\delta_1) \cap X_m^+ \cap Y_j \quad \text{for any } j,$$

$$\rho_i \circ \pi = \rho_i \quad \text{on } f_1^{-1}(\delta_1) \cap X_m^+ \cap U_i \text{ for any } i$$

by (2.14.6) and (2.14.7); here U_i are shrunk if necessary. It is easy to extend π by virtue of θ_m to a diffeomorphism of X_m^+ so that

(2.14.8) $$\pi = \text{ident on } X_m^+ \cap \{f_1 \ge \varepsilon' \text{ or } f_1 \le \delta_1^-\},$$

(2.14.3)′ $$\pi(X_m^+ - \{f_1 \le \delta_1\}) = X_m^+ - \{f_2 \le \delta_1\},$$

(2.14.4)′ $$\pi(X_m^+ \cap Y_j) = X_m^+ \cap Y_j \text{ for any } j,$$

(2.14.9) $$\rho_i \circ \pi = \rho_i \text{ on } X_m^+ \cap U_i \text{ for any } i > 1,$$

(2.14.10) $$\pi(x) \in \theta_m(D_m \cap (x \times \mathbf{R})) \text{ for any } x \in X_m^+,$$

where δ_1^- is an arbitrarily given positive number such that

$$X_m^+ \cap \{f_1 \le \delta_1^-\} \subset \{f_2 < \delta_1\} \quad \text{and} \quad \delta_1^- < \delta_1.$$

Then (2.14.2) follows from (2.14.8) and, moreover, π satisfies (2.14.3). To see this, we remark by (2.14.1) that $X_2^+ \cap \{\delta_1^- - \varepsilon'' \le f_1 \le \varepsilon' + \varepsilon''\}$ is compact for small $\varepsilon'' > 0$. Hence, we have a small $\varepsilon_2 > 0$ such that

$$\{x \in X_m^+ \mid \delta_1^- \le f_1(x) \le \varepsilon',\ \rho_2(x) \le \varepsilon_2\} \subset U_2.$$

Let $0 < \delta_2 \le \varepsilon_2$. Then it follows from (2.14.8) and (2.14.9) that

π(2.14.3)″ $$\pi(X_m^+ - \{f_1 \le \delta_1\} - \{\rho_2 \le \delta_2\}) = X_m^+ - \{f_2 \le \delta_1\} - \{\rho_2 \le \delta_2\}.$$

Repeating this argument for $\rho_3, \ldots, \rho_{m-1}$, we obtain equality (2.14.3). By the method of choice of $\delta_2, \ldots, \delta_{m-1}$, we clearly have positive functions $\varepsilon_2(t_1), \ldots, \varepsilon_{m-1}(t_1, \ldots, t_{m-2})$ required in the proposition.

Moreover, we extend π to a diffeomorphism of X_m by the use of θ_m so that (2.14.2) remains true. That is easily carried out by (2.14.10). Then (2.14.4) holds by (2.14.7). Hence the proposition is proved.

2.15. REMARK. In 2.14 the assumption $\dim X_i < \dim X_{i+1}$ is intended for clarifying the meaning of 2.14. If we replace it by the assumption $\dim X_1 < \dim X_i < \dim X_m$ for $i \neq 1, m$, then (2.14.3) must be changed by

$$\pi(F_1) = F_2, \tag{2.15.1}$$

where

$$F_1 = X_m^+ - \{f_1 \leq \delta_1\} - \{\rho_2 \leq \delta_{l_2}\} - \cdots - \{\rho_{m-1} \leq \delta_{l_{m-1}}\},$$
$$F_2 = X_m^+ - \{f_2 \leq \delta_1\} - \{\rho_2 \leq \delta_{l_2}\} - \cdots - \{\rho_{m-1} \leq \delta_{l_{m-1}}\},$$
$$l_i = \dim X_i - \dim X_1 + 1.$$

By the method of construction of π, the following is clear. For any subset $\{i_1, \ldots, i_k\}$ of $\{2, \ldots, m\}$,

$$\begin{aligned} &\pi(F_1 \cap \{\rho_{i_1} = \delta_{l_{i_1}}\} \cap \cdots \cap \{\rho_{i_k} = \delta_{l_{i_k}}\}) \\ &\quad = F_2 \cap \{\rho_{i_1} = \delta_{l_{i_1}}\} \cap \cdots \cap \{\rho_{i_k} = \delta_{l_{i_k}}\}. \end{aligned} \tag{2.15.2}$$

3. C^∞ triangulations and the Alexander trick. In this section K and L denote simplicial complexes, and M a C^∞ submanifold of $\mathbf{R}^n$.

3.1. DEFINITION. A *C^∞ map* $f\colon K \to M$ is a map $f\colon |K| \to M$ such that $f|_\sigma$, $\sigma \in K$, are of class C^∞. A *linear isomorphism* $g\colon K \to L$ is a homeomorphism $g\colon |K| \to |L|$ carrying each simplex of K linearly onto one of L. Let $f\colon K \to M$ be a C^∞ map and let $b \in |K|$. We define $df_b\colon \overline{\mathrm{St}}(b, K) \to \mathbf{R}^n$ by

$$df_b(x) = d(f|_\sigma)_b(x - b) \quad \text{for } x, b \in \sigma \in K,$$

where $\overline{\mathrm{St}}(b, K)$ denotes the closed star of b in K. We call f a *C^∞ imbedding* if f and df_b are homeomorphisms onto the images for any $b \in |K|$.

A locally finite family $\{X_i\}$ of closed sets in M is called *locally C^∞ triangulable* if each X_i is contained in a C^∞ submanifold Y_i of M of the same dimension so that

(3.1.1) X_i is the closure of the union of some connected components of $Y_i - \bigcup_{\dim Y_j < \dim Y_i} Y_j$,

(3.1.2) each point a of M has a C^∞ coordinate system $\varphi\colon U \to \mathbf{R}^m$ such that $\varphi(U \cap Y_i)$, with $a \in Y_i$, are linear subspaces of $\mathbf{R}^m$.

For example, M itself, or a C^∞ manifold with cornered boundary, is locally C^∞ triangulable.

The following shows that a locally C^∞ triangulable family is uniquely and globally C^∞ triangulable, which is a generalization of the Cairns-Whitehead theorem that a C^∞ manifold is uniquely C^∞ triangulable. The original proofs also prove the generalization, so we omit the proof (see [**11**, 10.5, 10.6]).

3.2. PROPOSITION. *Let $\{X_i\}$ be a locally C^∞ triangulable family in M. Then there exist K and a C^∞ imbedding $f\colon K \to M$ such that $f(|K|) = \bigcup_i X_i$ and each X_i is the union of some $f(\sigma)$'s, $\sigma \in K$ (we call f a C^∞ triangulation of $\{X_i\}$). Moreover, if $g\colon L \to M$ is another C^∞ triangulation of $\{X_i\}$, then K and L have subdivisions K' and L', respectively, and we have a linear isomorphism $h\colon K' \to L'$ such that*

$$f^{-1}(X_i) = (g \circ h)^{-1}(X_i) \quad \textit{for any } i.$$

In 3.2 if $\{X_i\}$ consists of one element and if there is no confusion, we also call $|K|$ a *C^∞ triangulation* of the element.

3.3. *The Alexander trick* (*e.g.* [**12**]). Let X, Y be compact polyhedrons, p, q be points, and $p * X$, $q * Y$ be cones. Let h: $p * X \to q * Y$ be a homeomorphism whose restriction to the base X is PL onto the base Y. Then there exists an isotopy h_t, $t \in [0,1]$, such that $h_t \equiv h$ on X, $t \in [0,1]$, $h_0 \equiv h$ and h_1 is a PL homeomorphism. Moreover, if h is subanalytic, h_t can be subanalytic, which is easy to see by the definition of subanalytic map.

The following is also clear. Let G: $p * X \to q * Y$ be a homeomorphism such that $G(X) = Y$. Let $g_t: X \to Y$, $t \in [0,1]$, be an isotopy of $G|_X$. Then there exists an isotopy G_t of G such that $G_t|_X = g_t$. In the case that G and g_t are subanalytic, G_t can be subanalytic.

4. Piecewise linearization of locally subanalytic homeorphism. In this section we prove the following.

4.1. THEOREM. *Let X, Y be polyhedrons, d a metric on Y, h: $X \to Y$ a locally subanalytic homeomorphism and ε a positive continuous function on X. Then there exists a locally subanalytic isotopy h_t: $X \to Y$, $t \in [0,1]$, of h such that*

(4.1.1) *h_1 is PL, and*

(4.1.2) *$d(h(x), h_t(x)) < \varepsilon(x)$ for any $x \in X$ and $t \in [0,1]$.*

Moreover, if h is PL on a neighborhood of a subpolyhedron X' of X, we can choose the isotopy so that

(4.1.3) *$h_t(x) = h(x)$ for any $t \in [0,1]$ and any x in a neighborhood of X'.*

PROOF. Let X, Y be contained in $\mathbf{R}^n$, and K, L be rectilinear triangulations of X, Y respectively. Apply 2.2 to the family of sets $h(\sigma)$, $\sigma \in K$, subanalytic in some open neighborhood of Y. Then we can reduce the problem to the case where for each $\sigma \in K$, $h(\sigma)$ is the underlying polyhedron of some subcomplex of L.

