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STABILITY OF GODUNOV'S METHOD FOR A CLASS OF 2×2 SYSTEMS OF CONSERVATION LAWS

BY

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ABSTRACT. We prove stability and convergence of the Godunov scheme for a special class of genuinely nonlinear 2×2 systems of conservation laws. The class of systems, which was identified and studied by Temple, is a subset of the class of systems for which the shock wave curves and rarefaction wave curves coincide. None of the equations of gas dynamics fall into this class, but equations of this type do arise, for example, in the study of multicomponent chromatography. To our knowledge this is the first time that a numerical method other than the random choice method of Glimm has been shown to be stable in the variation norm for a coupled system of nonlinear conservation laws. This implies that subsequences converge to weak solutions of the Cauchy problem, although convergence for 2×2 systems has been proved by DiPerna using the more abstract methods of compensated compactness.

1. Introduction. In **[9]** the class of 2×2 nonlinear conservation laws for which the shock and rarefaction curves coincide is characterized. For such equations each characteristic field is either a "line" field or a "contact" field (cf. **[9]**). Equations in which both characteristic fields are line fields arise in the study of multicomponent chromatography **[1]**. In the present paper we show that Godunov's numerical method converges to weak solutions of the Cauchy problem for 2×2 systems which have two line fields. For convenience we assume that the equations are strictly hyperbolic and that each characteristic field is genuinely nonlinear in the sense of Lax **[6]**. Under these assumptions we show that Godunov's method converges for arbitrary initial data of bounded total variation in a neighborhood of each point in state space.

In general, the analysis extends globally whenever the Riemann problem can be solved globally and the corresponding solutions consist of no more than two waves. We show this for any Courant number less than 1; i.e., for any fixed mesh ratio such that waves travel at most one mesh length per time step. This allows interaction between two neighboring Riemann problems as long as this interaction is confined to a single cell.

To our knowledge this is the first time that a numerical method other than the random choice method of Glimm **[4]** has been shown to be stable in the variation

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norm for a coupled system of nonlinear conservation laws. Convergence for such systems also follows from a more general theorem of DiPerna [**10**] but there a variation bound is not obtained. The variation bound is then a further regularity result for this numerical method. We believe our results will generalize to the case of n equations with n line fields, e.g. the equations of n-component chromatography [**11**].

Convergence is proved by a standard argument once one shows that, due to the geometry of the line fields, the total variation of the approximate solution is nonincreasing when measured in the plane of Riemann invariants. As usual, convergence is obtained modulo the extraction of a subsequence.

In §2 we discuss the class of equations considered. In §3 we describe Godunov's method and obtain the lemmas needed to prove convergence.

2. Preliminaries. We consider the Cauchy problem for a system of m hyperbolic partial differential equations in conservation form:

$$u_t + F(u)_x = 0, \tag{2.1a}$$

$$u(x,0) = u^0(x). \tag{2.1b}$$

Here $u = u(x,t)$ for $-\infty < x < \infty$, $t \geqslant 0$, and u, F are m-vector valued functions. We study solutions of (2.1) that take values in a region N_0 of u-space in which F is assumed to be strictly hyperbolic and genuinely nonlinear in the sense of Lax [**6**]. This requires that the Jacobian matrix $dF(u)$ have m real and distinct eigenvalues $\{\lambda_i\}_{i=1}^m$ for all $u \in N_0$ and that $\nabla\lambda_i \cdot R_i \neq 0$ for $i = 1,\ldots,m$, where R_i is the eigenvector corresponding to λ_i. We let $R_i(u_i)$ denote the unique integral curve of R_i in N_0 that passes through the point u_i.

Since solutions of (2.1) generally develop discontinuities, we look for weak solutions that satisfy

(2.2)
$$\int_{-\infty}^{\infty}\int_0^{\infty} u(x,t)\phi_t(x,t) + F(u(x,t))\phi_x(x,t)\,dx\,dt + \int_{-\infty}^{\infty} u(x,0)\phi(x,0)\,dx = 0$$

for all smooth test functions $\phi \in C_0^1((-\infty,\infty)\times[0,\infty))$ with compact support.

