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# DECOMPOSITIONS INTO CODIMENSION-TWO MANIFOLDS

BY

R. J. DAVERMAN AND J. J. WALSH[1]

ABSTRACT. Let $M$ denote an orientable $(n + 2)$-manifold and let $G$ denote an upper semicontinuous decomposition of $M$ into continua having the shape of closed, orientable $n$-manifolds. The main result establishes that the decomposition space $M/G$ is a 2-manifold.

**1. Introduction.** Let $M$ denote an orientable $(n + 2)$-manifold and let $G$ denote an upper semicontinuous decomposition of $M$ into continua having the shape of closed, orientable $n$-manifolds. The main result here establishes that the decomposition space $M/G$ is a 2-manifold.

This theorem lifts what is known about decompositions into codimension-two submanifolds to nearly the same level as what is known about decompositions into codimension-one submanifolds. Liem **[L]** proved that if $G$ is a decomposition of an $(n + 1)$-manifold $N$ into compacta having the shape of $S^n$, then $N/G$ is a 1-manifold, and for $n > 4$ the decomposition map $p: N \to N/G$ can be approximated by a locally trivial bundle map. Daverman **[D]** showed that if $G$ is a decomposition of $N$ into continua having the shape of arbitrary closed $n$-manifolds, then $N/G$ is a 1-manifold (possibly with boundary if various orientability conditions are not all met).

In an earlier paper **[DW1]** we derived the conclusion that $M/G$ is a 2-manifold in case $M$ is an $(n + 2)$-manifold and each $g$ in $G$ has the shape of $S^n$. Then the natural map $p: M \to M/G$ must be particularly nice, being an approximate fibration when $n > 1$ (the same is "almost" true when $n = 1$). The present work evolved from techniques similar to those of **[DW1]**, which were inspired for the most part by methods of Coram and Duvall **[CD1, CD2]**. However, the strong conclusion that $p: M \to M/G$ is an approximate fibration cannot be attained in the more general situation at hand.

In the fourth section we determine the structure of a certain sheaf $\mathscr{H}^n$ on $M/G$ induced by $p$. The properties of this sheaf can be used to recover the conclusions in **[DW1]** just mentioned, in the special situation where the elements of $G$ have the shape of a $n$-sphere.

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The manifold features present in such decomposition spaces $M/G$ do not persist through successively larger codimensions. If $G$ is a decomposition of a manifold $M$ into codimension-three submanifolds, $M/G$ need not be a manifold nor need it even be a generalized manifold. It seems that the fundamental question to address is whether $M/G$, if finite dimensional, is an ANR.

It should be pointed out that, in our terminology, a *manifold* is separable, metric and boundaryless. We will speak of a *manifold with boundary* as the need to permit a boundary occurs.

The symbol $\approx$ will be used to mean "is homeomorphic to" and the symbol $\cong$ will be used to mean "is isomorphic to." Unless expressly stated otherwise, homology and cohomology groups are determined with integral ($\mathbf{Z}$) coefficients.

**2. The winding function and its continuity set.** Throughout the remainder of the paper we shall employ the following notation: $M$ for an orientable $(n+2)$-manifold, $n \geqslant 1$, $G$ for a usc (the standard abbreviation of "upper semicontinuous") decomposition of $M$ into compact connected elements each having the shape of a closed orientable $n$-manifold, $B$ for the decomposition space $M/G$, and $p$ for the decomposition map $M \to B$.

We begin with a variation on the Coram-Duvall definition of local winding number function. Temporarily fix $b_0 \in B$. Since $g_0 = p^{-1}b_0$ is movable, there exist connected neighborhoods $U$ and $U_0$ of $b_0$ in $B$ such that $b_0 \in U_0 \subset U$ and (the inclusion induced)

$$\phi_0 \colon \check{H}_n(p^{-1}b_0) \to H_n(p^{-1}U)$$

is an isomorphism onto the image of

$$\phi \colon H_n(p^{-1}U_0) \to H_n(p^{-1}U).$$

For any $b \in U_0$, the image of

$$\phi_b \colon \check{H}_n(p^{-1}b) \to H_n(p^{-1}U)$$

is contained in $\operatorname{Im}\phi$. Hence,

$$\phi_0^{-1}\phi_b \colon \check{H}_n(p^{-1}b) \to \check{H}_n(p^{-1}b_0)$$

is a well-defined homomorphism between copies of $\mathbf{Z}$, meaning that it amounts to multiplication by some integer $q_b \geqslant 0$. Define the winding function $\alpha \colon U_0 \to \mathbf{Z}$ by $\alpha(b) = q_b$.

