TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 287, Number 2, February 1985

SOME ESTIMATES FOR NONDIVERGENCE STRUCTURE, SECOND ORDER ELLIPTIC EQUATIONS

BY

LAWRENCE C. EVANS[1]

ABSTRACT. We obtain various formal estimates for solutions of nondivergence structure, second order, uniformly elliptic PDE. These include interior lower bounds and also gradient estimates in L^p, for some $p > 0$.

1. Introduction. This paper presents some new estimates for solutions of nondivergence structure, second order, uniformly elliptic PDE of the form

$$\begin{cases} -a_{ij}u_{x_ix_j} = \chi_E & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and, more generally,

$$\begin{cases} -a_{ij}u_{x_ix_j} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Ω is a bounded, smooth, open subset of $\mathbf{R}^n$, $E \subset \Omega$ is measurable, f is bounded and measurable, and the symmetric coefficients a_{ij} satisfy the uniform ellipticity condition:

$$\begin{cases} \text{there exist real numbers } \Theta, \theta > 0 \text{ such that} \\ \theta|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \Theta|\xi|^2 \ (\xi \in \mathbf{R}^n, x \in \Omega). \end{cases} \tag{A}$$

Moreover we will implicity assume throughout that the a_{ij} are smooth on $\bar{\Omega}$, so that solutions of (1.1), (1.2) exist and belong at least to the Sobolev spaces $W^{2,p}(\Omega)$ $(1 \leqslant p < \infty)$. However all the bounds we derive are independent of the moduli of continuity of the a_{ij} and so depend on these coefficients only through θ and Θ.

Estimates for equation (1.2) which do not depend on the continuity of the coefficients have long been studied, both for their own sake and for applications to nonlinear PDE. The principal discoveries so far include the two dimensional estimates of Bernstein and of Nirenberg [**4**, §12.2], the supremum-norm bounds of Aleksandrov-Bakelman-Pucci [**4**, §9.1], and the recent, important Hölder and Harnack estimates of Krylov-Safonov [**6**]. (See also [**4**, §9.8, **7**, **9**].) One special case of these latter results states, roughly, that if $\Omega' \subset\subset \Omega$ is a connected and compactly

Received by the editors February 29, 1984 and, in revised form, May 23, 1984.
1980 *Mathematics Subject Classification*. Primary 35J15.
Key words and phrases. Second order elliptic PDE, nondivergence structure, mean sojourn time.
[1]Supported in part by NSF grant MCS-8301265.

contained subdomain, $E \subset \Omega'$, and u solves

$$(1.3)\qquad \begin{cases} -a_{ij}u_{x_ix_j} = 0 & \text{in } \Omega \setminus E, \\ u = 1 & \text{on } \partial E, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then

$$(1.4)\qquad \inf_{\Omega'} u \geqslant C|E|^l$$

for certain constants C, $l > 0$, depending only on θ, Θ, Ω' and Ω. This says probabilistically that a sample path of the diffusion associated with the above elliptic operator will with probability at least $C|E|^l$ hit the set E before it first exits Ω, provided it begins inside Ω'.

Our first theorem below provides a complementary estimate, this on the *mean sojourn time* of sample paths within the set E. In PDE terms we prove

THEOREM 1. *Let $\Omega' \subset\subset \Omega$ be a connected subdomain. There exist positive constants $m = m(n, \theta, \Theta)$ and $C(\Omega') = C(\Omega', \Omega, \theta, \Theta)$ such that*

$$\inf_{\Omega'} u \geqslant C(\Omega')|E|^m,$$

whenever E is a measurable subset of Ω' and u solves (1.1).

Probabilistically, a sample path of the diffusion will, if it begins in Ω', on the average thus spend at least $C(\Omega')|E|^m$ units of time in E before it first exits Ω.

Notice this is not a consequence of the Krylov-Safonov result: that a sample path hits the set E is no guarantee it will spend a positive amount of time within E. Nevertheless our proof in §§2 and 3 is strongly based on methods from [**6**], especially the idea of considering small cubes large fractions of whose interiors belong to E.