Assume for the present that $h(\sigma)$, $\sigma \in K$, are PL balls. Then there exists the required isotopy h_t which satisfies, moreover, $h_t(\sigma) = h(\sigma)$ for any $t \in [0,1]$ and $\sigma \in K$. We prove this by induction on $m = \dim X$. If $m = 0$, it is trivial. Hence we assume it for dim $\leq m - 1$. Let K^{m-1} be the $(m-1)$-skeleton of K, and $L^{*(m-1)}$ the subcomplex of L whose underlying polyhedron is $h(|K^{m-1}|)$. By the inductive assumption we have a locally subanalytic isotopy g_t: $|K^{m-1}| \to |L^{*(m-1)}|$ of $h|_{|K^{m-1}|}$ such that g_1 is PL and $g_t(\sigma) = h(\sigma)$ for any t and $\sigma \in K^{m-1}$, which implies $g_t(\partial\sigma) = h(\partial\sigma)$ for any t and $\sigma \in K - K^{m-1}$. Hence, 3.3 shows that g_t can be extended to a locally subanalytic isotopy G_t: $X \to Y$ of h and G_1 is locally subanalytically isotopic to a PL homeomorphism, so the isotopy fixes $|K^{m-1}|$. Thus we have a locally subanalytic isotopy h_t: $X \to Y$ of h satisfying (4.1.1) and $h_t(\sigma) = h(\sigma)$ for any $t \in [0,1]$ and $\sigma \in K$. (4.1.2) follows from (2.2.1) when we choose fine triangulations of X, Y, and (4.1.3) follows from (2.2.4). Therefore we only have to prove the weaker Theorem 4.4 below.

4.2. REMARK. In 4.1 if X, Y are contained in $\mathbf{R}^n$ as closed sets, then h_t is subanalytic.

4.3. COROLLARY. (*Hauptvermutung for triangulations of subanalytic or semi-algebraic sets*). *For any locally closed and* (*locally*) *subanalytic set $X \subset \mathbf{R}^n$, we have a polyhedron in $\mathbf{R}^n$, which is* (*locally resp.*) *subanalytically homeomorphic to X, uniquely up to a PL homeomorphism. The semialgebraic version also holds true* (*see* §6.1).

4.4. THEOREM. *Let X, $Y \subset \mathbf{R}^n$ be compact polyhedrons whose local dimensions at any point are constant m and which are subanalytically homeomorphic. Then X, Y are PL homeomorphic.*

We prove the theorem in a sequence of lemmas. Let K, L be rectilinear triangulations of X, Y, respectively, h: $X \to Y$ a subanalytic homeomorphism, Z its graph, and p_1: $Z \to X$, p_2: $Z \to Y$ the projections. Then we have

4.5. LEMMA. *There exist a subanalytic analytic Whitney stratification $\{Z_i\}$ of Z and a subanalytic homeomorphim χ_i: $\overline{Z}_i \to B^{m_i}$ for each i, where $B^{m_i} = \{x \in \mathbf{R}^{m_i} \mid |x| \le 1\}$ and $m_i = \dim Z_i$, such that:*

(4.5.1) *for each i, j, $p_j|_{Z_i}$ is a diffeomorphism;*

(4.5.2) *for each i, i' with $Z_{i'} \subset \overline{Z}_i$, $\chi_i|_{Z_{i'}}$ is a diffeomorphism, and $\chi_i(Z_i) = \operatorname{Int} B^{m_i}$;*

(4.5.3) *for each i, $\{\operatorname{graph} \chi_i|_{Z_{i'}} | Z_{i'} \subset \overline{Z}_i\}$ is a subanalytic analytic Whitney stratification; and*

(4.5.4) *each $\sigma \in K$ or $\in L$ is the union of some $p_j(Z_i)$'s, $j = 1$ or 2.*

PROOF. We remark that (4.5.4) is identical with

(4.5.4)′ if $\sigma \in K$ or $\sigma \in L$ intersects with $p_j(Z_i)$, then $p_j(Z_i) \subset \sigma$.

We inductively prove 4.5. Let $k \ge 0$. Assume

$(4.5)_k$ there is a closed subanalytic subset Z' of Z of dimension $\le k$, a subanalytic analytic Whitney stratification $\{Z_i\}$ of $Z - Z'$, and a subanalytic homeomorphism χ_i: $\overline{Z}_i \to B^{m_i}$ for each i such that (4.5.1)–(4.5.3) and (4.5.4)′ hold and $m_i > k$ for all i.

$(4.5)_m$ is trivial, so it is sufficient to show $(4.5)_{k-1}$. If $\dim Z' < k$, $(4.5)_{k-1}$ is identical to $(4.5)_k$. Hence we assume $\dim Z' = k$. Apply (2.7.1)–(2.7.5) to all $Z' \cap \overline{Z}_i$ and $\operatorname{graph} \chi_i|_{Z' \cap \overline{Z}_i}$. Then we have a closed subanalytic subset Z'' of Z' of dimension $< k$ such that: the connected components $Z'_1, \dots, Z'_l$ of $Z' - Z''$ are subanalytic analytic manifolds; for each i, j, $d(p_j|_{Z'_i})$ has rank k; if $Z'_{i'} \cap \overline{Z}_i \ne \emptyset$ then $Z'_{i'} \subset \overline{Z}_i$; for such a pair i, i', $\chi_i|_{Z'_{i'}}$ is a C^ω diffeomorphism; and for the same i, i', $(Z_i, Z'_{i'})$ and $(\operatorname{graph} \chi_i|_{Z_i}, \operatorname{graph} \chi_i|_{Z'_{i'}})$ satisfy the Whitney condition (b). If $p_j^{-1}(\operatorname{Int} \sigma) \cap Z'_i$, $\sigma \in K$ or $\sigma \in L$, is of dimension $< k$, we add such a set to Z''. Then (4.5.4)′ holds for $Z'_1, \dots, Z'_l$. Hence we have shown $(4.5)_{k-1}$ except for the existence of χ_i: $\overline{Z}'_i \to B^k$ for each i such that $\chi_i|_{Z'_i}$ is a C^ω diffeomorphism onto $\operatorname{Int} B^k$.

To see that apply 2.2 to each $(\overline{Z}'_i, \overline{Z}'_i \cap Z'' = \overline{Z}'_i - Z'_i)$. Then there exist a simplicial complex K_i and a subanalytic homeomorphism τ_i: $|K_i| \to \overline{Z}'_i$ such that for each $\sigma \in K_i$, $\tau_i|_{\operatorname{Int} \sigma}$: $\operatorname{Int} \sigma \to \tau_i(\operatorname{Int} \sigma)$ is an analytic diffeomorphism and $\overline{Z}'_i \cap Z''$ is the image under τ_i of the underlying polyhedron of a subcomplex of K_i. Denote by Z''_i the union of the images under τ_i of all simplexes of dimension $k - 1$ of K_i. Then we have $Z''_i \supset \overline{Z}'_i \cap Z''$. We remark that there is a subanalytic homeomorphism from a k-simplex to B^k whose restriction to the interior is of class C^ω. Hence, if we replace Z'' by $\bigcup Z''_i$, there exist required χ_i: $\overline{Z}'_i \to B^k$, which proves 4.5.

Let I be the index set of $\{Z_i\}$. Put

$$I^k = \{i \in I \mid \dim Z_i \le k\}, \qquad 0 \le k \le m - 1,$$
$$I^{-1} = \emptyset, \qquad X_i = p_1(Z_1) \quad \text{and} \quad Y_i = p_2(Z_i) \quad \text{for each } i \in I.$$

Then it follows that $\{X_i\}$ and $\{Y_i\}$ are subanalytic analytic Whitney stratifications of X and Y, respectively, and h maps C^ω diffeomorphically each X_i onto Y_i. Let $\rho_i^X, \rho_i, \rho_i^Y$ be the squares of the distance functions from X_i, Z_i, Y_i in $\mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n$, respectively, in the usual metrics, and let m_i denote dim X_i. Let δ be a sequence of positive numbers $\delta_0, \dots, \delta_{m-1}$, γ: $I^{m-1} \to \{=, \geq\}$ a map, and γ_0: $I^{m-1} \to \{\geq\}$ the constant map. Put

$$X_{\delta\gamma} = \bigcap_{i \in I^{m-1}} \{x \in X \mid \rho_i^X(x)\gamma(i)\delta_{m_i}\},$$

$$U_{i\delta}^X = \{x \in \mathbf{R}^n \mid \rho_i^X(x) < 2\delta_{m_i}\} - \bigcup_{m_j < m_i} \{\rho_j^X \leq \delta_{m_j}/2\} \quad \text{for } i \in i^{m-1},$$

