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NONVANISHING LOCAL COHOMOLOGY CLASSES[1]

BY

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ABSTRACT. We discuss the nonvanishing of a top-dimensional canonical cohomology class of the space $\overline{B\mathcal{D}iff}_\omega M$. We treat parallelizable and odd-dimensional stably parallelizable manifolds.

Conventions. The following hold unless stated otherwise:
(1) M is a closed, oriented, n-dimensional smooth manifold with volume form ω.
(2) All H^* and H_* are with $\mathbf{R}$-coefficients.
(3) $\deg f$ is the degree of f.
(4) $[\eta]$ is the cohomology class of the closed form η.

Introduction. Let $\mathcal{D}iff_\omega M$ denote the group of volume preserving (with respect to ω) diffeomorphisms of M with the C^∞ topology. We will investigate certain local cohomology classes of $\mathcal{D}iff_\omega M$, originally defined by McDuff **[M-1]**. These classes are related to the Gelfand-Fuks cohomology of divergence free (with respect to ω) vector fields on M. These classes also, in a certain sense, measure how $\mathcal{D}iff_\omega M$ twists M. In this paper we will show that the top-dimensional class in question is nonzero if M is parallelizable or if M is odd-dimensional and stably parallelizable.

Background. Let $\mathrm{Diff}^\delta_\omega M$ denote $\mathcal{D}iff_\omega M$ with the discrete topology. If $\mathcal{G}$ is a topological group, then $B\mathcal{G}$ denotes its classifying space. The inclusion $\bar{i}$: $\mathrm{Diff}^\delta_\omega M \to \mathcal{D}iff_\omega M$ passes to a continuous map

$$i\colon B\,\mathrm{Diff}^\delta_\omega M \to B\mathcal{D}iff_\omega M. \tag{1}$$

Let $\overline{B}\mathcal{D}iff_\omega M$ be the homotopy theoretic fibre of (1). This space is well-defined up to homotopy type. By the *local cohomology* of $\mathcal{D}iff_\omega M$ we mean the (real) singular cohomology of $\overline{B}\mathcal{D}iff_\omega M$.

The space $\overline{B}\mathcal{D}iff_\omega M$ is in its own right a classifying space. It classifies globally trivialized M-bundles with a flat structure. Sitting over $\overline{B}\mathcal{D}iff_\omega M$ we have the universal bundle $\overline{B}\mathcal{D}iff_\omega M \times M$. Let us now choose a specific model for $\overline{B}\mathcal{D}iff_\omega M$ **[Ma]**.

Let $\mathrm{Sing}\,\mathcal{D}iff_\omega M$ be the smooth singular complex of $\mathcal{D}iff_\omega M$. Note that $\mathcal{D}iff_\omega M$ acts freely on $\mathrm{Sing}\,\mathcal{D}iff_\omega M$ by multiplication on the right, so it also acts freely on the

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geometric realization $|\mathrm{Sing}\,\mathscr{D}\!\mathit{iff}_\omega M|$. We can now form the quotient space $|\mathrm{Sing}\,\mathscr{D}\!\mathit{iff}_\omega M|/\mathrm{Diff}^\delta_\omega M$. This will be our particular model for $\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M$ [**M-1**].

Thus $\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M$ has a PL-structure in which a k-simplex is a smooth map $\Delta^k \to \mathscr{D}\!\mathit{iff}_\omega M$, which is well defined up to composition on the right by an element of $\mathrm{Diff}^\delta_\omega M$. To get rid of this ambiguity we ask for the 0-vertex to go to the identity diffeomorphism;

$$(2) \qquad (\Delta^k, 0) \to (\mathscr{D}\!\mathit{iff}_\omega M, 1), \qquad t \to h_t.$$

We are now in a position to define a foliation $\mathscr{F}$ on $\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M \times M$.

