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SINGULAR INTEGRAL OPERATORS OF CALDERÓN TYPE AND RELATED OPERATORS ON THE ENERGY SPACES

BY

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ABSTRACT. We show the boundedness of some singular integral operators on the energy spaces.

1. Introduction. For a complex-valued continuous function $h(x)$ on the real line $\mathbf{R}$ and a real-valued continuous function $A(x)$ on $\mathbf{R}$, we define a kernel by

$$C[h, A](x, y) = \frac{1}{x-y} h\left\{\frac{A(x) - A(y)}{x-y}\right\}.$$

These kernels are important in harmonic analysis. Several authors have investigated the boundedness of these kernels as operators from L^p-spaces to L^p-spaces [**3, 7, 11**], and others have studied the boundedness on the Sobolev spaces only in the case $h(x) = x$ (generalizing $1/(x-y)$) [**1, 2**]. In this paper we study the boundedness of these kernels on the energy spaces for infinitely differentiable functions $h(x)$.

Let C_0^∞ be the totality of infinitely differentiable functions on $\mathbf{R}$ with compact support. For $0 < \alpha < 1$ we denote by E_α the Banach space of locally integrable functions on $\mathbf{R}$ obtained by the completion of C_0^∞ with respect to the norm

$$\|f\|_\alpha = \left\{\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{|f(x) - f(y)|^2}{|x-y|^{1+\alpha}} dx\, dy\right\}^{1/2}.$$

This is called the α-energy space [**10**, p. 77]. The α-capacity $\operatorname{cap}_\alpha(\cdot)$ is the capacity defined by the kernel $\kappa_\alpha(x) = 1/|x|^{1-\alpha}$ [**10**, p. 131]. A function $a(x)$ on $\mathbf{R}$ is called a multiplier on E_α if the multiplication operator $\mathbf{M}_a$: $f \in E_\alpha \to af \in E_\alpha$ is bounded [**10**, p. 38]. The totality of multipliers on E_α is denoted by $M(E_\alpha)$. The norm of $\mathbf{M}_a$ is simply denoted by $\|a\|_{M(E_\alpha)}$. We say that $C[h, A]$ is α-bounded if, for any $f \in E_\alpha$,

$$\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} C[h, A](x, y) f(y)\, dy \quad (= C[h, A]f(x))$$

exists α-q.e. (that is, the limit exists except for a set of α-capacity zero) and

$$\|C[h, A]\|_{\alpha,\alpha} = \sup\{\|C[h, A]f\|_\alpha / \|f\|_\alpha ; f \in E_\alpha\} < \infty.$$

We show

THEOREM 1. *Let* $0 < \alpha < 1$. *Then* $C[h, A]$ *is* α*-bounded as long as* $A' \in M(E_\alpha)$ *and* $h(x)$ *is infinitely differentiable.*

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We also study some operators closely related to $C[h, A]$. We define two families, $\mathfrak{P} = \{P_y(x)\}_{y>0}$, $\mathfrak{Q} = \{Q_y(x)\}_{y>0}$, of functions on $\mathbf{R}$ by

$$P_y(x) = (1/2y)e^{-|x|/y}, \quad Q_y(x) = (1/2y)\operatorname{sign}(x/y)e^{-|x|/y}.$$

For a complex-valued function $a(x)$ on $\mathbf{R}$ with $\int_{\mathbf{R}}|a(x)|/(1+x^2)\,dx < \infty$, we define four operators on C_0^∞ by

$$T_a[\mathfrak{U}, \mathfrak{V}]f(x) = \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon} U_y * a(x)V_y * f(x)\,dy/y \qquad (f \in C_0^\infty),$$

where $\mathfrak{U} = \mathfrak{P}$ or $\mathfrak{Q}$, $\mathfrak{V} = \mathfrak{P}$ or $\mathfrak{Q}$. Operators of this type were studied by Coifman-Meyer [**6**, Chapter VI]. These operators are considered weighted Hilbert transforms, and we see that, in the case where $h(x) = x$ and $A'(x) = a(x)$, $C[h, A] - T_a[\mathfrak{Q}, \mathfrak{P}]$ is a negligible operator in a sense [cf. Remark 21]. For $0 < \alpha < 1$ we say that $T_a[\mathfrak{U}, \mathfrak{V}]$ is E_α-bounded if it is extended as a bounded operator from E_α to itself. Let us remark that $T_a[\mathfrak{P}, \mathfrak{P}]$ is E_α-unbounded in the case where $a(x) \equiv 1$. We denote by L^∞ the Banach space of bounded functions on $\mathbf{R}$ and by BMO the Banach space of functions of bounded mean oscillation on $\mathbf{R}$, modulo constants [**9**, p. 141]. We show

THEOREM 2. *Let* $0 < \alpha < 1$. *Then*:
(1) *If* $a \in M(E_\alpha)$, *then* $T_a[\mathfrak{Q}, \mathfrak{P}]$ *is* E_α*-bounded.*
(2) *If* $a \in L^\infty$, *then* $T_a[\mathfrak{P}, \mathfrak{Q}]$ *is* E_α*-bounded.*
(3) *If* $a \in$ BMO, *then* $T_a[\mathfrak{Q}, \mathfrak{Q}]$ *is* E_α*-bounded.*

2. Preliminaries.

2.1. Throughout we fix $0 < \alpha < 1$ and use "Const." for constants depending only on α. The value of "Const." generally differs from one occasion to another. For a Banach space $\mathfrak{B}$, we denote by $\|\cdot\|_{\mathfrak{B}}$ its norm. Let L^2 denote the L^2-space of functions on $\mathbf{R}$ with respect to the Lebesgue measure dx. The maximal function of a locally dx-integrable function $f(x)$ on $\mathbf{R}$ is defined by

$$\mathfrak{M}f(x) = \sup m_I|f|, \quad \text{where } m_I|f| = \frac{1}{|I|}\int_I |f(s)|\,ds \left(|I| = \int_I ds\right)$$

and the supremum is taken over all finite intervals I containing x. We say that $f(x)$ is of bounded mean oscillation if $\|f\|_{\mathrm{BMO}} = \sup m_I|f - m_I f| < \infty$, where the supremum is taken over all finite intervals I. The α-capacity of a compact set W in $\mathbf{R}$ is defined by

$$\operatorname{cap}_\alpha(W) = \inf\left\{\|g\|_{L^2}^2;\ \kappa_{\alpha/2} * g(x) \geq 1\ (x \in W)\right\}$$

[**10**, p. 133, p. 138]. The α-capacities of Borel sets in $\mathbf{R}$ are usually defined [**10**, p. 143]. Let $f \in E_\alpha$. Then:
(4) $\|f\|_\alpha = \text{Const.}\{\int_{\mathbf{R}}|\xi|^\alpha|\hat{f}(\xi)|^2\,d\xi\}^{1/2}$ ($\hat{f}(\xi)$: the Fourier transform of $f(x)$).
(5) $f(x) = \kappa_{\alpha/2} * g(x)$ almost everywhere (a.e.) for some $g \in L^2$ with

$$\|g\|_{L^2} = \text{Const.}\,\|f\|_\alpha \qquad [\mathbf{10}, \text{p. } 80].$$

(6) $\lim_{\varepsilon\to 0}(1/\varepsilon)\int_x^{x+\varepsilon} f(s)\,ds = \kappa_{\alpha/2} * g(x)$ α-q.e.
(7) $m_{(-\varepsilon,\varepsilon)}|f| \leq \text{Const.}\ \varepsilon^{(\alpha-1)/2}\|f\|_\alpha$ ($\varepsilon > 0$).
(8) $\operatorname{cap}_\alpha(x;\ \mathfrak{M}f(x) > \varepsilon) \leq \text{Const.}/\varepsilon^2 \cdot \|f\|_\alpha^2$ [**4**, p. 70].

Equality (4) is deduced from Parseval's formula. Equality (6) is deduced from (5) and the Lebesgue dominated convergence theorem. Inequality (7) is also deduced from (5). Let $a \in M(E_\alpha)$. Then by (7) we have, for any $\varepsilon > 0$ and $s \in \mathbf{R}$,

$$m_{(-\varepsilon,\varepsilon)}|a\psi_{\varepsilon,s}| \leqslant \text{Const.}\,\varepsilon^{(\alpha-1)/2}\|a\psi_{\varepsilon,s}\|_\alpha$$
$$\leqslant \text{Const.}\,\varepsilon^{(\alpha-1)/2}\|a\|_{M(E_\alpha)}\|\psi_{\varepsilon,s}\|_\alpha = \text{Const.}/\varepsilon \cdot \|a\|_{M(E_\alpha)},$$

where $\psi_{\varepsilon,s}(x) = (1/\sqrt{\pi}\,\varepsilon)\exp(-(x-s)^2/\varepsilon^2)$. Since $\varepsilon > 0$, $s \in \mathbf{R}$ are arbitrary, we have $\|a\|_{L^\infty} \leqslant \text{Const.}\|a\|_{M(E_\alpha)}$. Let $b \in L^\infty$ satisfy $|b(x) - b(y)| \leqslant \eta|a(x) - a(y)|$ $(x, y \in \mathbf{R})$ for some $\eta > 0$. Then we have, for any $f \in E_\alpha$,

$$|b(x)f(x) - b(y)f(y)| \leqslant |b(x)|\,|f(x) - f(y)| + |b(x) - b(y)|\,|f(y)|$$
$$\leqslant \|b\|_{L^\infty}|f(x) - f(y)| + \eta|a(x) - a(y)|\,|f(y)|$$
$$\leqslant \{\|b\|_{L^\infty} + \eta\|a\|_{L^\infty}\}\,|f(x) - f(y)| + \eta|a(x)f(x) - a(y)f(y)|,$$

and hence

(9) $$\|b\|_{M(E_\alpha)} \leqslant \text{Const.}\left\{\|b\|_{L^\infty} + \eta\|a\|_{M(E_\alpha)}\right\}.$$