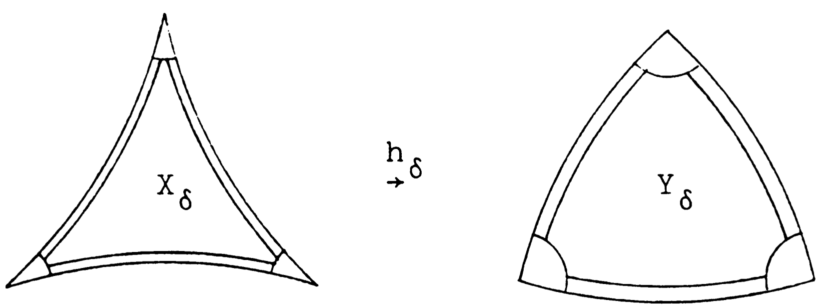
where $\rho_i^X(x)\gamma(i)\delta_{m_i}$ means $\rho_i^X(x) = \delta_{m_i}$ if $\gamma(i)$ is "=" and $\rho_i^X(x) \geq \delta_{m_i}$ if $\gamma(i)$ is "$\geq$". We write $X_{\delta\gamma_0}$ briefly as X_δ. We similarly define $Z_{\delta\gamma}, U_{i\delta}^Z, Z_\delta, Y_{\delta\gamma}, U_{i\delta}^Y$ and Y_δ. Then $X_{\delta\gamma}, Z_{\delta\gamma}$ and $Y_{\delta\gamma}$ are compact subsets of $\bigcup_{i \notin I^{m-1}} X_i, \bigcup_{i \notin I^{m-1}} Z_i$ and $\bigcup_{i \notin I^{m-1}} Y_i$, respectively. Moreover, they are C^∞ manifolds with cornered boundary for δ restricted as follows.

There exist a positive number and positive functions $\varepsilon_0, \varepsilon_1(t_0)$ on $\mathbf{R}, \dots,$ $\varepsilon_{m-1}(t_0, \dots, t_{m-2})$ on $\mathbf{R}^{m-1}$ such that for any sequence δ of positive numbers, $\delta_0, \dots, \delta_{m-1}$, with $\delta_0 \leq \varepsilon_0, \delta_1 \leq \varepsilon_1(\delta_0), \dots, \delta_{m-1} \leq \varepsilon_{m-1}(\delta_0, \dots, \delta_{m-2})$, and for any $i \in I^{m-1}$, ρ_i^X is analytic on $U_{i\delta}^X$ and C^∞ regular on $U_{i\delta}^X - X_i$, and for the same δ, i and any subset $\{j_1, \dots, j_k\}$ of I^{m-1}, $\{x \in U_{j_1\delta}^X \mid \rho_{j_1}^X(x) = \delta_{m_{j_1}}\}$ is transversal to

$$X_i \cap \{x \in U_{j_2\delta}^X \mid \rho_{j_2}^X(x) = \delta_{m_{j_2}}\} \cap \cdots \cap \{x \in U_{j_k\delta}^x \mid \rho_{j_k}^X(x) = \delta_{m_{j_k}}\}.$$

The existence of such $\varepsilon_0, \dots, \varepsilon_{m-1}$ is clear by the definition of analytic Whitney stratification, and similar statements hold for Z and Y.

4.6. LEMMA. *We can choose $\varepsilon_0, \dots, \varepsilon_{m-1}$ so that for any sequence δ bounded by $\varepsilon_0, \dots, \varepsilon_{m-1}$ as above, there exists, moreover, a C^∞ diffeomorphism h_δ: $X_\delta \to Y_\delta$ such that $h_\delta(X_{\delta\gamma}) = Y_{\delta\gamma}$ for any γ: $I^{m-1} \to \{=, \geq\}$.*



PROOF. We prove the lemma through the medium of Z, namely, we find C^∞ diffeomorphisms $p_{1\delta}$: $Z_\delta \to X_\delta$ and $p_{2\delta}$: $Z_\delta \to Y_\delta$ such that $p_{1\delta}(Z_{\delta\gamma}) = X_{\delta\gamma}$ and $p_{2\delta}(Z_{\delta\gamma}) = Y_{\delta\gamma}$ for any γ. For this we only have to construct a C^∞ diffeomorphism π_δ of $\bigcup_{i \neq I^{m-1}} Z_i$ such that

$$\pi_\delta(Z_{\delta\gamma}) = p_1^{-1}(X_{\delta\gamma}) \quad \text{for any } \gamma. \tag{4.6.1}$$

Put

$$\psi_i = \rho_i^X \circ p_1, \quad Z_{\psi\delta\gamma} = \bigcap_{i \in I^{m-1}} \{z \in Z \mid \psi_i(z)\gamma(i)\delta_{m_i}\}.$$

Then (4.6.1) equals $\pi_\delta(Z_{\delta\gamma}) = Z_{\psi\delta\gamma}$.

We inductively construct π_δ as follows. Let $1 \le l \le m$. Assume there are $\varepsilon_0, \dots, \varepsilon_{m-1}$ and a C^∞ diffeomorphism $\pi_{\delta l}$ of $\bigcup_{i \neq I^{m-1}} Z_i$ such that

$$(4.6.1)_l \qquad \pi_{\delta l}\left(\bigcap_{i \in I^{l-1}} \{\psi_i\gamma(i)\delta_{m_j}\} \cap \bigcap_{i \in I^{m-1}-I^{l-1}} \{\rho_i\gamma(i)\delta_{m_i}\}\right) = Z_{\psi\delta\gamma}.$$

$(4.6.1)_m$ is trivial, and $(4.6.1)_0 \equiv (4.6.1)$, so it is sufficient to construct $\pi_{\delta(l-1)}$, which satisfies $(4.6.1)_{l-1}$. For it, after lessening $\varepsilon_{l-1}, \dots, \varepsilon_{m-1}$, we will construct a C^∞ diffeomorphism $\pi'_{\delta l}$ of $\bigcup_{m-1} Z_i$ such that

$$(4.6.2) \qquad \begin{aligned} &\pi'_{\delta l}\left(\bigcap_{i \in I^{l-2}} \{\psi_i\gamma(i)\delta_{m_i}\} \cap \bigcap_{i \in I^{m-1}-I^{l-2}} \{\rho_i\gamma(i)\delta_{m_i}\}\right) \\ &\qquad = \bigcap_{i \in I^{l-1}} \{\psi_i\gamma(i)\delta_{m_i}\} \cap \bigcap_{i \in I^{m-1}-I^{l-1}} \{\rho_i\gamma(i)\delta_{m_i}\}, \end{aligned}$$

and

(4.6.3) the closure of $\{\pi'_{\delta l}(x) \neq x\}$ is contained in a small neighborhood of $\bigcup_{i \in I^{l-1}-I^{l-2}} Z_i$ because $\pi_{\delta(l-1)} = \pi_{\delta l} \circ \pi'_{\delta l}$ satisfies $(4.6.1)^{l-1}$.

We will choose this neighborhood of $\bigcup_{i \in I^{l-1}-I^{l-2}} Z_i$ so small as to be the disjoint union of some small neighborhoods U_i of Z_i, $i \in I^{l-1} - I^{l-2}$. Hence we can assume $I^{l-1} - I^{l-2}$ consists of one i.

Fix positive numbers $\delta_0, \dots, \delta_{l-2}$ so that $\delta_0 \le \varepsilon_0, \dots, \delta_{l-2} \le \varepsilon_{l-2}(\delta_0, \dots, \delta_{l-3})$. Consider a subanalytic analytic Whitney stratification

$$\{Z_j - V_{l-2}\}_{j \in I^{m-1}-I^{l-2}} \cup \left\{\bigcup_{j \notin I^{m-1}} Z_j - V_{l-2}\right\},$$

where

$$V_{l-2} = \bigcup_{j \in I^{l-2}} \{\psi_j \le \delta_{m_j}/2\},$$

a canonical tube system $\{T_j = (|T_j|, \pi_j, \rho_j|_{|T_j|})\}$ for $\{Z_j - V_{l-2}\}_{j \in I^{m-1}-I^{l-2}}$, $f_1 = \rho_i$, $f_2 = \psi_i$ and $\{\varphi_1, \dots, \varphi_k\} = \{\psi_j - \delta_{m_j}\}_{j \in I^{l-2}}$. It is easy to check that these satisfy the assumptions in 2.14 or 2.15. Hence, we have a positive number and positive functions $\varepsilon'_{l-1}, \varepsilon'_l(t_{l-1}), \dots, \varepsilon'_{m-1}(t_{l-1}, \dots, t_{m-2})$ such that for any sequence of positive numbers

$$\delta_{l-1} \le \varepsilon'_{l-1}, \delta_l \le \varepsilon'_l(\delta_{l-1}), \dots, \delta_{m-1} \le \varepsilon'_{m-1}(\delta_{l-1}, \dots, \delta_{m-2}),$$

there exists a C^∞ diffeomorphism $\pi'_{\delta l}$ of $\bigcup_{j \notin I^{m-1}} Z_j - V_{l-2}$ such that (4.6.2) is satisfied and