To each k-simplex in $\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M$ we can associate a foliation on $\Delta^k \times M$ with leaves

$$(3) \qquad \mathscr{L}_m = \{(t, h_t(m)) : t \in \Delta^k\}.$$

This foliation is the pull-back of the point foliation on M by the map $f\colon (t, m) \to h_t^{-1}(m)$. The foliation is volume preserving [**L**] and has $f^*\omega$ for its transverse volume form. The foliations and transverse volume forms on each $\{k\text{-simplex}\} \times M$ fit together to give a codimension-n volume preserving (generalized) foliation $\mathscr{F}$ on $\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M \times M$ with transverse volume form Ω.

For more background information see [**Mo**].

McDuff's classes. The cohomology class $[\Omega]$ may be decomposed via the Künneth formula. So

$$[\Omega] = \bigoplus_{i=0}^{n} [\Omega]_i, \quad \text{where } [\Omega]_i \in H^{n-i}(\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M) \otimes H^i(M).$$

The class $[\Omega]_i$ may be written as $\sum_j \alpha^j_{n-i} \otimes \beta^j_i$, where j is indexed over the rank of $H^i(M)$, $\alpha^j_{n-i} \in H^{n-i}(\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M)$, and $\beta^j_i \in H^i(M)$. Of course this representation is not unique.

We may now define the McDuff classes $c_k(M) \in H^k(\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M; H^{n-k}(M))$ by the formula

$$(4) \qquad c_k(M)\kappa = \sum_j \langle \alpha^j_k, \kappa\rangle \cdot \beta^j_{n-k},$$

where $\kappa \in H_k(\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M)$. Note that the class $c_k(M)$ is independent of the choices made. In fact $c_k(M)\kappa$ is just, neglecting sign, the slant product of $[\Omega]_{n-k}$ with κ. We will use absolute value signs to show that we are neglecting sign. Thus, we may also denote $c_k(M)\kappa$ by

$$(4') \qquad |[\Omega]/\kappa|.$$

In this paper we are concerned with the top class $c_n(M)$. We may express $[\Omega]_0$ as

$$\alpha_n \otimes 1 \in H^n(B\,\mathscr{D}\!\mathit{iff}_\omega M) \otimes H^0(M) \cong H^n(\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M) \otimes \mathbf{R}.$$

So we may identify $H^n(\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M)$ with $H^n(\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M) \otimes H^0(M)$ and thus identify $c_n(M)$ with $[\Omega]_0$. The class $[\Omega]_0$ is the same as $[\Omega]$ restricted to $\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M \times \mathrm{pt}$. Therefore,

$$(5) \qquad c_n(M) \neq 0 \Leftrightarrow [\Omega]|_{\overline{B}\,\mathscr{D}\!\mathit{iff}_\omega M \times \mathrm{pt}} \neq 0.$$

Fundamental diagram. The diagram we use follows from [T]. Recall that $B\Gamma^n_{sl}$ is the classifying space for codimension-n volume preserving Haefliger structures. There is a special cohomology class $\tilde{\mu} \in H^n(B\Gamma^n_{sl})$ called the *univeral transverse volume class*. If $f\colon X \to B\Gamma^n_{sl}$ classifies a codimension-n volume preserving Haefliger structure H, then $f^*\tilde{\mu} \in H^n(X)$ is called the *transverse volume class* of H. If H is actually the Haefliger structure coming from a codimension-n volume preserving foliation with transverse volume form λ, then $[\lambda] = f^*\tilde{\mu}$. The class $\tilde{\mu}$ can be constructed directly from the topological groupoid of germs Γ^n_{sl}, or it can be defined by a functorial principle.

