GRAPHS OF TANGLES

BY

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ABSTRACT. We prove that under necessary conditions a graph of tangles is a prime link. For this we generalize the result that the sum of 2-string prime L-tangles is a prime link. Some applications are found. We explore Property L for tangles in order to prove primeness of knots.

1. Introduction. We prove that under necessary conditions a graph of tangles is a prime link (Theorems 2 and 3). This extends work done by R. C. Kirby and W. B. R. Lickorish in [9]. For this we define (Definition 1) when a pair $(M, \mathcal{M})$ is a prime tangle, where M is a 3-manifold and $\mathcal{M}$ is a finite, pairwise disjoint collection of properly embedded simple arcs and closed curves in M. We prove in Theorem 1 a generalization of Theorem 1 in [10] and Theorem 1.10 in [15].

In Corollary 1, we prove that if for a link l in S^3 there exists a compact, connected (orientable or not) 2-manifold F embedded in S^3 such that $\partial F = l$ and $\overline{S^3 - V}$ has an incompressible boundary, where V is a regular neighbourhood of F in S^3, then l is a prime link. Afterwards, we exhibit in Corollary 2 a new family of prime (2-string) L-tangles $\mathcal{A}$ with Property L, i.e. for any $(A, t) \in \mathcal{A}$ there is at most one nonprime knot in the family of knots obtained by adding (A, t) to the untangle. We note that S. A. Bleiler has proved that not all prime tangles have this property. Corollaries 3 and 4 imply that a family of links somewhat related to the family of unknotting number one knots are prime. We say in Theorem 4 which M-tangles have Property L. The paper should be interpreted as being in the piecewise linear category. Standard definitions of 3-manifolds and knot theory may be found in [7] and [16], respectively. We follow [10] for definitions related to tangles not found here.

2. A generalization of a Lickorish theorem. For a *tangle*, we will understand here a pair $(M, \mathcal{M})$, where M is an orientable 3-manifold and $\mathcal{M} \subset M$ is a pairwise disjoint collection of properly embedded n simple arcs, $n \geq 0$, and m simple closed curves, $m \geq 0$. (Historically, the word "tangle" was first introduced in [2].) A tangle $(M, \mathcal{M})$, M a 3-cell, will be called an N-*tangle* (see [15]). An N-tangle with $m = 0$ will be called an (n-string) L-*tangle* (see [10]). Two tangles $(M_1, \mathcal{M}_1)$ and $(M_2, \mathcal{M}_2)$ are *equivalent* if there is a homeomorphism of pairs from $(M_1, \mathcal{M}_1)$ to $(M_2, \mathcal{M}_2)$.

Clearly, this is a true equivalence relation. When we talk about a tangle $(M, \mathcal{M})$, we will talk about its equivalence class. We will call $(D \times I, \{x, y\} \times I)$ the *untangle*, where D is a 2-cell and $x, y \in \mathring{D}$, $x \neq y$.

DEFINITION 1. A tangle $(M, \mathcal{M})$ is *prime* if it has the following properties:

(i) (Property I) $M - \mathcal{M}$ is irreducible, i.e. each 2-sphere in $M - \mathcal{M}$ bounds a 3-cell in $M - \mathcal{M}$.

(ii) (Property II) For each 2-sphere S in M that meets $\mathcal{M}$ transversely in two points, S bounds in M one and only one 3-cell intersecting $\mathcal{M}$ in an unknotted spanning arc.

(iii) (Property III) $M - \mathcal{M}$ has an incompressible boundary, i.e. each simple closed curve on $\partial M - \partial \mathcal{M}$ which bounds a properly embedded 2-cell in $M - \mathcal{M}$, bounds a 2-cell on $\partial M - \partial \mathcal{M}$.

(iv) (Property IV) For each properly embedded 2-cell D in M which meets $\mathcal{M}$ transversely in one point, $\overline{M}_1$ or $\overline{M}_2$ is a 3-cell which meets $\mathcal{M}$ in an arc (necessarily an unknotting spanning arc by Property II), where $M_1 \cup M_2 = M - D$.

Clearly, if (B, t) is a 2-string prime tangle as in [9], or a prime tangle as in [15], then (B, t) is a prime tangle as here. We will call a link l in S^3 *prime* if, as a tangle, (S^3, l) is prime. Note that this definition of a prime link coincides with the usual one. In particular, for links of one component it coincides with the usual definition of a prime knot. By the "only one" part of Property II, *the unknot is not prime*. We note that an arcbody as in [11] is not a prime tangle as defined here.

THEOREM 1. *Let (M, l) be a tangle, M closed and connected, and $\{F_i\}$ a finite, pairwise disjoint collection of closed connected surfaces embedded in M and transverse to l. Let $\{M_j\}$ be the closure of the connected components of $M - \cup F_i$ and $\mathcal{M}_j = M_j \cap l$. If $\{(M_j, \mathcal{M}_j)\}$ are prime tangles, then (M, l) is a prime tangle or $(M, l) = (S^3, \text{unknot})$. If furthermore, the number of points in $l \cap F_i$ is not two for those F_i homeomorphic to S^2, then (M, l) is not (S^3, unknot).*

PROOF. Let us set $F = \cup F_i$. First we will prove that $M - l$ is irreducible. Let S be a 2-sphere in $M - l$. It may be assumed that S intersects F transversely in a finite number of simple closed curves. We will show that after some isotopies we can suppose that $S \cap F = \varnothing$. Choose a simple loop α in $S \cap F$ such that $\alpha = \partial D$, where D is a 2-cell contained in S and $\mathring{D} \cap F = \varnothing$. Hence $D \subset M_j$ for some j. As $(M_j, \mathcal{M}_j)$ is a prime tangle there exists a 2-cell $E \subset \partial M_j - \partial \mathcal{M}_j$ such that $\partial E = \alpha$ (Property III) and $D \cup E = \partial C$, where C is a 3-cell contained in $M_j - \mathcal{M}_j$ (Property I). Thus, without moving l, D can be isotopped off M_j, reducing the number of components of $S \cap F$. Hence, it may be assumed that $S \cap F = \varnothing$, i.e. $S \subset \mathring{M}_j$ for some j. By Property I of $(M_j, \mathcal{M}_j)$, S bounds a 3-cell in $M_j - \mathcal{M}_j \subset M - l$.