The study of discontinuous solutions is centered on the so-called Riemann problem, the problem (2.1) with initial data of the form

$$u^0(x) = \begin{cases} u_L, & x \leqslant 0, \\ u_R, & x > 0. \end{cases} \tag{2.3}$$

The Hugoniot locus of u_L is the set of states u_R for which the jump condition $s[u] = [f]$ is satisfied for some scalar s [**6**]. Here $[u]$ denotes $u_L - u_R$ and $[f] = f(u_L) - f(u_R)$. If u_R lies in the Hugoniot locus of u_L then

$$u(x,t) = u^0(x - st) = \begin{cases} u_L, & x \leqslant st, \\ u_R, & x > st, \end{cases} \tag{2.4}$$

is a weak solution to the Riemann problem with initial data (2.3).

In a neighborhood of u_L the Hugoniot locus is composed of m curves $S_i(u_L)$ such that $S_i(u_L)$ makes C^2 contact with $R_i(u_L)$ at u_L and λ_i is monotonic on $S_i(u_L)$ (cf.

[5]). The i-shock curve of u_L is defined as

$$S_i^-(u_L) = \{w \in S_i(u_L) : \lambda_i(w) \leqslant \lambda_i(u_L)\} \tag{2.5}$$

and for $u_R \in S_i^-(u_L)$ we call the solution (2.4) an i-shock wave.

The i-rarefaction curve associated with u_L is defined as

$$R_i^+(u_L) = \{w \in R_i(u_L) : \lambda_i(w) \geqslant \lambda_i(u_L)\}. \tag{2.6}$$

For $u_R \in R_i^+(u_L)$ we call the solution

$$u(x,t) = \begin{cases} u_L, & x \leqslant \lambda_i(u_L)t, \\ u(\xi), & x = \xi t, \lambda_i(u_L) < \xi < \lambda_i(u_R), \\ u_R, & x \geqslant \lambda_i(u_R)t, \end{cases} \tag{2.7}$$

an i-rarefaction wave, where $u(\xi)$ denotes the parametrization of $R_i^+(u_L)$ with respect to λ_i. For general systems our assumptions imply that the Riemann problem can be solved uniquely in the class of shock and rarefaction waves in a neighborhood of any point in N_0, and this is the physically correct solution (cf. [6]).

In general, the curves $R_i(u_L)$ and $S_i(u_L)$ make only C^2 contact at u_L. If, however, $R_i(u_L) \equiv S_i(u_L)$ for any $u_L \in N_0$, then we say that the i-shock and rarefaction curves coincide in N_0. In [9] it is shown that if $\nabla\lambda_i \cdot R_i \neq 0$ in N_0, then $S_i(u_L) \equiv R_i(u_L)$ if and only if $R_i(u_L)$ is a straight line in u-space.

The the case of 2×2 systems with coinciding shock and rarefaction curves, the assumptions of strict hyperbolicity and genuine nonlinearity above imply that the Riemann problem can be solved globally in N_0 for any N_0 of the form

$$\begin{aligned} N_0 &= N([p_1, p_2]; [q_1, q_2]) \\ &\equiv \{u : p_1 \leqslant p(u) \leqslant p_2, q_1 \leqslant q(u) \leqslant q_2\} \end{aligned} \tag{2.8a}$$

subject to the condition that

$$s(u_L, u_M) \leqslant s(u_M, u_R) \tag{2.8b}$$

for all $u_L, u_M, u_R \in N_0$ such that $u_M \in S_1(u_L)$ and $u_R \in S_2(u_M)$. Here p is a 1-Riemann invariant, q is a 2-Riemann invariant, and $s(u,v)$ is the speed of the shock that connects u and v. (An i-Riemann invariant is a function $z_i(u)$ such that z_i is constant on R_i and $\nabla z_i \neq 0$.) Note that such an N_0 always exists locally assuming only that the equations are strictly hyperbolic. The Riemann problem for $u_L, u_R \in N_0$ is then solved by a 1-wave followed by a 2-wave, and since p (resp. q) is constant on 1-waves (resp. 2-waves), all states in the Riemann problem solution also lie in N_0. We state this as

