

A REFLEXIVITY THEOREM FOR WEAKLY CLOSED SUBSPACES OF OPERATORS

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ABSTRACT. It was proved in [4] that the ultraweakly closed algebras generated by certain contractions on Hilbert space have a remarkable property. This property, in conjunction with the fact that these algebras are isomorphic to H^∞ , was used in [3] to show that such ultraweakly closed algebras are reflexive. In the present paper we prove an analogous result that does not require isomorphism with H^∞ , and applies even to linear spaces of operators. Our result contains the reflexivity theorems of [3, 2 and 9] as particular cases.

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of (linear, bounded) operators acting on the Hilbert space $\mathcal{H}$, and let $\mathcal{M}$ denote a linear subspace of $\mathcal{L}(\mathcal{H})$. Then $\mathcal{M}$ is endowed with the weak and ultraweak topologies that it inherits from $\mathcal{L}(\mathcal{H})$ (cf. [6, §15]). For two arbitrary vectors $x, y \in \mathcal{H}$ we can define the (ultra) weakly continuous functional $[x \otimes y]$ on $\mathcal{M}$ by

$$[x \otimes y](A) = \langle Ax, y \rangle, \quad A \in \mathcal{M},$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathcal{H}$.

DEFINITION 1. Let n be a natural number, $n \geq 1$. The subspace $\mathcal{M}$ has property $(\tilde{\mathbf{B}}_n)$ [respectively $(\tilde{\mathbf{A}}_n)$] if for every positive number ε there exists a positive number $\delta = \delta(\varepsilon, n)$ such that for every system $\{\phi_{ij}: 1 \leq i, j \leq n\}$ of weakly [respectively ultraweakly] continuous functionals on $\mathcal{M}$ and every system $\{x_i, y_j: 1 \leq i, j \leq n\}$ of vectors in $\mathcal{H}$ satisfying the inequalities $\|\phi_{ij} - [x_i \otimes y_j]\| < \delta$ there exist vectors $\{x'_i, y'_j: 1 \leq i, j \leq n\}$ in $\mathcal{H}$ such that

$$\phi_{ij} = [x'_i \otimes y'_j], \quad 1 \leq i, j \leq n,$$

and

$$\|x_i - x'_i\| < \varepsilon, \quad \|y_j - y'_j\| < \varepsilon, \quad 1 \leq i, j \leq n.$$

Since every weakly continuous functional on $\mathcal{M}$ is also ultraweakly continuous, property $(\tilde{\mathbf{B}}_n)$ is weaker than $(\tilde{\mathbf{A}}_n)$. (ADDED IN PROOF. It was pointed out by C. Apostol that $(\tilde{\mathbf{B}}_n)$ and $(\tilde{\mathbf{A}}_n)$ are in fact equivalent. This fact is not used below.)

We recall now from [8] that a linear subspace $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ is said to be *reflexive* if it contains every operator $T \in \mathcal{L}(\mathcal{H})$ with the property that $Tx \in (\mathcal{M}x)^\perp$ for every

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$x \in \mathcal{H}$. Of course, reflexive subspaces are weakly closed. This definition coincides with the usual definition ($\mathcal{M} = \text{Alg Lat } \mathcal{M}$) if $\mathcal{M}$ is a subalgebra of $\mathcal{L}(\mathcal{H})$.

We state now the main result of this paper.

THEOREM 2. *Let $\mathcal{M}$ be a weakly closed subspace of $\mathcal{L}(\mathcal{H})$. If $\mathcal{M}$ has property $(\mathbf{B}_n^\sim)$ for every natural number n , then $\mathcal{M}$ is reflexive. Moreover, every weakly closed subspace of $\mathcal{M}$ is also reflexive.*

Before going into the proof, we relate this result with the reflexivity theorem from [3]. It was proved in [4] that, if T is a (BCP)-operator, the ultraweakly closed algebra A_T generated by T has property $(\mathbf{A}_n^\sim)$ for every $n = 1, 2, \dots$. The reflexivity of A_T follows then from Theorem 2 and the following lemma.