Theorem 1 has an immediate application in deriving a (quite weak) gradient estimate for the solution of (1.2):

THEOREM 2. *Assume u solves* (1.2). *Then there exist positive constants C and p, depending only on n, Ω, θ, and Θ, such that $\|Du\|_{L^p} \leqslant C\|f\|_{L^n}$.*

The proof, in §4, is based upon a duality argument of a type employed by Bensoussan, Lions and Papanicolaou in [**2**, §3.5.2]. As noted, this bound is rather poor; but in light of known pathologies for the operator $L = -a_{ij}\partial^2/\partial x_i \partial x_j$ (cf. Miller [**8**], Bauman [**1**]), it is not at all clear what kind of gradient estimate could be expected.

A consequence of Theorem 2 is this

COROLLARY. *Assume F: $\mathbf{R}^{n\times n} \to \mathbf{R}$ is smooth and satisfies the uniform ellipticity condition*

$$(1.5)\qquad \theta|\xi|^2 \leqslant \frac{\partial F}{\partial r_{ij}}(r)\xi_i\xi_j \leqslant \Theta|\xi|^2 \qquad (r \in \mathbf{R}^{n\times n}, \xi \in \mathbf{R}^n)$$

for some $0 < \theta < \Theta$. *Let u be a smooth solution of the fully nonlinear elliptic PDE*

$$F(D^2u) = 0 \quad \text{in } \Omega. \tag{1.6}$$

Then for each $\Omega' \subset\subset \Omega$ *there exist positive constants* p *and* C, *depending only on* $n, \Omega', \Omega, \theta$ *and* Θ *such that*

$$\|D^2u\|_{L^p(\Omega')} \leqslant C\|u\|_{W^{1,\infty}(\Omega)}.$$

This bound is fairly pitiful, but is to my knowledge the only a priori second derivative estimate for (1.6) available if $n \geqslant 3$ under just the natural hypothesis (1.5). Of course if F satisfies additional structure assumptions, far stronger estimates can be had: see [**4**, §17.4] for F concave, [**3**, §§2–3] for F "linear at infinity", and [**5**, p. 42] for F "close to linear". N. S. Trudinger has noted that a variant of the corollary provides *formal* estimates on the third derivatives in L^p_{loc} (for some $p > 0$) for a solution of Bellman's equation; this is explained in the concluding remark of §5.

Notation and terminology. $A(x) = ((a_{ij}(x)))$, $|E|$ = Lebesgue measure of the (measurable) set E.

All cubes mentioned below are closed and have sides parallel to some fixed coordinate axes; if Q is a cube with side R, αQ denotes the cube with the same center and side αR.

We employ the summation convention throughout; the letter C denotes various constants depending only on known quantities.

ACKNOWLEDGEMENT. I wish to thank A. Bensoussan for bringing to my attention to duality argument from [**2**] and N. S. Trudinger for several useful comments, including the remark mentioned above. I also thank the referee for a very thorough reading of this paper.

NOTED ADDED IN PROOF. A forthcoming paper of E. Fabes and D. Stroock presents a new approach to [**6, 7, 9**] and this paper, via the theory of A_∞ weights.

2. Estimates concerning cubes. We temporarily assume, in addition to hypothesis (A), that

$$\Omega \text{ and } \Omega' \text{ are cubes}, \quad \Omega = 4\Omega', \tag{B}$$

and

$$\begin{cases} \text{there exists a real number } \alpha \text{ such that} \\ \operatorname{tr} A(x) \geqslant \alpha > 2\Theta \qquad (x \in \Omega). \end{cases} \tag{C}$$

These restrictions will be removed in §3.

