

## EXACT SEQUENCES IN STABLE HOMOTOPY PAIR THEORY

BY

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**ABSTRACT.** A cylinder-web diagram with associated diagonal sequences is described in stable homotopy pair theory. The diagram may be used to compute stable homotopy pair groups and also stable track groups of two-cell complexes. For the stable Hopf class  $\eta$  the stable homotopy pair groups  $G_k(\eta, \eta)$  ( $k \leq 8$ ) are computed together with some of the additive structure of the stable homotopy ring of the complex projective plane.

**0. Introduction.** Let  $\alpha \in G_r$ ,  $\beta \in G_m$  denote stable classes of maps between spheres and let

$$\pi_k = \operatorname{Dir} \lim_{n \rightarrow \infty} \left[ S^{n+k} \cup_{\alpha} e^{n+r+k+1}, S^n \cup_{\beta} e^{n+m+1} \right]$$

denote the corresponding stable track group. The groups  $\pi_r$  have been studied by N. Yamamoto [13] in the case  $\alpha = \beta = p \neq 2$  and by J. Mukai [8, 9] in the case  $\alpha = \beta = 2$ . In these papers rather extensive computations are given, but they use special information concerning the stable homotopy of Moore spaces. Here we develop a technique for computation which relies on the (stable) Puppe and dual-Puppe sequences passing through  $\pi_k$  but which attempts to resolve the problems of group extension through the additional information contained in a cylinder-web diagram. Besides the rectangular mesh of Puppe and dual-Puppe sequences the cylinder-web diagram has three diagonal sequences (Theorem 3.3) that pass through the stable homotopy pair groups  $G_k(\alpha, \beta)$ . These groups are the natural "home" for (stable) Toda brackets of the form  $\langle \beta, \gamma, \alpha \rangle$  in the sense that the brackets live here with zero indeterminacy.

In §6 the technique is applied to compute, for the stable Hopf class  $\eta$ , the stable homotopy pair groups  $G_k(\eta, \eta)$  ( $k \leq 8$ ) and the stable track groups  $\pi_k$  ( $k \leq 8$ ) of the complex projective plane. It is interesting that besides knowledge of the stable homotopy groups of spheres, including composition and secondary composition operations, the method also requires information concerning the third order composition (quaternary Toda brackets). The computation presented as an illustration encountered a difficulty at  $\pi_9$ ; however, it seems reasonable to expect that a better understanding of the quaternary bracket will enable it to be continued.

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**1. Preliminaries.** Let  $f: X \rightarrow Y$ ,  $g: E \rightarrow B$  be pointed continuous maps (i.e. *pairs* in the sense of Eckmann and Hilton [1]). We recall that the morphism set  $\pi(f, g)$  in the category HPC of homotopy pairs and homotopy pair classes [3, 4] is obtained from the set of tracks from  $f$  to  $g$  by factoring out by the equivalence relation:

$$\begin{array}{ccccc}
 & & X & \xrightarrow{\psi_1} & E \\
 & & \parallel & \uparrow \{\psi_i\} & \parallel \\
 & & X & \xrightarrow{\psi_0} & E \\
 f \downarrow & \uparrow \{h_i\} & & \uparrow \{h_i\} & \downarrow g \\
 Y & \xrightarrow{\phi_0} & B & & B \\
 & \parallel & \parallel & \uparrow \{\phi_i\} & \parallel \\
 & & Y & \xrightarrow{\phi_1} & B
 \end{array}$$

(The square referred to on the right is the composite in the obvious sense of the three internal squares.) The equivalence class of the element in  $\pi(f, g)$  will be denoted  $\{\phi, \psi, h_i\}$  unless  $h_i$  is a constant homotopy in which case  $\{\phi, \psi\}$  will be used.

If  $X$  is a space,  $*X$ ,  $X*$  and  $X$ , respectively, will denote the inclusion of the basepoint  $*$  into  $X$ , the projection of  $X$  onto  $*$  and the identity map from  $X$  to  $X$ . Similarly, it will be convenient to use the notation  $X|f$ ,  $*X|f$ ,  $f|Y$ ,  $(f, *)$  and  $(*, f)$ , respectively, for the commutative squares:

$$\begin{array}{ccc}
 \begin{array}{ccc} X & \xrightarrow{X} & X \\ \parallel & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array} & \begin{array}{ccc} * & \xrightarrow{\quad} & X \\ \downarrow *X & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array} & \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow f & & \parallel \\ Y & \xrightarrow{Y} & Y \end{array} \\
 & & Y, \\
 \begin{array}{ccc} * & \xrightarrow{\quad} & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} & \text{and} & \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ * & \xrightarrow{\quad} & * \end{array}
 \end{array}$$

Following Puppe [10] we denote by  $Pf: Y \rightarrow C_f$  the inclusion of  $\text{codomain}(f)$  into  $\text{cofibre}(f)$ . We recall from [4] that  $P$  becomes an endofunctor of HPC if we set  $P\{\phi, \psi, h_i\} = \{\chi, \phi\}$ , where  $\chi = C(\psi, \phi, h_i): C_f \rightarrow C_g$  is the map defined by the rule [10, (9)]. It was proved in [4] that  $P^3$  is naturally isomorphic to  $\Sigma$  in HPC and that the Puppe operator  $P: \pi(f, g) \rightarrow \pi(Pf, Pg)$  eventually stabilizes. Let  $Qf: C_f \rightarrow \Sigma X$  be the map that shrinks to  $*$  the subset  $Y$  of  $C_f$  as described in [10, p. 308]. With morphisms defined as suggested [10, p. 312],  $Q$  becomes an endofunctor of HPC, naturally isomorphic to  $P^2$ . We denote by  $G_k(f, g)$  the stable group corresponding to  $\pi(\Sigma^k f, g)$ . As shown in [3],  $\pi(f, g)$  is an invariant of the homotopy classes of  $f$  and  $g$ . Hence  $G_k(f, g)$  is an invariant of the stable homotopy classes of  $f$  and  $g$ .

**2. The stable cylinder-web diagram.** Consider the following Diagram (2.1) of homotopy sets and induced functions. In the diagram,  $.f$  is the function induced by precomposition with the map  $f$ , and  $g.$  is the function induced by postcomposition

with the map  $g$ . The horizontal sequences are all exact, being portions of the Puppe sequences of  $f$  at  $E$ ,  $B$  or  $C_g$ . In the stable range, i.e. when  $\Sigma$  is bijective, the vertical sequences are exact, being instances of (stable) dual-Puppe sequences of  $g$ . Since all horizontal arrows are induced by precomposition and all vertical arrows are induced by postcomposition, the rectangles not of the types indicated by (A), (B) and (C) are all commutative. Naturality of the Puppe sequence and naturality of the suspension operator ensures that the rectangles of type (C) are commutative.