LEMMA 3. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{L}(\mathcal{H})$ having property $(\mathbf{A}_1^\sim)$. Then the weak and ultraweak closures of $\mathcal{M}$ coincide, and the weak and ultraweak topologies coincide on the weak closure of $\mathcal{M}$.*

PROOF. Since every ultraweakly continuous functional on $\mathcal{M}$ extends continuously to the ultraweak closure of $\mathcal{M}$, there is no loss of generality in assuming that $\mathcal{M}$ is ultraweakly closed. Let $\delta = \delta(1, 1)$ be as in Definition 1, and let ϕ be an arbitrary ultraweakly continuous functional on $\mathcal{M}$. Then $\|\delta\phi/(2\|\phi\|) - [0 \otimes 0]\| < \delta$ so that we can find vectors x' and y' such that $\|x'\| < 1$, $\|y'\| < 1$ and $\delta\phi/(2\|\phi\|) = [x' \otimes y']$ or, equivalently, $\phi = [x \otimes y]$ with $x = (2\|\phi\|/\delta)^{1/2}x'$, $y = (2\|\phi\|/\delta)^{1/2}y'$. Thus we can write ϕ as $[x \otimes y]$ with $\|x\| < (2/\delta)^{1/2}\|\phi\|^{1/2}$, $\|y\| < (2/\delta)^{1/2}\|\phi\|^{1/2}$. We can now apply, e.g., the proof of [3, Theorem 1] to conclude that $\mathcal{M}$ is weakly closed and the weak and ultraweak topologies coincide on $\mathcal{M}$.

We have therefore the following consequence of Theorem 2, which also implies the reflexivity results of [2 and 9].

COROLLARY 4. *Let $\mathcal{M}$ be an ultraweakly closed subspace of $\mathcal{L}(\mathcal{H})$. If $\mathcal{M}$ has property $(\mathbf{A}_n^\sim)$ for every natural number n , then $\mathcal{M}$ is weakly closed and reflexive. Moreover, every weakly closed subspace of $\mathcal{M}$ is also reflexive.*

For the proof of Theorem 2, we need two lemmas. The first was proved in [3] for the case in which $\mathcal{M}$ is a weakly closed algebra. The proof for linear subspaces of $\mathcal{L}(\mathcal{H})$ is identical (and easy) so we content ourselves with the statement.

LEMMA 5. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{L}(\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is in the weak closure of $\mathcal{M}$ if and only if for every natural n and every system $\{x_i, y_i; 1 \leq i \leq n\}$ of vectors in $\mathcal{H}$ such that $\sum_{i=1}^n [x_i \otimes y_i] = 0$, we have $\sum_{i=1}^n \langle Tx_i, y_i \rangle = 0$.*

LEMMA 6. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{L}(\mathcal{H})$. Assume that $\mathcal{M}$ has property $(\mathbf{B}_n^\sim)$ for every natural number n . Then for every natural number n , every system $\{x_i, y_i; 1 \leq i \leq n\}$ of vectors in $\mathcal{H}$ and every $\varepsilon > 0$, there exist vectors $\{x_{ij}, y_{ij}; 1 \leq i, j \leq n\}$ such that $[x_{ij} \otimes y_{kl}] = \delta_{jl}[x_i \otimes y_k]$, $1 \leq i, j, k, l \leq n$, and $\|x_i - x_{i1}\| < \varepsilon$, $\|y_j - y_{j1}\| < \varepsilon$, $1 \leq i, j \leq n$. (Here δ_{jl} denotes, as usual, Kronecker's symbol.)*

PROOF. Let $\delta = \delta(\varepsilon, n^2)$ be as in Definition 1. Set

$$\eta = \min\{\delta/(2\|x_i \otimes y_k\|) : [x_i \otimes y_k] \neq 0\}$$

and define $\phi_{ij,kl} = 0$ if $j \neq l$, $\phi_{i1,k1} = [x_i \otimes y_k]$, $\phi_{ij,kj} = \eta[x_i \otimes y_k]$, $j \geq 2$. The vectors $\{x_{ij}^0, y_{ij}^0 : 1 \leq i, j \leq n\}$ defined by $x_{ij}^0 = \delta_{j1}x_i$, $y_{ij}^0 = \delta_{j1}y_i$, $1 \leq i, j \leq n$, obviously satisfy the inequalities

$$\|\phi_{ij,kl} - [x_{ij}^0 \otimes y_{kl}^0]\| < \delta, \quad 0 \leq i, j, k, l \leq n,$$

and therefore, by property $(\mathbf{B}_{n^2})$, we can find vectors $\{x'_{ij}, y'_{ij} : 1 \leq i, j \leq n\}$ in $\mathcal{H}$ such that

$$\phi_{ij,kl} = [x'_{ij} \otimes y'_{kl}], \quad 0 \leq i, j, k, l \leq n,$$

and $\|x_{ij}^0 - x'_{ij}\| < \varepsilon$, $\|y_{ij}^0 - y'_{ij}\| < \varepsilon$, $0 \leq i, j \leq n$. Then the vectors $\{x_{ij}, y_{ij} : 1 \leq i, j \leq n\}$ defined by $x_{i1} = x'_{i1}$, $y_{i1} = y'_{i1}$, $x_{ij} = \eta^{-1/2}x'_{ij}$, $y_{ij} = \eta^{-1/2}y'_{ij}$, $1 \leq i \leq n$, $2 \leq j \leq n$, satisfy the requirements of the lemma.