In particular, if $a(x)$ is real-valued, then

$$\|1/(1 + ia)\|_{M(E_\alpha)} \leqslant \text{Const.}\left\{1 + \|a\|_{M(E_\alpha)}\right\}.$$

2.2. Let L_* be the Banach space of integrable functions on $\mathbf{R}$ with respect to the measure $dx/(1 + |x|)$. For a kernel $K(x, y)$ $(x, y \in \mathbf{R})$, we define $\Omega(K)$ by the minimum of C's satisfying the following two inequalities: $|K(x, y)| \leqslant C/|x - y|$, $|\partial K(x, y)/\partial x| + |\partial K(x, y)/\partial y| \leqslant C/(x - y)^2$ $(x \neq y)$. If such a C does not exist, we put $\Omega(K) = \infty$. For $f \in L_*$ and a kernel $K(x, y)$ with $\Omega(K) < \infty$, we put

$$K_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x, y)f(y)\,dy \quad (\varepsilon > 0), \qquad K^*f(x) = \sup_{\varepsilon>0}|K_\varepsilon f(x)|.$$

We say that $K(x, y)$ is a Calderón-Zygmund kernel (CZ-kernel) if $\Omega(K) < \infty$, $Kf(x) = \lim_{\varepsilon\to 0} K_\varepsilon f(x)$ exists a.e. for any $f \in L_*$, and

$$\|K^*\|_{L^2,L^2} = \sup\left\{\|K^*f\|_{L^2}/\|f\|_{L^2};\, f \in L^2\right\} < \infty.$$

We write simply $\|K^*\|_{CZ} = \Omega(K) + \|K^*\|_{L^2,L^2}$. Let $K(x, y)$ be a CZ-kernel, $f \in L_*$, and let $(f_n)_{n=1}^\infty$ be a sequence in L_* such that $\lim_{n\to\infty}\|f_n - f\|_{L_*} = 0$. Then the weak L^1-inequality [**6**, p. 90] yields that $\lim_{j\to\infty} Kf_{n_j}(x) = Kf(x)$ a.e. for some subsequence $(f_{n_j})_{j=1}^\infty$. We also have $K^*f(x) \leqslant \text{Const.}\{\mathfrak{M}(Kf)(x) + \|K^*\|_{CZ}\mathfrak{M}f(x)\}$ everywhere [**6**, p. 95]. Let $h \in C_0^\infty$ and $A(x)$ be a real-valued function with $A' \in L^\infty$. Then [**7**] (cf. [**11**, **6**, p. 98]):

$$\|C[h, A]^*\|_{CZ} \leqslant \text{Const.}\int_{\mathbf{R}}|\hat{h}(\xi)|(1 + r|\xi|)^9\,d\xi + \Omega(C[h, A]) \qquad (r = \|A'\|_{L^\infty}).$$

We may replace $h(x)$ by $(h\gamma)(x)$ for any $\gamma \in C_0^\infty$ with $\gamma(x) = 1$ $(|x| \leqslant r)$. Choosing $\gamma(x)$ suitably, we obtain

(10) $$\|C[h, A]^*\|_{CZ} \leqslant \text{Const.}\,d_h(r)(1 + r)^{10} \qquad (r = \|A'\|_{L^\infty}),$$

where

$$d_h(r) = \sum_{j=0}^{11} \max\left\{|h^{(j)}(x)|;\, |x| \leqslant 2r + 1\right\}.$$

Let $A_m(x) = \int_0^x A'*\phi_m(s)\,ds$ $(m \geqslant 1,\ \phi_m(x) = \sqrt{m/\pi}\, e^{-mx^2})$. Then the argument in [**11**, Lemma 11] yields that, for each $n \geqslant 0$,

$$\lim_{j\to\infty} C\left[t^n, A_{m_j}\right] f(x) = C[t^n, A] f(x) \quad \text{a.e.}$$

for some subsequence $(A_{m_j})_{j=1}^{\infty}$. Since

$$C[e^{it}, B](x, y) = \sum_{n=0}^{\infty} \left(\frac{i^n}{n!}\right) C[t^n, B](x, y)$$

and

$$C[h, B](x, y) = \text{Const.} \int_{\mathbf{R}} \hat{h}(\xi) C\left[e^{it}, \xi B\right](x, y)\, d\xi$$

$(x \neq y;\ B(x) = A(x), A_m(x)\ (m \geqslant 1))$, we have, by (10),

$$\lim_{j\to\infty} C\left[h, A_{m'_j}\right] f(x) = C[h, A] f(x) \quad \text{a.e.}$$

for some subsequence $(A_{m'_j})_{j=1}^{\infty}$. Inequality (7) shows that, for any $g \in E_\alpha$,

$$\|g\|_{L_*} \leqslant \text{Const.} \sum_{k=0}^{\infty} m_{(-2^k, 2^k)} |g| \leqslant \left\{\text{Const.} \sum_{k=0}^{\infty} 2^{(\alpha-1)k/2}\right\} \|g\|_\alpha .$$

This shows that $E_\alpha \subset L_*$ and $\|\cdot\|_{L_*} \leqslant \text{Const.} \|\cdot\|_\alpha$. Let $f \in E_\alpha$. Then, consequently, we have:

(11) $C[h, A] f(x)$ exists a.e.

(12) If $\lim_{n\to\infty} \|f_n - f\|_\alpha = 0$, then $\lim_{j\to\infty} C[h, A] f_{n_j}(x) = C[h, A] f(x)$ a.e. for some subsequence $(f_{n_j})_{j=1}^{\infty}$.

(13) $\lim_{j\to\infty} C[h, A_{m_j}] f(x) = C[h, A] f(x)$ a.e. for some subsequence $(A_{m_j})_{j=1}^{\infty}$.

(14) $C[h, A]^* f(x) \leqslant \text{Const.}\{\mathfrak{M}(C[h, A] f)(x) + d_h(r)(1 + r)^{10} \mathfrak{M} f(x)\}$ $(r = \|A'\|_{L^\infty})$ everywhere.

Let S be a dense set in E_α. Then (11) and (12) show that, for any $f \in E_\alpha$, $\lim_{n\to\infty} \|f_n - f\|_\alpha = 0$ and $\lim_{n\to\infty} C[h, A] f_n(x) = C[h, A] f(x)$ a.e. for some sequence $(f_n)_{n=1}^{\infty}$ in S. We have, by Fatou's lemma,

$$\begin{aligned}
\|C[h, A] f\|_\alpha &= \left\{\int_{\mathbf{R}} \int_{\mathbf{R}} \frac{|C[h, A] f(x) - C[h, A] f(y)|^2}{|x - y|^{1+\alpha}} dx\, dy\right\}^{1/2} \\
&\leqslant \liminf_{n\to\infty} \left\{\int_{\mathbf{R}} \int_{\mathbf{R}} \frac{|C[h, A] f_n(x) - C[h, A] f_n(y)|^2}{|x - y|^{1+\alpha}} dx\, dy\right\}^{1/2} \\
&= \liminf_{n\to\infty} \|C[h, A] f_n\|_\alpha \leqslant \sup\{\|C[h, A] g\|_\alpha / \|g\|_\alpha;\ g \in S\} \|f\|_\alpha .
\end{aligned}$$

Thus

$$\|C[h, A]\|_{\alpha,\alpha} = \sup\{\|C[h, A] f\|_\alpha / \|f\|_\alpha;\ f \in S\}. \tag{15}$$

2.3. Let $\mathfrak{X}$ be the totality of open squares in the complex plane $\mathbf{C}$ with sides parallel to the coordinate axes. For a nonnegative function $\omega(z)$ on $\mathbf{C}$, we denote by $L^2(\mathbf{C}, \omega)$ the L^2-space on $\mathbf{C}$ with respect to the measure $\omega(z)\, d\sigma(z)$, where $d\sigma(z)$

denotes the area element. For $p > 1$ we say that $\omega(z)$ satisfies (A_p) with a constant Ξ if

$$\sup_{X\in\mathfrak{X}} (\tilde{m}_X\omega)\left(\tilde{m}_X\omega^{-1/(p-1)}\right)^{p-1} \leqslant \Xi^p,$$

where

$$\tilde{m}_X\omega = \frac{1}{\sigma(X)}\int_X \omega(z)\,d\sigma(z) \qquad \left(\sigma(X) = \int_X d\sigma(z)\right).$$

We say that $\omega(z)$ satisfies (A_∞) with two constants Ξ, $0 < \delta \leqslant 1$, if, for any $X \in \mathfrak{X}$ and any Borel set E in X,

$$\omega(E)/\omega(X) \leqslant \Xi\{\sigma(E)/\sigma(X)\}^{\delta},$$

where $\omega(F) = \int_F \omega(z)\,d\sigma(z)$. The following two lemmas are well-known (cf. **[5]**).