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 \dots & [\Sigma^2 X, B] & \xrightarrow{\cdot \Sigma Q f} & [\Sigma C_f, B] & \xrightarrow{\cdot \Sigma P f} & [\Sigma Y, B] & \xrightarrow{\cdot \Sigma f} & [\Sigma X, B] \\
 & \downarrow Pg. & & \downarrow Pg. & & \downarrow Pg. & & \downarrow Pg. \\
 \dots & [\Sigma^2 X, C_g] & \xrightarrow{\cdot \Sigma Q f} & [\Sigma C_f, C_g] & \xrightarrow{\cdot \Sigma P f} & [\Sigma Y, C_g] & \xrightarrow{\cdot \Sigma f} & [\Sigma X, C_g] \\
 & \downarrow \Sigma^{-1} Q g. & & \downarrow \Sigma^{-1} Q g. & & \downarrow \Sigma^{-1} Q g. & & \downarrow \Sigma^{-1} Q g. \\
 \dots & [\Sigma X, E] & \xrightarrow{\cdot Q f} & [C_f, E] & \xrightarrow{\cdot P f} & [Y, E] & \xrightarrow{\cdot f} & [X, E] \\
 & \downarrow g. & & \downarrow g. & & \downarrow g. & & \downarrow g. \\
 \dots & [\Sigma X, B] & \xrightarrow{\cdot Q f} & [C_f, B] & \xrightarrow{\cdot P f} & [Y, B] & \xrightarrow{\cdot f} & [X, B] \\
 & \downarrow Pg. & & \downarrow Pg. & & \downarrow Pg. & & \downarrow Pg. \\
 \dots & [\Sigma X, C_g] & \xrightarrow{\cdot Q f} & [C_f, C_g] & \xrightarrow{\cdot P f} & [Y, C_g] & \xrightarrow{\cdot f} & [X, C_g]
 \end{array}$$

DIAGRAM 2.1

2.2. REMARK. There is a natural homeomorphism  $\lambda: \Sigma C_f \rightarrow C_{\Sigma f}$  inducing a diagram (for each space  $E$ )

$$\begin{array}{ccccc}
 [\Sigma^2 X, E] & \xrightarrow{\cdot Q \Sigma f} & [C_{\Sigma f}, E] & \xrightarrow{\cdot P \Sigma f} & [\Sigma Y, E] \\
 \parallel & & \downarrow \lambda & & \parallel \\
 [\Sigma^2 X, E] & \xrightarrow{\cdot \Sigma Q f} & [\Sigma C_f, E] & \xrightarrow{\cdot \Sigma P f} & [\Sigma Y, E]
 \end{array}$$

in which the right-hand square is commutative, but in which the left-hand square (D) is anticommutative through interchange of suspension coordinates in  $\Sigma^2 X$ . Hence identifying via  $\lambda$ , the rectangles (B) are commutative and the rectangles (A) are anticommutative. We define

$$\lambda_n: \Sigma^n C_f \rightarrow C_{\Sigma^n f} \quad \text{by} \quad \lambda_1 = \lambda, \lambda_n = \lambda_1 \cdot \Sigma \lambda_{n-1}.$$

One may now observe in Diagram 2.1 that objects and arrows separated by three diagonal shifts in a direction of positive slope are identical. (Two such arrows have been highlighted.) We have therefore another example of what has been described as a cylinder-web diagram [2, 3, 5]. The suspension operator provides a morphism of Diagram 2.1 into the corresponding diagram in which each arrow induced by a map

is replaced by the arrow induced by the suspension of that map. Taking direct limit with respect to suspension yields the stable cylinder-web diagram.

$$\begin{array}{ccccccc}
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \rightarrow G_{n+1}(X, E) & \xrightarrow{Qf} & G_n(C_f, E) & \xrightarrow{Pf} & G_n(Y, E) & \xrightarrow{f} & G_n(X, E) \rightarrow \\
 \downarrow g. & & \downarrow g. & & \downarrow g. & & \downarrow g. \\
 \rightarrow G_{n+1}(X, B) & \xrightarrow{Qf} & G_n(C_f, B) & \xrightarrow{Pf} & G_n(Y, B) & \xrightarrow{f} & G_n(X, B) \rightarrow \\
 \downarrow Pg. & & \downarrow Pg. & & \downarrow Pg. & & \downarrow Pg. \\
 \rightarrow G_{n+1}(X, C_g) & \xrightarrow{Qf} & G_n(C_f, C_g) & \xrightarrow{Pf} & G_n(Y, C_g) & \xrightarrow{f} & G_n(X, C_g) \rightarrow \\
 \downarrow Qg. & & \downarrow Qg. & & \downarrow Qg. & & \downarrow Qg. \\
 \rightarrow G_n(X, E) & \xrightarrow{Qf} & G_{n-1}(C_f, E) & \xrightarrow{Pf} & G_{n-1}(Y, E) & \xrightarrow{f} & G_{n-1}(X, E) \rightarrow \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow
 \end{array}$$

DIAGRAM 2.3

**2.4. REMARK.** The objects and arrows of 2.3 are invariants of the stable homotopy class of the map  $f: X \rightarrow Y$ . In the case of stable homotopy classes from  $X$  to  $Y$  which do not have representative maps from  $X$  to  $Y$ , Diagram 2.3 is still obtained as a direct limit over a cofinal subset. Hence  $f$  and  $g$  may be permitted to be virtual maps.

**3. The diagonal sequences.** The *domain* and *codomain* operators  $d: \pi(f, g) \rightarrow [X, E]$  and  $c: \pi(f, g) \rightarrow [Y, B]$  are defined by the rules  $d\{\phi, \psi, h_i\} = \{\psi\}$  and  $c\{\phi, \psi, h_i\} = \{\phi\}$ . Recalling [4, §5] that  $\Sigma\{\phi, \psi, h_i\} = \{\Sigma\phi, \Sigma\psi, \Sigma h_i\}$ , it follows that  $c$  and  $d$  are compatible with  $\Sigma$  and hence define stable operators  $c: G_n(f, g) \rightarrow G_n(X, E)$  and  $d: G_n(f, g) \rightarrow G_n(Y, B)$ . Further, recalling [5, §4] that the operator  $\Delta = \Delta(f, g): [Y, E] \rightarrow \pi(f, g)$  is defined by the rule  $\Delta\{h\} = \{gh, hf\}$ , it is also clear that  $\Delta$  is compatible with  $\Sigma$ , giving rise to an operator  $\Delta: G_n(Y, E) \rightarrow G_n(f, g)$ . Note that when the operators  $P$  and  $Q$  are bijective (as they are in the stable range [4]) the following composites are defined:

$$(3.1) \quad P^{-1}\Delta(P\Sigma^n f, Pg)(\cdot\lambda_n^{-1}): [\Sigma^n C_f, B] \rightarrow [\Sigma^n f, g],$$

$$(3.2) \quad Q^{-1}\Delta(Q\Sigma^n f, Qg)(\cdot\lambda_n^{-1}): [\Sigma^{n+1} X, C_g] \rightarrow [\Sigma^n f, g].$$

In the course of the proof of Theorem 3.3 it will be shown that (3.1) and (3.2) are compatible with  $\Sigma$  so that stable operators

$$\Delta': G_n(C_f, B) \rightarrow G_n(f, g) \quad \text{and} \quad \Delta'': G_{n+1}(X, C_g) \rightarrow G_n(f, g)$$

are also defined. A stable operator  $\chi: G_r(f, g) \rightarrow G_r(C_f, C_g)$  is obtained if we set  $\chi = \lambda_n^{-1} \cdot \lambda_n cP: \pi(\Sigma^m f, \Sigma^n g) \rightarrow [\Sigma^m C_f, \Sigma^n C_g]$ .