PROOF OF THEOREM 2. Let $T \in \mathcal{L}(\mathcal{H})$ satisfy the property that $Tx \in (\mathcal{M}x)^\perp$ for every $x \in \mathcal{H}$. We first note that the equality $[x \otimes y] = 0$, $x, y \in \mathcal{H}$, means that y is orthogonal to $(\mathcal{M}x)^\perp$, and hence it implies $\langle Tx, y \rangle = 0$.

In order to show that $T \in \mathcal{M}$, we must prove, according to Lemma 5, that the equality $\sum_{i=1}^n [x_i \otimes y_i] = 0$, $x_i, y_i \in \mathcal{H}$, $1 \leq i \leq n$, implies $\sum_{i=1}^n \langle Tx_i, y_i \rangle = 0$. By what has just been said, this property is satisfied for $n = 1$. Assume therefore that $n \geq 2$, $x_i, y_i \in \mathcal{H}$, $1 \leq i \leq n$, and $\sum_{i=1}^n [x_i \otimes y_i] = 0$. For every $\varepsilon > 0$ we can find, using Lemma 6, vectors $x_{ij} = x_{ij}(\varepsilon)$, $y_{ij} = y_{ij}(\varepsilon)$, $0 \leq i, j \leq n$, satisfying,

$$(1) \quad [x_{ij} \otimes y_{kl}] = \delta_{jl}[x_i \otimes y_k], \quad 1 \leq i, j, k, l \leq n,$$

and

$$(2) \quad \begin{aligned} \|x_i - x_{i1}\| &= \|x_i - x_{i1}(\varepsilon)\| < \varepsilon, \\ \|y_j - y_{j1}\| &= \|y_j - y_{j1}(\varepsilon)\| < \varepsilon, \quad 1 \leq i, j \leq n. \end{aligned}$$

We now remark that by (1)

$$\begin{aligned} \left[\sum_{i=1}^n x_{ii} \otimes \sum_{i=1}^n y_{ii} \right] &= \sum_{i=1}^n [x_{ii} \otimes y_{ii}] + \sum_{i \neq j} [x_{ii} \otimes y_{jj}] \\ &= \sum_{i=1}^n [x_i \otimes y_i] = 0 \end{aligned}$$

and therefore

$$(3) \quad \left\langle T \left(\sum_{i=1}^n x_{ii} \right), \sum_{i=1}^n y_{ii} \right\rangle = 0.$$

Since $[x_{ii} \otimes y_{jj}] = 0$ for $i \neq j$, we also have $\langle Tx_{ii}, y_{jj} \rangle = 0$ for $i \neq j$ so that (3) can be rewritten as

$$(4) \quad \sum_{i=1}^n \langle Tx_{ii}, y_{ii} \rangle = 0.$$

Assume now that $i \neq 1$. We have by (1)

$$\begin{aligned} [(x_{ii} - x_{i1}) \otimes (y_{ii} + y_{i1})] &= [x_{ii} \otimes y_{ii}] - [x_{i1} \otimes y_{i1}] + [x_{ii} \otimes y_{i1}] - [x_{i1} \otimes y_{ii}] \\ &= [x_i \otimes y_i] - [x_i \otimes y_i] = 0 \end{aligned}$$

and therefore

$$\begin{aligned} 0 &= \langle T(x_{ii} - x_{i1}), y_{ii} + y_{i1} \rangle \\ &= \langle Tx_{ii}, y_{ii} \rangle - \langle Tx_{i1}, y_{i1} \rangle + \langle Tx_{ii}, y_{i1} \rangle - \langle Tx_{i1}, y_{ii} \rangle. \end{aligned}$$