LEMMA 2.1. *There exist constants* $0 < \beta, \gamma < 1$ *such that if* $Q \subset \Omega'$ *is a cube,* $E \subset \Omega'$ *is measurable and*

$$|Q \cap E| \geqslant \beta|Q|, \tag{2.1}$$

then

$$u \geqslant \gamma v \quad \text{in } \Omega, \tag{2.2}$$

where u and v solve the PDE

$$(2.3)\qquad \begin{cases} -a_{ij}v_{x_ix_j} = \chi_{3Q} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

$$(2.4)\qquad \begin{cases} -a_{ij}u_{x_ix_j} = \chi_{E\cap Q} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The constants β and γ here depend only on n, θ, Θ and α, and not on the size of Q.

PROOF. Upon translation and scaling, we may suppose Q to be the unit cube, with side 2 and center 0. Let w be the solution of

$$(2.5)\qquad \begin{cases} -a_{ij}w_{x_ix_j} = \chi_Q & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Estimate on u and w from above. Define

$$r(x) \equiv 1/|x|^p \qquad (x \neq 0),$$

for $p > 0$ to be selected in a moment. We compute

$$\begin{aligned} -a_{ij}r_{x_ix_j} &= \frac{p}{|x|^{p+2}}\left(\operatorname{tr} A - (p+2)a_{ij}\frac{x_ix_j}{|x|^2}\right) \\ &\geqslant \frac{p}{|x|^{p+2}}(\operatorname{tr} A - (p+2)\Theta). \end{aligned}$$

From condition (C) it follows that $-a_{ij}r_{ij} \geqslant C_1/|x|^{p+2}$ for some $C_1 > 0$, provided $p > 0$ is taken sufficiently small. Thus (2.3) implies

$$(2.6)\qquad 0 \leqslant v \leqslant C_2/|x|^p \qquad (x \in \Omega,\ x \neq 0).$$

This bound and the estimate of Aleksandrov-Bakelman [**4**, p. 220] give also

$$(2.7)\qquad 0 \leqslant v \leqslant C_3 \qquad (x \in \Omega).$$

Since $0 \leqslant w \leqslant v$ in Ω, we have

$$(2.8)\qquad 0 \leqslant w \leqslant C_3/|x|^p, \qquad 0 \leqslant w \leqslant C_3.$$

Estimate on w from below. Set

$$s(x) \equiv \varepsilon\left(1 - |x|^2\right)$$

and fix $\varepsilon > 0$ so that

$$-a_{ij}s_{x_ix_j} \leqslant 1 \quad \text{in } B(1).$$

Since $B(1) \subset Q$ we have $w \geqslant s$ in $B(1)$, whence

$$(2.9)\qquad w(x) \geqslant C_4 \quad \text{in } B(1/2)$$

for some $C_4 > 0$. In view of the weak Harnack inequality for supersolutions [**4**, p. 246] it follows that

$$(2.10)\qquad w \geqslant C_5 \quad \text{in } 3Q$$

for some $C_5 > 0$.

Estimate on $w - u$. Extend $w - u \geqslant 0$ to be zero in $\mathbf{R}^n \setminus \Omega$; then

$$-a_{ij}(w-u)_{x_i x_j} \leqslant \chi_{Q\setminus E} \quad \text{in } \mathbf{R}^n. \tag{2.11}$$

For each $\mu > 0$ the Aleksandrov-Bakelman estimate gives

$$\begin{aligned} \max_{B(\mu)} |w-u| &\leqslant C\mu \|\chi_{Q\setminus E}\|_{L^n} + \max_{\partial B(\mu)} |w-u| \\ &\leqslant C\mu |Q\setminus E|^{1/n} + \max_{\partial B(\mu)} |w| \\ &\leqslant C_6\left[\mu(1-\beta)^{1/n} + \frac{1}{\mu^p}\right], \end{aligned} \tag{2.12}$$

by (2.1) and (2.8).

Completion of proof. In light of (2.10), (2.12) and (2.8) we have

$$\begin{aligned} u = w - (w-u) &\geqslant C_5 - C_6\left[\mu(1-\beta)^{1/n} + 1/\mu^p\right] \\ &\geqslant C_5/2 \geqslant C_7 w \quad \text{in } Q \end{aligned}$$

for some $C_7 > 0$, provided we first choose μ large enough and then β sufficiently close to 1.