Associated to $B\Gamma^n_{sl}$ we have the normal bundle of its universal Haefliger structure. This bundle can be classified by a map $d\colon B\Gamma^n_{sl} \to B\,\mathrm{Sl}(n, \mathbf{R})$. The map d has a homotopy theoretic fibre $\overline{B}\Gamma^n_{sl}$. We choose models and maps so that we get an actual Hurewicz fibration

$$\overline{B}\Gamma^n_{sl} \xrightarrow{i} B\Gamma^n_{sl} \xrightarrow{d} B\,\mathrm{Sl}(n, \mathbf{R}). \tag{6}$$

The space $\overline{B}\Gamma^n_{sl}$ is the classifying space for codimension-n volume preserving Haefliger structures whose normal bundle is framed. This space is $(n-1)$-connected with $\pi_n(\overline{B}\Gamma^n_{sl}) = \mathbf{R}$. The class $i^*\tilde{\mu} \equiv \mu \in H^n(\overline{B}\Gamma^n_{sl})$ corresponds to the identity homomorphism if we identify $H^n(\overline{B}\Gamma^n_{sl})$ with $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{R}, \mathbf{R})$ in the canonical manner.

The foliation $\mathscr{F}$ on $\overline{B}\mathscr{D}\!\mathit{iff}_\omega M \times M$ can be classified by the map Φ into $B\Gamma^n_{sl}$. Consider the following diagram:

$$\begin{array}{ccc} \overline{B}\mathscr{D}\!\mathit{iff}_\omega M \times M & \xrightarrow{\Phi} & B\Gamma^n_{sl} \\ \downarrow \pi & & \downarrow d \\ M & \xrightarrow{\tau} & B\,\mathrm{Sl}(n, R) \end{array} \tag{7}$$

Here π is projection, and τ classifies TM (the choice of ω gives an Sl-structure). The normal bundle $\nu(\mathscr{F})$ of $\mathscr{F}$ is just $\overline{B}\mathscr{D}\!\mathit{iff}_\omega M \times TM$ since $\mathscr{F}$ is transverse to the M-factors. We classify $\nu(\mathscr{F})$ by $d \circ \Phi$. Note that $\nu(\mathscr{F})$ is also classified by $\tau \circ \pi$. Everything is chosen so that (7) commutes and d is a Hurewicz fibration.

We are interested in $\mathscr{L}(M)$, the space of lifts of τ. Let us define $\Pi\colon \overline{B}\mathscr{D}\!\mathit{iff}_\omega M \to \mathscr{L}(M)$ by setting $\Pi(b) \equiv \Phi(b, \cdot)$ for each $b \in \overline{B}\mathscr{D}\!\mathit{iff}_\omega M$. Since M is compact and n-dimensional while the fibre of d is $\overline{B}\Gamma^n_{sl}$, which is $(n-1)$-connected, the space $\mathscr{L}(M)$ is not connected. Because $\overline{B}\mathscr{D}\!\mathit{iff}_\omega M$ is connected, its image under Π is in one component, $\mathscr{L}_0(M)$, of $\mathscr{L}(M)$. McDuff [M–2], in the spirit of Thurston [T], has shown that Π^* is a cohomology isomorphism. This enables us to view $c_n(M)$ as living in $H^n(\mathscr{L}_0(M))$. Let us exploit this philosophy.

Define $\varepsilon\colon \mathscr{L}_0(M) \to \overline{B}\Gamma^n_{sl}$ as evaluation at the fixed point m_0 in M. Consider $\mathscr{F}|$, $\mathscr{F}$ restricted to $\overline{B}\mathscr{D}\!\mathit{iff}_\omega M \times m_0$. The bundle $\nu(\mathscr{F}|)$ is isomorphic to $\overline{B}\mathscr{D}\!\mathit{iff}_\omega M \times R^n$. This tell us that $\mathscr{F}|$ is a codimension-n Haefliger structure with trivial normal bundle. Due to this $\Phi|$, the map classifying $\mathscr{F}|$, is homotopic to a map with image in $\overline{B}\Gamma^n_{sl}$. Without loss of generality we may assume $\Phi|$ actually maps into $\overline{B}\Gamma^n_{sl}$. The following diagram homotopy commutes.