The last two terms are zero because $[x_{ii} \otimes y_{i1}] = [x_{i1} \otimes y_{ii}] = 0$ and we conclude that $\langle Tx_{ii}, y_{ii} \rangle = \langle Tx_{i1}, y_{i1} \rangle$. Therefore (4) can now be written as $\sum_{i=1}^n \langle Tx_{i1}, y_{i1} \rangle = 0$. We now let ε approach zero. We have $\lim_{\varepsilon \rightarrow 0} x_{i1}(\varepsilon) = x_i$, $\lim_{\varepsilon \rightarrow 0} y_{i1}(\varepsilon) = y_i$ so that

$$\sum_{i=1}^n \langle Tx_i, y_i \rangle = \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^n \langle Tx_{i1}(\varepsilon), y_{i1}(\varepsilon) \rangle = 0$$

and the reflexivity of $\mathcal{M}$ is proved by Lemma 5. The last statement of the theorem follows from [8, Theorem 2.3] (cf. also [7]).

We conclude with a condition implying property (A_n^*) and which is sometimes easier to verify. For an arbitrary linear subspace $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ we will denote by $\mathcal{M}_*$ the Banach space of all ultraweakly continuous functionals on $\mathcal{M}$. It is well known that the dual space of $\mathcal{M}_*$ coincides with the ultraweak closure of $\mathcal{M}$; we will not use this fact here. The following two definitions were given in [1] for ultraweakly closed algebras $\mathcal{M}$ (cf. [1, Definitions 1.4 and 1.5]).

DEFINITION 7. Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace and $0 \leq \theta < +\infty$. We denote by $X_\theta(\mathcal{M})$ the set of all ϕ in $\mathcal{M}_*$ such that there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ in $\mathcal{H}$ satisfying the following conditions:

$$\begin{aligned} \|x_i\| \leq 1, \quad \|y_i\| \leq 1, \quad 1 \leq i \leq \infty, \\ \limsup_{i \rightarrow +\infty} \|\phi - [x_i \otimes y_i]\| \leq \theta, \end{aligned}$$

and

$$\lim_{i \rightarrow \infty} (\|[x_i \otimes z]\| + \|[z \otimes x_i]\| + \|[y_i \otimes z]\| + \|[z \otimes y_i]\|) = 0, \quad z \in \mathcal{H}.$$

DEFINITION 8. Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace and $0 \leq \theta < \gamma < +\infty$. We say that $\mathcal{M}$ has property $X_{\theta, \gamma}$ if the closed absolutely convex hull of the set $X_\theta(\mathcal{M})$ contains the closed ball of radius γ centered at the origin in $\mathcal{M}_*$:

$$\overline{\text{aco}} X_\theta(\mathcal{M}) \supset \{\phi \in \mathcal{M}_* : \|\phi\| \leq \gamma\}.$$

The following result coincides with [1, Theorem 1.9] if $\mathcal{M}$ is an ultraweakly closed algebra. However, neither the algebra structure, nor the ultraweak closedness of $\mathcal{M}$ has been used in the proof of that theorem, so that we refer to [1] for the proof.

THEOREM 9. Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace with property $X_{\theta, \gamma}$ for some $\gamma > \theta \geq 0$. Then for every $\phi \in \mathcal{M}_*$ there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ in $\mathcal{H}$ such that

$$\phi = [x_i \otimes y_i], \quad 1 \leq i < \infty,$$

$$\limsup_{i \rightarrow \infty} \|x_i\| \leq (\gamma - \theta)^{-1/2} \|\phi\|^{1/2}, \quad \limsup_{i \rightarrow \infty} \|y_i\| \leq (\gamma - \theta)^{-1/2} \|\phi\|^{1/2},$$

and

$$\lim_{i \rightarrow \infty} (\| [x_i \otimes z] \| + \| [z \otimes x_i] \| + \| [y_i \otimes z] \| + \| [z \otimes y_i] \|) = 0, \quad z \in \mathcal{H}.$$

It was seen in [1] that this theorem implies that $\mathcal{M}$ has property (A_n) for each n ; we recall that property (A_n) requires the solvability for x_i and y_i of arbitrary systems of the form $[x_i \otimes y_j] = \phi_{ij}$, $\phi_{ij} \in \mathcal{M}_*$, $1 \leq i, j \leq n$. In order to prove the stronger property (A_n) we need the following lemma, whose proof is reminiscent of the techniques of Robel [9].