Let $K$ denote the set of all points $b_0$ in $B$ such that every neighborhood $W$ of $b_0$ (with $W \subset U_0$) contains some point $w$ for which $\alpha(w) = 0$. As in previous analyses in [**CD1, CD2, DW1**], the set $K$ is a troublesome spot. Clearly, $K$ is closed in $B$.

LEMMA 2.1. *$K$ is nowhere dense in $B$.*

PROOF. See [**CD2**, Lemma 3.1].

Define $B_+$ as $B - K$. In the rest of this section we outline an argument showing that the complement in $B_+$ of a certain countable closed subset is a 2-manifold. Although operating under less limiting hypotheses, we are able to extract enough from arguments in [**CD1** and **DW1**, §3] to satisfy our needs.

We shall continue to work locally throughout most of this section. Assume that $b_0 \in B$ mentioned near the beginning lies in $B_+$ and that the neighborhood $U_0$ mentioned there satisfies $U_0 \subset B_+$ and

$$\operatorname{Im}\{\check{H}_1(p^{-1}b_0) \to H_1(p^{-1}U)\} = \operatorname{Im}\{H_1(p^{-1}U_0) \to H_1(p^{-1}U)\}$$

as well. The three facts stated below are direct analogues of Lemmas 1–3 in **[CD1]**.

(1) If $b \in U_0$, then there is a neighborhood $V$ of $b$ in $U_0$ such that to each $b' \in V$ there corresponds a positive integer $k$ such that $\alpha(b') = k\alpha(b)$.

(2) $\alpha$ is lower semicontinuous.

(3) The continuity set $C = \{b \in U_0\colon \alpha \text{ is continuous at } b\}$ is open and dense in $U_0$; its complement $D = U_0 - C$ can be written as $D_1 \cup D_2$ where $D_2$ is countable, $D_1$ is closed, and each neighborhood of $d \in D_1$ contains uncountably many points of $D_1$.

(Here $D_2 = \{d \in D\colon \text{some neighborhood of } d \text{ in } D \text{ is countable}\}$ and $D_1 = D - D_2$.)

LEMMA 2.2. *Let $A$ be an arc in $C$. Then, using either $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$-coefficients,*

(i) $\check{H}^{n+1}(p^{-1}A) \cong 0$;

(ii) *for $x \in A$, $i^*\colon \check{H}^n(p^{-1}A) \to \check{H}^n(p^{-1}x)$ is surjective; and*

(iii) *for $x, y \in A$, $\ker\{\check{H}^n(p^{-1}A) \to \check{H}^n(p^{-1}x)\} = \ker\{\check{H}^n(p^{-1}A) \to \check{H}^n(p^{-1}y)\}$.*

PROOF. We assume the coefficient group is $\mathbf{Z}$, the argument being the same for other coefficient groups. For each $x \in A$, there is a closed subarc $A_x$, a neighborhood of $x$ in $A$, for which there is a shape strong deformation retraction of $p^{-1}A_x$ to $p^{-1}x$, the deformation occurring in a neighborhood of $p^{-1}x$. Consequently, the diagram

$$\begin{array}{ccccc} & & \check{H}^n(p^{-1}A_x) & & \\ & i^* \swarrow & & \searrow i^* & \\ \check{H}^n(p^{-1}y) & & \overset{\cong}{\underset{\psi}{\longleftarrow}} & & \check{H}^n(p^{-1}x) \end{array}$$

commutes for each $y \in A_x$, where $\psi$ is induced by the retraction and is an isomorphism as we are working in the continuity set. It follows easily that the arc $A_x$ satisfies conclusions (ii) and (iii).

If a closed subarc $A'$ of $A$ satisfies conclusion (ii), then obviously so does each closed subarc $A''$; moreover, inclusion induces a surjection

$$\check{H}^n(p^{-1}A') \to \check{H}^n(p^{-1}A'').$$

To see why this is so, express $A' = E_1 \cup A'' \cup E_2$ as the union of closed subarcs (one of $E_1$ or $E_2$ being empty, should $A''$ contain an endpoint of $A'$), where $E_1 \cap A''$ and $E_2 \cap A''$ are endpoints of $A''$. Given an element $z \in \check{H}^n(p^{-1}A'')$, use conclusion (ii) to find $z_j \in \check{H}^n(p^{-1}E_j)$ for $j \in \{1,2\}$ such that

$$\operatorname{Im}\{z_j \to \check{H}^n(p^{-1}(A'' \cap E_j))\} = \operatorname{Im}\{z \to \check{H}^n(p^{-1}(A'' \cap E_j))\}.$$