Furthermore

$$\begin{cases} -a_{ij}(u - C_7 w)_{x_i x_j} = 0 & \text{in } \Omega \setminus Q, \\ u - C_7 w = 0 & \text{on } \partial\Omega, \end{cases}$$

and so

$$u \geqslant C_7 w \quad \text{in } \Omega. \tag{2.13}$$

Next observe from (2.7) and (2.10) that for some $C_8 > 0$

$$u \geqslant C_8 v \quad \text{in } 3Q;$$

whence follows (2.2). □

Notation. If Q is a cube, we denote by $\hat{Q}$ any one of the 2^n subcubes formed by bisecting the sides of Q.

LEMMA 2.2. *Let E be a measurable subset of the cube Ω' satisfying $|E| < \beta|\Omega'|$. Then there exists a countable collection of cubes $\{Q_l\}$ in Ω' such thąt*

(a) $\operatorname{int} Q_k \cap \operatorname{int} Q_l = \varnothing$ $(k \neq l)$,

(b) $|E \setminus \bigcup_l Q_l| = 0$,

(c) $\sum_l |Q_l| \geqslant |E|/\beta$, *and*

(d) *for each cube Q_l there exists at least one subcube $\hat{Q}_l$ satisfying $|E \cap \hat{Q}_l| \geqslant \beta|\hat{Q}_l|$.*

This result is essentially due to Krylov and Safonov [**6**]; see also [**4**, pp. 226–227, **9**, p. 853].

LEMMA 2.3. *Let $E \subset \Omega'$, $|E| < \beta|\Omega'|$ and set $F \equiv \bigcup_l Q_l$. Then*

$$u \geqslant \gamma v \quad \textit{in } \Omega, \tag{2.14}$$

where u and v solve the PDE

$$(2.15)\qquad \begin{cases} -a_{ij}v_{x_ix_j} = \chi_F & \textit{in } \Omega, \\ v = 0 & \textit{on } \partial\Omega, \end{cases}$$

$$(2.16)\qquad \begin{cases} -a_{ij}u_{x_ix_j} = \chi_E & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Here β, γ are the constants from Lemma 2.1 *and* $\{Q_l\}$ *the cubes described in Lemma* 2.2.

PROOF. We have

$$u = \sum_l u^l, \qquad v = \sum_l v^l,$$

where

$$\begin{cases} -a_{ij}v^l_{x_ix_j} = \chi_{Q_l} & \text{in } \Omega, \\ v^l = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -a_{ij}u^l_{x_ix_j} = \chi_{E\cap Q_l} & \text{in } \Omega, \\ u^l = 0 & \text{on } \partial\Omega. \end{cases}$$

For each Q_l fix some subcube $\hat{Q}_l$ satisfying

$$(2.17)\qquad |E \cap \hat{Q}_l| \geqslant \beta|\hat{Q}_l|.$$

Introduce the functions $\hat{u}^l$ and $\hat{v}^l$ solving

$$\begin{cases} -a_{ij}\hat{v}^l_{x_ix_j} = \chi_{3\hat{Q}_l} & \text{in } \Omega, \\ \hat{v}^l = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -a_{ij}\hat{u}^l_{x_ix_j} = \chi_{E\cap \hat{Q}_l} & \text{in } \Omega, \\ \hat{u}^l = 0 & \text{on } \partial\Omega. \end{cases}$$

According to Lemma 2.1 and (2.17) $\hat{u}^l \geqslant \gamma\hat{v}^l$. But $u^l \geqslant \hat{u}^l$ and $\hat{v}^l \geqslant v^l$ (since $Q_l \subset 3\hat{Q}_l$). Thus $u^l \geqslant \gamma v^l$ and so

$$u = \sum_l u^l \geqslant \gamma \sum_l v^l = \gamma v. \quad \square$$

3. Proof of Theorem 1. Continue for the time being in assuming conditions (B) and (C).

We claim there exist constants m, $C(\Omega') > 0$ such that for $k = 1, 2, \ldots$

$$(3.1)_k\qquad \begin{cases} \text{if } E \subset \Omega' \text{ is measurable, } u \text{ solves } (1.1) \text{ and } |E| \geqslant \beta^k|\Omega'|, \text{ then} \\ \qquad \inf_{\Omega'} u \geqslant C(\Omega')|E|^m. \end{cases}$$