(8)
$$\begin{array}{ccc} \bar{B}\,\mathscr{D}\!\mathit{iff}_\omega M & & \\ \downarrow \Pi & \searrow \Phi| & \\ \mathscr{L}_0(M) & \overset{\varepsilon}{\to} & \bar{B}\Gamma^n_{sl} \end{array}$$

Recall that $c_n(M)$ is $[\Omega]|_{\bar{B}\mathscr{D}\mathit{iff}_\omega M \times m_0}$. Therefore, $\Phi^*\mu$ is $c_n(M)$. Since Π^* is an isomorphism,

(9)
$$c_n(M) \neq 0 \Leftrightarrow \varepsilon^*\mu \neq 0.$$

So the problem is now one of understanding the behavior of $\mathscr{L}_0(M) \overset{\varepsilon}{\to} \bar{B}\Gamma^n_{sl}$. Inherent in $\mathscr{L}_0(M)$ is the twisting of TM. The simpler that TM is, the easier it is to understand ε. Such a case is provided when M is *stably parallelizable*. By this we mean that $TM \oplus \varepsilon^1 \cong \varepsilon^{n+1}$. The most obvious example of such a manifold is S^n. In fact spheres "classify" stably parallelizable manifolds.

(10) PROPOSITION. *The manifold M is stably parallelizable if and only if $TM \cong \gamma^*(TS^n)$ for some map γ from $M \to S^n$.*

Before proceeding further we need a technical lemma which follows easily from obstruction theory.

(11) LEMMA. *If f_0 and f_1 are two lifts in $\mathscr{L}(M)$, they are in the same component of $\mathscr{L}(M)$ if and only if $f_0^*\tilde{\mu} = f_1^*\tilde{\mu}$.*

Since $\Pi(b)$ is $\Phi(b,\cdot)$ and $\Phi(b,\cdot)^*\tilde{\mu}$ is $[\Omega]|_{b\times M} = [\omega]$, we have

(12)
$$f \in \mathscr{L}_0(M) \Leftrightarrow f^*\tilde{\mu} = [\omega].$$

(13) THEOREM. *If M is an odd-dimensional stably parallelizable manifold, then $c_n(M) \neq 0$.*

REMARK. It is essential that n be odd, for McDuff **[M-1]** has shown that $c_{2n}(S^{2n}) = 0$ and $c_{2n+1}(S^{2n+1}) \neq 0$.

PROOF. Case 1. Suppose that M is stably parallelizable but not parallelizable. Then by (10) $TM \cong \gamma^*(TS^n)$, and γ has nonzero degree. Certainly we can give M a volume form ω_M such that $[\omega_M] = \gamma^*[\omega_S]$, where ω_S is the standard volume form on S^n. Consider the following diagram.

$$\begin{array}{ccccc} & & & & B\Gamma^n_{sl} \\ & \nearrow & & \nearrow & \downarrow \\ M & \overset{\gamma}{\to} & S^n & \overset{\tau}{\to} & B\,\mathrm{Sl}(n,R) \end{array}$$

The map τ classifies TS^n and therefore $\tau \circ \gamma$ classifies TM. Say $f \in \mathscr{L}_0(S^n)$; then $f^*\tilde{\mu} = [\omega_S]$. It follows that $f \circ \gamma \in \mathscr{L}_0(M)$. Hence, we have a map η: $\mathscr{L}_0(S^n) \to \mathscr{L}_0(M)$ given by $\eta(f) = f \circ \gamma$. Consider the next diagram, where ε_m is evaluation at $m_0 \in M$ and ε_s is evaluation at $\gamma(m_0)\varepsilon S^n$.