LEMMA 3. *Let $1 < p < 2$ and let $\omega(z)$ be a nonnegative function in $\mathbf{C}$ satisfying (A_p) with a constant Ξ and satisfying, for any $X \in \mathfrak{X}$, $\omega(X^*) \leqslant$ Const. $\omega(X)$, where X^* is the square in $\mathfrak{X}$ with the same center as X and $\sigma(X^*) = 2\sigma(X)$. Then, for any $F \in L^2(\mathbf{C}, \omega)$,*

$$\|\mathfrak{M}F\|_{L^2(\mathbf{C},\omega)} \leqslant C_p\Xi\|F\|_{L^2(\mathbf{C},\omega)},$$

where $\mathfrak{M}F(z) = \sup_{z\in X,\, X\in\mathfrak{X}} \tilde{m}_X|F|$ and C_p is a constant depending only on p.

LEMMA 4. *Let $\omega(z)$ satisfy (A_∞) with two constants Ξ, $0 < \delta \leqslant 1$. Then, for any $F \in L^2(\mathbf{C}, \omega)$,*

$$\|\mathfrak{G}F\|_{L^2(\mathbf{C},\omega)} \leqslant C_\delta\Xi\|F\|_{L^2(\mathbf{C},\omega)},$$

where

$$\mathfrak{G}F(z) = \lim_{\varepsilon\to 0}\int_{|z-\zeta|>\varepsilon} \frac{F(\zeta)}{(z-\zeta)^2}\,d\sigma(\zeta)$$

and C_δ is a constant depending only on δ.

3. Proof of Theorem 1.

3.1. Let $h(x)$ be an infinitely differentiable function on $\mathbf{R}$, $A(x)$ a real-valued function on $\mathbf{R}$ with $A' \in M(E_\alpha)$, and $B(x)$ an infinitely differentiable real-valued function on $\mathbf{R}$ with $\|B'\|_{M(E_\alpha)} \leqslant \|A'\|_{M(E_\alpha)}$. We write simply $N = 1 + \|A'\|_{L^\infty} + \|A'\|_{M(E_\alpha)}$. Let $\Gamma = \{(x, B(x));\ x \in \mathbf{R}\}$ and $V = \{x + iy \in \mathbf{C};\ y > B(x)\}$. We define

$$\omega(x + iy) = |y - B(x)|^{1-\alpha} \qquad (x + iy \in \mathbf{C}).$$

Then $\omega(X^*) \leqslant$ Const.$\omega(X)$ $(X \in \mathfrak{X})$. We say that a function $g(z)$ on Γ is differentiable if $\lim_{\zeta\to 0,\, z+\zeta\in\Gamma}\{g(z+\zeta) - g(z)\}/\zeta$ exists everywhere on Γ. We denote by $E_{\alpha\Gamma}$ the Banach space obtained by the completion of differentiable functions on Γ with compact support with respect to the norm

$$\|g\|_{\alpha\Gamma} = \left\{\int_\Gamma\int_\Gamma \frac{|g(z) - g(\zeta)|^2}{|z-\zeta|^{1+\alpha}}|dz|\,|d\zeta|\right\}^{1/2},$$

where dz is the curvilinear integral element. This is the α-energy space on Γ. In this section we estimate $\|C[h, A]\|_{\alpha,\alpha}$.

LEMMA 5. *The function* $\omega(z)$ *satisfies* $(A_{(4-\alpha)/2})$ *with a constant* Const.N.

PROOF. We write simply $p = (4 - \alpha)/2$. For $X \in \mathfrak{X}$ we put $l = \sqrt{\sigma(X)}$. First we suppose that $l > \operatorname{dis}(X, \Gamma)$ $(= \eta)$, where $\operatorname{dis}(\cdot, \cdot)$ denotes the distance. Since $\|B'\|_{L^\infty} \leqslant$ Const. N, we have $\operatorname{dis}(z, z_\Gamma) \leqslant$ Const. Nl $(z = x + iy \in X,\ z_\Gamma = x + iB(x))$. Hence,

$$\tilde{m}_X\omega \leqslant \left\{\frac{\text{Const.}}{l}\right\}\int_0^{\text{Const. }Nl} s^{1-\alpha}\,ds = \text{Const. } N^{2-\alpha}l^{1-\alpha}$$

and

$$\tilde{m}_X\omega^{-1/(p-1)} \leqslant \left\{\frac{\text{Const.}}{l}\right\}\int_0^{\text{Const. }Nl} s^{(\alpha-1)/(p-1)}\,ds$$
$$= \text{Const. } N^{1+\{(\alpha-1)/(p-1)\}}l^{(\alpha-1)/(p-1)}.$$

Thus,

$$(\tilde{m}_X\omega)\left(\tilde{m}_X\omega^{-1/(p-1)}\right)^{p-1} \leqslant \text{Const. } N^p.$$

Next we suppose that $l \leqslant \eta$. Then $\eta \leqslant \operatorname{dis}(z, z_\Gamma) \leqslant$ Const. $N\eta$ $(z \in X)$. Hence,

$$\tilde{m}_X\omega \leqslant \text{Const. } N^{1-\alpha}\eta^{1-\alpha} \quad\text{and}\quad \tilde{m}_X\omega^{-1/(p-1)} \leqslant \text{Const. } \eta^{(\alpha-1)/(p-1)}.$$

Thus

$$(\tilde{m}_X\omega)\left(\tilde{m}_X\omega^{-1/(p-1)}\right)^{p-1} \leqslant \text{Const. } N^{1-\alpha} \leqslant \text{Const. } N^p. \qquad \text{Q.E.D.}$$

LEMMA 6. *The function* $\omega(z)$ *satisfies* (A_∞) *with two constants* Const. $N^{1-\alpha}$, 1.

PROOF. Let $X \in \mathfrak{X}$ and $E \subset X$. We put $l = \sqrt{\sigma(X)}$ and $\eta = \operatorname{dis}(X, \Gamma)$. If $l \geqslant \eta$, then $\omega(X) \geqslant$ Const. $l^{3-\alpha}$, $\omega(E) \leqslant$ Const. $N^{1-\alpha}l^{1-\alpha}\sigma(E)$, and hence, $\omega(E)/\omega(X) \leqslant$ Const. $N^{1-\alpha}\sigma(E)/\sigma(X)$. If $l < \eta$, then $\omega(X) \geqslant \eta^{1-\alpha}l^2$, $\omega(E) \leqslant$ Const. $N^{1-\alpha}\eta^{1-\alpha}\sigma(E)$, and hence, $\omega(E)/\omega(X) \leqslant$ Const. $N^{1-\alpha}\sigma(E)/\sigma(X)$. Q.E.D.

LEMMA 7 (CARLESON [**4**, p. 55]). *Let* $F(z)$ *be a differentiable function in* $U = \{x + iy \in \mathbf{C};\ y > 0\}$ *such that*

$$\|F\|_{\alpha U} = \left\{\int_U |\nabla F(x+iy)|^2 y^{1-\alpha}\,d\sigma(x+iy)\right\}^{1/2} < \infty.$$

Then $F_+(x) = \lim_{y\downarrow 0} F(x+iy)$ *exists a.e. on* $\mathbf{R}$ *and* $\|F_+\|_\alpha \leqslant$ Const.$\|F\|_{\alpha U}$.

LEMMA 8. *Let* $\tilde{G}(z)$ *be a differentiable function in* V *such that*

$$\|\tilde{G}\|_{\alpha V} = \left\{\int_V |\nabla\tilde{G}(z)|^2\omega(z)\,d\sigma(z)\right\}^{1/2} < \infty.$$

Then $\tilde{G}_+(z) = \lim_{y\downarrow 0}\tilde{G}(z+iy)$ *exists a.e. on* Γ *and* $\|\tilde{G}_+\|_{\alpha\Gamma} \leqslant$ Const. $N^2\|\tilde{G}\|_{\alpha V}$.

PROOF. Let $F(x+iy) = \tilde{G}(x + i(y + B(x)))$ $(x + iy \in U)$. Then $\|F\|_{\alpha U} \leqslant$ Const. $N\|\tilde{G}\|_{\alpha V}$. Lemma 7 shows that $F_+(x)$ exists a.e. on $\mathbf{R}$ and $\|F_+\|_\alpha \leqslant$ Const.$\|F\|_{\alpha U}$. Since $A' \in L^\infty$, $\tilde{G}_+(z) = F_+(\operatorname{Re} z)$ exists a.e. on Γ and

$$\|\tilde{G}_+\|_{\alpha\Gamma} \leqslant \text{Const. } N\|F_+\|_\alpha \leqslant \text{Const. } N\|F\|_{\alpha U} \leqslant \text{Const. } N^2\|\tilde{G}\|_{\alpha V}. \qquad \text{Q.E.D.}$$

LEMMA 9. *For* $g \in E_{\alpha\Gamma}$ *we define*

$$\mathfrak{C}g(z) = \int_\Gamma \frac{g(\zeta)}{z-\zeta}\,d\zeta \qquad (z \in V).$$

Then $\mathfrak{C}_+g(z) = \lim_{y\downarrow 0}\mathfrak{C}g(z+iy)$ *exists a.e. on* Γ *and*

$$\|\mathfrak{C}_+g\|_{\alpha\Gamma} \leqslant \text{Const. } N^{4-\alpha}\|g\|_{\alpha\Gamma}.$$