To check the compatibility with  $\Sigma$ , observe commutativity in the following diagram:

$$\begin{array}{ccccc}
 \pi(\Sigma^m f, \Sigma^n g) & \xrightarrow{\Sigma} & & \pi(\Sigma^{m+1} f, \Sigma^{n+1} g) & \\
 \downarrow cP & & (A) & \downarrow cP & \\
 [C_{\Sigma^m f}, C_{\Sigma^n g}] & \xrightarrow{\Sigma} & [\Sigma C_{\Sigma^m f}, \Sigma C_{\Sigma^n g}] & \xleftarrow{\lambda^{-1} \cdot \lambda} & [C_{\Sigma^{m+1} f}, C_{\Sigma^{n+1} g}] \\
 \downarrow \lambda_n^{-1} \cdot \lambda_n & & \searrow \Sigma \lambda_n^{-1} \cdot \Sigma \lambda_n & & \downarrow \lambda_{n+1}^{-1} \cdot \lambda_{n+1} \\
 [\Sigma^m C_f, \Sigma^n C_g] & \xrightarrow{\Sigma} & & & [\Sigma^{m+1} C_f, \Sigma^{n+1} C_g]
 \end{array}$$

To verify the commutativity in (A), it is sufficient to consider  $m = n = 0$  and to verify the desired equality directly from the definition of the  $P$  operator going back to [10, (9)].

Finally we shall need to refer to the diagonal composites

$$\begin{aligned}\beta &= g..Qf: G_{n+1}(X, E) \rightarrow G_n(C_f, B), \\ \beta' &= Pg..f: G_{n+1}(Y, B) \rightarrow G_{n+1}(X, C_g)\end{aligned}$$

and

$$\beta'' = Qg..Pf: G_{n+1}(C_f, C_g) \rightarrow G_n(Y, E).$$

**3.3. THEOREM.** *The following sequences are exact:*

$$(3.4) \quad \cdots \rightarrow G_{n+1}(f, g) \xrightarrow{d} G_{n+1}(X, E) \xrightarrow{\beta} G_n(C_f, B) \xrightarrow{\Delta'} G_n(f, g) \rightarrow \cdots,$$

$$(3.5) \quad \cdots \rightarrow G_{n+1}(f, g) \xrightarrow{c} G_{n+1}(Y, B) \xrightarrow{\beta'} G_{n+1}(X, C_g) \xrightarrow{\Delta''} G_n(f, g) \rightarrow \cdots,$$

$$(3.6) \quad \cdots \rightarrow G_{n+1}(f, g) \xrightarrow{x} G_{n+1}(C_f, C_g) \xrightarrow{\beta''} G_n(Y, E) \xrightarrow{\Delta} G_n(f, g) \rightarrow \cdots.$$

**PROOF.** The sequence (3.4) is a stable version of the exact sequence [5, 4.3]

$$\cdots \rightarrow \pi(\Sigma^{n+1}f, g) \xrightarrow{d} [\Sigma^{n+1}X, E] \xrightarrow{\beta} [\Sigma^n C_f, B] \xrightarrow{\delta} \pi(\Sigma^n f, g) \rightarrow \cdots,$$

where  $\delta$  (for  $n \geq 0$ ) is given by the following composition:

$$\begin{array}{c} [\Sigma^n C_f, B] \xleftarrow{\lambda_n} [C_{\Sigma^n f}, B] \xleftarrow{cP} \pi(\Sigma^n f, *B) \xrightarrow{*B|g.} \pi(\Sigma^n f, g) \\ \underbrace{\hspace{10em}}_{\delta} \end{array}$$

The argument for compatibility of  $\delta$  with suspension is similar to that given above for the operator  $\chi$ . To check that  $P^{-1}\Delta(P\Sigma^n f, Pg)(\lambda_n^{-1})\{k\} = \delta\{k\}$ , first observe that  $\Delta(f, g)\{h\} = \{(E|g)(h, h)(f|Y)\}$ , so that

$$\begin{aligned}P^{-1}\Delta(P\Sigma^n f, Pg)(\lambda_n^{-1})\{k\} &= P^{-1}\{(B|Pg)(k\lambda_n^{-1}, k\lambda_n^{-1})(P\Sigma^n f|C_{\Sigma^n f})\} \\ &= *B|g.P^{-1}\{(k\lambda_n^{-1}, k\lambda_n^{-1})(P\Sigma^n f|C_{\Sigma^n f})\}.\end{aligned}$$

It is now sufficient to note that  $(\lambda_n)c\{(k\lambda_n^{-1}, k\lambda_n^{-1})(P\Sigma^n f|C_{\Sigma^n f})\} = \{k\}$ .

The sequence (3.5) has essentially been given in [4, 8.1]. However, this derivation is inconvenient for the purpose of recognition of the arrows. We sketch briefly an alternative. One may begin with the sequence

$$(3.7) \quad \begin{aligned} \cdots \rightarrow \pi((\Sigma^n Y)*, *B) &\xrightarrow{*B|g.(\cdot, \Sigma^n f)} \pi((\Sigma^n X)*, g) \xrightarrow{\Sigma^n f|(\Sigma^n X)*} \\ &\pi(\Sigma^n f, g) \xrightarrow{c} [\Sigma^n Y, B] \rightarrow \cdots \end{aligned}$$

The exactness can be verified directly, essentially dualising the proof of [5, Proposition 4.3]. In the form (3.7) the compatibility of the arrows with  $\Sigma$  is obvious. The bijection

$$\chi: \pi((\Sigma^n Y)*, *B) \rightarrow [\Sigma^{n+1}Y, B]$$



Hence

$$\begin{aligned} P\Delta(.Pf)\{h\} &= \{(B|Pg)(gh\lambda_n^{-1}, gh\lambda_n^{-1})(P\Sigma^n f|C_{\Sigma^n f})\} \\ &= \Delta(P\Sigma^n f, Pg)\{gh\lambda_n^{-1}\}, \end{aligned}$$

establishing the desired equality. Since it is straightforward to check that (3.6) is differential at each object, an application of the lemma [12, (1), p. 97] completes the proof of Theorem 3.3.