LEMMA 2.1. *Assume N_0 satisfies* (2.8a) *and* (2.8b). *If u_L and u_R lie in N_0, then all states in the solution of the Riemann problem* (2.3) *also lie in N_0.*

For the remainder of this paper we restrict our attention to such systems of two equations and write $u = (u, v)$, $F = (f, g)$. We assume that N_0 is of the form (2.8) and that the integral curves of R_i are straight lines with monotonically varying slopes, so that each characteristic field is a "line" field in the sense of [8].

In this case we take p, q to be the Riemann invariants defined by

$$p(u) = \text{slope of } R_1(u), \qquad q(u) = \text{slope of } R_2(u),$$

so that each satisfies the Burgers equation $z_u + zz_v = 0$ with $\nabla p \neq 0$, $\nabla q \neq 0$.

In **[9]** it is shown that a system of two equations has a pair of line fields if and only if

$$f(u, v) = \frac{H_1(p) - H_2(q)}{q - p}, \qquad g(u, v) = \frac{qH_1(p) - pH_2(q)}{q - p}, \tag{2.9}$$

where the H_i are any smooth functions, although genuine nonlinearity and strict hyperbolicity place further restrictions on the H_i.

As an example, the following equations arise in chromatography **[1]**:

$$u_t + \left(\frac{u}{1 + u + v}\right)_x = 0, \qquad v_t + \left(\frac{Kv}{1 + u + v}\right)_x = 0. \tag{2.10}$$

Here K is a constant, $0 < K < 1$. One can verify that (2.10) satisfies (2.9) with

$$H_1(z) = H_2(z) = (z - Kz)/(z + 1) \tag{2.11}$$

when p and q are the two solutions of Burgers equation that satisfy

$$uz^2 + [K(u + 1) - (v + 1)]z - Kv = 0$$

in z and are smooth in $u > 0$, $v > 0$. In fact, the physical range of the variables u, v for (2.10) is $(0, \infty) \times (0, \infty)$ and this set satisfies the assumptions placed on N_0.

3. Godunov's method. Let h be a mesh length in x and k a time step such that

$$\frac{k}{h} \sup_{u \in N_0} |\lambda_i(u)| < \frac{1}{2}. \tag{3.1}$$

Later we will relax this restriction to allow 1 on the right-hand side. The quantity on the left-hand side is called the Courant number. In discussing convergence we will always assume that the mesh ratio k/h is fixed as $h \to 0$.

Let $x_j = jh$ for $j \in \mathbf{Z}$ and $t_n = nk$ for $n \in \mathbf{Z}^+$. For fixed h and k we will obtain a grid function approximation u_j^n to $u(x_j, t_n)$ as follows. For $n = 0$ set

$$u_j^0 = \frac{1}{h} \int_{x_{j-1}}^{x_j} u^0(x)\, dx. \tag{3.2}$$

Note that by convexity if $u^0(x) \in N_0$ for all x, then $u_j^0 \in N_0$ as well. Now suppose that $u_j^{n-1} \in N_0$ is known for all j and set

$$u^n(x, t_{n-1}) = u_j^{n-1} \quad \text{for } x_{j-1} \leqslant x < x_j. \tag{3.3}$$

The piecewise constant initial data (3.3) poses Riemann problems for (2.1a) at $t = t_n$. By (3.1) these do not interact for $t - t_n < k$, and we can solve (2.1a) to obtain $u^n(x, t)$ for $t_{n-1} \leqslant t < t_n$. We then average this solution to obtain

$$u_j^n = \frac{1}{h} \int_{x_{j-1}}^{x_j} u^n(x, t_n -)\, dx. \tag{3.4}$$

In Lemma 3.1 we show that $u_j^n \in N_0$ so that the process can be repeated.