LEMMA 10. Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace with property $X_{\theta, \gamma}$ for some $\gamma > \theta \geq 0$. If n is a natural number, $a > 0$, $\varepsilon > 0$, and $\phi_{ij} \in \mathcal{M}_*$, $x_i, y_j \in \mathcal{H}$, $1 \leq i, j \leq n$, are such that

$$\| \phi_{ij} - [x_i \otimes y_j] \| < a, \quad 1 \leq i, j \leq n,$$

then there exist $\{x'_i, y'_j: 1 \leq i, j \leq n\}$ in $\mathcal{H}$ such that

$$\| \phi_{ij} - [x'_i \otimes y'_j] \| < \varepsilon, \quad 1 \leq i, j \leq n,$$

and

$$\| x_i - x'_i \| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad \| y_j - y'_j \| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i, j \leq n.$$

PROOF. Let $\delta > 0$ be such that $(n^2 + 2n - 1)\delta < \varepsilon$. An application of Theorem 9 to $\phi = \phi_{ij} - [x_i \otimes y_j]$ yields sequences $\{\xi_{ij}(k)\}_{k=1}^\infty, \{\eta_{ij}(k)\}_{k=1}^\infty$ such that

$$(6) \quad \phi_{ij} - [x_i \otimes y_j] = [\xi_{ij}(k) \otimes \eta_{ij}(k)], \quad 1 \leq k < \infty,$$

$$(7) \quad \|\xi_{ij}(k)\| < (\gamma - \theta)^{-1/2} a^{1/2}, \quad \|\eta_{ij}(k)\| < (\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq k < \infty,$$

and

$$(8) \quad \lim_{k \rightarrow \infty} (\| [\xi_{ij}(k) \otimes z] \| + \| [z \otimes \eta_{ij}(k)] \|) = 0, \quad z \in \mathcal{H}.$$

An easy induction using (8) shows that we can find natural numbers k_{ij} , $1 \leq i, j \leq n$, such that the vectors $\xi_{ij} = \xi_{ij}(k_{ij})$ and $\eta_{ij} = \eta_{ij}(k_{ij})$ satisfy the inequalities

$$(9) \quad \begin{cases} \| [\xi_{ij} \otimes \eta_{kl}] \| < \delta & \text{if } (i, j) \neq (k, l), \\ \| [x_i \otimes \eta_{kl}] \| < \delta, & 1 \leq i, k, l \leq n, \\ \| [\xi_{ij} \otimes y_k] \| < \delta, & 1 \leq i, j, k \leq n. \end{cases}$$

We can now set

$$x'_i = x_i + \sum_{k=1}^n \xi_{ik}, \quad y'_j = y_j + \sum_{l=1}^n \eta_{lj}$$

and note that we obviously have from (7)

$$\| x'_i - x_i \| \leq \sum_{k=1}^n \|\xi_{ik}\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i \leq n,$$

and similarly

$$\|y'_j - y_j\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq j \leq n.$$

Finally, we observe that

$$\begin{aligned} \phi_{ij} - [x'_i \otimes y'_j] &= \phi_{ij} - [x_i \otimes y_j] - [\xi_{ij} \otimes \eta_{ij}] - \sum_{l=1}^n [x_i \otimes \eta_{lj}] \\ &\quad - \sum_{k=1}^n [\xi_{ik} \otimes y_j] - \sum_{(l,k) \neq (i,j)} [\xi_{ik} \otimes \eta_{lj}] \end{aligned}$$

and we obtain, using (6) and (9),

$$\|\phi_{ij} - [x'_i \otimes y'_j]\| \leq n\delta + n\delta + (n^2 - 1)\delta < \varepsilon.$$

The lemma follows.

A routine argument shows now that Lemma 10 is self-improving to yield the following result.

THEOREM 11. *Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace with property $X_{\gamma, \theta}$ for some $\gamma > \theta \geq 0$. If n is a natural number, $a > 0$ and $\phi_{ij} \in \mathcal{M}_*$, $x_i, y_j \in \mathcal{H}$, $1 \leq i, j \leq n$, are such that*

$$\|\phi_{ij} - [x_i \otimes y_j]\| < a, \quad 1 \leq i, j \leq n,$$

then there exist $\{x'_i, y'_j: 1 \leq i, j \leq n\}$ in $\mathcal{H}$ such that

$$\phi_{ij} = [x'_i \otimes y'_j], \quad 1 \leq i, j \leq n,$$

and

$$\|x_i - x'_i\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad \|y_j - y'_j\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i, j \leq n.$$

PROOF. Choose a positive number b such that

$$\|\phi_{ij} - [x_i \otimes y_j]\| < b < a, \quad 1 \leq i, j \leq n,$$

and let ε be a positive number to be specified later (ε will only depend on a and b).