The proof is by induction. First note that, upon our redefining β to be closer to 1 if necessary, the methods for §2 show $\inf_{\Omega'} u \geqslant C(\Omega')$ for some constant $C(\Omega') > 0$

provided $|E| \geqslant \beta|\Omega'|$ and u solves (1.1). Now set

$$m = \log\gamma/\log\beta > 0 \tag{3.2}$$

and observe then that $(3.1)_1$ is valid for some possibly new constant $C(\Omega') > 0$.

Now assume $(3.1)_k$ and suppose u solves (1.1), where

$$\beta^k|\Omega'| > |E| \geqslant \beta^{k+1}|\Omega'|.$$

Define $F \equiv \bigcup_l Q_l$, the cubes $\{Q_l\}$ taken from Lemma 2.2. We have

$$|F| = \sum |Q_l| \geqslant \frac{|E|}{\beta} \geqslant \beta^k|\Omega'|; \tag{3.3}$$

whence by the induction hypothesis

$$\inf_{\Omega'} v \geqslant C(\Omega')|F|^m,$$

v solving (2.15). Then Lemma 2.3 implies

$$\begin{aligned}\inf_{\Omega'} u \geqslant \gamma \inf_{\Omega'} v &\geqslant C(\Omega')\gamma|F|^m\\ &\geqslant C(\Omega')\frac{\gamma}{\beta^m}|E|^m \qquad \text{by (3.3)}\\ &= C(\Omega')|E|^m \quad \text{by (3.2)}.\end{aligned}$$

This proves $(3.1)_{k+1}$ and completes the induction.

We have consequently established Theorem 1 under the additional assumptions (B) and (C), which we now remove.

First we retain hypothesis (C), but now drop (B). Thus assume $\Omega' \subset\subset \Omega$ are open, bounded subsets of $\mathbf{R}^n$, Ω' connected. There exists a finite collection of cubes $\{Q_k\}_{k=1}^N$ with equal sides and disjoint interiors such that

$$\Omega' \subset \bigcup_{k=1}^N Q_k \subset \bigcup_{k=1}^N 4Q_k \subset \Omega.$$

If $E \subset \Omega'$, then $|E \cap Q_k| \geqslant |E|/N$ for some $k \in \{1,\dots,N\}$. Let v solve

$$\begin{cases} -a_{ij}v_{x_ix_j} = \chi_{E\cap Q_k} & \text{in } 4Q_k,\\ v = 0 & \text{on } \partial(4Q_k).\end{cases}$$

Then the calculations before show

$$\inf_{Q_k} v \geqslant C(Q_k)|E \cap Q_k|^m \geqslant C_9|E|^m$$

for an appropriate constant C_9. Since $u \geqslant v$ and Ω' is connected the estimates of Krylov and Safanov **[6]** give

$$\inf_{\Omega'} u \geqslant C(\Omega')|E|^m.$$

Finally we eliminate the restrictive hypothesis (C). Set

$$\tilde{\Omega} \equiv \Omega \times (-1,1) \times (-1,1) \subset \mathbf{R}^{n+2},$$

$$\tilde{\Omega}' \equiv \Omega' \times (-\tfrac{1}{2},\tfrac{1}{2}) \times (-\tfrac{1}{2},\tfrac{1}{2}), \quad \tilde{E} \equiv E \times (-\tfrac{1}{2},\tfrac{1}{2}) \times (-\tfrac{1}{2},\tfrac{1}{2}),$$

and then define

$$\tilde{A} \equiv \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & \vdots & \vdots \\ 0 & \cdots & \cdots & \Theta & \vdots \\ & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \Theta \end{pmatrix}_{(n+2)\times(n+2)}.$$