From the Mayer-Vietoris sequence of $(p^{-1}(A'' \cup E_1), p^{-1}A'', p^{-1}E_1)$,

$$\check{H}^n(p^{-1}(A'' \cup E_2)) \to \check{H}^n(p^{-1}A'') \oplus \check{H}^n(p^{-1}E_1) \xrightarrow{\Delta} \check{H}^n(p^{-1}(A'' \cap E_1)),$$

we see that $\Delta(z \oplus z_1) = 0$, which gives $z'' \in \check{H}^n(p^{-1}(A'' \cup E_1))$, whose restrictions (= inclusion-induced images) in $\check{H}^n(p^{-1}A'')$ and $\check{H}^n(p_1^{-1})$ are $z$ and $z_1$, respectively. Similarly, starting from part of the Mayer-Vietoris sequence for

$$\left(p^{-1}(E_1 \cup A'' \cup E_2), p^{-1}(E_1 \cup A''), p^{-1}E_2\right),$$

we obtain $z' \in H^n(p^{-1}A')$ whose restriction to $H^n(p^{-1}A'')$ is $z$. It now follows easily that if, in addition, $A$ satisfies conclusion (iii), then so does each closed subarc.

Let $A_1$ and $A_2$ denote closed subarcs of $A$ such that $A_1 \cap A_2 \neq \varnothing$. By the preceding argument, if $A_1$ and $A_2$ satisfy conclusion (ii), so does their union $A_1 \cup A_2$. If both $A_1$ and $A_2$ satisfy conclusion (iii), we can verify that the same is true of $A$ by supposing $y \in A_1 \cap A_2$ and $x \in A_1$, say, and considering the commutative diagram:

$$\begin{array}{ccccc} & & \check{H}^n(p^{-1}A) & & \\ & \swarrow & \downarrow & \searrow & \\ \check{H}^n(p^{-1}x) & \leftarrow & \check{H}^n(p^{-1}A_1) & \rightarrow & \check{H}^n(p^{-1}y) \end{array}$$

Let $A'$ be a maximal subarc of $A$ such that conclusions (ii) and (iii) are valid for each proper closed subarc of $A'$. Each endpoint of $A'$ is contained in $A'$, else the arc $A_x$, where $x$ is an endpoint of $A'$, together with the conclusions of the previous paragraph would lead to a contradiction of the maximality of $A'$. For the same reason, $A' = A$.

Finally, conclusion (i) is a consequence of conclusion (ii) since repeated applications of the Mayer-Vietoris sequence reveal

$$\check{H}^{n+1}(p^{-1}A) \cong \check{H}^{n+1}(p^{-1}A_1) \oplus \cdots \oplus \check{H}^{n+1}(p^{-1}A_k)$$

for any "partition" $A_1, \ldots, A_k$ of $A$ into subarcs. The nontriviality of $\check{H}^{n+1}(p^{-1}A)$ would ultimately detect the nontriviality of $\check{H}^{n+1}(p^{-1}x)$ for some $x \in A$, but this is not possible.

LEMMA 2.3. *Neither a point nor an arc in $C$ separates $C$, but every simple closed curve there does separate $C$.*

PROOF. Let $A$ denote an arc in $C$ and $V$ the component of $C$ containing $A$. From the exact sequence of the pair $(p^{-1}V, p^{-1}V - p^{-1}A)$, we obtain

$$H_1(p^{-1}V, p^{-1}V - p^{-1}A) \to \tilde{H}_0(p^{-1}V - p^{-1}A) \to \tilde{H}_0(p^{-1}A) \cong 0$$

and, by duality [**S**, p. 296] and Lemma 2.2,

$$H_1(p^{-1}V, p^{-1}V - p^{-1}A) \cong \check{H}^{n+1}(p^{-1}A) \cong 0.$$

Thus $p^{-1}V - p^{-1}A$ and $V - A$ are connected.

Similarly, no point separates $C$.