Now suppose that S is a 2-sphere in M meeting l transversely in two points. We will show that, on one side, S bounds a 3-cell meeting l in an unknotted spanning arc. It may be assumed that S intersects F transversely in a finite number of simple closed curves and $S \cap F \cap l = \varnothing$. We will show that after some isotopies we can suppose that $S \cap F = \varnothing$. We can choose a simple loop α in $S \cap F$ such that $\alpha = \partial D$, where D is a 2-cell contained in S and $\mathring{D} \cap F = \varnothing$. Hence $D \subset M_j$ for
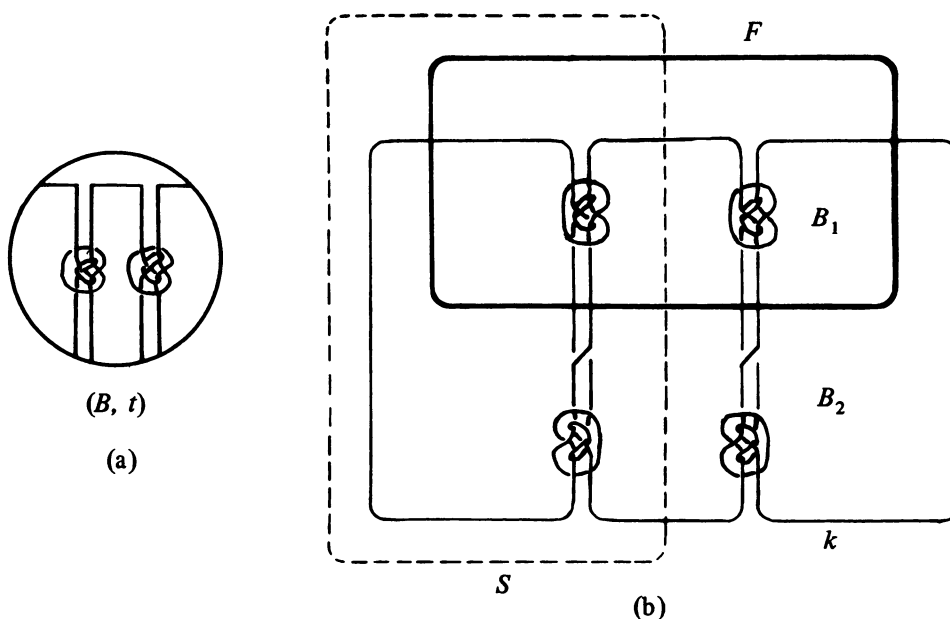
FIGURE 1

some j. Furthermore, α can be choosen so that $D \cap l = \varnothing$ or $D \cap l$ is a single point. If $D \cap l = \varnothing$, we can reduce the number of components of $S \cap F$ as before. If $D \cap l$ is a single point, then by Property IV of $(M_j, \mathcal{M}_j)$ there is an isotopy that reduces the number of components of $S \cap F$ and which keeps l set-wise fixed. Hence, it may be assumed that $S \cap F = \varnothing$, i.e. $S \subset \mathring{M}_j$ for some j. By Property II of $(M_j, \mathcal{M}_j)$, S bounds, in M_j, one and only one 3-cell C intersecting l in an unknotted spanning arc. Now, if for some S, $\overline{M - C}$ is a 3-cell and intersects l in an unknotted spanning arc, then $(M, l) = (S^3, \text{unknot})$. Otherwise, (M, l) is a prime tangle.

Now, suppose that the number of points in $l \cap F_i$ is not two for those F_i homeomorphic to S^2. If $(M, l) = (S^3, \text{unknot})$, then $l = \partial D$, where D is a 2-cell. It may be assumed that D intersescts F transversely in a finite number of simple closed curves and arcs. As before, we can suppose that $D \cap F$ are just arcs. If $D \cap F = \varnothing$, then $D \subset \mathring{M}_j$ for some j. By Property I of $(M_j, \mathcal{M}_j)$, it is clear that $M_j = S^3$. This is a contradiction as (S^3, unknot) is not a prime tangle. If $D \cap F \neq \varnothing$, let α be an arc in $D \cap F$ such that one of the 2-cells (say $\overline{D}_1$) $\overline{D}_1$, $\overline{D}_2$ does not intersects F, where $D - \alpha = D_1 \cup D_2$. Let F_i be such that $\alpha \subset F_i$ and M_j such that $\overline{D}_1 \subset M_j$. Choose a regular neighbourhood C of $\overline{D}_1$ in M_j such that $C \cap D = \overline{D}_1$ and $C \cap F_i$ is a 2-cell H. In any case, it is clear that ∂H contradicts Property III of $(M_j, \mathcal{M}_j)$. Therefore (M, l) is not (S^3, unknot). This completes the proof.

It can be seen that Theorem 1 generalizes Theorem 1 in [10] and Theorem 1.10 in [15]. Let us also emphasize that Property IV in Definition 1 is necessary in order that Theorem 1 is true. In Figure 1(a) a 3-string L-tangle (B, t) with Properties I–III, but not IV, is given. Figure 1(b) shows how Theorem 1 fails if we do not ask for Property IV. There, the 2-sphere S shows that k is composite (i.e. k is a nontrivial,

nonprime knot), there is just one F_i, shown as the 2-sphere F, and (B_i, t_i), $i = 1, 2$, are as in Figure 1(a). Similar counterexamples with (B, t) an n-string L-tangle, $n > 2$, can be constructed.