Since

$$\operatorname{tr}\tilde{A} = \operatorname{tr}A + 2\Theta \geqslant n\theta + 2\Theta \equiv \alpha > 2\Theta,$$

condition (C) holds for $\tilde{A} = ((\tilde{a}_{ij}))$. Thus if $\tilde{u}$ is the solution of

$$\begin{cases} -\tilde{a}_{ij}\tilde{u}_{x_ix_j} = \chi_{\tilde{E}} & \text{in } \tilde{\Omega}, \\ \tilde{u} = 0 & \text{on } \partial\tilde{\Omega}, \end{cases}$$

we have

$$\inf_{\tilde{\Omega}'} \tilde{u} \geqslant C|\tilde{E}|^m \tag{3.4}$$

for certain constants $C = C(\Omega')$, m. Now if u solves (1.1) and we regard u as really being a function of $(x_1, \ldots, x_{n+2})$, we have

$$\begin{cases} -\tilde{a}_{ij}u_{x_ix_j} = \chi_{\tilde{F}} & \text{in } \tilde{\Omega}, \\ u \geqslant 0 & \text{on } \partial\tilde{\Omega}, \end{cases}$$

for

$$\tilde{F} \equiv E \times (-1,1) \times (-1,1) \supseteq \tilde{E}.$$

Hence $u \geqslant \tilde{u}$ in $\tilde{\Omega}$; and so (3.4) implies $\inf_{\Omega'} u \geqslant C|E|^m$. □

4. Proof of Theorem 2. Choose some smooth, bounded, connected domain $\Omega \subset\subset \Omega'$. According to the Fredholm alternative there exists a solution w of

$$\begin{cases} -(a_{ij}w)_{x_ix_j} = 1 & \text{in } \Omega', \\ w = 0 & \text{on } \partial\Omega'. \end{cases} \tag{4.1}$$

Estimate on w from above. Select any $f \in L^n(\Omega')$ and let v solve

$$\begin{cases} -a_{ij}v_{x_ix_j} = f & \text{in } \Omega', \\ v = 0 & \text{on } \partial\Omega'. \end{cases}$$

Then

$$\begin{aligned} \int_{\Omega'} wf\,dx &= \int_{\Omega'} w(-a_{ij}v_{x_ix_j})\,dx \\ &= \int_{\Omega'} -(a_{ij}w)_{x_ix_j} v\,dx \\ &= \int_{\Omega'} v\,dx \leqslant C\|f\|_{L^n(\Omega')}, \end{aligned}$$

and so

$$\|w\|_{L^{n/(n-1)}(\Omega')} \leqslant C_{10}. \tag{4.2}$$

Estimate on w from below. Let $E \subset \Omega$ be measurable and suppose v now solves

$$\begin{cases} -a_{ij}v_{x_ix_j} = \chi_E & \text{in } \Omega', \\ v = 0 & \text{on } \partial\Omega'. \end{cases}$$

Then

$$\int_E w\,dx = \int_{\Omega'} w(-a_{ij}v_{x_ix_j})\,dx = \int_{\Omega'} v\,dx \geqslant C_{11}|E|^m$$

for certain constants $m, C_{11} > 0$, according to Theorem 1. Thus, in particular, $w > 0$ a.e. in Ω'. Fix $\lambda > 0$ and take $E \equiv \{x \in \Omega \mid 1/w(x) > \lambda\}$. Then

$$|E|/\lambda \geqslant \int_E w\,dx \geqslant C_{11}|E|^m.$$

Consequently

$$|E| \leqslant C_{12}\lambda^{-1/(m-1)},$$

and this in turn implies

$$\int_\Omega \frac{dx}{w(x)^\alpha} \leqslant C_{13} \tag{4.3}$$

for C_{13} and $0 < \alpha < 1/(m-1)$.