(15)
$$\begin{array}{ccc} \mathscr{L}_0(S^n) & \overset{\eta}{\to} & \mathscr{L}_0(M) \\ \downarrow \varepsilon_s & \swarrow \varepsilon_m & \\ \bar{B}\Gamma^n_{sl} & & \end{array}$$

Since $\varepsilon_m \circ \eta(f) = \varepsilon_m(f \circ \gamma) = (f \circ \gamma)(m_0) = \varepsilon_s \circ f$, the above diagram commutes. Since $c_n(S^n) \neq 0$ for n odd, (9) tells us that $\varepsilon_s^* \mu \neq 0$; therefore $\varepsilon_n^* \mu \neq 0$ and $c_n(M) \neq 0$.

Case 2. M is parallelizable. Let τ now stand for a map classifying TM into $B\,\mathrm{Sl}(n, R)$. Without loss of generality τ may be taken as a constant map. In this case (7) becomes

$$\begin{array}{ccccc} \overline{B}\mathscr{D}\!\mathit{iff}_\omega M \times M & \xrightarrow{\Phi} & \overline{B}\Gamma^n_{sl} & \to & B\Gamma^n_{sl} \\ \downarrow \pi & & \downarrow d & & \downarrow d \\ M & \xrightarrow{\tau} & * & \to & B\,\mathrm{Sl}(n, R) \end{array}$$

Since M is parallelizable, $\mathscr{L}(M) = \mathrm{Maps}(M, \overline{B}\Gamma^n_{sl})$ and we will designate the component corresponding to $\mathscr{L}_0(M)$ as $\mathrm{Maps}_1(M, \overline{B}\Gamma^n_{sl})$. Therefore, $s \in \mathrm{Maps}_1(M, \overline{B}\Gamma^n_{sl})$ if and only if $s^*\mu = [\omega]$. The evaluation map $\varepsilon\colon \mathscr{L}_0(M) \to \overline{B}\Gamma^n_{sl}$ becomes $\varepsilon\colon \mathrm{Maps}_1(M, \overline{B}\Gamma^n_{sl}) \to \overline{B}\Gamma^n_{sl}$. We wish to showthat $\varepsilon^*\mu \neq 0$. Choose ω so that $\langle[\omega], [M]\rangle = 1$.

Following McDuff's proof [M–1] that $\pi_n(\overline{B}\Gamma^n_{sl}) \cong \mathbf{R}$, we choose a map $f\colon S^n \to \overline{B}\Gamma^n_{sl}$ such that $[f] = 1 \in \pi_n(\overline{B}\Gamma^n_{sl})$, i.e. $\langle f^*\mu, [S^n]\rangle = 1$. Let us give S^n a volume form ω_S so that $f^*\mu = [\omega_S]$. We will say that $g\colon M \to S^n$ is of degree one if $g^*[\omega_S] = [\omega]$. Let $\mathrm{Maps}_1(M, S^n)$ be all the maps of degree one. Now define a map $\hat{f}\colon \mathrm{Maps}_1(M, S^n) \to \mathrm{Maps}_1(M, \overline{B}\Gamma^n_{sl})$ by setting $\hat{f}(g) = f \circ g$. Let ε and ε' be evaluation at m_0. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{Maps}_1(M, S^n) & \xrightarrow{\hat{f}} & \mathrm{Maps}_1(M, \overline{B}\Gamma^n_{sl}) \\ \downarrow \varepsilon' & & \downarrow \varepsilon \\ S^n & \xrightarrow{f} & \overline{B}\Gamma^n_{sl} \end{array} \tag{16}$$

Since $f^*\mu = [\omega_S]$, if we can show that $\varepsilon'^*[\omega_S] \neq 0$, we will have shown that $\varepsilon^*\mu \neq 0$.