3. Graphs of tangles. We will say now when a link l in S^3 is *a graph of tangles*. *From now on, each time that we talk about a graph of tangles we will use notation and conventions as defined here.* At some stage, we use a little handle theory terminology that can be found in [17]. Let Γ be a finite connected graph. We allow Γ to have multiple edges and loops, that is several edges from one vertex to another or edges connecting a vertex to itself. *We will always suppose that the degree of any vertex of* Γ *is greater than one*, that is the number of incidents edges at any vertex (with loops at the vertex counted twice) is greater than one. Now, embed Γ in S^3 and construct a regular neighbourhood V of Γ in S^3 (V is a cube with handles) attaching one 0-handle for each vertex and one 1-handle for each edge in the obvious way.

Let us construct a link l in S^3 as follows. For each 0-handle B_i insert an N-tangle (B_i, t_i) such that, for any j, if D_j^k, $k = 1, 2$, are the connected components of the attaching tube of the 1-handle H_j (i.e. $H_j \cap (\cup B_i) = \cup_{k=1}^2 D_j^k$) then the number n_j of points in $P_j^k = D_j^k \cap (\cup t_i)$ is the same for $k = 1, 2$ and $n_j \geqslant 2$. Now, for each j embed (properly) a collection u_j of n_j pairwise disjoint simple arcs in H_j, each arc joining one point in P_j^1 to one point in P_j^2. Furthermore, we will suppose that each of these arcs can be considered as the core of H_j (i.e. each arc is "parallel" to $H_j \cap \Gamma$, the core of H_j). Let us make $l = (\cup t_i) \cup (\cup u_j)$. We will call any link constructed in this way a *graph of tangles*. If furthermore, l can be constructed as before with (B_i, t_i) a prime tangle (L-tangle) for all i, then l will be called a *graph of prime tangles (L-tangles)*.

It was proved in [9] that if we construct a graph of prime L-tangles l starting with a graph Γ which is an unknotted circuit with more than one vertex and l is a knot, then l is prime. This nice and simple method of creating and detecting prime knots
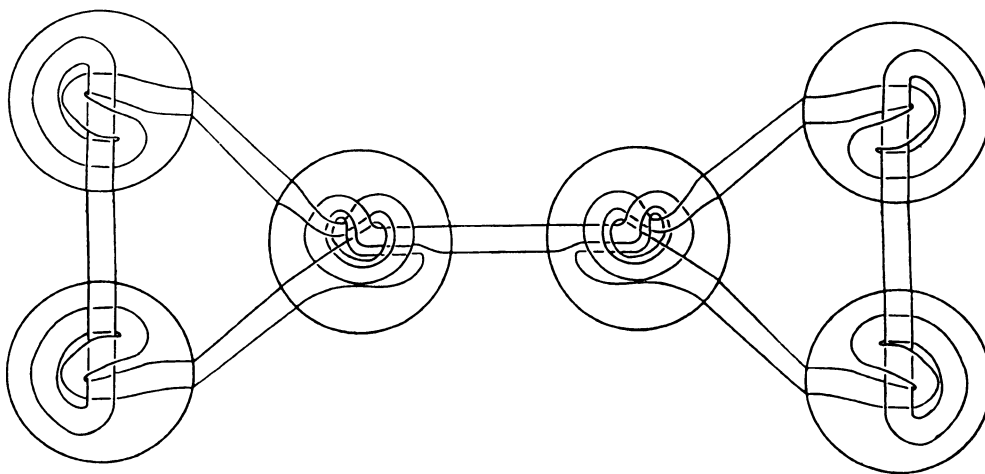

FIGURE 2. A graph of prime tangles that is a composite knot

has already been used in the literature (see e.g. [8 and 14]). These papers also pointed out the possibility of extending this method to graphs of prime L-tangles in general. Theorem 2 implies that this can be done if we add the necessary condition (see Figure 2) that the graph does not have cutting points. Let Γ be a graph embedded in S^3. We will say that *a 2-sphere S in S^3 intersects Γ transversely in one point p* if there are neighbourhoods V_1, V_2, V_3 of p in Γ, S, S^3 such that (V_1, V_2, V_3) is homeomorphic to a neighbourhood of 0 in $(A_{nm}, 0 \times \mathbf{R}^2, \mathbf{R}^3)$, $n > 0$, $m > 0$, where

$$A_{nm} = \bigcup_{i=1}^{n} \{ x(1,0,i) | x \geqslant 0 \} \cup \bigcup_{i=1}^{m} \{ x(1,0,i) | x \leqslant 0 \}.$$

THEOREM 2. *Let Γ be a graph without loops and such that for any 2-sphere S in S^3, S does not intersect the embedding of Γ in S^3 transversely in one point. Let l be a graph of prime tangles constructed from Γ. Then l is a prime link.*

PROOF. By Theorem 1, all we have to prove is that if we set $\overline{S^3 - \bigcup B_i} = M$ and $M \cap l = \mathcal{M}$, then $(M, \mathcal{M})$ is a prime tangle. Clearly $A = \bigcup B_i \cup \mathcal{M}$ is homotopically equivalent to a graph Γ' such that Γ is a subgraph of Γ' and the vertices of Γ and Γ' coincide. As Γ is connected, Γ' is connected and so is A. Therefore $(M, \mathcal{M})$ has Property I. Let α_1 be any arc of the collection $\mathcal{M}$. Suppose that α_1 is a member of the collection u_j for some j and $\partial \alpha_1 \subset \bigcup_{k=1}^{2} B_{i_k}$. As $n_j \geqslant 2$, there exists an arc α_2 in u_j, $\alpha_2 \neq \alpha_1$. Consequently, $\partial \alpha_2 \subset \bigcup_{k=1}^{2} B_{i_k}$. Let β_k be any unknotted arc in B_{i_k} joining α_1 to α_2, $k = 1, 2$. Clearly $(\bigcup \alpha_i) \cup (\bigcup \beta_k)$ is the trivial knot in S^3. Therefore $(M, \mathcal{M})$ has Property II.