Estimate on Du. Now assume u solves (1.2) in Ω. We multiply both sides of the equation by wu and integrate by parts to obtain

$$\int_\Omega (a_{ij}wu)_{x_j}u_{x_i}\,dx = \int_\Omega fwu\,dx. \tag{4.4}$$

The term on the left equals

$$\begin{aligned} &\int_\Omega wa_{ij}u_{x_i}u_{x_j} + (a_{ij}w)_{x_j}\left(\frac{u^2}{2}\right)_{x_i} dx \\ &= \int_\Omega wa_{ij}u_{x_i}u_{x_j} - (a_{ij}w)_{x_ix_j}\frac{u^2}{2}\,dx \\ &\geqslant \int_\Omega \theta w|Du|^2 + \frac{u^2}{2}\,dx. \end{aligned}$$

Since $\|u\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^n(\Omega)}$, the calculation above, (4.4) and (4.2) yield

$$\int_\Omega w|Du|^2\,dx \leqslant C_{14}\|f\|^2_{L^n(\Omega)}. \tag{4.5}$$

Finally observe that

$$\int_\Omega |Du|^p\,dx = \int_\Omega |Du|^p\frac{w^q}{w^q}\,dx \leqslant \left(\int_\Omega |Du|^{pr}w^{qr}\,dx\right)^{1/r}\left(\int_\Omega \frac{dx}{w^{qs}}\right)^{1/s}$$

for any $p, q, r, s > 0$, $1/r + 1/s = 1$. Choose

$$p = \frac{2\alpha}{\alpha + 1}, \quad q = \frac{\alpha}{\alpha + 1}, \quad r = \frac{\alpha + 1}{\alpha}, \quad s = \alpha + 1,$$

where α is the constant from (4.3). Then

$$\begin{aligned} \int_\Omega |Du|^p \, dx &\leqslant \left(\int_\Omega |Du|^2 w \, dx \right)^{\alpha/(\alpha+1)} \left(\int_\Omega \frac{dx}{w^\alpha} \right)^{1/(\alpha+1)} \\ &\leqslant \left(C_{14} \|f\|_{L^n}^2 \right)^{\alpha/(\alpha+1)} C_{13}^{1/(\alpha+1)} = C_{15} \|f\|_{L^n(\Omega)}^p, \end{aligned}$$

owing to (4.3) and (4.5). □

5. Proof of the corollary. Now assume u to be a smooth solution of (1.6) in Ω and choose $\Omega' \subset\subset \Omega$. Let w solve

$$\begin{cases} -(a_{ij} w)_{x_i x_j} = 1 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Select a smooth cutoff function ζ which equals 1 in Ω', 0 near $\partial\Omega$. Differentiate (1.6) with respect to x_k:

$$\frac{\partial F}{\partial r_{ij}}(D^2 u) u_{x_k x_i x_j} = 0.$$

We set

$$v \equiv u_{x_k}, \qquad a_{ij} \equiv \frac{\partial F}{\partial r_{ij}}(D^2 u)$$

to obtain

$$-a_{ij} v_{x_i x_j} = 0 \quad \text{in } \Omega.$$

Multiply by $\zeta^2 w v$ and integrate by parts to find

$$\int_\Omega \zeta^2 w a_{ij} v_{x_i} v_{x_j} + (a_{ij} w)_{x_j} \zeta^2 v v_{x_i} \, dx = -\int_\Omega 2 a_{ij} w \zeta \zeta_{x_j} v v_{x_i} \, dx.$$

The second term on the left equals

$$\begin{aligned} &-\int_\Omega (a_{ij} w)_{x_i x_j} \zeta^2 \frac{v^2}{2} + (a_{ij} w)_{x_j} \zeta \zeta_{x_i} v^2 \, dx \\ &\qquad = \int_\Omega \zeta^2 \frac{v^2}{2} + a_{ij} w \left((\zeta \zeta_{x_i})_{x_j} v^2 + 2 \zeta \zeta_{x_i} v v_{x_j} \right) dx. \end{aligned}$$