One can show by integrating (2.1a) over the rectangle $[x_{j-1}, x_j) \times [t_{n-1}, t_n)$ that the grid function u_j^n can equivalently be defined by the finite difference approximation

$$u_j^n = u_j^{n-1} - \frac{k}{h}\left[\bar{F}\left(u_{j-1}^{n-1}, u_j^{n-1}\right) - \bar{F}\left(u_j^{n-1}, u_{j+1}^{n-1}\right)\right], \tag{3.5}$$

where the numerical flux function $\bar{F}(u_L, u_R)$ is defined as

$$\bar{F}(u_L, u_R) = \frac{1}{k}\int_0^k F(V(0, t))\, dt.$$

Here $V(x, t)$ for $t \geqslant 0$ is defined as the solution to the Riemann problem (2.1) with initial data (2.3). Since $V(x, t) = \tilde{V}(x/t)$, $\bar{F}$ is simply

$$\bar{F}(u_L, u_R) = F(\tilde{V}(0)).$$

If $u_L = u_R$, then $\tilde{V} \equiv u_L$ and so the numerical flux is "consistent" with the flux F in the sense that $\bar{F}(u_L, u_L) = F(u_L)$. Moreover $\bar{F}$ can easily be shown to be Lipschitz continuous.

From the grid function u_j^n we obtain the piecewise constant approximate solution

$$u_h(x, t) = u_j^n \quad \text{for } (x, t) \in [x_{j-1}, x_j) \times [t_n, t_{n+1}).$$

Lax and Wendroff **[8]** have shown that for any method in the conservation form (3.5) with a Lipschitz continuous consistent flux, $\bar{F}$, if $u_h(x, t)$ approaches a limit $u(x, t)$ for some sequence $h_l \to 0$, then $u(x, t)$ is a weak solution of (2.1). This is obtained by performing summation by parts on the identity

$$\sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \left\{\left(u_j^n - u_j^{n-1}\right)\Phi_j^n + \frac{k}{h}\left(\bar{F}\left(u_{j-1}^{n-1}, u_j^{n-1}\right) - \bar{F}\left(u_j^{n-1}, u_{j+1}^{n-1}\right)\right)\Phi_j^n\right\} = 0,$$

where $\Phi_j^n = \phi(x_j, t_n)$, and observing that the resulting expression approaches (2.2) as $h \to 0$ (see also Lemma 4.2 of **[3]**).

In fact, by this same argument one has that if u_h is any sequence of approximate solutions with total variation $\mathrm{TV}(u_h(\cdot, t))$ uniformly bounded in h and t, then it weakly converges in the sense that the integral condition (2.2) for $u_h(x, t)$ tends to zero as $h \to 0$.

Here we prove a stronger statement for systems of the type discussed in §2.

THEOREM 3.1. *Suppose Godunov's method is applied to a 2×2 system of conservation laws satisfying* (2.9). *Assume that the system is strictly hyperbolic and genuinely nonlinear in some N_0 of the form* (2.8) *and that the initial data $u^0(x) \in N_0$ has bounded total variation. Let $\{h_l\}$ be a sequence of mesh lengths approaching zero and suppose* (3.1) *holds. Then $u_h(x, t)$ converges weakly in the above sense and, moreover, there is a subsequence of $\{h_l\}$ for which $u_h(x, t)$ converges boundedly, a.e., to a weak solution $u(x, t)$ of* (2.1).

As a corollary this gives a global existence theorem for systems of this type. This existence theorem can also be obtained by showing convergence of the Glimm scheme as noted in **[9]**.

We must show that the total variation of $u_h(\cdot, t)$ is uniformly bounded in t and h and, to obtain a strongly convergent subsequence, that $u_h(x, t)$ is L_1 continuous in time (modulo the time step). To obtain these results we work in the Riemann invariant coordinate system and take as components of u the two Riemann invariants p and q. The l_1 norm of the vector $u_L - u_R$ is then well defined by

$$\|u_L - u_R\| = |p(u_L) - p(u_R)| + |q(u_L) - q(u_R)|. \tag{3.6}$$

Recall that $u^n(x_j, t)$ is constant for $t_{n-1} < t < t_n$. Denote this constant value by $u_j^{n-1/2} = u^n(x_j, t_{n-1}+)$.