Let D be a properly embedded 2-cell in $M - \mathcal{M}$. We can suppose that $\partial D \subset \partial B_i = S$ for some i. Let $\{D_k\}_{k=1}^{2}$ be the closure of the connected components of $S - \partial D$. If $D_k \cap \mathcal{M} = \varnothing$ for some k, then clearly ∂D bounds a 2-cell on $\partial M - \partial \mathcal{M}$. Hence, suppose that $D_k \cap \mathcal{M} \neq \varnothing$, $k = 1, 2$. Note that, as Γ does not have loops, if an arc of u_j intersects D_k then all the arcs in u_j intersect D_k. Therefore, if we choose just one arc α_j in u_j for all j and make $\mathcal{M}' = \bigcup \alpha_j$, then $D_k \cap \mathcal{M}' \neq \varnothing$, $k = 1, 2$. Furthermore, if in S^3 we collapse each B_i to a point, we obtain S^3 again and the image of $\mathcal{M}'$ becomes the embedding of Γ and D becomes a 2-sphere intersecting Γ transversely in one point. This is a contradiction. Therefore $(M, \mathcal{M})$ has Property III.

Let D be a properly embedded 2-cell in M which meets $\mathcal{M}$ transversely in one point. We can suppose that $\partial D \subset \partial B_{i_1} = S$. Let $\{D_i\}_{i=1}^{2}$ be the closure of the connected components of $S - \partial D$. Let β_1 be the arc of $\mathcal{M}$ that intersects D. Suppose that β_1 is a member of the collection u_s. As there are no loops in Γ there are just two cases.

Case 1. $\partial \beta_1 \subset \bigcup_{k=2}^{3} B_{i_k}$ and $\{i_k\}_{k=1}^{3}$ are all different (see Figure 3(a)). Clearly $M - D$ has two connected components. As $n_s \geqslant 2$ there exists a member β_2 of the collection u_s, $\beta_2 \neq \beta_1$. Hence, it is clear that β_2 intersects both connected components of $M - D$. As $\beta_2 \cap D = \varnothing$, $\beta_2 \subset M - D$. This is a contradiction.

Case 2. $\partial \beta_1 = \bigcup_{k=1}^{2} B_{i_k}$ (see Figure 3(b)). Suppose that $\beta_1 \cap S \subset D_1$. It is clear that there exists at least one member β of u_s different from β_1 and for any such β we have that $\beta \cap S \subset D_2$.

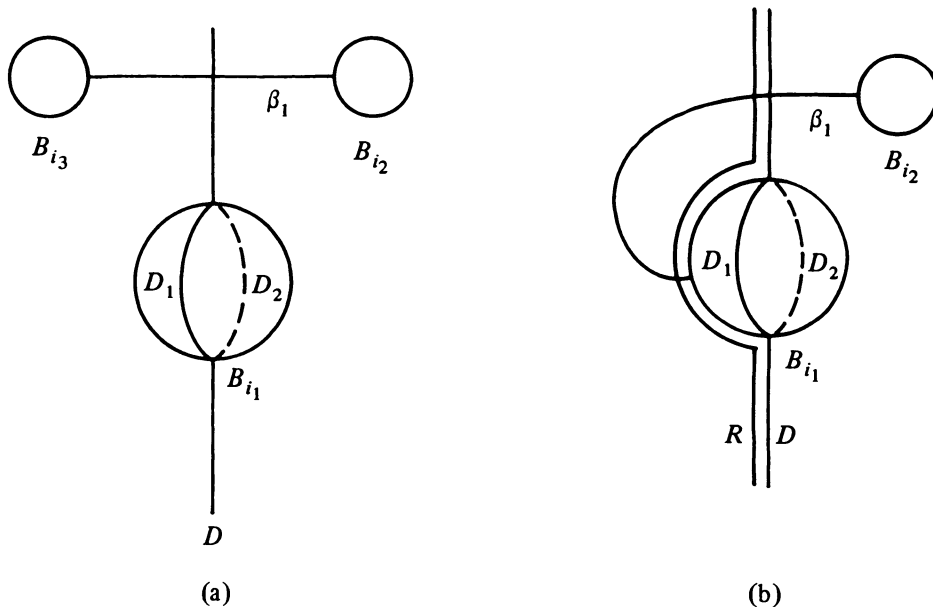$$\text{(a)} \qquad\qquad\qquad\qquad\qquad \text{(b)}$$

FIGURE 3

Case 2.1. Suppose that there exists γ_1, a member of the collection $\mathscr{M}$, $\gamma_1 \neq \beta_1$, such that $\gamma_1 \cap D_1 \neq \varnothing$. It is clear that γ_1 is a member of u_t, $t \neq s$, and for any member γ of u_t we have that $\gamma \cap D_1 \neq \varnothing$. Choose just one arc α_j in u_j for all $j \neq s$. Choose α_s to be any member in u_s different from β_1. Let us set $\mathscr{M}' = \bigcup \alpha_j$. Note that $\mathscr{M}' \cap D = \varnothing$ and $D_i \cap \mathscr{M}' \neq \varnothing$, $i = 1, 2$. Now, it is clear that, continuing as in the proof of Property III, we obtain a contradiction.

Case 2.2. β_1 is the only arc in $\mathscr{M}$ intersecting D_1. Let M_1 be the connected component of $M - D$ such that $M_1 \cap D_1 = \mathring{D}_1$. We claim that $\overline{M}_1 = A$ is a 3-cell. Otherwise, it is clear that there exists B_{i_3} with $\partial B_{i_3} \subset A$. Choose $\mathscr{M}'$ as in Case 2.1. It is clear that $\mathscr{M}' \cap (D \cup D_1) = \varnothing$. Let R be a copy of $D \cup D_1$ pushed inside A slightly in such a way that, still, $\mathscr{M}' \cap R = \varnothing$ (see Figure 3(b)). Let us do the same collapsing as in the proof of Property III. It is clear that R becomes a 2-sphere that separates Γ. This contradicts the assertion that Γ is connected. Therefore A is a 3-cell. Clearly A intersects $\mathscr{M}$ in an arc. Hence $(M, \mathscr{M})$ has Property IV. Therefore $(M, \mathscr{M})$ is a prime tangle. This completes the proof.