PROOF. Let $f(x) = g(x+iB(x))$ and $F(x+iy) = R_y * f(x)$ $(y>0)$, where $R_y(x) = y/\{\pi(x^2+y^2)\}$. Then we have

$$(16) \qquad \|F\|^2_{\alpha U} = \text{Const.}\int_0^\infty y^{1-\alpha}\,dy\int_{\mathbf{R}}|\hat{R}_y(\xi)\xi\hat{f}(\xi)|^2\,d\xi$$

$$= \text{Const.}\int_{\mathbf{R}}|\xi|^\alpha|\hat{f}(\xi)|^2\,d\xi = \text{Const.}\|f\|^2_\alpha \leqslant \text{Const. } N^{1+\alpha}\|g\|^2_{\alpha\Gamma}.$$

We put $G(x+iy) = F(x+i(y-B(x)))$ $(x+iy \in V)$. Then

$$g(z) = \lim_{y\downarrow 0} G(z+iy) \quad \text{a.e. on } \Gamma.$$

By (16) we have

$$\|G\|_{\alpha V} \leqslant \text{Const. } N\|F\|_{\alpha U} \leqslant \text{Const. } N^{(3+\alpha)/2}\|g\|_{\alpha\Gamma}.$$

We fix $z \in V$, $\varepsilon > 0$, and apply Green's formula to $G(\zeta)$ and $1/(z-\zeta)$ in $V_s = V - \{\zeta;\, |z-\zeta| \leqslant s\}$ $(0 < s \leqslant \varepsilon)$. Then

$$(17) \qquad \mathfrak{C}g(z) = \int_{|z-\zeta|=s}\frac{G(\zeta)}{z-\zeta}\,d\zeta - 2i\int_{V_s}\frac{\partial G(\zeta)/\partial\bar{\zeta}}{z-\zeta}\,d\sigma(\zeta)$$

as long as $\operatorname{dis}(z,\Gamma) > \varepsilon$. Integrating each quantity in (17) by $(2/\varepsilon)\,ds$ in $(\varepsilon/2, \varepsilon)$, we have

$$(18) \qquad \mathfrak{C}g(z) = -\frac{2i}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon} ds\int_0^{2\pi} G(z+se^{it})\,dt$$

$$-2i\int_V \frac{\rho_\varepsilon(z-\zeta)\big(\partial G(\zeta)/\partial\bar{\zeta}\big)}{z-\zeta}\,d\sigma(\zeta),$$

where $\rho_\varepsilon(w) = 0$ $(|w| \leqslant \varepsilon/2)$, $\rho_\varepsilon(w) = (2/\varepsilon)(|w| - \varepsilon/2)$ $(\varepsilon/2 < |w| \leqslant \varepsilon)$ and $\rho_\varepsilon(w) = 1$ $(|w| > \varepsilon)$. We denote by $I^\varepsilon(z)$ and $J^\varepsilon(z)$ the first and second quantities, respectively, on the right side of (18). Note that $(\partial G/\partial\bar{\zeta})\tilde{\chi}_V \in L^2(\mathbf{C},\omega)$, where $\tilde{\chi}_V(z)$ denotes the characteristic function of V. Lemmas 3–6 show that

$$\|\mathfrak{S}\big(\partial G/\partial\bar{\zeta}\cdot\tilde{\chi}_V\big)\|_{L^2(\mathbf{C},\omega)} \leqslant \text{Const. } N\|\tilde{m}\,\big(\partial G/\partial\bar{\zeta}\cdot\tilde{\chi}_V\big)\|_{L^2(\mathbf{C},\omega)}$$

$$\leqslant \text{Const. } N^{2-\alpha}\|\partial G/\partial\bar{\zeta}\cdot\tilde{\chi}_V\|_{L^2(\mathbf{C},\omega)} \leqslant \text{Const. } N^{2-\alpha}\|G\|_{\alpha V}.$$

Note that $\lim_{\varepsilon\to 0}(\partial I^\varepsilon/\partial\xi)(x+iy) = -2\pi i(\partial G/\partial\xi)(x+iy)$ a.e. in V $(\xi = x, y)$ and $\lim_{\varepsilon\to 0}(\partial J^\varepsilon/\partial\xi)(x+iy) = 2iq\mathfrak{S}(\partial G/\partial\bar{\zeta}\cdot\tilde{\chi}_V)(x+iy)$ a.e. in V, where $q = 1$ if $\xi = x$ and $q = i$ if $\xi = y$. Hence, we have

$$(\partial\mathfrak{C}g/\partial\xi)(x+iy) = -2\pi i(\partial G/\partial\xi)(x+iy) + 2iq\mathfrak{S}\big(\partial G/\partial\bar{\zeta}\cdot\tilde{\chi}_V\big)(x+iy)$$

a.e. in V $(\xi = x, y)$. Thus

$$\|\mathfrak{C}g\|_{\alpha V} = \||\nabla\mathfrak{C}g|\tilde{\chi}_V\|_{L^2(\mathbf{C},\omega)} \leqslant \text{Const.}\||\nabla G|\tilde{\chi}_V\|_{L^2(\mathbf{C},\omega)}$$

$$+\text{Const.}\|\mathfrak{S}\big(\partial G/\partial\bar{\zeta}\cdot\tilde{\chi}_V\big)\|_{L^2(\mathbf{C},\omega)} \leqslant \text{Const. } N^{2-\alpha}\|G\|_{\alpha V}.$$

Using Lemma 8 with $\tilde{G}(z) = \mathfrak{C}g(z)$, we obtain the required assertion. Q.E.D.

LEMMA 10. *Let*

$$C[B](x, y) = 1/\{(x-y) + i(B(x) - B(y))\}.$$

Then $\|C[B]\|_{\alpha,\alpha} \leqslant$ Const. $N^{(13-\alpha)/2}$.

PROOF. For $f \in C_0^\infty$ we put $\bar{f}(x) = f(x)(1 + iB'(x))$. Then

$$C[B]\bar{f}(x) = \mathfrak{C}_+ g(x + iB(x)) - \pi i f(x),$$

where $g(z) = f(\operatorname{Re} z)$ $(z \in \Gamma)$ (cf. [3]). Thus Lemma 9 shows that

$$\begin{aligned} \|C[B]\bar{f}\|_\alpha &\leqslant \|\mathfrak{C}_+ g(\cdot + iB(\cdot))\|_\alpha + \pi\|f\|_\alpha \\ &\leqslant N^{(1+\alpha)/2}\|\mathfrak{C}_+ g\|_{\alpha\Gamma} + \pi N\|\bar{f}\|_\alpha \leqslant \text{Const. } N^{(9-\alpha)/2}\|g\|_{\alpha\Gamma} + \pi N\|\bar{f}\|_\alpha \\ &\leqslant \text{Const. } N^{(11-\alpha)/2}\|f\|_\alpha + N\|\bar{f}\|_\alpha \leqslant \text{Const. } N^{(13-\alpha)/2}\|\bar{f}\|_\alpha. \end{aligned}$$

Since $\{f(1 + iB');\ f \in C_0^\infty\}$ is dense in E_α, (15) shows the required inequality. Q.E.D.

LEMMA 11. $\|C[e^{i\cdot}, B]\|_{\alpha,\alpha} \leqslant$ Const. $N^{(15-\alpha)/2}$.

PROOF. We consider the anticlockwise contour $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$, where $\Lambda_1 = \{x + iy;\ |x| \leqslant 2N,\ y = -1\}$, $\Lambda_2 = \{x + iy;\ x = 2N,\ |y| \leqslant 1\}$, $\Lambda_3 = \{x + iy;\ |x| \leqslant 2N,\ y = 1\}$ and $\Lambda_4 = \{x + iy;\ x = -2N,\ |y| \leqslant 1\}$. Then

$$\begin{aligned} C[e^{i\cdot}, B](x, y) &= \frac{1}{2\pi i}\int_\Lambda \frac{e^{i\zeta}}{\zeta(x-y) - (B(x) - B(y))}\,d\zeta \\ &= \frac{1}{2\pi i}\sum_{j=1}^{4}\int_{\Lambda_j} \left(= \frac{1}{2\pi i}\sum_{j=1}^{4} C[\Lambda_j](x, y), \text{ say}\right). \end{aligned}$$

Let $f \in C_0^\infty$. Then we have

$$C[\Lambda_1]f(x) = e\int_{-2N}^{2N} C[s \cdot - B(\cdot)]\,f(x)\,ds \quad \text{a.e.},$$

and hence, by Lemma 10,

$$\|C[\Lambda_1]f\|_\alpha \leqslant \text{Const.}\int_{-2N}^{2N}\|C[s \cdot - B(\cdot)]\,f\|_\alpha\,ds \leqslant \text{Const. } N^{(15-\alpha)/2}\|f\|_\alpha.$$