LEMMA 3.1. *If $u^0(x) \subset N_0 = N([p_1, p_2]; [q_1, q_2])$, then for fixed k and h, $u_j^n \in N_0$ for all j, n and moreover*

$$\begin{aligned} &\left\|u_{j-1}^{n-1/2} - u_j^n\right\| + \left\|u_j^n - u_j^{n-1/2}\right\| \\ &\qquad \leqslant \left\|u_{j-1}^{n-1/2} - u_j^{n-1}\right\| + \left\|u_j^{n-1} - u_j^{n-1/2}\right\|. \end{aligned} \tag{3.7a}$$

$$\left\|u_j^{n-1} - u_{j-1}^{n-1/2}\right\| + \left\|u_{j-1}^{n-1/2} - u_{j-1}^{n-1}\right\| = \left\|u_j^{n-1} - u_{j-1}^{n-1}\right\|. \tag{3.7b}$$

PROOF. Suppose $u_j^{n-1} \in N_0$ for all j. Then $u_j^{n-1/2} \in N_0$ for all j by Lemma 2.1. Set

$$\begin{aligned} p_m &= \min\left(p\left(u_{j-1}^{n-1/2}\right), p\left(u_j^{n-1}\right), p\left(u_j^{n-1/2}\right)\right), \\ p_M &= \max\left(p\left(u_{j-1}^{n-1/2}\right), p\left(u_j^{n-1}\right), p\left(u_j^{n-1/2}\right)\right), \end{aligned} \tag{3.8}$$

and similarly for q_m, q_M. Define

$$N_j^{n-1} = N([p_m, p_M]; [q_m, q_M]) \subset N_0.$$

This convex set contains $u^n(x, t_n -)$ for $x_{j-1} \leqslant x \leqslant x_j$ by Lemma 2.1. Since by (3.4) u_j^n is an average of these values,

$$u_j^n \in N_j^{n-1} \tag{3.9}$$

and hence by induction $u_j^n \in N_0$ for all j, n provided $u_j^0 \in N_0$ for all j. This shows that Godunov's method is well defined for such systems.

It also follows that

$$p_m \leqslant p\left(u_j^n\right) \leqslant p_M \tag{3.10}$$

and therefore by (3.8), (3.10) and general inequalities for real numbers,

$$\begin{aligned} &\left|p\left(u_{j-1}^{n-1/2}\right) - p\left(u_j^n\right)\right| + \left|p\left(u_j^n\right) - p\left(u_j^{n-1/2}\right)\right| \\ &\qquad \leqslant \left|p\left(u_{j-1}^{n-1/2}\right) - p\left(u_j^{n-1}\right)\right| + \left|p\left(u_j^{n-1}\right) - p\left(u_j^{n-1/2}\right)\right|. \end{aligned}$$

Adding this to the corresponding bound for q and using (3.6) gives (3.7a)

By Lemma 2.1 $u_{j-1}^{n-1/2}$ must satisfy

$$\min\left(p\left(u_j^{n-1}\right), p\left(u_{j-1}^{n-1}\right)\right) \leqslant p\left(u_{j-1}^{n-1/2}\right) \leqslant \max\left(p\left(u_j^{n-1}\right), p\left(u_{j-1}^{n-1}\right)\right)$$

and hence

$$\left|p\left(u_j^{n-1}\right) - p\left(u_{j-1}^{n-1/2}\right)\right| + \left|p\left(u_{j-1}^{n-1/2}\right) - p\left(u_{j-1}^{n-1}\right)\right| = \left|p\left(u_j^{n-1}\right) - p\left(u_{j-1}^{n-1}\right)\right|.$$

Adding this to the corresponding expression for q gives (3.7b).