From now on, let us set $E = \bigcup_{j,k} D_j^k$. If l is a graph of tangles with $n_j = 2$ for all j and each tangle (B_i, t_i) has Properties I, II and Property III': $C_i = \overline{\partial B_i - E}$ is incompressible in $B_i - t_i$, then l will be called a *graph of weakly prime tangles*. It can be seen that the following result implies Theorem 2 in [4].

THEOREM 3. *Let* Γ *be a graph such that for a regular neighbourhood* V *of the embedding of* Γ *in* S^3, $\overline{S^3 - V}$ *has an incompressible boundary. Let* l *be a graph of weakly prime tangles constructed from* Γ. *Then* l *is a prime link.*

PROOF. By the isotopy uniqueness of regular neighbourhoods (see e.g. [17]), it may be assumed that V is constructed as in the definition of a graph of tangles. By Theorem 1, it is clear that all we have to prove is that (V, l) is a prime tangle. For this, we will mention the following facts. It is easy to see that (H_j, u_j) has Properties I, II and III': $C_j = \overline{\partial H_j - E}$ is incompressible in $H_j - u_j$. Now, let us denote by (B, t) any (B_i, t_i) or (H_j, u_j), then (B, t) has Property IV': there is not a properly embedded 2-cell D in B such that $\partial D \subset \overline{\partial B - E}$, which meets t in exactly one point. Otherwise, choose a 2-cell $H \subset \partial B$ such that $\partial H = \partial D$. Clearly, $H \cap t$ is an even number of points. Therefore, $D \cup H$ is a 2-sphere that meets l in an odd number of points. This is a contradiction. Summing up, we can say that any (B, t) has Properties I, II, III' and IV'. Finally, it is also easy to see that any (B, t) has Property III'': $B \cap E - t$ is incompressible in $B - t$.

By Properties I and III'' of $\{(B, t)\}$, we can make a straightforward innermost disc argument to prove that for any 2-sphere S in $V - l$ there is an isotopy that makes $S \cap E = \varnothing$. By Property I of $\{(B, t)\}$, (V, l) has Property I.

Let S be a 2-sphere in V that meets l transversely in two points. It may be assumed that $S \cap E$ is a finite collection of simple closed curves. Let α be such a curve innermost on S that bounds a disc D in S. This α may be chosen so that $D \cap l = \varnothing$ or $D \cap l$ is a single point. If $D \cap l = \varnothing$, by Properties I and III'' there is an isotopy that reduces the number of components of $S \cap E$. If $D \cap l$ is a single point, choose the 2-cell $D' \subset E$ such that $\partial D' = \alpha$. Clearly $D' \cap l$ has to be a single point. Therefore, $D \cup D'$ is a 2-cell that intersects l in two points and is contained in some (B, t). By Property II of $\{(B, t)\}$, there is an isotopy that reduces the number of components of $S \cap E$. Therefore, it may be assumed that $S \cap E = \varnothing$. By Property II of $\{(B, t)\}$, (V, l) has Property II.

Let D be a properly embedded 2-cell in $V - l$. It may be assumed that $D \cap E$ is a finite collection of simple closed curves and arcs. As before, we can see that there is an isotopy which reduces the number of simple closed curves in $D \cap E$ to zero. Let α be an outermost arc on D. Let F be the closure of a connected component of $D - \alpha$ such that $F \cap E = \alpha$. Let us suppose that $F \subset B$ for some (B, t) and $\alpha \subset D_j^k = A$ for some j and k.

Case 1. Both closures of the connected components of $A - \alpha$ intersect t in a single point. Let us call one of these components G. Clearly $F \cup G$ is a 2-cell contained in B meeting t in a single point and $\partial(F \cup G) \subset \overline{\partial B - E}$. By Property IV' of (B, t), it can be seen that this is a contradiction.

Case 2. For one of the closures of the connected components of $A - \alpha$, call it G, we have that $G \cap t = \varnothing$. Clearly $F \cup G$ is a 2-cell contained in $B - t$ and $\partial(F \cup G) \subset \overline{\partial B - E}$. By Property III' of (B, t), it can be seen that there exists a 2-cell $H \subset \overline{\partial B - E}$ such that $\partial H = \partial(F \cup G)$. Clearly $F \cup G \cup H$ is a 2-sphere contained in $B - t$. By Property I of (B, t), there is an isotopy that reduces the number of components of $D \cap E$. Therefore, it may be assumed that $D \cap E = \varnothing$. By Property III' of (B, t), (V, l) has Property III.

Let D be a properly embedded 2-cell in V that meets l transversely in a single point. With similar arguments to those already used in this proof, it may be assumed

that $D \cap E = \varnothing$. This contradicts Property IV' of $\{(B, t)\}$. Therefore, there are no such 2-cells in V. Hence, (V, l) has Property IV. Therefore (V, l) is a prime tangle. This completes the proof.

It is not difficult to see that Theorem 3 implies the following result.

COROLLARY 1. *Let l be a link in S^3. If there exists a compact, connected (orientable or not) 2-manifold F embedded in S^3 such that $\partial F = l$ and $\overline{S^3 - V}$ has an incompressible boundary, where V is a regular neighbourhood of F in S^3, then l is a prime link.*