$\mathrm{Maps}_1(S^n, S^n)$ is the space of maps from S^n to S^n such that $f^*[\omega_S] = [\omega_S]$ (as before). Let ξ be a fixed element of $\mathrm{Maps}_1(M, S^n)$ such that $\xi(m_0) = s$, the south pole of S^n. Define $\hat{\xi}\colon \mathrm{Maps}_1(S^n, S^n) \to \mathrm{Maps}_1(M, S^n)$ by $\hat{\xi}(h) = h \circ \xi$. The following diagram, with ε'' being evaluation at s, commutes.

$$\begin{array}{ccc} \mathrm{Maps}_1(S^n, S^n) & \xrightarrow{\hat{\xi}} & \mathrm{Maps}_1(M, S^n) \\ \downarrow \varepsilon'' & & \downarrow \varepsilon' \\ S^n & \xrightarrow{\text{identity}} & S^n \end{array} \tag{17}$$

If we can show that $\varepsilon''^*[\omega_S] \neq 0$, then we will have shown that $\varepsilon'^*[\omega_S] \neq 0$ and we will be done. The left-hand side of (17) is from the fibration

$$\Omega^n S^n_1 \to \mathrm{Maps}_1(S^n, S^n) \to S^n, \tag{18}$$

where $\Omega^n S^n_1$ is the obvious component. Using the spectral sequence of (18) and the fact that n is odd, we have $\varepsilon''^*[\omega_S] \neq 0$. Q.E.D.

We will now prove

(19) THEOREM. *If M is parallelizable then $c_n(M) \neq 0$.*

REMARK. Here we have removed the condition of n being odd but yet we still use S^n in our argument.

PROOF. Since M is parallelizable it may be immersed in $\mathbf{R}^{n+1}$ [H]. We will consider M as being immersed in $\mathbf{R}^{n+1}$ with a fixed immersion. We will also freely identify M with its image in $\mathbf{R}^{n+1}$. This causes no trouble, as will become apparent. Our goal is to define a map $\Psi\colon M \to \mathrm{Maps}_1(M, S^n)$ such that if ε' is evaluation at m_0 then $\varepsilon' \circ \Psi\colon M \to S^n$ has nonzero degree. If this is true then $\varepsilon'^*[\omega_S]$ is nonzero, where ω_S is the volume form on S^n. Let $\overline{\Psi}$ be the composite $\varepsilon' \circ \Psi$.

(20)
$$\begin{array}{ccc} M & \xrightarrow{\Psi} & \mathrm{Maps}_1(M, S^n) \\ & \searrow^{\overline{\Psi}} & \downarrow \varepsilon' \\ & & S^n \end{array}$$

Choose positive ε less than the injectivity radius of M and small enough so that every (open) ball $B(p, \varepsilon)$ with center $p \in M$ and radius ε is embedded by the immersion. Consider $\mathrm{Exp}^{-1}\colon B(p, \varepsilon) \to TM_p$. In fact, the image is in $B(O_p, \varepsilon) \subset TM_p$. Let γ be the Gauss map from M to S^n. By parallel translation in $\mathbf{R}^{n+1}$ we get a congruence from $TM_p \to TS^n_{\gamma(p)}$. Remembering that π is the injectivity radius of the sphere we see that Exp maps $B(\vec{O}_{\gamma(p)}, \pi) \subset TS^n_{\gamma(p)}$ diffeomorphically onto $S^n - A(\gamma(p))$, where A is the antipodal map on S^n. Let $\zeta\colon TS^n \to TS^n$ by $\zeta(\vec{v}) = \pi/\varepsilon \cdot \vec{v}$. We are now in a position to define $\Psi\colon M \to \mathrm{Maps}_1(M, S^n)$.