There are many ways of displaying the cylinder-web diagram together with its diagonal sequences. The following conveniently separates the diagonal sequences (the labels of the horizontal and vertical arrows as shown in Diagram 2.3 have been suppressed):

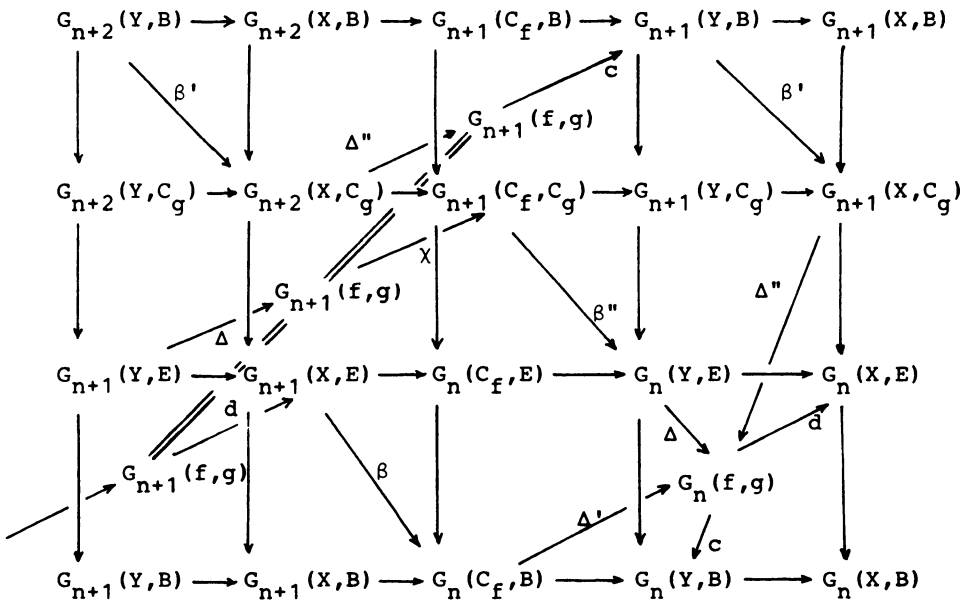


DIAGRAM 3.9

It is intended that the relationship between the cylinder-web diagram and the diagrams of the type considered by C. T. C. Wall [12, p. 100] will be examined in a subsequent paper.

**4. The stable Toda bracket.** Let  $f: X \rightarrow Y$ ,  $h: \Sigma^r Y \rightarrow E$ ,  $g: E \rightarrow B$  be maps, and suppose that, for some  $n$ , null-homotopies  $m_i$  and  $n_i$  of  $\Sigma^n(h\Sigma^r f)$  and  $\Sigma^n(gh)$  exist. The composite square

$$(4.1) \quad \begin{array}{ccccc} \Sigma^{r+n} X & \xrightarrow{\Sigma^{r+n} f} & \Sigma^{r+n} Y & \xrightarrow{\quad} & * \\ \downarrow & \nearrow m_{1-i} & \downarrow \Sigma^n h & \nearrow n_i & \downarrow \\ * & \rightarrow & \Sigma^n E & \xrightarrow{\Sigma^n g} & \Sigma^n B \end{array}$$

represents an element of  $G_r(X^*, * B)$ , and applying  $\chi$  we obtain an element of  $G_{r+1}(X, B)$ . The set of elements obtained in this way for all choices of  $m_i$  and  $n_i$  is denoted

$$(4.2) \quad \langle \{g\}, \{h\}, \{f\} \rangle \subset G_{r+1}(X, B).$$

The definition is consistent with the definition of the stable Toda bracket as given in [7, p. 136].

**4.3. THEOREM.** *Suppose that  $\Sigma^n(gh) \simeq *$  and  $\Sigma^n(h\Sigma'f) \simeq *$ . Then:*

(a) *for each element  $\tau \in \langle \{g\}, \{h\}, \{f\} \rangle \subset G_{r+1}(X, B)$ , we have*

$$\Delta'(.Qf)\tau = -\Delta\{h\} \in G_r(f, g);$$

(b)  $\Delta\{h\} = 0$  in  $G_r(f, g)$  if and only if  $0 \in \langle \{g\}, \{h\}, \{f\} \rangle$ .

**PROOF.** Let  $\tau \in \langle \{g\}, \{h\}, \{f\} \rangle$ . Then  $-\tau$  has a representative of the form  $\chi\alpha \in [\Sigma^{r+n+1}X, \Sigma^n B]$  where  $\alpha$  is a composite square of the form:

$$\begin{array}{ccc} \Sigma^{r+n}X & \xrightarrow{\quad} & * \\ \Sigma^{r+n}f \downarrow & \nearrow m_i & \downarrow \\ \Sigma^{r+n}Y & \xrightarrow{\Sigma^n h} & \Sigma^n E \\ \downarrow & \nearrow n_{1-i} & \downarrow \Sigma^n g \\ * & \rightarrow & \Sigma^n B \end{array}$$

Hence  $-\Delta'(.Qf)\tau$  has a representative

$$\begin{aligned} \delta(. \Sigma^{r+n}Qf)\chi\alpha &= \delta\chi((. \Sigma^{r+n}f|(\Sigma^{r+n}X)*)\alpha) \\ &= * \Sigma^n B | \Sigma^n g.. \Sigma^{r+n}f|(\Sigma^{r+n}X)*(\alpha) = \Delta\{\Sigma^n h\}, \end{aligned}$$

the last step by the argument in the proof of [5, Theorem 5.3]. For part (b), observe that by [12, (2), p. 97] commutativity of the diagram

$$(4.4) \quad \begin{array}{ccccccc} & & \xrightarrow{\beta'} & & \xrightarrow{Qg.} & & \\ & G_{r+1}(Y, B) & & G_{r+1}(X, C_g) & & G_r(X, E) & \\ & \nearrow c & \searrow .f & \nearrow Pg. & \searrow \Delta'' & \nearrow d & \searrow \\ G_{r+1}(f, g) & & G_{r+1}(X, B) & & G_r(f, g) & & G_r(X, B) \\ & \searrow d & \nearrow g. & \searrow .Qf & \nearrow \Delta' & \searrow c & \nearrow \\ & G_{r+1}(X, E) & & G_r(C_f, B) & & G_r(Y, B) & \\ & \xrightarrow{\beta} & & \xrightarrow{.Pf} & & & \end{array}$$

would yield a Mayer-Vietoris sequence

$$\rightarrow G_{r+1}(f, g) \rightarrow G_{r+1}(X, E) \oplus G_{r+1}(Y, B) \rightarrow G_{r+1}(X, B) \xrightarrow{\Delta'(.Qf)} G_r(f, g) \rightarrow$$



in which the kernel of the arrow  $\Delta'(\mathcal{Q}f)$  is precisely the indeterminacy of  $\langle \{g\}, \{h\}, \{f\} \rangle$ . The reader may now easily check that the triangles (A) are commutative for  $r$  even and anticommutative for  $r$  odd, and further that the remaining triangles and rectangles of (4.4) are commutative.