4. Property L. In this section we explore Property L for tangles in order to prove primeness of knots. Let (A, t) be a (2-string) L-tangle with Property II. Clearly (A, t) is the untangle or a prime tangle. We will say that (A, t) has *Property* L if there is at most one nonprime knot in the family of knots obtained by adding (A, t) to the untangle (compare with [10, §4]). The untangle has Property L as the resulting knots have bridge number less than or equal to two. It can be seen that Theorem 2 in
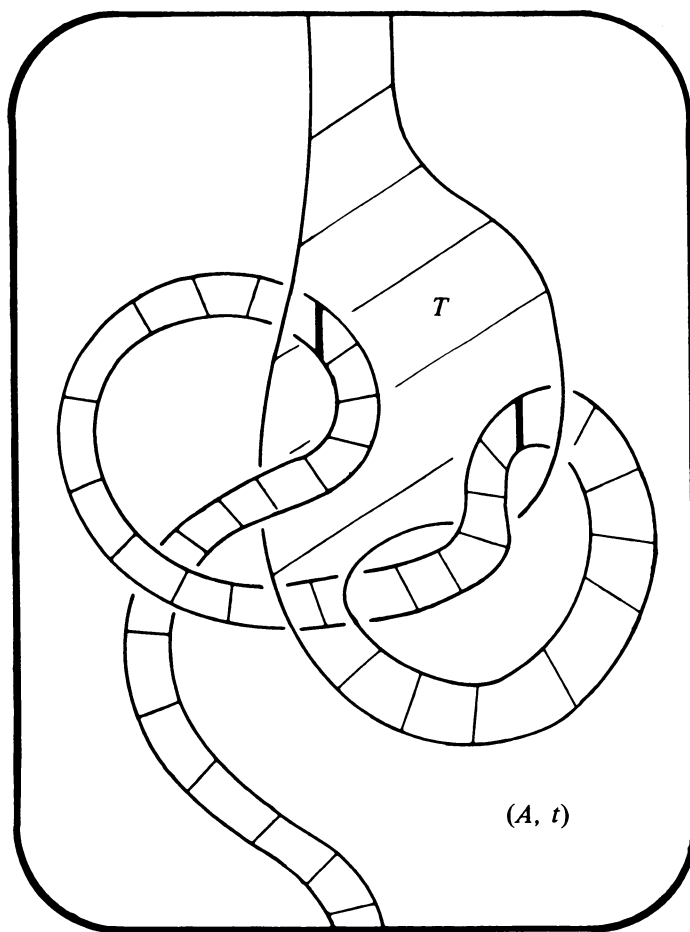

FIGURE 4

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[4] proves that a certain tangle has Property L (Theorem 4 generalizes this result). S. A. Bleiler proved in [1] that not all prime tangles have Property L.

We almost rephrase from [10, p. 331] in the following lines. A nontrivial knot has unknotting number one if it has a presentation in which the changing of one crossover from an overpass to an underpass changes the knot to an unknot. It is an open question if unknotting number one knots are prime [5]. Let B be a small 3-ball "enclosing a neighbourhood of such a crossover" that meets the knot k in an untangle. Let (A, t) be the complementary tangle. Certainly (A, t) has Property II since the untangle can be added to it to create the unknot. Thus (A, t) is either the untangle, in which case k is prime, or (A, t) is prime in which case the primeness of k would follow if (A, t) has Property L. Therefore, if we call $\mathscr{B}$ the family of such (A, t)'s (prime or not), it would be interesting to know which members of $\mathscr{B}_p = \{(A, t) \in \mathscr{B} | (A, t)$ is prime$\}$ have Property L. The referee pointed out to the author that not all members of B_p have Property L (see Figure 6 in [1]).

It is easy to see that if $(A, t) \in \mathscr{B}$, then "t bounds a band T with <u>ribbon singularities</u>" (see Figure 4). Let $\mathscr{B}_I$ be the family of tangles in $\mathscr{B}$ such that $S^3 - V$ has an incompressible boundary where V is a regular neighbourhood of $T \cup B$ in S^3.

COROLLARY 2. $\mathscr{B}_I \subset \mathscr{B}_p$ and any member of $\mathscr{B}_I$ has Property L.

PROOF. (SKETCH). Using Theorem 3, note that V is a regular neighbourhood of a convenient Γ. Also, note that all the ways, except exactly one, of adding (A, t) to the untangle (see [10, §4] for an exact definition of the "ways") gives a graph of weakly prime tangles. Also, $\mathscr{B}_I \subset \mathscr{B}_p$ because otherwise we could create (Theorem 3) a closed incompressible surface $(= \partial V)$ in the complement of a two-bridge knot. This is impossible by [6]. This completes the proof.

From now on, *we will use terminology and conventions of rational tangles as in* [12]. We note that from our point of view, any rational tangle is the untangle. An *M-tangle* is any (2-string) *L*-tangle (A, t), as Figure 5 shows, where (α_i, β_i) is a rational tangle different from $(0, 1)$ and $(1, b)$; $i = 1, \ldots, n$, $b \in \mathbf{Z}$. If we add any *M*-tangle (A, t) to the untangle in such a way that the result is a knot k, we can picture k, as Figure 5 shows, where $(\alpha_{n+1}, \beta_{n+1})$ is a rational tangle.

LEMMA 1. *If* $(\alpha_{n+1}, \beta_{n+1}) \neq (0, 1)$, *then k is prime or the trivial knot. If, furthermore, $n > 2$, then k is nontrivial.*

PROOF. It follows from [12] that the Seifert manifold

$$M = (O \circ 0 | 0; (\alpha_1, \beta_1), \ldots, (\alpha_{n+1}, \beta_{n+1}))$$

is the 2-fold cyclic covering of S^3 branched over k. Now M is a prime manifold, otherwise it can be seen from [18, Theorem 22] that $\Pi_1(M) = \mathbf{Z}_2 * \mathbf{Z}_2$ and this is impossible as the order of the homology of a 2-fold cyclic covering of S^3 branched over a knot k is an odd number (see e.g. [3]). Therefore, by [19], k is prime or the trivial knot. Finally, by [18, p. 205] we know that if $\Pi_1(M) = 1$ then $n \leqslant 2$. This completes the proof.

LEMMA 2. *Let (A, t) be an M-tangle with $n = 2$. (A, t) does not have Property* L *iff there exists an integer b such that $b\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 = \pm 1$ and $\alpha_1\alpha_2$ is an odd number.*

PROOF. We can picture the whole situation as in Figure 5 with $n = 2$ and (α_3, β_3) any rational tangle. We will call k the resulting knot. If (α_3, β_3) is different from $(0, 1)$ and $(1, b)$, $b \in \mathbf{Z}$, then k is a prime knot by Lemma 1. Therefore, by Lemma 1, (A, t) does not have Property L iff there exists an integer b such that for $(\alpha_3, \beta_3) = (1, b)$, k is trivial and for $(\alpha_3, \beta_3) = (0, 1)$, k is composite.