In the same manner $\|C[\Lambda_3]f\|_\alpha \leqslant$ Const. $N^{(15-\alpha)/2}\|f\|_\alpha$. We have

$$C[\Lambda_2]f(x) = e^{2Ni}\int_{-1}^{1}\tilde{C}^s f(x)e^{-s}\,ds \quad \text{a.e.},$$

where

$$\tilde{C}^s(x, y) = 1/\{(2Nx - B(x)) - (2Ny - B(y)) + is(x-y)\}.$$

We now consider a mapping $x \to \bar{x} = 2Nx - B(x)$, its inverse mapping $\tau(\bar{x}) = x$, and put $h(\bar{x}) = f \circ \tau(\bar{x})/(2N - B' \circ \tau(\bar{x}))$. Then $\tilde{C}^s f(x) = C[s\tau]h(2Nx - B(x))$ a.e. Hence, we have

$$\begin{aligned} (19)\quad \|C[\Lambda_2]f\|_\alpha &\leqslant \text{Const.}\int_{-1}^{1}\|\tilde{C}^s f\|_\alpha\,ds = \text{Const.}\int_{-1}^{1}\|C[s\tau]h(2N \cdot - B(\cdot))\|_\alpha\,ds \\ &\leqslant \text{Const. } N^{(\alpha-1)/2}\int_{-1}^{1}\|C[s\tau]h\|_\alpha\,ds \\ &\leqslant \text{Const. } N^{(\alpha-1)/2}\int_{-1}^{1}\|C[s\tau]\|_{\alpha,\alpha}\,ds\,\|h\|_\alpha. \end{aligned}$$

By (9)

$$\|\tau'\|_{M(E_\alpha)} = \|1/(2N - B'\circ\tau)\|_{M(E_\alpha)}$$
$$\leqslant \text{Const.}\left\{1/N + \|B'\circ\tau\|_{M(E_\alpha)}/N^2\right\} \leqslant \text{Const.}/N.$$

Using Lemma 10 with $B(x) = s\tau(x)$, we have

$$\int_{-1}^{1} \|C[s\tau]\|_{\alpha,\alpha}\, ds \leqslant \int_{-1}^{1}\left\{1 + \|s\tau\|_{L^\infty} + \|s\tau\|_{M(E_\alpha)}\right\}^{(13-\alpha)/2} ds \leqslant \text{Const.}$$

We also have

$$\|h\|_\alpha \leqslant \|\tau'\|_{M(E_\alpha)}\|f\circ\tau\|_\alpha \leqslant \text{Const. } N^{-(1+\alpha)/2}\|f\|_\alpha.$$

Thus the last quantity in (19) is dominated by Const.$/N \cdot \|f\|_\alpha$. In the same manner

$$\|C[\Lambda_4]f\|_\alpha \leqslant \text{Const.}/N\cdot\|f\|_\alpha.$$

Consequently,

$$\|C[e^{i\cdot}, B]f\|_\alpha \leqslant \text{Const. } N^{(15-\alpha)/2}\|f\|_\alpha.$$

Since C_0^∞ is dense in E_α, (15) yields the required inequality. Q.E.D.

LEMMA 12.

$$\|C[h, A]\|_{\alpha,\alpha} \leqslant \text{Const.} d_h^*(N) N^{(17-\alpha)/2},$$

where $d_h^*(N) = \sum_{j=0}^{9}\max\{|h^{(j)}(x)|;\ |x| \leqslant 2N\}$.

PROOF. We choose a function $\gamma(x)$ in C_0^∞ so that $\gamma(x) = 1$ ($|x| \leqslant 1$), $\gamma(x) = 0$ ($|x| \geqslant 2$), and put $\gamma^*(x) = \gamma(x/N)$. Then

$$|(\widehat{h\gamma^*})(\xi)| \leqslant \frac{\text{Const.}}{(1+|\xi|)^9}\int_{\mathbf{R}} |(h\gamma^*)^{(9)}(x)|\, dx \leqslant \frac{\text{Const. } d_h^*(N)N}{(1+|\xi|)^9}.$$

Let $A_m(x) = \int_0^x A' * \phi_m(s)\, ds$ ($m \geqslant 1$, $\phi_m(x) = \sqrt{m/\pi}\, e^{-mx^2}$). Then we have, for any $m \geqslant 1$ and $f \in C_0^\infty$,

$$C[h, A_m]f(x) = C[h\gamma^*, A_m]f(x) = \text{Const.}\int_{\mathbf{R}} (\widehat{h\gamma^*})(\xi) C[e^{i\cdot}, \xi A_m] f(x)\, d\xi \quad \text{a.e.}$$

Since $\|A'(\cdot - y)\|_{M(E_\alpha)} = \|A'\|_{M(E_\alpha)}$ for any $y \in \mathbf{R}$, we have $\|A_m'\|_{M(E_\alpha)} \leqslant \|A'\|_{M(E_\alpha)}$, and hence,

$$\tilde N(\xi A_m) = 1 + \|\xi A_m'\|_{L^\infty} + \|\xi A_m'\|_{M(E_\alpha)} \leqslant (1 + |\xi|)N.$$

Using Lemma 11, with $B(x) = \xi A_m(x)$, we have

$$\|C[h, A_m]f\|_\alpha \leqslant \text{Const.}\int_{\mathbf{R}} |(\widehat{h\gamma^*})(\xi)|\|C[e^{i\cdot}, \xi A_m]f\|_\alpha\, d\xi$$
$$\leqslant \text{Const.} d_h^*(N) N \int_{\mathbf{R}} \frac{\tilde N(\xi A_m)^{(15-\alpha)/2}}{(1+|\xi|)^9}\, d\xi\, \|f\|_\alpha$$
$$\leqslant \text{Const.} d_h^*(N) N^{(17-\alpha)/2}\|f\|_\alpha.$$

Thus Fatou's lemma and (13) show that, for some subsequence $(A_{m_j})_{j=1}^{\infty}$,

$$\|C[h, A]f\|_\alpha \leqslant \liminf_{j\to\infty}\left\|C\left[h, A_{m_j}\right]f\right\|_\alpha \leqslant \text{Const.}\, d_h^*(N)N^{(17-\alpha)/2}\|f\|_\alpha.$$

Since C_0^∞ is dense in E_α, we have the required inequality. Q.E.D.

3.2. In this section we discuss the pointwise existence of $C[h, A]f(x)$ for $f \in E_\alpha$.

LEMMA 13. $\operatorname{cap}_\alpha(x;\, C[h, A]^*f(x) > \lambda) \leqslant (\text{Const.}/\lambda^2)d_h(N)N^{10}\|f\|_\alpha$ $(\lambda > 0, f \in E_\alpha)$.

PROOF. We write simply $d^* = d_h(N)N^{10}$. Inequality (14) gives that, for any $x \in \mathbf{R}$,

$$\begin{aligned} C[h, A]^*f(x) &\leqslant \text{Const.}\left\{\mathfrak{M}(C[h, A]f)(x) + d_h(\|A'\|_{L^\infty})(1 + \|A'\|_{L^\infty})^{10}\mathfrak{M}f(x)\right\} \\ &\leqslant \text{Const.}\{\mathfrak{M}(C[h, A]f)(x) + d^*\mathfrak{M}f(x)\}. \end{aligned}$$

By Lemma 12 and (8) we have, with two absolute constants c_1, c_2,

$$\begin{aligned} &\operatorname{cap}_\alpha(x;\, C[h, A]^*f(x) > \lambda) \\ &\quad\leqslant \operatorname{cap}_\alpha(x;\, \mathfrak{M}(C[h, A]f)(x) > c_1\lambda) + \operatorname{cap}_\alpha(x;\, d^*\mathfrak{M}f(x) > c_2\lambda) \\ &\quad\leqslant (\text{Const.}/\lambda^2)\{\|C[h, A]f\|_\alpha^2 + d^*\|f\|_\alpha^2\} \\ &\quad\leqslant (\text{Const.}/\lambda^2)d^*\|f\|_\alpha^2. \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 14. *Let $n \geqslant 0$. Then*

$$(*)_n \qquad C[t^n, A]f(x) \textit{ exists } \alpha\textit{-q.e. for any } f \in E_\alpha.$$

PROOF. We inductively prove $(*)_n$. We have, for any $f \in C_0^\infty$,

$$C[t^0, A]_\varepsilon f(x) = -\int_{|x-y|>\varepsilon}\frac{f(x) - f(y)}{x - y}\,dy \qquad (\varepsilon > 0).$$

Hence $C[t^0, A]f(x)$ exists everywhere. From this and Lemma 13, $(*)_0$ is deduced. Let $f \in C_0^\infty$. Then we have by integration by parts,

$$\begin{aligned} (20)\quad C[t^n, A]_\varepsilon f(x) &= C[t^{n-1}, A]_\varepsilon(A'f)(x) \\ &\quad - \frac{1}{n}\left\{\left(\frac{A(x+\varepsilon) - A(x)}{\varepsilon}\right)^n f(x+\varepsilon) - \left(\frac{A(x) - A(x-\varepsilon)}{\varepsilon}\right)^n f(x-\varepsilon)\right\} \\ &\quad - \int_{|x-y|>\varepsilon}\frac{(A(x) - A(y))^n}{(x-y)^n}f'(y)\,dy \qquad (\varepsilon > 0). \end{aligned}$$

By $(*)_{n-1}$, $C[t^{n-1}, A](A'f)(x)$ exists α-q.e. By (6) the second quantity on the right side of (20) tends to zero as $\varepsilon \to 0$ except for a set of α-capacity zero. The third quantity tends to zero as $\varepsilon \to 0$ everywhere. Thus $C[t^n, A]f(x)$ exists α-q.e. From this and Lemma 13, $(*)_n$ is deduced. Q.E.D.