Now let  $\gamma = \{h\} \in G_r(Y, E)$  denote the stable homotopy class of  $h$ . The stable homotopy type of  $C_h$  is an invariant of  $\gamma$ . Accordingly we shall denote by  $C_\gamma$  a space  $C_h$  for some  $h$  in  $\gamma$  and refer (with some abuse) to stable classes  $P\gamma = \{Ph\} \in G_0(E, C_\gamma)$  and  $Q\gamma = \{Qh\} \in G_{-r-1}(C_\gamma, Y)$ . Moreover, if  $\alpha = \{f\} \in G_0(X, Y)$  and  $\beta = \{g\} \in G_0(E, B)$  are as above, we define

$$(4.5) \quad \text{Coext}_\gamma \alpha = \left\{ \chi \left( \begin{array}{ccc} \Sigma^{r+n} X & \xrightarrow{\Sigma^{r+n} f} & \Sigma^{r+n} Y \\ \downarrow & \nearrow m_{1-t} & \downarrow \Sigma^n h \\ * & \xrightarrow{\quad} & \Sigma^n E \end{array} \right) \left| \begin{array}{l} \text{null-homotopies} \\ m_t \text{ of } \Sigma^n(h\Sigma^r f), \\ \text{for some } n \end{array} \right. \right\},$$

$$(4.6) \quad \text{Ext}_\gamma \beta = \left\{ \chi \left( \begin{array}{ccc} \Sigma^{r+n} Y & \xrightarrow{\quad} & * \\ \downarrow \Sigma^n h & \nearrow n_t & \downarrow \\ \Sigma^n E & \xrightarrow{\Sigma^n g} & \Sigma^n B \end{array} \right) \left| \begin{array}{l} \text{null-homotopies } n_t \\ \text{of } \Sigma^n(gh), \text{ for} \\ \text{some } n \end{array} \right. \right\}.$$

We have the following

4.7. LEMMA. (i)  $\langle \beta, \gamma, \alpha \rangle = \text{Ext}_\gamma \beta \text{Coext}_\gamma \alpha \subset G_{r+1}(X, B)$ .

(ii)  $\text{Ext}_\gamma \beta = (.P\gamma)^{-1}\beta \subset G_0(C_\gamma, B)$ .

(iii)  $\text{Coext}_\gamma \alpha = (Q\gamma)^{-1}\alpha \subset G_{r+1}(X, C_\gamma)$ .

PROOF. Part (i) follows immediately from the definitions. Part (ii) is an easy consequence of the fact that  $\chi(*(\Sigma^n E)|\Sigma^n h) = P\gamma$  and (iii) is a consequence of the equality  $\chi(\Sigma^n h(\Sigma^{r+n} Y)*) = Q\gamma$ .

Using the definition of (4.2) one may derive various standard properties of the stable Toda bracket. Particularly useful for the solution of group-extension problems that arise in computation is the following stable version of [11, Proposition 1.9]. A special case is given in [7, Lemma 1.2].

4.8. PROPOSITION. If  $\beta\gamma = 0$  and  $\gamma\alpha = 0$  then  $(\beta.)\text{Ext}_\alpha \gamma = -(.Q\alpha)\langle \beta, \gamma, \alpha \rangle$  as a coset of  $(\beta.)\langle .Q\alpha \rangle G_{r+1}(X, E)$ .

PROOF. Since  $\chi(\Sigma^{r+s} f(\Sigma^{r+s} X)*) = Q\alpha$ , the desired result is a consequence of the following equality in HPC:

$$\begin{array}{ccccc} \Sigma^{r+s} X & \xrightarrow{\quad} & \Sigma^{r+s} X & \xrightarrow{\quad} & * \\ \downarrow \Sigma^{r+s} f & & \downarrow \Sigma^{r+s} f & \nearrow m_t & \downarrow \Sigma^s h \\ \Sigma^{r+s} Y & \xrightarrow{\quad} & \Sigma^{r+s} Y & \xrightarrow{\Sigma^s h} & \Sigma^s E \\ \downarrow n_t & \nearrow n_{1-t} & \downarrow n_{1-t} & & \downarrow \Sigma^s g \\ \Sigma^{r+s} Y & \xrightarrow{\quad} & * & \xrightarrow{\quad} & \Sigma^s B \\ \downarrow \Sigma^s h & & \downarrow \Sigma^s h & & \downarrow \Sigma^s g \\ \Sigma^{r+s} Y & \xrightarrow{\quad} & \Sigma^s E & \xrightarrow{\quad} & \Sigma^s B \end{array} = \begin{array}{ccccc} \Sigma^{r+s} X & \xrightarrow{\quad} & * & \xrightarrow{\quad} & * \\ \downarrow \Sigma^{r+s} f & \nearrow m_t & \downarrow & & \downarrow \\ \Sigma^{r+s} Y & \xrightarrow{\Sigma^s h} & \Sigma^s E & \xrightarrow{\Sigma^s g} & \Sigma^s B \end{array}$$

Proposition 4.8 is frequently used in the case  $\beta = n$ .

**4.9. COROLLARY.** *If  $n\gamma = 0$  and  $\gamma\alpha = 0$  then  $n \operatorname{Ext}_\alpha \gamma = -(Q\alpha)\langle n, \gamma, \alpha \rangle$  as a coset of  $n(Q\alpha)G_{r+1}(X, E)$ .*

By a dual argument one may prove the following

**4.10. PROPOSITION.** *If  $\beta\gamma = 0$  and  $n\gamma = 0$  then  $n \operatorname{Coext}_\beta \gamma = -(P\beta)\langle \beta, \gamma, n \rangle$  as a coset of  $n(P\beta)G_{r+1}(Y, B)$ .*

**5. The stable Puppe sequence of  $\eta$ .** In §§5 and 6 the preceding theory will be applied to compute some stable homotopy of the cofibre of the stable Hopf class  $\eta$ . If we choose  $f = g: S^3 = X = E \rightarrow S^2 = Y = B$  to be the Hopf map, then

$$G_r(C_f, B) = \lim_{n \rightarrow \infty} [S^{r+n+2} \cup_\eta e^{r+n+4}, S^{n+2}] = \pi_{r+2}(\eta)$$

and

$$G_r(X, C_g) = \lim_{n \rightarrow \infty} [S^{r+n+3}, S^{n+2} \cup_\eta e^{n+4}] = \pi_{r+1}^*(\eta)$$

in a standard notation [8]. The stable Puppe sequence of  $\eta$  becomes as follows:

$$G_{-1} \xleftarrow{\cdot\eta} G_{-2} \xleftarrow{\cdot P} \pi_0(\eta) \xleftarrow{\cdot Q} G_0 \xleftarrow{\cdot\eta} G_{-1} \xleftarrow{\cdot P} \pi_1(\eta) \xleftarrow{\cdot Q} G_1 \xleftarrow{\cdot\eta} G_0 \leftarrow \cdots$$

As essentially observed in [7, p. 141] the groups  $\pi_r(\eta)$  ( $r \leq 22$ ) can be counted. We use the notation for stable elements given in [11] and denote by  $\operatorname{ext} \alpha$  an element at random in the coset  $\operatorname{Ext}_\eta \alpha$ .