By [19], k is trivial iff M, the 2-fold cyclic covering of S^3 branched over k, is S^3. By [12], $M = (O \circ 0 | b; (\alpha_1, \beta_1), (\alpha_2, \beta_2))$. By [18, p. 206], M is S^3 iff $b\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 = \pm 1$. Therefore, k is trivial iff there exists an integer b such that $b\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 = \pm 1$. It is easy to see that for $(\alpha_3, \beta_3) = (0, 1)$, k is composite iff $\alpha_1\alpha_2$ is an odd number. This completes the proof.
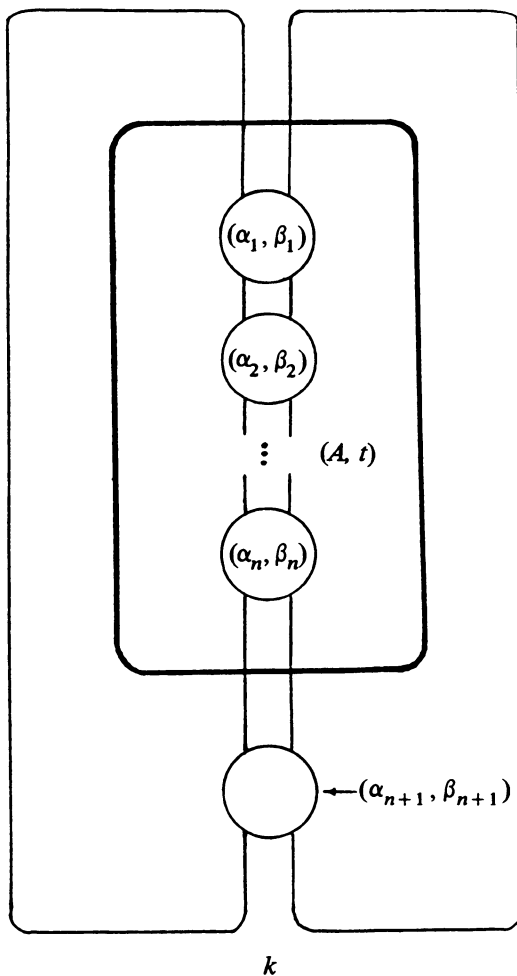

k

FIGURE 5

If (A, t) is an M-tangle with $n = 1$, then (A, t) is a rational tangle. Therefore, as mentioned before, (A, t) has Property L. This and Lemmas 1 and 2 imply Theorem 4. This result generalizes Theorem 2 in [4] and gives some more examples of prime tangles with and without Property L. We note that in [1] the first prime tangle without Property L is given, answering a question in [10].

THEOREM 4. *Let (A, t) be an M-tangle. If $n \neq 2$, then (A, t) has Property L. If $n = 2$, (A, t) does not have Property L iff there exists an integer b such that $b\alpha_1\alpha_2 + \alpha_1\beta_2 + \beta_1\alpha_2 = \pm 1$ and $\alpha_1\alpha_2$ is an odd number.*

5. Families of prime links related to unknotting number one knots. Let $\mathcal{B}_n$, $n \geqslant 0$, be the family of tangles constructed as those of $\mathcal{B}$ but inserting n simple closed curves in each member of $\mathcal{B}$ as Figure 6 shows. Let $(\alpha, \beta)_n$ be a tangle as Figure 7 shows and $\mathcal{A} = \{(\alpha, \beta)_n | (\alpha, \beta) \neq (0, 1)$ and $n > 0\}$.

LEMMA 3. *Any member of $\mathcal{A}$ is a prime tangle.*
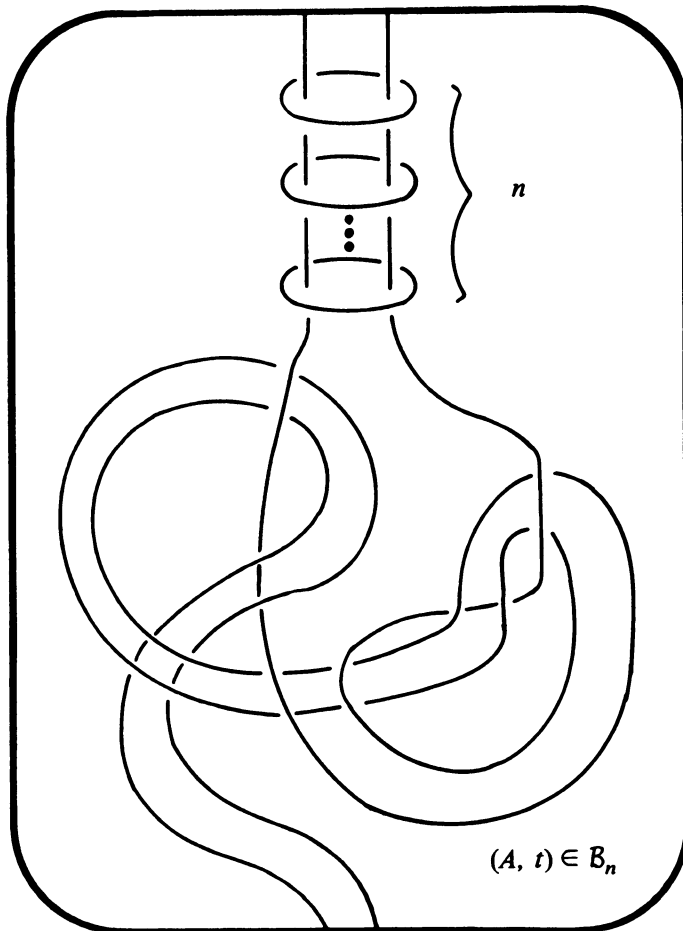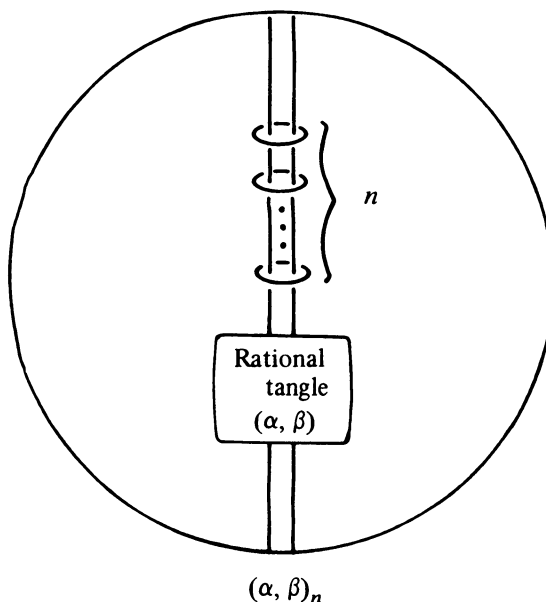