LEMMA 15. *Let $f \in E_\alpha$. Then $C[h, A]f(x)$ exists α-q.e.*

PROOF. We put

$$D_m(x, y) = \int_{m-1<|\xi|\leqslant m}\hat{h}(\xi)C[e^{i\cdot}, \xi A](x, y)\,d\xi = \sum_{n=0}^{\infty}u_n^{(m)}C[t^n, A](x, y)$$

$$\left(x \neq y, m \geqslant 1;\, u_n^{(m)} = \frac{i^n}{n!}\int_{m-1<|\xi|\leqslant m}\hat{h}(\xi)\xi^n\,d\xi\right).$$

Then we have, for any $f \in C_0^\infty$ and $m \geqslant 1$,

$$D_m^* f(x) \leqslant \sum_{n=0}^{\infty} v_n^{(m)} C[t^n, A]^* f(x)$$

everywhere ($v_n^{(m)} = |u_n^{(m)}|$). Since $d_{t^n}(N) \leqslant \text{Const.}(n+1)^{11} N^n$, we have, by (5) and (14),

$$\begin{aligned} C[t^n, A]^* f(x) &\leqslant \text{Const.}\left\{ \mathfrak{M}(C[t^n, A]f)(x) + d_{t^n}(N) N^{10} \mathfrak{M} f(x) \right\} \\ &\leqslant \text{Const. } \kappa_{\alpha/2} * \left\{ \mathfrak{M} g_n + (n+1)^{11} N^{n+10} \mathfrak{M} g \right\}(x) \quad \text{everywhere } (n \geqslant 0), \end{aligned}$$

where $g_n(x)$ and $g(x)$ are defined by $C[t^n, A]f(x) = \kappa_{\alpha/2} * g_n(x)$ and $f(x) = \kappa_{\alpha/2} * g(x)$ a.e. We have, for any $x \in \mathbf{R}$ and $m \geqslant 1$,

$$D_m^* f(x) \leqslant \text{Const.}\left\{ \kappa_{\alpha/2} * \sum_{n=0}^{\infty} v_n^{(m)} \left[\mathfrak{M} g_n + (n+1)^{11} N^{n+10} \mathfrak{M} g \right] \right\}(x)$$

$$\left(= \text{Const. } \kappa_{\alpha/2} * h_m(x), \text{ say}\right).$$

Since

$$\begin{aligned} \|\kappa_{\alpha/2} * h_m\|_\alpha &= \text{Const.} \|h_m\|_{L^2} \\ &\leqslant \text{Const. } \sum_{n=0}^{\infty} v_n^{(m)} \left\{ \|\mathfrak{M} g_n\|_{L^2} + (n+1)^{11} N^{n+10} \|\mathfrak{M} g\|_{L^2} \right\} \\ &\leqslant \text{Const. } \sum_{n=0}^{\infty} v_n^{(m)} \left\{ \|g_n\|_{L^2} + (n+1)^{11} N^{n+10} \|g\|_{L^2} \right\} \\ &= \text{Const. } \sum_{n=0}^{\infty} v_n^{(m)} \left\{ \|C[t^n, A]f\|_\alpha + (n+1)^{11} N^{n+10} \|f\|_\alpha \right\} \\ &\leqslant \text{Const. } \sum_{n=0}^{\infty} v_n^{(m)} \left\{ d_{t^n}^*(N) N^{(17-\alpha)/2} + (n+1)^{11} N^{n+10} \right\} \|f\|_\alpha \\ &\leqslant \text{Const.}\left\{ \sum_{n=0}^{\infty} v_n^{(m)} (n+1)^{11} N^{n+10} \right\} \|f\|_\alpha < \infty, \end{aligned}$$

Lemma 13 shows that $D_m^* f(x) < \infty$ α-q.e. ($m \geqslant 1$). Since the α-capacity of a set where $C[t^n, A]f(x)$ does not exist for some $n \geqslant 0$ is equal to zero according to Lemma 14, the Lebesgue dominated convergence theorem gives that $D_m f(x)$ exists α-q.e. ($m \geqslant 1$). We also have

$$\begin{aligned} C[h, A]^* f(x) &\leqslant \sum_{m=1}^{\infty} D_m^* f(x) \\ &\leqslant \text{Const.} \int_{\mathbf{R}} |\hat{h}(\xi)| C\left[e^{i\cdot}, \xi A\right]^* f(x)\, d\xi \quad \text{everywhere.} \end{aligned}$$

In the same manner as above, we have $\int_{\mathbf{R}} |\hat{h}(\xi)| C[e^{i\cdot}, \xi A]^* f(x)\, d\xi < \infty$ α-q.e. Since the α-capacity of a set where $D_m f(x)$ does not exist for some $m \geqslant 1$ is equal to zero, $C[h, A]f(x)$ exists α-q.e. Q.E.D.

Lemmas 12 and 15 show that $C[h, A]$ is α-bounded. This completes the proof of Theorem 1.

4. Proof of Theorem 2.

4.1. We begin by showing some lemmas. Let $E_{-\alpha}$ be the Banach space of distributions on $\mathbf{R}$ obtained by the completion of C_0^∞ with respect to the norm

$$\|f\|_{-\alpha} = \left\{ \int_{\mathbf{R}} |\xi|^{-\alpha} |\hat{f}(\xi)|^2 \, d\xi \right\}^{1/2}.$$

This is the dual space of E_α [**10**, p. 354]. We denote by $(\cdot, \cdot)$ the inner product of functions on $\mathbf{R}$ with respect to the measure dx. Let $\mathbf{P}_y$, $\mathbf{Q}_y$ be the operators defined by $\mathbf{P}_y f = P_y * f$, $\mathbf{Q}_y f = Q_y * f$ ($y > 0$, $f \in C_0^\infty$). Let $\mathbf{M}_a$ denote the multiplication operator associated with a function $a(x)$ on $\mathbf{R}$. For $h_0(x) = x$ and a complex-valued continuous function $A(x)$ on $\mathbf{R}$, we denote by $C[h_0, A]$ the operator associated with a kernal $(A(x) - A(y))/(x - y)^2$. We denote the identity operator by $\mathbf{I}$.

LEMMA 16 (COIFMAN - MCINTOSH - MEYER [**8**]). *Let $a \in L^\infty$. Then*

$$C[h_0, A] = \lim_{\varepsilon \to 0} \int_\varepsilon^{1/\varepsilon} \{ \mathbf{P}_y \mathbf{M}_a \mathbf{Q}_y + \mathbf{Q}_y \mathbf{M}_a \mathbf{P}_y \} \frac{dy}{y} \quad \text{on } C_0^\infty,$$

where $A(x) = \int_0^x a(s)\, ds$.

LEMMA 17. *Let $a \in$ BMO. Then, as operators on C_0^∞:*

(21) $$\mathbf{M}_{(Q_y * a)} \mathbf{P}_y = \mathbf{Q}_y \mathbf{M}_a \mathbf{P}_y - \mathbf{P}_y \mathbf{M}_{(P_y * a)} \mathbf{Q}_y - \mathbf{Q}_y \mathbf{M}_{(Q_y * a)} \mathbf{Q}_y.$$

(22) $$\mathbf{M}_{(P_y * a)} \mathbf{Q}_y = \mathbf{P}_y \mathbf{M}_{(P_y * a)} \mathbf{Q}_y - \mathbf{Q}_y \mathbf{M}_{(Q_y * a)} \mathbf{Q}_y + \mathbf{Q}_y \mathbf{M}_{(P_y * a)} (\mathbf{I} - \mathbf{P}_y).$$

(23) $$\mathbf{M}_{(Q_y * a)} \mathbf{Q}_y = \mathbf{P}_y \mathbf{M}_{(Q_y * a)} \mathbf{Q}_y + \mathbf{Q}_y \mathbf{M}_{(a - P_y * a)} \mathbf{Q}_y + \mathbf{Q}_y \mathbf{M}_{(Q_y * a)} (\mathbf{I} - \mathbf{P}_y).$$

PROOF. Equality (21) is already known [**8**, p. 371]. Since the proofs of other equalities are analogous, we give only the sketch of the proof of (22). We may assume that $a(x) = e^{isx}$ ($s \in \mathbf{R}$). Note that

$$\hat{P}_y(\xi) = 1/(1 + \xi^2 y^2), \quad \hat{Q}_y(\xi) = -i\xi y/(1 + \xi^2 y^2).$$

We have

$$\begin{aligned} \frac{1}{1 + s^2 y^2} \frac{-i\xi y}{1 + \xi^2 y^2} &= \frac{1}{1 + (s + \xi)^2 y^2} \frac{1}{1 + s^2 y^2} \frac{-i\xi y}{1 + \xi^2 y^2} \\ &\quad - \frac{-i(s + \xi) y}{1 + (s + \xi)^2 y^2} \frac{-isy}{1 + s^2 y^2} \frac{-i\xi y}{1 + \xi^2 y^2} \\ &\quad + \frac{-i(s + \xi) y}{1 + (s + \xi)^2 y^2} \frac{1}{1 + s^2 y^2} \left\{ 1 - \frac{1}{1 + \xi^2 y^2} \right\}, \end{aligned}$$

which gives (22). Q.E.D.