**5.1. PROPOSITION.**

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$\pi_k(\eta) \approx$	$Z$	$0$	$Z$	$Z_4 + Z_3$	$0$
Generators	$Q$		$\operatorname{ext} 2$	$\nu Q \alpha_1 Q$	
	$k = 5$	$k = 6$	$k = 7$	$k = 8$	
	$Z_8 + Z_3$	$Z_2$	$Z_{16} + Z_3 + Z_5$	$Z_4$	
	$\operatorname{ext} \nu \operatorname{ext} \alpha_1$	$\nu^2 Q$	$\sigma Q \alpha_2 Q \alpha_1 Q$	$\operatorname{ext} \nu^2$	
	$k = 9$	$k = 10$	$k = 11$	$k = 12$	
	$Z_{16} + Z_3 + Z_5$	$Z_3$	$Z_8 + Z_2 + Z_9 + Z_7$	$Z_3$	
	$\operatorname{ext} 2 \sigma \operatorname{ext} \alpha_2 \operatorname{ext} \alpha_1$	$\beta_1 Q$	$\operatorname{ext} \eta \epsilon \operatorname{ext} \nu^3 \alpha' Q \alpha_1 Q$	$\operatorname{ext} \beta_1$	

**PROOF.** The computation via the Puppe sequence, given knowledge of the groups  $G_k$  ( $k < 19$ ) as in [11], is quite straightforward except in cases where a problem of group extension arises. The first such case is  $\pi_8(\eta)$ . From the sequence

$$\begin{array}{ccccc} G_8 & \xrightarrow{\cdot Q} & \pi_8(\eta) & \xrightarrow{\cdot P} & G_6 \\ Z_2 + Z_2 & & & & Z_2 \\ \bar{\nu} \quad \epsilon & & & & \nu^2 \end{array}$$

and knowing that  $\eta\sigma = \bar{\nu} + \epsilon$  and  $\nu^2\eta = 0$  [11] it follows that  $\pi_8(\eta)$  is an extension of  $Z_2$  by  $Z_2$ . It is claimed that  $\epsilon \in \langle 2, \nu^2, \eta \rangle$  from which Corollary 4.9 yields that  $\epsilon Q \in 2 \operatorname{ext} \nu^2$ , so that  $\pi_8(\eta) \approx Z_4$ , generated by  $\operatorname{ext} \nu^2$ . To see that  $\epsilon \in \langle 2, \nu^2, \eta \rangle$

observe, firstly that  $\varepsilon \in \langle \nu^2, 2, \eta \rangle$  [11, p. 189] and, secondly, that the cosets  $\langle \nu^2, 2, \eta \rangle$ ,  $\langle \nu, 2\nu, \eta \rangle$ ,  $\langle 2\nu, \nu, \eta \rangle$  and  $\langle 2, \nu^2, \eta \rangle$  overlap and all have the same indeterminacy, namely  $G_7 \circ \eta$ . The other cases in which extension problems arise are as follows:

$$\begin{array}{ccc} G_9 & \xrightarrow{\cdot Q} & \pi_9(\eta) \xrightarrow{\cdot P} G_7 \\ Z_2 + Z_2 + Z_2 & & Z_{16} + Z_3 + Z_5 \\ \nu^3 \quad \mu \quad \eta \varepsilon & & \sigma \quad \alpha_2 \quad \alpha_1 \end{array}$$

Here  $\nu^3 Q = \eta \varepsilon Q = 0$  and  $\pi_9(\eta)$  is an extension of  $Z_2$  by  $Z_8 + Z_3 + Z_5$ . It is claimed that  $\mu \in \langle 8, 2\sigma, \eta \rangle$  from which Corollary 4.9 yields  $\mu Q \in 8 \text{Ext } 2\sigma$ , settling the group extension. To see that  $\mu \in \langle 8, 2\sigma, \eta \rangle$  observe that  $\mu \in \langle 8\sigma, 2, \eta \rangle$  [11, p. 189], and that  $\langle 8\sigma, 2, \eta \rangle \subset \langle 8, 2\sigma, \eta \rangle$  [7, Proposition 1 (iii)]:

$$\begin{array}{ccc} G_{11} & \xrightarrow{\cdot Q} & \pi_{11}(\eta) \xrightarrow{\cdot P} G_9 \\ Z_8 + Z_9 + Z_7 & & Z_2 + Z_2 + Z_2 \\ \zeta \quad \alpha' \quad \alpha_1 & & \nu^3 \quad \mu \quad \eta \varepsilon \end{array}$$

We have  $\eta^2 \mu = 4\zeta$  and  $\nu^3 \eta = \eta \varepsilon \eta = 0$  so that  $\pi_{11}(\eta)$  is an extension of  $Z_4 + Z_9 + Z_7$  by  $Z_2 + Z_2$ . Since  $\zeta \in \langle 2, \eta \varepsilon, \eta \rangle$  [11, p. 91, Lemma 9.1], we have  $\zeta Q \in 2 \text{Ext } \eta \varepsilon$ , so that  $\pi_{11}(\eta) \approx Z_9 + Z_7 + Z_8 + Z_2$ , with the last two summands generated by  $\text{ext } \eta \varepsilon$  and  $\text{ext } \nu^3$ , respectively.

5.2. REMARK. The groups  $\pi_r^*(\eta)$  may be computed through the dual Puppe sequence of  $\eta$  in a dual manner. Precisely the same extension problems arise and they are resolved via 4.10 so that  $\pi_r(\eta) \approx \pi_r^*(\eta)$ .