(4.6.4) $\{\pi'_{\delta l}(x) \neq x\}$ is contained in $U_i - \bigcup_{j \in I^{l-2}}\{\psi_j \le 2\delta_{m_j}/3\}$. Here (4.6.2) follows easily from (2.14.4), (2.15.1) and (2.15.2), and (4.6.4) follows from (2.14.2). Clearly (4.6.4) implies that $\pi'_{\delta l}$ is extensible to $\bigcup_{i \notin I^{m-1}} Z_i$. As ε'_{l-1} is determined by $\delta_0,\dots,\delta_{l-2}$, we regard it as a positive function in $(\delta_0,\dots,\delta_{l-2})$-variables, hence ε'_l is regarded as a positive function in $\delta_0,\dots,\delta_{l-1}$, and so on. Thus, if we take $\varepsilon_0,\dots,\varepsilon_{l-2},\min\{\varepsilon_{l-1},\varepsilon'_{l-1}\},\dots,\min\{\varepsilon_{m-1},\varepsilon'_{m-1}\}$ in place of $\varepsilon_0,\dots,\varepsilon_{m-1}$, then the required $\pi'_{\delta l}$ exists.

For each i,δ and γ as above, put

$$X_{i\delta\gamma} = \bigcap_{j\in I^{m_i-1}} \{x\in X_i \mid \rho_j^X(x)\gamma(j)\delta_{m_j}\},$$

$$Y_{i\delta\gamma} = \bigcap_{j\in I^{m_i-1}} \{y\in Y_i \mid \rho_j^Y(y)\gamma(j)\delta_{m_j}\},$$

$$X_{i\delta} = X_{i\delta\gamma_0} \quad \text{and} \quad Y_{i\delta} = Y_{i\delta\gamma_0}.$$

Then, in the same way as 4.6, we choose $\varepsilon_0,\dots,\varepsilon_{m-1}$ so that for any i and any δ bounded by $\varepsilon_0,\dots,\varepsilon_{m-1}$, there exists a C^∞ diffeomorphism $h_{i\delta}\colon X_{i\delta}\to Y_{i\delta}$ such that $h_{i\delta}(X_{i\delta\gamma}) = Y_{i\delta\gamma}$ for any γ, and $X_{i\delta\gamma}$ and $Y_{i\delta\gamma}$ are C^∞ manifolds with cornered boundary.

Put

$$X^+_{i\delta\gamma} = \bigcap_{j\in I^{m_i-1}} \{x\in \mathbf{R}^n \mid \rho_j^X(x)\gamma(j)\delta_{m_j}\}\cap\{x\in\mathbf{R}^n \mid \rho_i^X(x)\le\delta_{m_i}\},$$

$$Y^+_{i\delta\gamma} = \bigcap_{j\in I^{m_i-1}} \{y\in \mathbf{R}^n \mid \rho_i^Y(y)\gamma(j)\delta_{m_j}\}\cap\{y\in\mathbf{R}^n \mid \rho_i^Y(y)\le\delta_{m_i}\}.$$

Then, for any δ bounded as above, they are C^∞ manifolds with cornered boundary and, moreover, we have the following. Let δ be fixed, and $\mathcal{X}_\delta$ (or $\mathcal{Y}_\delta$) be the family of $X_{i\delta\gamma}$ and $X^+_{i\delta\gamma}$ (or $Y_{i\delta\gamma}$ and $Y^+_{i\delta\gamma}$ respectively) for all $i\in I$ and for all $\gamma\colon I^{m-1}\to\{=,\ge\}$. Then the union of $\mathcal{X}_\delta$ or $\mathcal{Y}_\delta$ contains X or Y, respectively, and $\mathcal{X}_\delta$ or $\mathcal{Y}_\delta$ is locally C^∞ triangulable for the following reason. Let $\mathcal{D}$ be the family of all C^∞ manifolds $\mathbf{R}^n$, $X_i\cap U^X_{i\delta}$ and $\{x\in U^X_{i\delta} \mid \rho_i^X(x)=\delta_{m_i}\}$, $i\in I$, and $\mathcal{D}'$ the family of all intersections of elements of $\mathcal{D}$. Then for any $A_1,\dots,A_r$ of $\mathcal{D}$, A_1 is transversal to $A_2\cap\dots\cap A_r$ and, hence, $\mathcal{D}'$ satisfies (3.1.2). By definition of $X_{i\delta\gamma}$ and $X^+_{i\delta\gamma}$, they are defined by $\mathcal{D}'$ by means (3.3.1). Hence $\mathcal{X}_\delta$ is locally C^∞ triangulable.

Furthermore we can easily prove by (4.5.4) that $\mathcal{X}_\delta\cup K$ and $\mathcal{Y}_\delta\cup L$ are locally C^∞ triangulable.

4.7. LEMMA. *For small $\varepsilon_0,\dots,\varepsilon_{m-1}$, let i, δ and γ be the same as above. A C^∞ triangulation of $X_{i\delta\gamma}$ (3.2) is a PL ball.*

PROOF. We prove this by induction on m_i. Let $k\ge 1$. As the case $m_i=0$ is trivial, we assume 4.7 for $m_i<k$. Let $i_0\in I^k-I^{k-1}$. Put

$$J=\{i\in I \mid X_i\subset \overline{X}_{i_0}\},\qquad J^l=J\cap I^l \quad \text{for } -1\le l\le k.$$

Let $\chi_{i_0}\colon \overline{Z}_{i_0}\to B=B^k\subset\mathbf{R}^k$ be the homeomorphism in 4.5. Put

$$B_i=\chi_{i_0}(Z_i)\quad \text{for } i\in J,\qquad B_{\delta\gamma}=\bigcap_{i\in J^{k-1}}\{b\in B \mid \rho_i^B(b)\gamma(i)\delta_{m_i}\},$$

where ρ_i^B are the squares of the distance functions from B_i in $\mathbf{R}^k$ in the usual metric. Then (4.5.2), (4.5.3) and 4.6 show that for small $\varepsilon_0,\dots,\varepsilon_{k-1}$, $X_{i_0\delta\gamma}$ and

$B_{\delta\gamma}$ are C^∞ diffeomorphic. Hence it is sufficient to prove that a C^∞ triangulation of $B_{\delta\gamma}$ is a PL ball. In the same way as $\{X_i\}$, we define $B_{i\delta\gamma}$, $B^+_{i\delta\gamma}$, $\mathcal{B}_\delta$. We remark that $\mathcal{B}_\delta$ and $\mathcal{B}_\delta \cup B$ are locally C^∞ triangulable and $\partial B = \bigcup_{i\in J^{k-1}} B_i$.

The case $\gamma = \gamma_0$: It is clear that $B \subset \bigcup_{i\in J} B^+_{i\delta\gamma_0}$, and $B^+_{i_0\delta_{\gamma_0}} = B_{\delta\gamma_0}$. For each $i \in J^{k-1}$, we also have

$$(4.7.1)\qquad (B^+_{i\delta\gamma_0}, B\cap B^+_{i\delta\gamma_0}, B\cap B^+_{i\delta\gamma_0} \cap \{\rho^B_i = \delta_{m_i}\}, B_{i\delta\gamma_0})$$

is C^∞ diffeomorphic to

$$(B^{k-m_i}, B^{k-m_i}\cap\{x_1\ge 0\}, \partial B^{k-m_i}\cap\{x_1\ge 0\}, 0)\times B_{i\delta\gamma_0},$$

where

$$B^{k-m_i} = \{(x_1,\dots,x_{k-m_i})\in \mathbf{R}^{k-m_i}\,|\,|x|\le 1\}.$$

The reason is the following. Let $q\colon T \to B_{i\delta\gamma_0}$ be the orthogonal projection of a small tubular neighborhood in $\mathbf{R}^k$. Recall that for any subset $i_1,\dots,i_l$ of J^{m_i-1}, $\{\rho^B_{i_1} = \delta_{m_{i_1}}\}$ is transversal to

$$\left(B_i - \bigcup_{m_j<m_i}\{\rho_j \le \delta_{m_j}/2\}\right)\cap\{\rho^B_{i_2} = \delta_{m_{j_2}}\}\cap\cdots\cap\{\rho_{i_l} = \delta_{m_{i_l}}\}.$$

Hence, using C^∞ vector fields ξ near $B_{i\delta\gamma_0}$ with $d\rho^B_i\xi = 0$ and their C^∞ flows, we obtain, in the same way as 2.14 and 4.6, but more easily, a C^∞ diffeomorphism τ of $\mathbf{R}^k$ such that $\tau|_{B_{i\delta\gamma_0}} = \text{ident}$, $\rho_i\circ\tau = \rho_i$ near $B_{i\delta\gamma_0}$,

$$\tau\{\rho_j = \delta_{m_j}\}\cap T = q^{-1}(B_{i\delta\gamma_0}\cap\{\rho_j = \delta_{m_j}\})\quad\text{for any } j\in J^{m_i-1},$$

and for any $x \in B_{i\delta\gamma_0}$ and $y \in \tau(\partial B)\cap q^{-1}(x)$, the vector xy is a tangent vector of ∂B at x. Now $B_{i\delta\gamma_0}$ is homeomorphic to a ball by the induction hypothesis since $B_{i\delta\gamma_0}$ and $X_{i\delta\gamma_0}$ are C^∞ diffeomorphic. Hence the normal bundles of $B_{i\delta\gamma_0}$ in $\mathbf{R}^k$ and ∂B are trivial, which proves (4.7.1).