FIGURE 6

$(\alpha, \beta)_n$

FIGURE 7

PROOF. Let $(A, t) \in \mathscr{A}$. It is easy to check that (A, t) has Properties I and II. Let D be a properly embedded disc in $A - t$. If $[\partial D] \neq 0$ in $\Pi_1(\partial A - \partial t)$, then D separates the arcs of t', where t' is both arcs of t. Let A' be the closure of one of the connected components of $A - D$ such that A' contains a simple closed curve c of t, and call t'' the arc of t' contained in A'. $[c] \neq 0$ in $\Pi_1(A' - t'') \approx H_1(A' - t'')$ as $[c] \neq 0$ in $\Pi_1(A - t')$. Hence, $[c]$ is a nonzero multiple of one of the generators of $H_1(A - t')$. This is a contradiction. Therefore, (A, t) has Property III. Let D be a properly embedded 2-cell in A that meets t transversely in one point (necessarily contained in t'). Let $\bar{A}_1$ and $\bar{A}_2$ be the 3-cells such that $A_1 \cup A_2 = A - D$. Using the same argument as before, we can see that if $\bar{A}_1$ contains the arc of t' not intersecting D, Then $t - t' \subset \bar{A}_1$. Therefore (A, t) has Property IV. This completes the proof.

COROLLARY 3. *The unlink of $n + 1$ components is the only nonprime link in the family of links obtained by adding any $(A, t) \in \mathscr{B}_n$, $n > 1$, to the untangle.*

PROOF. We can picture the resulting link l as in Figure 8. Suppose that $(\alpha, \beta) \neq (0, 1)$. If (A, t') is a prime tangle, then, by Lemma 3 and Theorem 1, l is a prime link. If (A, t') is not prime, then it is clear that it is a rational tangle not $(0, 1)$. By Lemma 3, sliding one of the simple closed curves of (A, t) inside (A, t'), (A, t') becomes a prime tangle (A, t''). As $n > 1$, by Lemma 3 and Theorem 1, l is a prime link.

COROLLARY 4. *The unlink of two components and the link L (Figure 8 with $n = 1$ and $(B, u) = (1, 0)$) are the only nonprime links in the family of links obtained by adding any $(A, t) \in \mathscr{B}_1$ to the untangle.*
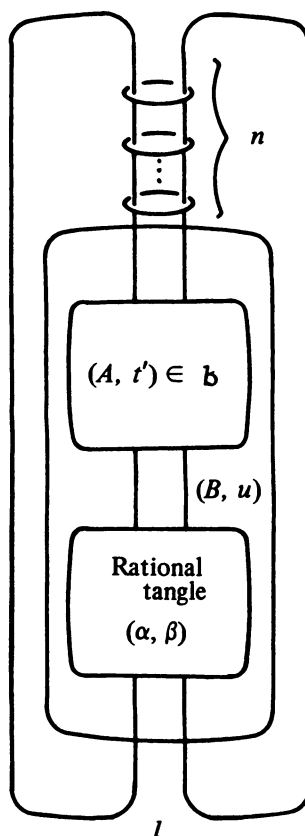
FIGURE 8

PROOF. We can picture the resulting link l as in Figure 8 with $n = 1$. Suppose that $(\alpha, \beta) \neq (0, 1)$ and $(B, u) \neq (1, 0)$. If (B, u) is a prime tangle, then, by Lemma 3 and Theorem 1, l is prime. If (B, u) is not prime, then it can be seen that it is a rational tangle. Let us make $l' = l - c$, where c is the simple closed curve of t. Clearly, (S^3, l') has Property I. Hence, if there is a separating 2-sphere S in $S^3 - l$, S separates c from l'. Hence, we can separate a link l^* obtained from l by adding more simple closed curves as c in t. By Corollary 3, this is a contradiction. Therefore (S^3, l) has Property I.

Let S be a 2-sphere that intersects l transversely in two points. If $S \cap l' \neq \varnothing$ and none of the 3-cells that bounds S intersects l in an unknotted spanning arc, then the same is true for a link l^* formed as above. By Corollary 3, this is a contradiction. Therefore, one (and clearly just one) of these 3-cells intersects l as desired. If $S \cap c \neq \varnothing$ then l' is contained in one component of $S^3 - S$. Hence, the closure of the other component intersects l as desired. Therefore (S^3, l) has Property II. Therefore (S^3, l) is a prime link. This completes the proof.

It is clear that Corollary 4 says that if k is an unknotting number one knot, then we can find in a very simple way a simple closed curve c in $S^3 - k$ such that $k \cup c$ is

a prime link. Let us point out that such a c exists for any link l. By [15], if l is a link, then there exists a 2-sphere S embedded in S^3 meeting l transversely in four points and separating S^3 into 3-balls A and B such that $(A, A \cap l)$ is the untangle and $(B, B \cap l)$ is a prime tangle. By Lemma 3, we can find a simple closed curve c in $A - (A \cap l)$ such that $(A, (A \cap l) \cup c)$ is a prime tangle. By Theorem 1, $l \cup c$ is a prime link. Finally, let us mention that by the work of Myers in [13] we know even more: for any link l there is a *simple* knot c in $S^3 - l$, i.e. the complement M of a regular neighbourhood of $l \cup c$ in S^3 is irreducible, boundary irreducible and contains no nonboundary-parallel incompressible annuli or tori.

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