LEMMA 3.2. *The total variation is nonincreasing*:

$$\sum_j \|u_j^n - u_{j-1}^n\| \leqslant \sum_j \|u_j^{n-1} - u_{j-1}^{n-1}\| \quad \textit{for all } n. \tag{3.11}$$

PROOF. By the triangle inequality

$$\begin{aligned}\sum_j \|u_j^n - u_{j-1}^n\| &\leqslant \sum_j \left[\|u_j^n - u_{j-1}^{n-1/2}\| + \|u_{j-1}^{n-1/2} - u_{j-1}^n\|\right]\\ &= \sum_j \left[\|u_j^n - u_{j-1}^{n-1/2}\| + \|u_j^{n-1/2} - u_j^n\|\right].\end{aligned}$$

We now apply (3.7a), regroup the terms again and use (3.7b) to obtain

$$\begin{aligned}\sum_j \|u_j^n - u_{j-1}^n\| &\leqslant \sum_j \left[\|u_j^{n-1} - u_{j-1}^{n-1/2}\| + \|u_j^{n-1/2} - u_j^{n-1}\|\right]\\ &= \sum_j \left[\|u_j^{n-1} - u_{j-1}^{n-1/2}\| + \|u_{j-1}^{n-1/2} - u_{j-1}^{n-1}\|\right]\\ &= \sum_j \|u_j^{n-1} - u_{j-1}^{n-1}\|.\end{aligned}$$

We now show that $u_h(x, t)$ is L_1 continuous in time.

LEMMA 3.3. *There exists a constant C such that*

$$\int_{-\infty}^{\infty} |u_h(x, \tau_2) - u_h(x, \tau_1)| dx \leqslant C\big(|\tau_2 - \tau_1| + k\big) \cdot \mathrm{TV}(u^0),$$

where $\mathrm{TV}(u^0)$ *is the total variation of the initial data* (2.1b) *measured in the norm* (3.6).

PROOF. Assume without loss of generality that $\tau_1 < \tau_2$ and let m and M be integers such that

$$t_m \leqslant \tau_1 < t_{m+1} \leqslant t_M \leqslant \tau_2 < t_{M+1}.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} |u_h(x, \tau_2) - u_h(x, \tau_1)| dx &= \sum_{j=-\infty}^{\infty} |u_j^M - u_j^m|\\ &\leqslant h \sum_{j=-\infty}^{\infty} \sum_{n=m+1}^{M} |u_j^n - u_j^{n-1}|.\end{aligned} \tag{3.12}$$

By (3.5) we can continue

$$= k \sum_{j=-\infty}^{\infty} \sum_{n=m+1}^{M} \left|\bar{F}\big(u_{j-1}^{n-1}, u_j^{n-1}\big) - \bar{F}\big(u_j^{n-1}, u_{j+1}^{n-1}\big)\right|. \tag{3.13}$$

Since $\bar{F}$ is Lipschitz continuous,

$$\sum_{j=-\infty}^{\infty} \left|\bar{F}\big(u_{j-1}^{n-1}, u_j^{n-1}\big) - \bar{F}\big(u_j^{n-1}, u_{j+1}^{n-1}\big)\right| \leqslant C \cdot \mathrm{TV}(u^{n-1}) \leqslant C \cdot \mathrm{TV}(u^0), \tag{3.14}$$

where we have applied Lemma 3.2 and the compatibility of norms. Interchanging the order of summation in (3.13) and using (3.14) we conclude that

$$\int_{-\infty}^{\infty} |u_h(x, \tau_2) - u_h(x, \tau_1)| \leqslant C \cdot (M - m)k \cdot \mathrm{TV}(u^0), \tag{3.15}$$

where C depends only on N_0. Since $k(M-m) \leqslant |\tau_2 - \tau_1| + k$, this completes the proof of Lemma 3.3.