We write $\beta = \alpha/2$. We say that a nonnegative measure $d\mu(z)$ on $U = \{x + iy \in \mathbf{C};\, y > 0\}$ is a $(\beta, 1/y)$-measure with a constant Ξ if, for any $\lambda \geqslant 1$ and any finite interval I in $\mathbf{R}$,

$$\int_{S(\lambda, I)} d\mu(z) \leqslant \Xi \lambda^\beta |I|,$$

where

$$S(\lambda, I) = \{ x + iy \in \mathbf{C};\, x \in I, \lambda |I| < y \leqslant 2\lambda |I| \}.$$

LEMMA 18. *Let* $a \in \mathrm{BMO}$. *Then* $|a(x) - P_y * a(x)|^2\, d\sigma(x+iy)/y$ *and* $|Q_y * a(x)|^2\, d\sigma(x+iy)/y$ *are* $(\beta, 1/y)$*-measures with a constant* $\mathrm{Const.}\|a\|_{\mathrm{BMO}}^2$.

PROOF. We have, for any $x + iy \in S(\lambda, I)$,

$$a(x) - P_y * a(x) = \{a(x) - m_J a\} - P_y * (a - m_J a)(x),$$

where J is an interval with the same midpoint as I and of length $2|I|$. We have

$$\int_{S(\lambda, I)} |a(x) - m_J a|^2 \frac{d\sigma(x+iy)}{y} = \log 2 \int_I |a(x) - m_J a|^2\, dx$$
$$\leqslant \mathrm{Const.}\|a\|_{\mathrm{BMO}}^2 |I|$$

(cf. [9, p. 141]). Since

$$\int_{|x-s|>|J|} \frac{|a(s) - m_J a|}{|x-s|^{1+\beta/2}}\, ds \leqslant \mathrm{Const.}\|a\|_{\mathrm{BMO}} |J|^{-\beta/2} \qquad (x \in I)$$

(cf. [9, p. 142]), we have

$$\int_{S(\lambda, I)} |P_y * (a - m_J a)(x)|^2 \frac{d\sigma(x+iy)}{y}$$
$$\leqslant \int_{S(1, I)} P_{\lambda y} * |a - m_J a|(x)^2 \frac{d\sigma(x+iy)}{y}$$
$$\leqslant \mathrm{Const.} \int_{S(1, I)} \frac{d\sigma(x+iy)}{y} \left\{ \int_{\mathbf{R}} \frac{\lambda y}{(x-s)^2 + (\lambda y)^2} |a(s) - m_J a|\, ds \right\}^2$$
$$\leqslant \mathrm{Const.} \int_{S(1, I)} \frac{d\sigma(x+iy)}{y}$$
$$\cdot \left\{ \frac{1}{|J|} \int_{|x-s| \leqslant |J|} |a(s) - m_J a|\, ds + (\lambda y)^{\beta/2} \int_{|x-s|>|J|} \frac{|a(s) - m_J a|}{|x-s|^{1+\beta/2}}\, ds \right\}^2$$
$$\leqslant \mathrm{Const.}\|a\|_{\mathrm{BMO}}^2 \int_{S(1, I)} \{1 + (\lambda y)^\beta |J|^{-\beta}\} \frac{d\sigma(x+iy)}{y}$$
$$\leqslant \mathrm{Const.}\|a\|_{\mathrm{BMO}}^2 \lambda^\beta |I|.$$

Hence, $|a(x) - P_y * a(x)|^2\, d\sigma(x+iy)/y$ is a $(\beta, 1/y)$-measure with a constant $\mathrm{Const.}\|a\|_{\mathrm{BMO}}^2$.

We have

$$\int_{S(\lambda, I)} |Q_y * a(x)|^2 \frac{d\sigma(x+iy)}{y}$$
$$= \int_{S(1, I)} |Q_{\lambda y} * (a - m_J a)(x)|^2 \frac{d\sigma(x+iy)}{y}$$
$$\leqslant \int_{S(1, I)} P_{\lambda y} * |a - m_J a|(x)^2 \frac{d\sigma(x+iy)}{y} \leqslant \mathrm{Const.}\|a\|_{\mathrm{BMO}}^2 \lambda^\beta |I|,$$

and hence, $|Q_y * a(x)|^2\, d\sigma(x+iy)/y$ is also a $(\beta, 1/y)$-measure with a constant $\mathrm{Const.}\|a\|_{\mathrm{BMO}}^2$. Q.E.D.

LEMMA 19. *For $f \in C_0^\infty$, we put*

$$(24)\qquad u(x, y) = \int_{\mathbf{R}} R_y(x-s)|f(s) - f(x)|\, ds \qquad (x + iy \in U),$$

$$(25)\qquad v(x, y) = \int_{\mathbf{R}} R_y(x-s)|f(s) - R_{y/2} * f(x)|\, ds \qquad (x + iy \in U),$$

where $R_y(x) = y/\{\pi(x^2 + y^2)\}$. Then, for any $n \geqslant 0$ and any finite interval I,

$$\sup_{x+iy \in S_n(I)} v(x, 2^{-n}y) \leqslant \text{Const.} \inf_{x+iy \in S_n(I)} u(x, 2^{-n}y) \qquad (S_n(I) = S(2^n, I)).$$

PROOF. Let $x + iy,\ \bar{x} + i\bar{y} \in S_n(I)$. Then $R_{2^{-n}y}(x-s) \leqslant \text{Const.}\, R_{2^{-n}\bar{y}}(\bar{x} - s)$ $(s \in \mathbf{R})$. We have

$$\begin{aligned} v(x, 2^{-n}y) &\leqslant \text{Const.} \int_{\mathbf{R}} R_{2^{-n}\bar{y}}(\bar{x} - s)|f(s) - R_{2^{-n-1}y} * f(x)|\, ds \\ &\leqslant \text{Const.}\, u(\bar{x}, 2^{-n}\bar{y}) + \text{Const.} \int_{\mathbf{R}} R_{2^{-n}\bar{y}}(\bar{x} - s)|R_{2^{-n-1}y} * f(x) - f(\bar{x})|\, ds \\ &\leqslant \text{Const.}\, u(\bar{x}, 2^{-n}\bar{y}) + \text{Const.} \int_{\mathbf{R}} R_{2^{-n-1}y}(x - s)|f(s) - f(\bar{x})|\, ds \\ &\leqslant \text{Const.}\, u(\bar{x}, 2^{-n}\bar{y}), \end{aligned}$$

and hence, the required inequality holds. Q.E.D.

LEMMA 20. *Let $d\mu(z)$ be a $(\beta, 1/y)$-measure with a constant Ξ. Then for any $f \in C_0^\infty$:*

$$(26)\qquad \int_U |f(x) - P_y * f(x)|^2 \frac{d\mu(x+iy)}{y^\alpha} \leqslant \text{Const.}\, \Xi \|f\|_\alpha^2.$$

$$(27)\qquad \int_U |Q_y * f(x)|^2 \frac{d\mu(x+iy)}{y^\alpha} \leqslant \text{Const.}\, \Xi \|f\|_\alpha^2.$$

PROOF. We have

$$\begin{aligned} I &= \int_U |f(x) - P_y * f(x)|^2 \frac{d\mu(x+iy)}{y^\alpha} \\ &\leqslant \text{Const.} \int_U u(x, y)^2 \frac{d\mu(x+iy)}{y^\alpha} \\ &\leqslant \text{Const.} \int_U \left\{ \sum_{n=0}^{\infty} v(x, 2^{-n}y) \right\}^2 \frac{d\mu(x+iy)}{y^\alpha} \\ &\leqslant \text{Const.} \sum_{n=0}^{\infty} (n+1)^2 \int_U v(x, 2^{-n}y)^2 \frac{d\mu(x+iy)}{y^\alpha} \\ &\left(= \text{Const.} \sum_{n=0}^{\infty} (n+1)^2 I_n, \text{ say} \right). \end{aligned}$$

We fix $n \geqslant 0$ and divide U into countable rectangles $\{\tilde{S}_{j,k}\}_{j,k=-\infty}^{\infty}$, where $\tilde{S}_{j,k} = S_n(I_{j,k})$, $I_{j,k} = (j2^{k-n}, (j+1)2^{k-n}]$ $(j, k = 0, \pm 1, \pm 2, \ldots)$. Since $d\mu(z)$ is a $(\beta, 1/y)$-measure with a constant Ξ, we have, by Lemma 19,

$$
\begin{aligned}
(28)\quad l_n &= \sum_{j,k=-\infty}^{\infty} \int_{\tilde{S}_{jk}} v(x, 2^{-n}y)^2 \frac{d\mu(x+iy)}{y^\alpha} \\
&\leqslant \text{Const.} \sum_{j,k=-\infty}^{\infty} \left(2^n |I_{jk}|\right)^{-\alpha} \int_{\tilde{S}_{jk}} d\mu(z) \sup_{x+iy \in \tilde{S}_{jk}} v(x, 2^{-n}y)^2 \\
&\leqslant \text{Const.}\, \Xi 2^{n\beta} \sum_{j,k=-\infty}^{\infty} \left(2^n |I_{jk}|\right)^{-\alpha} |I_{jk}| \inf_{x+iy \in \tilde{S}_{jk}} u(x, 2^{-n}y)^2 \\
&\leqslant \text{Const.}\, \Xi 2^{n\beta} \sum_{j,k=-\infty}^{\infty} \int_{\tilde{S}_{jk}} u(x, 2^{-n}y)^2 \frac{d\sigma(x+iy)}{y^{1+\alpha}} \\
&= \text{Const.}\, \Xi 2^{n\beta} \int_U u(x, 2^{-n}y)^2 \frac{d\sigma(x+iy)}{y^{1+\alpha}} \\
&= \text{Const.}\, \Xi 2^{(\beta-\alpha)n} \int_U u(x, y)^2 \frac{d\sigma(x+iy)}{y^{1+\alpha}} \\
&\leqslant \text{Const.}\, \Xi 2^{(\beta-\alpha)n} \int_0^\infty \frac{1}{y^{1+\alpha}} dy \int_{\mathbf{R}} \int_{\mathbf{R}} R_y(x-s) |f(s) - f(x)|^2 \, ds \, dx \\
&= \text{Const.}\, \Xi 2^{(\beta-\alpha)n} \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{|f(s) - f(x)|^2}{|x-s|^{1+\alpha}} ds \, dx \\
&= \text{Const.}\, \Xi 2^{(\beta-\alpha)n} \|f\|_\alpha^2.
\end{aligned}
$$