Consequently

$$\begin{aligned} \theta \int_\Omega \zeta^2 w |Dv|^2 \, dx &\leqslant -\int_\Omega 4 a_{ij} w \zeta \zeta_{x_j} v v_{x_i} + a_{ij} w (\zeta \zeta_{x_i})_{x_j} v^2 \, dx \\ &\leqslant \frac{\theta}{2} \int_\Omega \zeta^2 w |Dv|^2 \, dx + C \int_\Omega w v^2 \left(|D\zeta|^2 + \zeta |D^2 \zeta| \right) dx; \end{aligned}$$

whence

$$\int_{\Omega'} w|Dv|^2\,dx \leqslant C\int_{\Omega} wv^2\,dx \leqslant C\|v\|^2_{L^\infty(\Omega)}.$$

Since $v = u_{x_k}$ is any derivative of u, the estimate above and the calculations in §4 show

$$\int_{\Omega'} |D^2u|^p\,dx \leqslant C_{16}\|u\|^p_{W^{1,\infty}(\Omega)}. \quad \square$$

REMARK. The following observation is due to N. S. Trudinger.

Suppose in addition to (1.5) that F is concave. Differentiate (1.6) twice in some direction η and discard the term involving D^2F to find

$$-\frac{\partial F}{\partial r_{ij}}(D^2u)u_{\eta\eta x_i x_j} \leqslant 0 \quad \text{in } \Omega.$$

Define a_{ij} as above and set $v = u_{\eta\eta}$. This gives

$$-a_{ij}v_{x_i x_j} \leqslant 0 \quad \text{in } \Omega. \tag{5.1}$$

Set $K \equiv \|D^2u\|_{L^\infty}$, take ζ and w as above, multiply (5.1) by $\zeta^2 w(v+K)$ and mimic then the foregoing calculations to find

$$\int_{\Omega'} |Dv|^p\,dx \leqslant C \qquad (C, p > 0).$$

We then derive the bound

$$\int_{\Omega'} |D^3u|^p\,dx \leqslant C_{17}, \tag{5.2}$$

C_{17} depending on $\|u\|_{W^{2,\infty}}$ and the other usual quantities.

A review of the proof shows that the various constants here depend on F only through θ, Θ and $\|DF\|_{L^\infty}$. Estimate (5.2) thus *formally* applies to the Bellman equation

$$\inf_{\alpha\in A}\{L^\alpha u\} = 0 \quad \text{in } \Omega,$$

where each L^α is a linear, constant coefficient elliptic operator with ellipticity bounds θ, Θ.

It is not at all clear if any of this can be made rigorous. $\square$

REFERENCES

1. P. Bauman, *Equivalence of the Green's functions for diffusion operators in* $\mathbf{R}^n$: *A counterexample*, Proc. Amer. Math. Soc. **91** (1984), 64–68.

2. A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, New York, 1978.

3. L. C. Evans, *On solving certain nonlinear PDE by accretive operator methods*, Israel J. Math. **36** (1980), 225–247.

4. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, Berlin and New York, 1983.

5. S. Hildebrandt, *Nonlinear elliptic systems and harmonic mappings*, Vorlesungsreiche SFB 72, Universität Bonn, 1980.

6. N. V. Krylov and M. V. Safonov, *An estimate of the probability that a diffusion hits a set of positive measure*, Soviet Math. Dokl. **20** (1979), 253–256.

7. ______, *A certain property of solutions of parabolic equations with measurable coefficients*, Math. USSR-Izv. **16** (1981), 151–164.

8. K. Miller, *Nonexistence of an a priori bound at the center in terms of an L^1 bound on the boundary for solutions of uniformly elliptic equations in a sphere*, Ann. Mat. Pura Appl. **73** (1966), 11–16.

9. M. V. Safonov, *Harnack's inequality for elliptic equations and the Hölder property of their solutions*, J. Soviet Math. **21** (1983), 851–863.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742