By 3.2 we have a C^∞ triangulation $g\colon C \to \mathbf{R}^k$ of $\mathcal{B}_\delta \cup B$. Let C_i, C'_i, C''_i be the subcomplexes of C whose underlying polyhedrons are mapped by g onto $B_{i\delta\gamma_0}$, $B\cap B^+_{i\delta\gamma_0}$, and $B\cap B^+_{i\delta\gamma_0}\cap\{\rho^B_i = \delta_{m_i}\}$ respectively. Then the induction hypothesis shows that $|C_i|$ are PL balls for $i\in J^{k-1}$. Moreover, (4.7.1) and the uniqueness of C^∞ triangulation (3.2) imply that $|C'_i|$ and $|C''_i|$ are PL balls of dimension k and $k-1$, respectively, for $i\in J^{k-1}$.

We want to see that $|C'_{i_0}| = |C_{i_0}|$ is a PL ball. Order $B_1,\dots,B_{i_0}$ in such a way that $m_i \le m_{i'}$ if $i \le i'$, and regard $B_{\delta\gamma_0}$ as obtained by the orderly excisions of $B\cap B^+_{i\delta\gamma_0}$, $i = 1,\dots,i_0-1$, from B. Put

$$W_i = \bigcup_{j\ge i}(B\cap B^+_{j\delta\gamma_0}),\qquad i = 1,\dots,i_0.$$

Then

$$W_{i_0} = B_{\delta\gamma_0},\qquad W_i = (B\cap B^+_{i\delta\gamma_0})\cup W_{i+1},$$

$$(B\cap B^+_{i\delta\gamma_0})\cap W_{i+1} = B\cap B^+_{i\delta\gamma_0}\cap\{\rho^B_i = \delta_{m_i}\},\qquad i = 1,\dots,i_0-1.$$

It is clear that the interior of $B\cap B^+_{i\delta\gamma_0}\cap\{\rho^B_i = \delta_{m_i}\}$ is contained in the interior of W_i for $i = 1,\dots,i_0-1$. Recall the following fact (e.g. [**12**, 3.13, 3.16]). Let

$A_1, A_2 \subset \mathbf{R}^k$ be polyhedrons such that $A_1 \cup A_2$ and A_2 are PL k-balls and $A_1 \cap A_2$ is a PL $(k-1)$-ball whose interior is contained in the interior of $A_1 \cup A_2$. Then A_1 is a PL k-ball. From this, it inductively follows that $\bigcup_{j \geq i} |C'_j|$ for any i is a PL k-ball and, in particular, that $|C'_{i_0}|$ is a PL ball. Since a C^∞ triangulation is unique, we have shown that any C^∞ triangulation of $B_{\delta\gamma_0}$ is a PL ball.

The general case of γ: We assume $\gamma \neq \gamma_0$. At first we remark that if there are $i \neq i' \in J$ with $m_i = m_{i'}$ and with $\gamma(i) = \gamma(i') = =$ then $B_{\delta\gamma}$ is empty. Hence we exclude such a case. Let $i_1 \in J^{k-1}$ be such that $\gamma(i_1) = =$ and $\gamma(i) = \geq$ for any $i \in J^{k-1} - J^{m_{i_1}}$. Then we have already proved by the induction hypothesis that a C^∞ triangulation of $B \cap B^+_{i_1\delta\gamma} \cap \{\rho^B_{i_1} = \delta_{m_{i_1}}\}$ is a PL ball. As

$$B \cap B^+_{i_1\delta\gamma} \cap \{\rho^B_{i_1} = \delta_{m_{i_1}}\} = \bigcap_{j \in I^{m_{i_1}-1}} \{b \in B \mid \rho^B_j \gamma(j) \delta_{m_j}\} \cap \{\rho^B_{i_1} = \delta_{m_{i_1}}\},$$

we obtain $B_\delta\gamma$ by removing $B \cap B^+_{i\delta\gamma}, i \in J^{k-1} - J^{m_{i_1}}$, in order from $B \cap B^+_{i_1\delta\gamma} \cap \{\rho^B_{i_1} = \delta_{m_{i_1}}\}$ in exactly the same way as in the case $\gamma = \gamma_0$. Hence we see that a C^∞ triangulation of $B_{\delta\gamma}$ is a PL ball. We complete the proof of 4.7.

4.8. REMARK. For each $i \in I^{m-1}, \delta$ and γ as above, a C^∞ triangulation of $X^+_{i\delta\gamma}$ is a PL ball, and that of $X \cap X^+_{i\delta\gamma}$ is PL homeomorphic to a PL cone whose base corresponds to the union of $X \cap X^+_{i\delta\gamma} \cap \{\rho^X_i = \delta_{m_i}\}$ and all $X \cap X^+_{i\delta\gamma'}$ such that $X_{i\delta\gamma'} \subset \partial X_{i\delta\gamma}$. We also have

$$X \cap X^+_{i\delta\gamma} \cap \{\rho^X_i = \delta_{m_i}\} = X \cap X^+_{i\delta\gamma} \cap \bigcup_{m_i < m_j} X^+_{j\delta\gamma_0}.$$

PROOF. Let q: $T \to X_{i\delta\gamma_0}$ be the orthogonal projection of a small tubular neighborhood in $\mathbf{R}^n$, and x_0 a point of $X_{i\delta\gamma_0}$. Then we have seen in the proof of 4.7 that q is trivial and that there is a C^∞ diffeomorphism τ of $\mathbf{R}^n$ such that

$$\tau|_{X_{i\delta\gamma_0}} = \text{ident}, \qquad \rho_i \circ \tau = \rho_i \quad \text{near } X_{i\delta\gamma_0},$$
$$\tau\{\rho^X_j = \delta_{m_j}\} \cap T = q^{-1}(X_{i\delta\gamma_0} \cap \{\rho^X_j = \delta_{m_j}\}) \quad \text{for } j \in I^{m_i - 1}.$$

Now (4.5.4) means that X_i is contained in one open simplex of K, say $\operatorname{Int}\sigma_0$. Hence, by the method of construction of τ, we can assume $\tau(\sigma) = \sigma$ for $\sigma \in K$, which implies $\tau(X) = X$. Therefore we have a C^∞ diffeomorphism τ': $X^+_{i\delta\gamma_0} \to q^{-1}(x_0) \times X_{i\delta\gamma_0}$ such that for any γ,

$$\tau'|_{X_{i\delta\gamma_0}} = x_0 \times \text{ident},$$
$$\tau'(\operatorname{Int}\sigma_0 \cap X^+_{i\delta\gamma}) = (\operatorname{Int}\sigma_0 \cap q_i^{-1}(x_0)) \times X_{i\delta\gamma}, \tag{4.8.1}$$
$$\tau'(X^+_{i\delta\gamma}) = q^{-1}(x_0) \times X_{i\delta\gamma},$$

$$\tau'(X \cap X^+_{i\delta\gamma}) = (X \cap q^{-1}(x_0)) \times X_{i\delta\gamma}. \tag{4.8.2}$$

Here (4.8.2) easily follows from (4.8.1) and the homogeneity property of X at $\operatorname{Int}\sigma_0$. Hence a C^∞ triangulation of $X^+_{i\delta\gamma}$ is a PL ball, and that of $X \cap X^+_{i\delta\gamma}$ is PL homeomorphic to a PL cone since a product of PL cones is a PL cone. The other statements in the remark are clear.