Decompose M as $B(p, \varepsilon) \cup \{M - B(p, \varepsilon)\}$:

(21)
$$\begin{array}{ccccc} B(p,\varepsilon) & \xrightarrow{\mathrm{Exp}^{-1}} & B(\vec{O}_p, \varepsilon) & \xrightarrow{\|\text{-trans}} & B(\vec{O}_{\gamma(p)}, \varepsilon) \\ & \searrow & & & \downarrow \zeta \\ & & \text{diffeomorphism} & & B(\vec{O}_{\gamma(p)}, \pi) \\ & & & \searrow & \downarrow \mathrm{Exp} \\ & & & & S^n - A(\gamma(p)) \end{array}$$

If $m \in B(p, \varepsilon)$ define $\Psi(p)m$ to be the image of m under the above composition. At this stage we want the orientation on $B(p, \varepsilon)$, induced from the orientation of M, to go to the usual orientation on $S^n - A(\gamma(p))$. If it does not we give M a different orientation and volume form. If $m \notin B(p, \varepsilon)$ set $\Psi(p)m \equiv A(\gamma(p))$.

As we vary p we get a continuous map $\Psi\colon M \to \mathrm{Maps}_1(M, S^n)$. Each $\Psi(p)$ is of degree one, for $\Psi(p)$ is just a standard collapsing map of degree one.

(22) LEMMA. $\deg \overline{\Psi} = (-1)^{n+1} \deg \gamma + (-1)^n$, *where* deg *stands for degree of the map.*

PROOF OF LEMMA. We just sketch the proof since the techniques are standard. If necessary, the first thing we do is adjust ε in our definition of Ψ to make sure that in a neighborhood D containing $B(m_0, \varepsilon)$ we can slightly deform M so that D is flat.

Thus, the Gauss map γ is constant on D, hence $\gamma(D) \equiv \gamma(m_0)$. On $M - D$, $\overline{\Psi}$ sends x to $\Psi(x)m_0 = A(\gamma(x))$. If $x \in D$ then we must be careful. If $x \in D - B(M_0, \varepsilon)$, then $\overline{\Psi}(x) = \Psi(x)m_0 = A(\gamma(x)) = A(\gamma(m_0))$ since D is flat. The map $\overline{\Psi}$ restricted to $B(m_0, \varepsilon)$ has degree $(-1)^n$. $\overline{\Psi}$ is not smooth on $\overline{\partial B(m_0, \varepsilon)}$. Therefore, we replace it by a map $\overline{\Psi}'$ in the same homotopy class and very close to $\overline{\Psi}$ and calculate the Brouwer degree. The $(-1)^{n+1}\deg\gamma$ part comes from $A \circ \gamma$, while $(-1)^n$ is due to the fact that $\overline{\Psi}$, near m_0, is a local diffeomorphism. By a judicious choice of a regular value it is easy to show that $\overline{\Psi}'$, and hence $\overline{\Psi}$, has the proper degree. The *proof* of the *lemma* is *complete*.

Remember that we are trying to show that if M is parallelizable then $c_n(M) \neq 0$. By (16) if we can show that $\varepsilon'^*[\omega_s] \neq 0$ we will be done. By the previous lemma $\overline{\Psi}$ has degree $(-1)^n + (-1)^{n+1}\deg\gamma$. If we can show that this is nonzero, then $\varepsilon'^*[\omega_s] \neq 0$. This is our plan. We will vary our immersion of M so that we get $\deg\gamma$ to our liking. To accomplish this we appeal to some results of Hopf **[Ho, Mi]**.

(23) THEOREM (HOPF). *Let L: $M^n \to \mathbf{R}^{n+1}$ be an immersion with corresponding Gauss map $\gamma(L)$: $M^n \to S^n$.*

(a) *If n is even*, $\deg\gamma(L) = \frac{1}{2}\chi(M)$, *where $\chi(M)$ is the Euler characteristic of M.*

(b) *If n is odd and* $\deg\gamma(L) = k$, *then given any m one can find an immersion $j = j(m)$ such that* $\deg\gamma(j) = k + 2m$.

Thus $\overline{\Psi}$ can be taken to have nonzero degree. Q.E.D.

REMARK. It is worth pointing out that all 3-manifolds and products of spheres with one factor being odd are parallelizable and therefore have $c_n(M) \neq 0$.

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