Now, let $J$ denote a simple closed curve in some component $V$ of $C$. Express $J$ as the union of two arcs $A_1$ and $A_2$ whose intersection is the boundary $\{a_1, a_2\}$ of each. From the Mayer-Vietoris sequence for $p^{-1}J = p^{-1}A_1 \cup p^{-1}A_2$, we analyse with $\mathbf{Z}/2\mathbf{Z}$-coefficients

$$\begin{array}{c} \cdots \to \check{H}^n(p^{-1}A_1) \oplus \check{H}^n(p^{-1}A_2) \xrightarrow{\beta} \check{H}^n(p^{-1}\{a_1, a_2\}) \to \check{H}^{n+1}(p^{-1}J) \to 0 \\ \cong \downarrow \phi \\ \check{H}^n(p^{-1}a_1) \oplus \check{H}^n(p^{-1}a_2) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \end{array}$$

(where the zero at the right comes from Lemma 2.2). Write $\phi \circ \beta = \delta_1 \oplus \delta_2$, where, for $i = 1, 2$, $\delta_i$: $\check{H}^n(p^{-1}A_1) \oplus \check{H}^n(p^{-1}A_2) \to \check{H}^n(p^{-1}a_i)$. The conclusion in part (iii) of Lemma 2.2 reveals that $\delta_1(z, 0) = 0$ if and only if $\delta_2(z, 0) = 0$ and, likewise, that $\delta_1(0, z) = 0$ if and only if $\delta_2(0, z) = 0$. It follows that the image of $\phi\beta$ is contained in the diagonal subgroup $\{(0,0), (1,1)\}$ and, in light of part (ii) of Lemma 2.2, is equal to this subgroup. Consequently,

$$\check{H}^{n+1}(p^{-1}J; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}.$$

Continuing throughout the remainder of the proof to use $\mathbf{Z}/2\mathbf{Z}$-coefficients, we establish that $p^{-1}J$ separates $p^{-1}V$ by showing that the homomorphism $\gamma$ in the sequence below is trivial:

$$\begin{array}{ccccccc} H_1(p^{-1}V) & \overset{\gamma}{\to} & H_1(p^{-1}V, p^{-1}V - p^{-1}J) & \to & \tilde{H}_0(p^{-1}V - p^{-1}J) & \to & 0 \\ & & & & \downarrow \cong & & \\ & & & & \check{H}^{n+1}(p^{-1}J) \cong \mathbf{Z}/2\mathbf{Z} & & \end{array}$$

A simple chase through the diagram

$$\begin{array}{ccccc} H_1(p^{-1}V - p^{-1}J) & \to & H_1(p^{-1}V) & \overset{\gamma}{\to} & H_1(p^{-1}V, p^{-1}V - p^{-1}J) \\ \downarrow & & \downarrow & & \downarrow \cong \\ H_1(p^{-1}U_0 - p^{-1}J) & \to & H_1(p^{-1}U_0) & \to & H_1(p^{-1}U_0, p^{-1}U_0 - p^{-1}J) \\ \downarrow & & j\downarrow & & \downarrow \cong \\ H_1(p^{-1}U - p^{-1}J) & \overset{\eta}{\to} & H_1(p^{-1}U) & \overset{\gamma'}{\to} & H_1(p^{-1}U, p^{-1}U - p^{-1}J) \end{array}$$

establishes the triviality of $\gamma$ provided $\operatorname{Im} \eta \supset \operatorname{Im} j$. One can detect the last containment by choosing $c_0 \in U_0 - J$ so close to $b_0$ that $c_0$ lies in a neighborhood $W$ of $b_0$ for which there is a shape deformation retraction in $p^{-1}U_0$, say $\underline{r}$: $p^{-1}W \to p^{-1}b_0$. Since the restriction of $\underline{r}$ induces an isomorphism $\check{H}_n(p^{-1}c_0) \to \check{H}_n(p^{-1}b_0)$ with $\mathbf{Z}$ and, hence, with $\mathbf{Z}/2\mathbf{Z}$-coefficients (as we are working in the continuity set), $\underline{r}$ induces a surjection $\check{H}_1(p^{-1}c_0) \to \check{H}_1(p^{-1}b_0)$. It follows that

$$\operatorname{Im}\{\check{H}_1(p^{-1}c_0) \to H_1(p^{-1}U)\} = \operatorname{Im}\{\check{H}_1(p^{-1}b_0) \to H_1(p^{-1}U)\}.$$

We stipulated earlier (for $\mathbf{Z}$ and, hence, $\mathbf{Z}/2\mathbf{Z}$-coefficients) that

$$\operatorname{Im}\{\check{H}_1(p^{-1}b_0) \to H_1(p^{-1}U)\} = \operatorname{Im}\{H_1(p^{-1}U_0) \to H_1(p^{-1}U)\},$$

and, as

$$\operatorname{Im} \eta \supset \operatorname{Im}\{\check{H}_1(p^{-1}c_0) \to H_1(p^{-1}U)\},$$

we conclude that $\operatorname{Im} \eta \supset \operatorname{Im} j$.