Thus,

$$
l \leqslant \text{Const.} \left\{ \sum_{n=0}^{\infty} (n+1)^2 2^{(\beta-\alpha)n} \right\} \Xi \|f\|_\alpha^2.
$$

We have

$$
\begin{aligned}
&\int_U |Q_y * f(x)|^2 \frac{d\mu(x+iy)}{y^\alpha} \\
&\qquad = \int_U \left| \int_{\mathbf{R}} Q_y(x-s) \{ f(s) - R_{y/2} * f(x) \} \, ds \right|^2 \frac{d\mu(x+iy)}{y^\alpha} \\
&\qquad \leqslant \text{Const.} \int_U v(x, y)^2 \frac{d\mu(x+iy)}{y^\alpha} = \text{Const.}\, l_0,
\end{aligned}
$$

and hence, we obtain (27) by (28). Q.E.D.

4.2. In this section we prove Theorem 2. Let $a \in M(E_\alpha)$, f, $g \in C_0^\infty$. Then we have, with $A(x) = \int_0^x a(s)\, ds$,

(29)

$$
\begin{aligned}
(T_a[\mathfrak{Q}, \mathfrak{P}]f, g) &= \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}\big(\mathbf{M}_{(Q_y * a)}\mathbf{P}_y f, g\big)\,\frac{dy}{y}\\
&= \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}\Big\{\big(\mathbf{M}_a\mathbf{P}_y f, \mathbf{Q}_y g\big) - \big(\mathbf{M}_{(P_y * a)}\mathbf{Q}_y f, \mathbf{P}_y g\big) - \big(\mathbf{M}_{(Q_y * a)}\mathbf{Q}_y f, \mathbf{Q}_y g\big)\Big\}\,\frac{dy}{y}\\
&= (C[h_0, A]f, g) - \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}\big(\mathbf{M}_a\mathbf{Q}_y f, \mathbf{P}_y g\big)\,\frac{dy}{y}\\
&\quad - \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}\big(\mathbf{M}_{(P_y * a)}\mathbf{Q}_y f, \mathbf{P}_y g\big)\,\frac{dy}{y}\\
&\quad - \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}\big(\mathbf{M}_{(Q_y * a)}\mathbf{Q}_y f, \mathbf{Q}_y g\big)\,\frac{dy}{y}\\
&= L_1 - L_2 - L_3 - L_4,
\end{aligned}
$$

according to Lemma 16 and (21). Theorem 1 shows that

$$|L_1| \leqslant \|C[h_0, A]f\|_\alpha \|g\|_{-\alpha} \leqslant \|C[h_0, A]\|_{\alpha,\alpha}\|f\|_\alpha\|g\|_{-\alpha}.$$

Since $a \in L^\infty$ we have

$$
\begin{aligned}
|L_2|^2 &\leqslant \int_U |a(x)Q_y * f(x)|^2\frac{d\sigma(x+iy)}{y^{1+\alpha}}\int_U |P_y * g(x)|^2\frac{d\sigma(x+iy)}{y^{1-\alpha}}\\
&\leqslant \|a\|_{L^\infty}^2\int_U |Q_y * f(x)|^2\frac{d\sigma(x+iy)}{y^{1+\alpha}}\int_U |P_y * g(x)|^2\frac{d\sigma(x+iy)}{y^{1-\alpha}}\\
&= \text{Const.}\|a\|_{L^\infty}^2\|f\|_\alpha^2\|g\|_{-\alpha}^2.
\end{aligned}
$$

In the same manner we have

$$|L_3|^2 + |L_4|^2 \leqslant \text{Const.}\|a\|_{L^\infty}^2\|f\|_\alpha^2\|g\|_{-\alpha}^2.$$

Since $g \in C_0^\infty$ is arbitrary, we have

$$\|T_a[\mathfrak{Q}, \mathfrak{P}]f\|_\alpha \leqslant \text{Const.}\{\|C[h_0, A]\|_{\alpha,\alpha} + \|a\|_{L^\infty}\}\|f\|_\alpha$$

for any $f \in C_0^\infty$. Hence (1) holds.

In the same manner as in the estimate of L_2, we have (2) by (22) and

$$\int_U |f(x) - P_y * f(x)|^2\frac{d\sigma(x+iy)}{y^{1+\alpha}} = \text{Const.}\|f\|_\alpha^2.$$

At last we prove (3). Let $a \in \text{BMO}$. Equality (23) shows that, for any $f, g \in C_0^\infty$,

$$
\begin{aligned}
(T_a[\mathfrak{Q}, \mathfrak{Q}]f, g) &= \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}\big(\mathbf{M}_{(Q_y * a)}\mathbf{Q}_y f, g\big)\frac{dy}{y}\\
&= \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}\big(\mathbf{M}_{(Q_y * a)}\mathbf{Q}_y f, \mathbf{P}_y g\big)\frac{dy}{y}\\
&\quad + \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}\big(\mathbf{M}_{(a - P_y * a)}\mathbf{Q}_y f, \mathbf{Q}_y g\big)\,\frac{dy}{y}\\
&\quad + \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}\big(\mathbf{M}_{(Q_y * a)}(\mathbf{I} - \mathbf{P}_y)f, \mathbf{Q}_y g\big)\frac{dy}{y} \quad (= L_1' + L_2' + L_3', \text{ say}).
\end{aligned}
$$

Since $|Q_y * a(x)|^2\, d\sigma(x+iy)/y$ is a $(\beta, 1/y)$-measure with a constant Const.$\|a\|^2_{\mathrm{BMO}}$, we have, by (27),

$$|L_1'|^2 \leqslant \int_U |Q_y * f(x)|^2 |Q_y * a(x)|^2 \frac{d\sigma(x+iy)}{y^{1+\alpha}} \int_U |P_y * g(x)|^2 \frac{d\sigma(x+iy)}{y^{1-\alpha}}$$

$$\leqslant \text{Const.}\|a\|^2_{\mathrm{BMO}} \|f\|^2_\alpha \|g\|^2_{-\alpha}.$$

Since $|a(x) - P_y * a(x)|^2\, d\sigma(x+iy)/y$ is a $(\beta, 1/y)$-measure with a constant Const.$\|a\|^2_{\mathrm{BMO}}$, we have, by (27), $|L_2'|^2 \leqslant$ Const.$\|a\|^2_{\mathrm{BMO}}\|f\|^2_\alpha\|g\|^2_{-\alpha}$. We have, by (26), $|L_3'|^2 \leqslant$ Const.$\|a\|^2_{\mathrm{BMO}}\|f\|^2_\alpha\|g\|^2_{-\alpha}$. Thus (3) holds. This completes the proof of Theorem 2.

5. Remarks.

REMARK 21. We denote by F_α the totality of functions $a(x)$ in L^∞ such that $af \in E_\alpha$ for any $f \in C_0^\infty$. We easily see that $F_\alpha \subsetneqq L^\infty$. Let us show that if, for $a \in L^\infty$ with compact support, $T_a[\mathfrak{Q}, \mathfrak{P}]$ is E_α-bounded, then $a \in F_\alpha$.

Without loss of generality, we may assume that $\int_{\mathbf{R}} a(x)\,dx = 0$. Since $a \in L^\infty$, (29) and the estimates of L_2, L_3, L_4 show that $T_a[\mathfrak{Q}, \mathfrak{P}] - C[h_0, A]$ is E_α-bounded, where $A(x) = \int_0^x a(s)\,ds$. Hence, $C[h_0, A]$ is E_α-bounded according to our assumption. Let $f \in C_0^\infty$. Then we have, by (20),

$$C[h_0, A]f(x) = H(af)(x) - A(Hf')(x) + H(Af')(x) \quad \text{a.e.},$$

where $Hg(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} g(y)/(x-y)\,dy$. Note that $Hg \in E_\alpha$ if and only if $g \in E_\alpha$. Since $A \in M(E_\alpha)$, we have $A(Hf')$, $H(Af') \in E_\alpha$. Hence, $H(af) \in E_\alpha$, which shows $af \in E_\alpha$. Since $f \in C_0^\infty$ is arbitrary, we have $a \in F_\alpha$.

REMARK 22. The 1-energy space E_1 is analogously defined. Let $f(x) = 0$ $(x \leqslant 0)$, $= x/e$ $(0 < x \leqslant e)$ and $= 1/\log x$ $(x > e)$. Then $f \in E_1$. Since

$$\lim_{\eta\to\infty} \int_{1<|x-y|<\eta} \frac{f(y)}{x-y}\,dy = -\infty \quad \text{for all } x \in \mathbf{R},$$

$1/(x-y)$ is 1-unbounded according to our definition. On the other hand, we easily see that $1/(x-y)$ is E_1-bounded.

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