4.9. PROOF OF 4.4. Let δ be fixed so that the above statements hold. As $\mathcal{X}_\delta \cup K$ and $\mathcal{Y}_\delta \cup L$ are locally C^∞ triangulable, by 3.2 we have simplicial complexes K', L'

and C^∞ imbeddings $f: K' \to \mathbf{R}^n$, g: $L' \to \mathbf{R}^n$ such that $f(|K'|) = X$, $g(|L'|) = Y$, and each of $X_{i\delta\gamma}$, $X \cap X^+_{i\delta\gamma}$, $\sigma \in K$, $Y_{i\delta\gamma}$, $Y \cap Y^+_{i\delta\gamma}$ and $\sigma \in L$ is a union of some $f(\sigma'), \sigma' \in K'$, or $g(\sigma'), \sigma' \in L'$. In particular, we can regard f and g as C^∞ triangulations of K and L respectively. Hence, by the uniqueness of C^∞ triangulation (3.2), $|K'|$ and $|L'|$ are PL homeomorphic to X and Y respectively. Therefore we need only prove that $|K'|$ and $|L'|$ are PL homeomorphic. Let $K_{i\delta\gamma}, K^+_{i\delta\gamma}$ be the subcomplexes of K' whose underlying polyhedrons are mapped by f to $X_{i\delta\gamma}$, $X \cap X^+_{i\delta\gamma}$, respectively, and let $L_{i\delta\gamma}, L^+_{i\delta\gamma}$ be similarly defined. Then 4.7 says that $|K_{i\delta\gamma}|$ and $|L_{i\delta\gamma}|$ are PL balls. Order the indexes in I, $1, 2, \dots$ so that $m_1 \geq m_2 \geq \cdots$.

We inductively construct a PL homeomorphism from $|K'|$ to $|L'|$ as follows. Let $i \geq 1$. Assume a PL homeomorphism

$$h_i\colon \bigcup_{j\leq i} |K^+_{j\delta\gamma_0}| \to \bigcup_{j\leq i} |L^+_{j\delta\gamma_0}|$$

such that for any $j \leq i$ and any γ,

$$h_i(|K^+_{j\delta\gamma}|) = |L^+_{j\delta\gamma}|. \tag{4.9.1}$$

If $m_i = m$, the existence of h_i follows from 4.6 and from the uniqueness of the C^∞ triangulation. Hence, we assume $m_{i+1} < m$, and it is sufficient to extend h_i to

$$h_{i+1}\colon \bigcup_{j\leq i+1} |K^+_{j\delta\gamma_0}| \to \bigcup_{j\leq i+1} |K^+_{j\delta\gamma_0}|$$

so that h_{i+1} satisfies (4.9.1) for any $j \leq i+1$ and any γ. For any γ, 4.8 shows that $|K^+_{(i+1)\delta\gamma}|$ is PL homeomorphic to a PL cone so that the union of $|K^+_{(i+1)\delta\gamma}| \cap \bigcup_{j\leq i} |K^+_{j\delta\gamma_0}|$ and all $|K^+_{(i+1)\delta\gamma'}|$ such that $|K_{(i+1)\delta\gamma'}| \subset \partial|K_{(i+1)\delta\gamma}|$ corresponds to the base of the cone. Hence we regard $|K^+_{(i+1)\delta\gamma}|$ as a PL cone with base the above union. We remark that

$$|K^+_{(i+1)\delta\gamma}| \cap \bigcup_{j\leq i} |K^+_{j\delta\gamma_0}| = \bigcup_{j\leq i} |K^+_{j\delta\gamma_{i+1}}|,$$

where

$$\gamma_{i+1}(j) = \begin{cases} \gamma(j) & \text{for } j > i+1, \\ = & \text{for } j = i+1, \\ \geq & \text{for } j < i+1. \end{cases}$$

Clearly these hold true for $L_{j\delta\gamma}$ and $L^+_{j\delta\gamma}$, and we also regard $|L^+_{(i+1)\delta\gamma}|$ as PL cones. Order the elements of $\{\gamma\colon I^{m-1} \to \{=, \geq\} \mid K_{(i+1)\delta\gamma} \neq \emptyset\}$, $\gamma^1, \gamma^2, \dots$, so that $\dim K_{(i+1)\delta\gamma^1} \leq \dim K_{(i+1)\delta\gamma^2} \leq \cdots$.

We also use induction for the extension of h_i. Let $0 \leq l$, and assume a PL extension of h_i,

$$h_{il}\colon \bigcup_{k\leq l} |K^+_{(i+1)\delta\gamma^k}| \cup \bigcup_{j\leq i} |K^+_{j\delta\gamma_0}| \to \bigcup_{k\leq l} |L^+_{(i+1)\delta\gamma^k}| \cup \bigcup_{j\leq i} |L^+_{j\delta\gamma_0}|,$$

such that

$$h_{il}(|K^+_{(i+1)\delta\gamma^k}|) = |L^+_{(i+1)\delta\gamma^k}| \quad \text{for } k \leq l.$$

Now it follows from the cone structures of $|K^+_{(i+1)\delta\gamma^{l+1}}|$ and $|L^+_{(i+1)\delta\gamma^{l+1}}|$ that the bases of the cones are contained in the domain or the target of h_{il}. Hence 3.3 shows a PL extension of h_{il} :

$$h_{i(l+1)}\colon \bigcup_{k\le l+1} |K^+_{(i+1)\delta\gamma^k}| \cup \bigcup_{j\le i} |K^+_{j\delta\gamma_0}| \to \bigcup_{k\le l+1} |L^+_{(i+1)\delta\gamma^k}| \cup \bigcup_{j\le i} |L^+_{j\delta\gamma_0}|.$$

It is trivial that

$$h_{i(l+1)}(|K^+_{(i+1)\delta\gamma^{l+1}}|) = |L^+_{(i+1)\delta\gamma^{l+1}}|.$$

Thus by induction we obtain a required PL extension h_{i+1} of h_i, which completes the proof of 4.4.

4.10. REMARK. In 4.1, given a locally finite point set $\{a_1, \ldots\}$ in X, we can clearly choose h_t, $t \in [0,1]$, so that $h_t(a_i) = h(a_i)$ for any i.

5. Locally subanalytic manifolds. It is clear by definition that the set of all locally subanalytic homeomorphisms of open subsets of a Euclidean space is a pseudo-group. Denote it by Γ.

5.1. DEFINITION. A manifold with Γ-structure is called a *locally subanalytic manifold.* A continuous map $f\colon M_1 \to M_2$ of locally subanalytic manifolds is called *locally subanalytic* if for each pair $h_1\colon U_1 \to \mathbf{R}^n$ and $h_2\colon U_2 \to \mathbf{R}^m$ of coordinate systems of M_1 and M_2, $h_2 \circ f \circ h_1^{-1}\colon h_1(U_1 \cap f^{-1}(U_2)) \to \mathbf{R}^m$ is locally subanalytic.

5.2. REMARK. (i) A PL manifold is a locally subanalytic manifold. (ii) The composition of two subanalytic homeomorphisms is not necessarily subanalytic. Hence the set of subanalytic homeomorphisms of open subsets of $\mathbf{R}^n$ is not a pseudo-group. (iii) A topological manifold contained in $\mathbf{R}^n$ as a subanalytic subset is not necessarily a locally subanalytic manifold, e.g., Edwards' example of a polyhedron [**1**] which is homeomorphic to $\mathbf{R}^n$ but has no PL manifold structure is not a locally subanalytic manifold by 4.1 and 5.3.

5.3. THEOREM. *For any locally subanalytic manifold M, M is locally subanalytically homeomorphic to a PL manifold. If there are two locally subanalytic homeomorphisms $f_1\colon M \to N_1$ and $f_2\colon M \to N_2$ onto PL manifolds, $f_2 \circ f_1^{-1}\colon N_1 \to N_2$ is locally subanalytically isotopic to a PL homeomorphism.*

For the proof we need

5.4. LEMMA. *M is locally subanalytically homeomorphic to some closed subanalytic subset of a Euclidean space.*

PROOF. The case of compact M: Clearly we have a finite open covering $\{U_i\}$ of M and a locally subanalytic homeomorphism $h_i\colon U_i \to \mathbf{R}^n$ for each i. We can assume that $\{h_i^{-1}(O)\}$ is a covering of M where $O = \{x \in \mathbf{R}^n \mid |x| < 1\}$. Let $\varphi\colon \mathbf{R}^n \to \mathbf{R}^{n+1}$ be a subanalytic map such that $\varphi|_O\colon O \to \varphi(O)$ is a homeomorphism and $\varphi(\mathbf{R}^n - O)$ is a point. Let $H_i\colon M \to \mathbf{R}^{n+1}$ be the extension of $\varphi \circ h_i$ defined by $H_i(M - U_i) = \varphi(\mathbf{R}^n - O)$. Then $\prod_i H_i(M)$ is a subanalytic subset of a Euclidean space, and $\prod_i H_i$ is a subanalytical homeomorphism.

The noncompact case: The case is shown in the same way as 2.10 in [**11**] (a C^∞ imbedding of a C^∞ manifold in a Euclidean space).

PROOF OF 5.3. By 2.2 and 5.4, M is locally subanalytically homeomorphic to a polyhedron N. We need only prove that N is a PL manifold. For any point x

of N, by Definition 5.1 there is an open neighborhood U of x locally subanalytically homeomorphic to $\mathbf{R}^n$. As U itself is a polyhedron, 4.1 shows that U is PL homeomorphic to $\mathbf{R}^n$. Hence U and, consequently, N are PL manifolds.

6. Semialgebraic manifolds. See [**7**] for the fundamental properties of semialgebraic sets. Let $\mathrm{St}(x, K)$ denote the open star of x in a simplicial complex K.