6. Computation of  $\pi_k$ . In this section the cylinder-web diagram together with 4.3 will be used to compute

$$\pi_k = G_k(C_\eta, C_\eta) = \lim_{n \rightarrow \infty} [S^{n+k} \cup_\eta e^{n+k+2}, S^n \cup e^{n+2}] \quad (k \leq 8)$$

and, concurrently,  $G_k(\eta, \eta)$  ( $k \leq 8$ ). The diagram begins as indicated below (further to the right all the groups are trivial):

$$\begin{array}{ccccccccccccccc} \pi_0 & \longrightarrow & \pi_0^*(\eta) & \longrightarrow & \pi_1^*(\eta) & \longrightarrow & \pi_{-1} & \longrightarrow & \pi_{-1}^*(\eta) & \longrightarrow & \pi_0^*(\eta) & \longrightarrow & \pi_{-2} \\ \downarrow & \searrow \beta'' & \downarrow & & \downarrow & & \downarrow & \searrow \beta'' & \downarrow & & \downarrow & & \downarrow Q. \\ \pi_0(\eta) & \longrightarrow & G_{-2} & \longrightarrow & G_{-1} & \longrightarrow & \pi_{-1}(\eta) & \longrightarrow & G_{-3} & \longrightarrow & G_{-2} & \longrightarrow & \pi_{-2}(\eta) \\ \downarrow & & \downarrow \Delta & \nearrow d & \downarrow & & \downarrow & & \downarrow \Delta & \nearrow d & \downarrow & & \downarrow \\ \pi_1(\eta) & \longrightarrow & G_{-1} & \longrightarrow & G_0 & \longrightarrow & \pi_0(\eta) & \longrightarrow & G_{-2} & \longrightarrow & G_{-1} & \longrightarrow & \pi_{-1}(\eta) \\ \downarrow \cdot P & & \downarrow \cdot \eta & & \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow \cdot \eta & & \downarrow \cdot Q & & \downarrow \cdot Q \end{array}$$

Note that all vertical arrows in the same row have the same label (either  $\eta$  or  $Q$ .) and all horizontal arrows in the same column have the same label (namely  $\cdot P$ ,  $\cdot Q$  or  $\cdot \eta$ ). The diagonal sequence

$$\cdot \xrightarrow{\beta'} \cdot \xrightarrow{\Delta''} \cdot \xrightarrow{c} \cdot$$

has not been shown and the arrows  $\chi: G_k(\eta, \eta) \rightarrow \pi_k$  have also been suppressed. The symmetry of the diagram gives rise to a very desirable simplification:

6.1. LEMMA. *The arrows  $\beta$  are all trivial.*

PROOF. The commutation property [11, (3.4), p. 33] implies that  $\eta = \pm .\eta$  hence  $\beta = (.Q)(\eta) = \pm (.Q)(\eta) = 0$ .

As a consequence of 6.1 the diagonal  $\Delta' - d - \beta$  sequence becomes a collection of short exact sequences

$$(6.2) \quad 0 \rightarrow \pi_k(\eta) \xrightarrow{\Delta'} G_{k-2}(\eta, \eta) \xrightarrow{d} G_{k-2} \rightarrow 0.$$

A similar (dual) argument shows that the arrows  $\beta'$  are all trivial so that embedded in the cylinder-web diagram are subdiagrams as follows:

$$(6.3) \quad \begin{array}{ccccc} 0 & \rightarrow & \pi_k(\eta) & \xrightarrow{\Delta'} & G_{k-2}(\eta, \eta) & \xleftarrow{\Delta''} & \pi_k^*(\eta) & \leftarrow 0 \\ & & \downarrow .P & & \downarrow d & & \downarrow Q & \\ & & G_{k-2} & \xleftarrow{c} & G_{k-2} & \xrightarrow{d} & G_{k-2} & \rightarrow 0 \end{array}$$

Clearly the short exact sequences split whenever  $.P$  or  $Q$  is an isomorphism. From the nature of the nonsplit extensions it is also easy to see that a diagram

$$(6.4) \quad \begin{array}{ccc} Z_{2^n} & & Z_{2^n} \\ & \searrow & \nearrow \\ & G & \\ & \swarrow & \searrow \\ Z_{2^n} & & Z_{2^n} \end{array} \quad \begin{array}{c} \downarrow 2 \\ \downarrow 2 \end{array}$$

is only possible in the case  $G \approx Z_{2^n} + Z_{2^n}$ . The results of the computation are contained in the following theorems.

6.5. THEOREM.

$$G_k(\eta, \eta) = \begin{array}{cccc} k < -2 & k = -2 & k = -1 & k = 0 \\ 0 & Z & 0 & Z + Z \\ k = 1 & k = 2 & k = 3 & k = 4 \\ Z_2 + Z_4 + Z_3 & Z_2 & Z_8 + Z_8 + Z_3 + Z_3 & Z_2 \\ k = 5 & k = 6 & k = 7 & k = 8 \\ Z_{16} + Z_3 + Z_5 & Z_4 + Z_2 & Z_{16} + Z_{16} + Z_3 + Z_3 + Z_5 + Z_5 & Z_2 + Z_2 + Z_3 \end{array}$$

6.6. THEOREM.

$$\pi_k = \begin{array}{cccc} k < -2 & k = -2 & k = -1 & k = 0 \\ 0 & Z & 0 & Z + Z \\ k = 1 & k = 2 & k = 3 & k = 4 \\ Z_2 + Z_3 & Z & Z_8 + Z_4 + Z_3 + Z_3 & 0 \\ k = 5 & k = 6 & k = 7 & k = 8 \\ Z_{16} + Z_4 + Z_3 + Z_3 + Z_5 & Z_4 + Z_2 & Z_{16} + Z_{16} + Z_3 + Z_3 + Z_5 + Z_5 & Z_4 + Z_3 \end{array}$$