PROOF OF THEOREM 3.1. Using Lemma 3.2 and the argument of Lax and Wendroff [**8**] we obtain weak convergence of any sequence. By also using Lemma 3.3 and a standard compactness argument (cf. p. 714 of Glimm [**4**]) we obtain a subsequence converging boundedly a.e. to a limit $u(x,t)$ which must be a weak solution of (2.1) by [**8**].

REMARK. In Theorem 3.1 the restriction (3.1) can be replaced by the weaker condition

$$\frac{k}{h} \sup_{u \in N_0} |\lambda_i(u)| < 1. \tag{3.16}$$

For Courant numbers between $\frac{1}{2}$ and 1 the solutions to neighboring Riemann problems may interact. In this case $u^n(x, t_n -)$ is no longer simply the concatenation of Riemann problem solutions. Nonetheless, Godunov's method remains well defined and easy to implement using the conservation form (3.5) since $\tilde{V}(x/t)$ remains constant at $x=0$ for all $t<k$ if k satisfies (3.16).

In order to prove stability and convergence of the method in the more general case (3.16), we can still define u_j^n as the average (3.4), where $u^n(x,t)$ is a solution to the problem with piecewise constant initial data (3.3). Such a solution exists because our proof of the convergence of Godunov's method for Courant number less than $\frac{1}{2}$ gives as a corollary a global existence theorem for these equations. Uniqueness of solutions is not important here because any solution has the same integral (3.4) in view of the derivation of (3.5). Thus the only obstacle to the proof assuming (3.16) is that in this case it is not a priori obvious that (3.9) holds for all u_j^n. However, this follows immediately because N_j^{n-1} is an invariant region for the solutions constructed in Theorem 3.1. Once we have (3.9) the proofs of Lemmas 3.1 and 3.2 and hence of the convergence theorem proceed exactly as before.

Finally, note that the solutions constructed by the above method satisfy the entropy condition of Lax [**7**]. Suppose the system (2.1) has a convex entropy function $U(u)$ and associated entropy flux $Q(u)$ such that

(i) $U_{uu} > 0$,

(ii) $U_u F_u = Q_u$.

By Lax, the physically relevant solutions of (2.1) will satisfy the entropy condition

$$\int_0^\infty \int_{-\infty}^\infty U(u(x,t)) w_t(x,t) + Q(u(x,t)) w_x(x,t)\, dx\, dt + \int_{-\infty}^\infty U(u(x,0)) w(x,0)\, dx \geqslant 0 \tag{3.17}$$

for smooth nonnegative test functions $w \in C_0^1$. As pointed out by Harten, Lax and van Leer [**5**], Godunov's method is consistent with the entropy condition. Set

$$V_j^n = V(u_j^n), \qquad \overline{Q}(u_{j-1}^n, u_j^n) = Q\big(u^{n+1}(x_{j-1}, t_n +)\big).$$

Then using Jensen's inequality for convex functions,

$$\begin{aligned} V_j^n &= V\left(\frac{1}{h}\int_{x_{j-1}}^{x_j} u^n(x, t_n -)\,dx\right) \\ &\leqslant \frac{1}{h}\int_{x_{j-1}}^{x_j} V(u^n(x, t_n -))\,dx \\ &\leqslant \frac{1}{h}\left[\int_{x_{j-1}}^{x_j} V(u^n(x, t_{n-1}))\,dx - \int_{t_{n-1}}^{t_n} Q\big(u^n(x_{j-1}, t)\big) - Q\big(u^n(x_{j-1}, t)\big)\,dt\right], \end{aligned}$$

where we use the fact that the Riemann problem solutions satisfy the entropy condition. This is equivalent to

$$V_j^n \leqslant V_j^{n-1} - \lambda\left[\overline{Q}\left(u_j^{n-1}, u_{j+1}^{n-1}\right) - \overline{Q}\left(u_{j-1}^{n-1}, u_j^{n-1}\right)\right].$$

Since $\overline{Q}$ is consistent with Q ($\overline{Q}(u, u) = Q(u)$), it follows from Theorem 1.1 of [**5**] that any limit solution obtained with Godunov's method satisfies the entropy condition (3.17).

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