6.1. DEFINITION. A subset of $\mathbf{R}^n$ is called *semialgebraic* if it is a finite union of sets of the form

$$\{x \in \mathbf{R}^n \mid f_1 x) > 0, \dots, f_k(x) > 0,\ f_{k+1}(x) = 0, \dots, f_l(x) = 0\},$$

where f_i are polynomials on $\mathbf{R}^n$. A continuous map of semialgebraic sets is called *semialgebraic* if the graph is semialgebraic.

For example, a compact polyhedron in a Euclidean space is a semialgebraic set, and a PL map of compact polyhedrons in Euclidean spaces is semialgebraic. We can define a *semialgebraic map* of compact polyhedrons without particular imbedding of the polyhedrons in Euclidean spaces (see 2.1).

Let Γ denote the pseudo-group of all semialgebraic homeomorphisms of semialgebraic open subsets of a Euclidean space. A manifold with Γ-structure is called a *semialgebraic manifold* if there is a finite system of coordinate neighborhoods. A *semialgebraic* map of semialgebraic manifolds is defined in the same way as a locally subanalytic map (5.1).

A compact PL manifold or the interior of a compact PL manifold with boundary is a semialgebraic manifold. Conversely, we have

6.2. THEOREM. *Let M be a semialgebraic manifold. Then there exists a compact PL manifold possibly with boundary in some Euclidean space whose interior is semialgebraically homeomorphic to M.*

PROOF. Assume M is not compact, because the compact case follows more easily. Let $h_i\colon U_i \to \mathbf{R}^n$, $U_i \subset M$, $i = 1, \dots, k$, be a system of coordinate neighborhoods of M. For each i put

$$\varphi_i(x) = 1/\mathrm{dist}(x, \overline{h_i(U_i)} - h_i(U_i)), \qquad x \in h_i(U_i),$$
$$h'_i = (h_i, \varphi_i \circ h_i).$$

Then φ_i and, hence, h'_i are semialgebraic, h'_i is a homeomorphism from U_i onto the image, and the image is closed in $\mathbf{R}^{n+1}$. Recall that $\mathbf{R}^{n+1}$ is algebraically homeomorphic to S^{n+1}-*a point* by the stereographic projection. Hence we have a semialgebraic map $h''_i\colon U_i \to \mathbf{R}^{n+2}$ such that $h''_i\colon U_i \to h''_i(U_i)$ is a homeomorphism, $h''_i(U_i)$ is bounded in $\mathbf{R}^{n+2}$, and for every point sequence $\{x_j\}$ of U_i such that any subsequence does not converge in U_i, $\{h''_i(x_j)\}$ converges to a point $a_i \notin h''_i(U_i)$. By the last condition we can extend h''_i to M by putting $h''_i(x) = a_i$ for $x \notin U_i$. Let $\tilde{h}''_i$ be the extension, and put $h = \prod_{i=1}^k \tilde{h}''_i\colon M \to \mathbf{R}^{k(n+2)}$. Then h is semialgebraic and a homeomorphism onto the image, $h(M)$ is bounded and semialgebraic, and $\overline{h(M)} - h(M)$ is a point. Hence we can reduce the problem to the case where M is contained in $\mathbf{R}^m$ as a bounded semialgebraic subset and where $\overline{M} - M$ is a point a.

By Theorem 3 of [**6**] we have a compact polyhedron $X \subset \mathbf{R}^m$ and a semialgebraic homeomorphism $f\colon \overline{M} \to X$. The proof of 5.3 says that $X - f(a)$ is a PL manifold.

Let K be a rectilinear triangulation of X such that $f(a) \in K$, and K' is the barycentric subdivision of K. Then $X - \mathrm{St}(f(a), K')$ is a compact PL manifold with boundary whose interior is semialgebraically homeomorphic to M.

In 6.2 we denote by $C(M)$ the PL manifold possibly with boundary.

6.3. THEOREM. *Let M_1, M_2 be semialgebraic manifolds. The following are equivalent.*

(i) *M_1, M_2 are semialgebraically homeomorphic.*

(ii) *$C(M_1)$, $C(M_2)$ are semialgebraically homeomorphic.*

(iii) *$C(M_1)$, $C(M_2)$ are PL homeomorphic.*

PROOF. (iii)⇒(ii)⇒(i) are trivial. Hence we prove (i)⇒(iii). We assume M_1 and M_2 are not compact, because the compact case is trivial by 4.1. At first we imbed $C(M_1)$, $C(M_2)$ in $\mathbf{R}^m$ for some m so that there are cones N_1, N_2 with vertexes $a_1, a_2 \in \mathbf{R}^m$ and bases $\partial C(M_1)$, $\partial C(M_2)$, respectively, such that $C(M_1) \cap N_1 = \partial C(M_1)$ and $C(M_2) \cap N_2 = \partial C(M_2)$. Then we have PL maps ψ_i: $C(M_1) \to C(M_1) \cup N_1$ and ψ_2: $C(M_2) \to C(M_2) \cup N_2$ such that $\psi_1|_{C(M_1)-\partial C(M_1)}$ and $\psi_2|_{C(M_2)-\partial C(M_2)}$ are homeomorphisms onto $C(M_1) \cup N_1 - a_1$ and $C(M_2) \cup N_2 - a_2$, respectively, and $\psi_1(\partial C(M_1)) = a_1$ and $\psi_2(\partial C(M_2)) = a_2$. As ψ_1 and ψ_2 are semialgebraic, so are $\psi_1|_{C(M_1)-\partial C(M_1)}$ and $\psi_2|_{C(M_2)-\partial C(M_2)}$. Hence (i) implies a semialgebraic homeomorphism h: $C(M_1) \cup N_1 - a_1 \to C(M_2) \cup N_2 - a_2$.

Clearly the extension $\tilde{h}$: $C(M_1) \cup N_1 \to C(M_2) \cup N_2$ of h defined by $\tilde{h}(a_1) = a_2$ is a homeomorphism, and its graph is the closure of the graph of h. Now the closure of a semialgebraic set is semialgebraic [**7**]. Accordingly $\tilde{h}$ is semialgebraic. Apply 4.1 and 4.10 to $\tilde{h}$. Then we have rectilinear triangulations K_1, K_2 of $C(M_1) \cup N_1, C(M_2) \cup N_2$, respectively, and a linear isomorphism (3.1) $g: K_1 \to K_2$ such that $a_1 \in K_1$, $a_2 \in K_2$ and $g(a_1) = a_2$. Hence there is a PL homeomorphism from $C(M_1) \cup N_1 - \mathrm{St}(a_1, K_1)$ to $C(M_2) \cup N_2 - \mathrm{St}(a_2, K_2)$ because of $g(\mathrm{St}(a_1, K_1)) = \mathrm{St}(a_2, K_2)$. It is clear that $C(M_1)$ and $C(M_2)$ are PL homeomorphic to $C(M_1) \cup N_1 - \mathrm{St}(a_1, K_1)$ and $C(M_2) \cup N_2 - \mathrm{St}(a_2, K_2)$ respectively. Therefore $C(M_1)$ and $C(M_2)$ are PL homeomorphic.

6.4. COROLLARY. *Let $\mathcal{S}$ be the semialgebraic homeomorphism classes of all semialgebraic manifolds, $\mathcal{P}$ the PL homeomorphism classes of all compact PL manifolds possibly with boundary, and $\mathcal{C}: \mathcal{S} \to \mathcal{P}$ the map induced by $C(\ )$. Then $\mathcal{C}$ is one-to-one and onto.*

PROOF. Trivial by 6.2 and 6.3.

6.5. REMARK. Let $\mathcal{P}'$ be the PL homeomorphism classes of interiors of compact PL manifolds possibly with boundary, and $i: \mathcal{P} \to \mathcal{P}'$ the natural map. Then i is not one-to-one [**10**]. In other words there are two compact PL manifolds with boundary in a Euclidean space whose interiors are PL homeomorphic but semialgebraically homeomorphic.

6.6. THEOREM. *Let M, N be semialgebraic manifolds, f: $M \to N$ a semialgebraic map, and h: $N \to C(N) - \partial C(N)$ a semialgebraic homeomorphism* (6.2). *Then we have a semialgebraic homeomorphism g: $M \to C(M) - \partial C(M)$ such that $h \circ f \circ g^{-1}$ is extensible to $C(M) \to C(N)$ as a semialgebraic map.*

PROOF. Imbedding $C(N)$ in a Euclidean space, we may assume that $N = \mathbf{R}^k$ and $f(M)$ is bounded, and it is sufficient to prove that $f \circ g^{-1}$ is extensible to $C(M)$.