**PROOF OF 6.5 AND 6.6.** Since  $\pi_0(\eta) = Z$ , the sequence (6.3) with  $k = 0$  yields  $G_{-2}(\eta, \eta) = 0$ , and the diagonal sequence  $0 = G_{-3} \rightarrow G_{-2}(\eta, \eta) \rightarrow \pi_{-2} \rightarrow 0$  implies  $\pi_{-2} = Z$ . Again, (6.3) with  $k = 1$  clearly yields  $G_{-1}(\eta, \eta) = 0$  and clearly  $G_{-1}(\eta, \eta) \approx \pi_{-1}$ . With  $k = 2$ , (6.3) is  $0 \rightarrow Z \rightarrow G_0(\eta, \eta) \rightarrow Z \rightarrow 0$ , necessarily split. Moreover,  $G_0(\eta, \eta) \approx \pi_0$  as in the previous case. With  $k = 3$ , (6.2) becomes  $Z_4 + Z_3 \rightarrow G_1(\eta, \eta) \rightarrow Z_2$ . As has been discussed in [5, Proposition 6.3], the extension is split and the image of  $\Delta$  is an element of order 4. It follows that  $\pi_1$ , the cokernel of  $\Delta$ , is isomorphic with  $Z_2 + Z_3$  [5, Corollary 6.4]. With  $k = 4$ , (6.2) becomes  $0 \rightarrow G_2(\eta, \eta) \rightarrow Z_2$  and hence  $G_2(\eta, \eta) = Z_2$ . Moreover,  $\Delta: G_1 \rightarrow G_2(\eta, \eta)$  is surjective (since  $\eta: G_1 \rightarrow G_2$  factors through  $\Delta$ ) and hence  $\pi_2$  is isomorphic with the kernel of  $\Delta: G_0 \rightarrow G_1(\eta, \eta)$  which is  $Z$ . With  $k = 5$ ,  $\eta: P: \pi_5(\eta) \rightarrow G_3 \approx Z_8 + Z_3$  is an isomorphism, and hence (6.3) implies (6.2) splits yielding  $G_3(\eta, \eta) = Z_8 + Z_8 + Z_3 + Z_3$ . Since  $\Delta: G_2 \rightarrow G_3(\eta, \eta)$  is nontrivial, we have  $\pi_3 = \text{cokernel } \Delta = Z_8 + Z_4 + Z_3 + Z_3$ . With  $k = 6$ , (6.2) becomes  $Z_2 \rightarrow G_4(\eta, \eta) \rightarrow 0$ , hence  $G_4(\eta, \eta) = Z_2$ . Since  $\nu^2 = \langle \eta, \nu, \eta \rangle \in G_6$ , 4.3 implies that  $\Delta: G_3 \rightarrow G_4(\eta, \eta)$  is surjective. Since also  $\Delta: G_2 \rightarrow G_3(\eta, \eta)$  is nontrivial the exactness of the  $\Delta - \chi - \beta''$  sequence implies that  $\pi_4 = 0$ . Since  $G_5 = 0$ , (6.2) with  $k = 7$  implies  $G_5(\eta, \eta) \approx \pi_7(\eta) = Z_{16} + Z_3 + Z_5$ . The Puppe sequence passing through  $\pi_5$  is

$$\rightarrow \pi_7^*(\eta) \rightarrow \pi_5 \rightarrow \pi_5^*(\eta) \rightarrow$$

which raises the extension problem:

$$\begin{array}{ccccccc}
 0 & \rightarrow & Z_{16} + Z_3 + Z_5 & \rightarrow & \pi_5 & \rightarrow & Z_4 + Z_3 & \rightarrow & 0 \\
 & & P\sigma & P\alpha_2 & P\alpha_1 & & \text{coext } 2\nu & \text{coext } \alpha_1 & 
 \end{array}$$

Let  $C_\eta$  denote the identity class of the cofibre of  $\eta$ .

**6.7. LEMMA.**  $0 \in \langle 4C_\eta, \text{coext } 2\nu, \eta \rangle \subset \pi_7^*(\eta)$ .

**PROOF.** Since  $C_\eta = \text{ext } P$  we have

$$\langle 4C_\eta, \text{coext } 2\nu, \eta \rangle = \langle \text{ext } 4P, \text{coext } 2\nu, \eta \rangle = \langle 4P, \eta, -2\nu, \eta \rangle,$$

by the definition of the quaternary bracket [6, p. 174]. Since  $\langle \eta, 2\nu, \eta \rangle$  has zero indeterminacy and  $0 \in \langle 2\eta, \nu, \eta \rangle \subset \langle \eta, 2\nu, \eta \rangle$ , we have  $\langle \eta, 2\nu, \eta \rangle = 0$ . Also  $0 \in \langle 2P, 2\eta, 2\nu \rangle$ , which has the same indeterminacy as  $\langle 4P, \eta, 2\nu \rangle$  so that  $0 \in \langle 2P, 2\eta, 2\nu \rangle$ . It follows that  $\langle 4P, \eta, -2\nu, \eta \rangle$  is well defined. Moreover, we have

$$0 \in \langle 4P, \eta, -2, \nu \circ \eta \rangle \subset \langle 4P, \eta, -2\nu, \eta \rangle$$

by [6, Proposition 2.9(ii)]. Lemma 6.7 implies that  $0 \in 4\text{ext } (\text{coext } 2\nu)$  so that  $\pi_5 \approx Z_{16} + Z_4 + Z_3 + Z_3 + Z_5$ . If  $k = 8$  then (6.3) becomes

$$\begin{array}{ccccc}
 & Z_4 & & & Z_4 \\
 & \swarrow & & \searrow & \\
 & & G_6(\eta, \eta) & & \\
 & \swarrow & & \searrow & \\
 Z_2 & & & & Z_2
 \end{array}$$

in which the vertical arrows are nontrivial. The split extension is the only one possible. Moreover, since  $G_4 = G_5 = 0$ , we have that  $\chi: G_6(\eta, \eta) \rightarrow \pi_6$  is an isomorphism. If  $k = 9$ , the corner groups of (6.3) are each  $Z_{16} + Z_3 + Z_5$  and the 2-primary components yield an instance of (6.4). It follows that  $G_7(\eta, \eta) = Z_{16} + Z_{16} + Z_3 + Z_3 + Z_5 + Z_5$ . Since  $0 \in \langle \eta, \nu, \nu\eta \rangle \subset \langle \eta, \nu^2, \eta \rangle$ , an application of 4.3 yields that  $\Delta: G_6 \rightarrow G_7(\eta, \eta)$  is trivial, and since  $G_5 = 0$  the  $\Delta - \chi - \beta''$  sequence yields  $\pi_7 \approx G_7(\eta, \eta)$ . Moreover,  $\beta'': \pi_8 \rightarrow G_6$  is epic from which it follows that  $Q: \pi_8 \rightarrow \pi_8(\eta) = Z_4$  is epic. Since also  $\pi_{10}(\eta) = Z_3 = \{\beta_1 Q\}$  and since  $\eta: \pi_9(\eta) \rightarrow \pi_{10}(\eta)$  is trivial, we have  $\pi_8 = Z_4 + Z_3$  and  $G_8(\eta, \eta) = Z_2 + Z_2 + Z_3$ . Attempting to determine  $\pi_9$ , we have  $\pi_{11}^*(\eta) = Z_8 + Z_2 + Z_9 + Z_7$ ,  $\pi_9^*(\eta) = Z_{16} + Z_3 + Z_5$  and a short exact sequence

$$\pi_{11}^*(\eta) \twoheadrightarrow \pi_9 \twoheadrightarrow \pi_9^*(\eta).$$

Since the 2-primary component of  $\pi_9^*(\eta)$  is generated by  $\text{coext } 2\sigma$ , the extension problem will be solved if we determine  $\langle 16C_\eta, \text{coext } 2\sigma, \eta \rangle \subset \pi_{11}^*(\eta)$ . In this case the brackets  $\langle 16P, \eta, 2\sigma \rangle$  and  $\langle 16, \eta, 2\sigma \rangle$  each have nontrivial indeterminacy so that in the sense of [6, p. 174] the quaternary bracket  $\langle 16P, \eta, 2\sigma, \eta \rangle$  is not well defined. It is reasonable to expect that with a deeper understanding of the quaternary bracket this difficulty can be overcome.

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