ADDENDUM. Having established that $p^{-1}J$ separates, we can conclude that $\check{H}^{n+1}(p^{-1}J; \mathbf{Z}) \cong \mathbf{Z}$. The only point in the preceding argument that made essential use of $\mathbf{Z}/2\mathbf{Z}$-coefficients arose in the conclusion that the image of $\beta$ equalled a diagonal subgroup. For $\mathbf{Z}$-coefficients, we can draw the analogous conclusion provided we know $\check{H}^{n+1}(p^{-1}J; \mathbf{Z}) \neq 0$, which holds because $p^{-1}J$ separates $V$ (and since the homomorphism $\gamma$ analysed in the proof is trivial for $\mathbf{Z}$-coefficients as well).

THEOREM 2.4. *The space $C$ is a 2-manifold.*

PROOF. This is a consequence of Lemma 2.3 and [**Wi**, p. 95].

LEMMA 2.5. *The set $D$ is countable.*

PROOF. By fact (3), $D = D_1 \cup D_2$ where $D_1$ is closed in $D$ and dense-in-itself, and where $D_2$ is countable. We shall prove that $D_1 = \varnothing$.

Suppose $D_1 \neq \varnothing$. Since $\alpha|D_1$ is lower semicontinuous, the set where $\alpha|D_1$ is continuous forms a dense open subset of $D_1$. Thus, we can find $b \in D_1$ and some neighborhood $W$ of $b$ in $U_0$ such that $\alpha|D_1 \cap W$ is constant. Accordingly, as in fact (1), $W$ can be restricted so that for all $w \in W$ there exists an integer $k(w) > 0$ such that $\alpha(w) = k(w) \cdot \alpha(b)$. Since $b$ is a discontinuity of $\alpha$, there is some $w \in W - D_1$ for which $k(w) > 1$. This yields $k(w') \geqslant k(w) > 1$ for all $w'$ in some neighborhood of $w$, so we can assume $w \in W - D$.

Using the countability of $D_2$, we build an arc $A$ in $W$ with endpoints $c$, $d$ in $D_1$ and with $A - \{c, d\} \subset W - D$ such that, for $x \in A - \{c, d\}$,

$$\alpha(x) = k\alpha(c) = k\alpha(d) \quad (\text{some } k > 1).$$

It suffices to compute that $\check{H}^{n+1}(p^{-1}A) \cong \mathbf{Z}/k\mathbf{Z}$, for one can verify that this is impossible by inspecting the sequence

$$\begin{array}{c} H_1(p^{-1}W) \xrightarrow{\gamma} H_1(p^{-1}W, p^{-1}W - p^{-1}A) \to \tilde{H}_0(p^{-1}W - p^{-1}A) \to 0 \\ \downarrow \cong \\ \check{H}^{n+1}(p^{-1}A) \cong \mathbf{Z}/k\mathbf{Z} \end{array}$$

while recalling the argument in Lemma 2.3 that $\gamma$ is the trivial homomorphism.

Express $A$ as the union of two subarcs $A_1$, $A_2$ having a common endpoint $e$. It suffices to determine that, for $i = 1, 2$, $\check{H}^{n+1}(p^{-1}A_i) \cong 0$ and that

$$\operatorname{Im}\{\check{H}^n(p^{-1}A_i) \to \check{H}^n(p^{-1}e)\} = k\mathbf{Z}.$$

Then the tail of a Mayer-Vietoris sequence

$$\check{H}^n(p^{-1}A_1) \oplus \check{H}^n(p^{-1}A_2) \to \check{H}^n(p^{-1}e) \to \check{H}^{n+1}(p^{-1}A) \to 0$$

detects that $\check{H}^{n+1}(p^{-1}A) \cong \mathbf{Z}/k\mathbf{Z}$.