Moreover, by 6.2 we can assume that M is contained in $\mathbf{R}^m$ as a semialgebraic bounded subset. Let M_1 be the graph of f, and p_1, p_2 be the projections of M_1 to $\mathbf{R}^m$, $\mathbf{R}^k$ respectively. Then by [**6**] we have a compact polyhedron $X \subset \mathbf{R}^m \times \mathbf{R}^k$, a subpolyhedron Y and a semialgebraic homeomorphism h: $(X, Y) \to (\overline{M}_1, \overline{M}_1 - M_1)$. In the proof of 6.2 we have already shown that $X - Y$ is a PL manifold. Let (K, L) be a rectilinear triangulation of (X, Y) such that L is a full subcomplex of K (see [**12**] for the definition). Define a simplicial map η: $K \to \{[0,1], 0, 1\}$ by putting $\eta(v) = 0$ for vertexes $v \in L$ and $\eta(v) = 1$ for other vertexes. Then [**12**, 3.10] and the proof of [**12**, 3.8] tell us that $|\eta^{-1}([\varepsilon, 1])|$ for any $0 < \varepsilon < 1$ is a compact PL manifold with boundary $|\eta^{-1}(\varepsilon)|$. It is easy to see also that there is a PL map χ: $|\eta^{-1}([\varepsilon, 1])| \to X$ such that $\chi|_{|\eta^{-1}((\varepsilon,1])|}$ is a homeomorphism onto $X - Y$. Then $p_1 \circ h \circ \chi|_{|\eta^{-1}((\varepsilon,1])|}$ is a semialgebraic homeomorphism onto M, and $f \circ p_1 \circ h \circ \chi|_{|\eta^{-1}((\varepsilon,1])|}$ is extensible to $|\eta^{-1}([\varepsilon, 1])|$ because of $f \circ p_1 \circ h \circ \chi = p_2 \circ h \circ \chi$ on $|\eta^{-1}((\varepsilon, 1])|$. Hence, $g = (p_1 \circ h \circ \chi)^{-1}$ is a required homeomorphism.

7. Semialgebraic polyhedrons and semialgebraic homeomorphisms.

7.1. DEFINITION. A set of the form

$$\{x \in \mathbf{R}^n \mid f_1(x) \geq 0, \ldots, f_k(x) \geq 0, f_{k+1}(x) = 0, \ldots, f_l(x) = 0\},$$

where f_i are linear functions, is called a *cell* (we do not assume compactness as usual). We define a *cell complex* in the same way as the case of compact cells.

7.2. LEMMA. *Let $X \subset \mathbf{R}^n$ be a finite union of cells. Then the closure of a connected component of $\mathbf{R}^n - X$ is also a finite union of cells.*

PROOF. There is a finite cell complex K such that $|K| = \mathbf{R}^n$ and X is a union of cells of K for the following reason. Let X be the union of cells $X_1, \ldots, X_m$, and let each X_i be described by linear functions $f_{i1}, \ldots, f_{il}$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a partition of $\{1, \ldots, m\} \times \{1, \ldots, l\}$. Put

$$X_\alpha = \cap\{f_{ij} \leq 0 \text{ for } (i,j) \in \alpha_1\} \cap \{f_{ij} = 0 \text{ for } (i,j) \in \alpha_2\}$$
$$\cap\{f_{ij} \geq 0 \text{ for } (i,j) \in \alpha_3\}.$$

Then the collection K of all X_α satisfies the required properties (see [**11**, 7.3, 7.5]). Since a cell is closed and connected, and since $\mathbf{R}^n$ is the disjoint union of all open cells of K, a connected component of $\mathbf{R}^n - X$ is a union of open cells of K. Hence, the lemma follows.

We prove in the same way

7.3. LEMMA. *Let $X_1, \ldots, X_m$ be cells in $\mathbf{R}^n$. Then we have a finite cell complex K such that $|K| = \bigcup_{i=1}^m X_i$ and each X_i is a union of cells of K.*

7.4. THEOREM. *Let $X, Y \subset \mathbf{R}^n$ be semialgebraic polyhedrons and f: $X \to Y$ a semialgebraic PL map. If X is closed in $\mathbf{R}^n$, it is a finite union of cells X_i. In the nonclosed case, X is a finite union of connected components X_i of sets of the form $C - (\overline{X} - X)$, C being cells. We can choose such decompositions $\{X_i\}$ and $\{Y_i\}$ of X and Y, respectively, so that for each i, $f(X_i) \subset Y_j$ and $f|_{X_i}$ is linear for some j.*

PROOF. We treat only the case where X and Y are closed in $\mathbf{R}^n$, because the nonclosed case needs more complicated notation. Let the dimension of X be r. We prove the first statement of the theorem by induction on r. If $r = 0$, it is trivial, so we assume the statement for dimension $< r$. Let K be a rectilinear triangulation of X, and X' the singular set of X, namely, consisting of points where the germ of X is of dimension $< r$ or not C^∞ smooth. Clearly X' is closed, and we know it is semialgebraic [**7**]. Let $\sigma \in K$, and assume a point of $\operatorname{Int}\sigma$ is contained in X'. Then, by the homogeneity of X at $\operatorname{Int}\sigma$, any point of $\operatorname{Int}\sigma$ is contained in X', which means X' is a subpolyhedron of X. Hence, by the induction hypothesis X' is a finite union of cells.

Now we will see that the closure of each connected component of $X - X'$ is a finite union of cells. Let W be such a component. Since W is at once a connected C^∞ manifold of dimension r and a union of open simplexes of K, we have a plane P of dimension r containing W. Hence, W is a connected component of $P - X'$. Obviously, $P \cap X'$ is a semialgebraic polyhedron of dimension $< r$. Therefore, by the induction hypothesis and 7.2, the closure of W is a finite union of cells. Now the number of connected components of a semialgebraic set is finite. Hence X is a finite union of cells.

For the last statement, consider the graph Z of f. As Z is a closed semialgebraic polyhedron in $\mathbf{R}^n \times \mathbf{R}^n$, the first statement implies a cell decomposition $\{Z'_i\}$ of Z. Let p_1 or p_2 be the projection of $\mathbf{R}^n \times \mathbf{R}^n$ to the first or second factor, respectively, and put $Y'_i = p_2(Z'_i)$ for each i. Now we can prove in the same way as the compact cell case that Y'_i are cells. By 7.3 we have a cell decomposition $\{Y_i\}$ of Y such that each Y'_i is a union of some Y_i's. Put $\{Z_i\} = \{Z'_j \cap P_2^{-1}(Y_k)\}$ and $X_i = p_1(Z_i)$ for each i. Then $\{X_i\}$ is a cell decomposition of X, each $f(X_i)$ is contained in some one Y_j, and $f|_{X_i}$ is linear. Hence the theorem is proved.

7.5. THEOREM. *Let $X_1, X_2 \subset \mathbf{R}^n$ be compact polyhedrons. If $X_1 \times \mathbf{R}$, $X_2 \times \mathbf{R} \subset \mathbf{R}^n \times \mathbf{R}$ are semialgebraically homeomorphic, X_1 and X_2 are PL homeomorphic.*

PROOF. Since $\mathbf{R}$ is semialgebraically homeomorphic to $(0,1) \subset \mathbf{R}$, $X_1 \times (0,1)$ and $X_2 \times (0,1) \subset \mathbf{R}^n \times \mathbf{R}$ are semialgebraically homeomorphic. Let Y_1, Y_2 be open cones in $\mathbf{R}^n \times \mathbf{R}$ with vertex $0 = 0 \times 0$ and bases $X_1 \times 1$, $X_2 \times 1$ respectively. Then we have natural semialgebraic homeomorphisms from $X_1 \times (0,1)$, $X_2 \times (0,1)$ to $Y_1 - 0$, $Y_2 - 0$ respectively. Hence there is a semialgebraic homeomorphism $f\colon Y_1 - 0 \to Y_2 - 0$. Here we can assume without loss of generality that $f(x) \to 0$ as $x \to 0$. Let $\tilde{f}$ be the extension of f to Y_1 defined by $\tilde{f}(0) = 0$. Then we see in the same way as the proof of 6.2 that $\tilde{f}$ is semialgebraic. Hence, by 4.1 and 4.10, $(Y_1, 0)$ and $(Y_2, 0)$ are PL homeomorphic. Now the links of 0 in rectilinear triangulations of Y_1 and Y_2 are PL homeomorphic to X_1 and X_2 respectively. Therefore the theorem follows.

7.6. EXAMPLE. In 7.5, if X_1 and X_2 are not compact, they are not necessarily PL homeomorphic as follows. Let Y be Mazur's example [**9**] of a compact contractible PL manifold with boundary of dimension 4 whose boundary is not simply connected, and let X_1 be the interior of Y. Assume Y is PL imbedded in $\mathbf{R}^n$. Let X_2 be an open 4-simplex in $\mathbf{R}^n$. Then it is known that $Y \times [0,1]$ and $\overline{X}_2 \times [0,1]$ are PL and, hence, semialgebraically homeomorphic. Hence $X_1 \times (0,1)$, $X_2 \times (0,1) \subset \mathbf{R}^n \times \mathbf{R}$ are semialgebraically homeomorphic. But X_1 and X_2 are not homeomorphic, because X_1 is not simply connected at infinity.

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