By Lemma 10, we can find vectors $\{x_i^1, y_j^1: 1 \leq i, j \leq n\}$ such that

$$\|\phi_{ij} - [x_i^1 \otimes y_j^1]\| < \varepsilon, \quad 1 \leq i, j \leq n,$$

and

$$\|x_i^1 - x_i\| < n(\gamma - \theta)^{-1/2} b^{1/2}, \quad \|y_j^1 - y_j\| < n(\gamma - \theta)^{-1/2} b^{1/2}.$$

We can then use Lemma 10 to construct inductively sequences $\{x_i^k\}_{k=2}^\infty, \{y_j^k\}_{k=2}^\infty$, $1 \leq i, j \leq n$, such that

$$\|\phi_{ij} - [x_i^k \otimes y_j^k]\| < \varepsilon^k, \quad 1 \leq i, j \leq n, 2 \leq k < \infty,$$

and

$$\begin{aligned} \|x_i^{k+1} - x_i^k\| &< n(\gamma - \theta)^{-1/2} \varepsilon^{k/2}, \quad \|y_j^{k+1} - y_j^k\| < n(\gamma - \theta)^{-1/2} \varepsilon^{k/2}, \\ &1 \leq i, j \leq n, 1 \leq k < \infty. \end{aligned}$$

It is obvious that the sequences $\{x_i^k\}_{k=1}^\infty$ and $\{y_j^k\}_{k=1}^\infty$, $1 \leq i, j \leq n$, are Cauchy and $\phi_{ij} = [x_i' \otimes y_j']$, $1 \leq i, j \leq n$, if

$$x_i' = \lim_{k \rightarrow \infty} x_i^k, \quad y_j' = \lim_{k \rightarrow \infty} y_j^k, \quad 1 \leq i, j \leq n.$$

Finally,

$$\begin{aligned} \|x_i' - x_i\| &\leq \|x_i^1 - x_i\| + \sum_{k=1}^\infty \|x_i^{k+1} - x_i^k\| \\ &< n(\gamma - \theta)^{-1/2} \left(b^{1/2} + \sum_{k=1}^\infty \varepsilon^{k/2} \right) \\ &= n(\gamma - \theta)^{-1/2} \left(b^{1/2} + \varepsilon^{1/2} (1 - \varepsilon^{1/2})^{-1} \right), \quad 1 \leq i \leq n, \end{aligned}$$

and analogously

$$\|y_j' - y_j\| < n(\gamma - \theta)^{-1/2} \left(b^{1/2} + \varepsilon^{1/2} (1 - \varepsilon^{1/2})^{-1} \right), \quad 1 \leq j \leq n.$$

It suffices therefore to choose ε so small that $b^{1/2} + \varepsilon^{1/2}(1 - \varepsilon^{1/2})^{-1} < a^{1/2}$. The theorem is proved.

We are now able to prove the promised criterion.

COROLLARY 12. *Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace with property $x_{\theta, \gamma}$ for some $\gamma > \theta \geq 0$. Then $\mathcal{M}$ has property $(\tilde{\mathbf{A}}_n)$ for every natural number n . In particular the ultraweak closure $\mathcal{M}^-$ of $\mathcal{M}$ is weakly closed and reflexive.*

PROOF. The last part of the statement follows from the first part, combined with Lemma 3 and Corollary 4. To prove the first part we only have to use Theorem 11. Observe that we can take $\delta(\varepsilon, n) = \varepsilon^2 n^{-2}(\gamma - \theta)$.

We finally note that one could give a definition analogous to Definition 8, in which the space $\mathcal{M}_*$ is replaced by the set $\mathcal{M}_-$ of all weakly continuous functionals on $\mathcal{M}$. The property thus defined would however be stronger than $X_{\theta, \gamma}$ since $\mathcal{M}_*$ coincides with the norm closure of $\mathcal{M}_-$; this is why we restricted ourselves to the space $\mathcal{M}_*$ and the properties $(\tilde{\mathbf{A}}_n)$. We do not know whether the weaker properties $(\mathbf{A}_n)$ imply reflexivity. Property $(\mathbf{A}_1)$ alone does not imply reflexivity. Indeed, the algebra $\mathcal{M}$ of 2×2 matrices defined as

$$\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

is not reflexive, but it has property $(\mathbf{A}_1)$ (and even $(\tilde{\mathbf{A}}_1)$, as can be seen by an easy computation).

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