SUBLEMMA 2.6. *Let $W$ and $D$ be as above and let $A$ be an arc with one endpoint $c \in D$, with $A \setminus \{c\} \subset W$, and with $\alpha(x) = k\alpha(c)$ for $x \in A - \{c\}$ (some $k > 1$). Then*

(i) $\check{H}^{n+1}(p^{-1}A) = 0$, *and*

(ii) $\operatorname{Im}\{\check{H}^n(p^{-1}A) \to \check{H}^n(p^{-1}e)\} = k\mathbf{Z}$, *where $e$ is the endpoint of $A - \{c\}$.*

PROOF. Conclusion (i) follows from conclusion (ii) of Lemma 2.2, just as conclusion (i) of Lemma 2.2 followed. (The key observation is that, expressing $A = A_1 \cup A_2$ as the union of subarcs having just an endpoint in common,

$$\check{H}^n(p^{-1}A_1) \oplus \check{H}^n(p^{-1}A_2) \to \check{H}^n(p^{-1}(A_1 \cap A_2))$$

is surjective, since one of $A_1$ and $A_2$ is contained in the continuity set.)

Take an arbitrary neighborhood $N$ of $p^{-1}A$. Let $A_1$ be a proper subarc of $A$ containing $c$ for which there exists a shape strong deformation of $p^{-1}A_1$ to $p^{-1}c$, the deformation taking place in $N$. Let $d$ denote the other endpoint of $A_1$ and $A' = \mathrm{Cl}(A - A_1)$. It follows that

$$\mathrm{Im}\{\check{H}^n(p^{-1}A_1) \to \check{H}^n(p^{-1}d)\} = k\mathbf{Z}$$

because $\alpha(d) = k\alpha(c)$. From the cohomology ladder

$$\begin{array}{ccccccc}
\check{H}^n(p^{-1}A) & \to & \check{H}^n(p^{-1}A') & \to & \check{H}^{n+1}(p^{-1}A, p^{-1}A') & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \cong & & \\
\check{H}^n(p^{-1}A_1) & \to & \check{H}^n(p^{-1}d) & \to & \check{H}^{n+1}(p^{-1}A_1, p^{-1}d) & \to & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \\
 & & \mathbf{Z} & & \mathbf{Z}/k\mathbf{Z} & &
\end{array}$$

we see that $\mathrm{Im}\{\check{H}^n(p^{-1}A) \to \check{H}^n(p^{-1}d)\} = k\mathbf{Z}$. The desired conclusion that

$$\mathrm{Im}\{\check{H}^n(p^{-1}A) \to \check{H}^n(p^{-1}e)\} = k\mathbf{Z}$$

follows from the following diagram and the conclusion of Lemma 2.2 that the surjections $i_e^*$ and $i_d^*$ have equal kernels:

$$\begin{array}{ccccc}
 & & \check{H}^n(p^{-1}A) & & \\
 & \swarrow & \downarrow & \searrow & \\
\check{H}^n(p^{-1}e) & \overset{i_e^*}{\leftarrow} & \check{H}^n(p^{-1}A') & \overset{i_d^*}{\rightarrow} & \check{H}^n(p^{-1}d)
\end{array}$$

This completes the proof that $D$ is countable.

Now it is appropriate to let the point $b_0$, identified shortly after the statement of Lemma 2.1, take on different values in $B_+ = B - K$. For each $b \in B_+$ we have a neighborhood $U_b \subset B_+$, a closed and countable subset $D_b$ of $U_b$, a dense open subset $C_b = U_b - D_b$ of $U_b$, and a winding function $\alpha_b$: $U_b \to \mathbf{Z}$ with the properties described at the outset of this section. In case $c \in U_{b(1)} \cap U_{b(2)}$, then $c \in C_{b(1)}$ iff $c \in C_{b(2)}$, since continuity of $\alpha_{b(i)}$ at $c$ depends only on the behavior of point inverses near $p^{-1}c$. Select a countable subcollection $\{U_{b(i)}\}$ of $\{U_b\}$ covering $B_+$ and define $C_+$ as $\bigcup_i C_{b(i)}$. From Theorem 2.4 and Lemma 2.5 we obtain

THEOREM 2.7. *The set $C_+$ is open and dense in $B$ and $D_+ = B_+ - C_+$ is a countable, relatively closed subset of $B_+$. Furthermore, $C_+$ is a 2-manifold.*

**3. The structure of the decomposition space.** At this point we launch an investigation of the subset $K = B - B_+$, with an eye toward ultimately concluding it is locally finite. The crucial step is detecting that $K$ is totally disconnected, for we have

LEMMA 3.1. *If $K$ is totally disconnected, then $B$ is a 2-manifold.*

SKETCH OF THE PROOF. The argument in Lemma 2.3 shows each $y_0 \in K \cup D_+$ to have a connected neighborhood $U_0$ in $B$ such that every simple closed curve in $U_0 \cap C_+$ separates $U_0 \cap C_+$. Thus, as argued in **[DW1**, Lemma 4.1], $U_0 \cap C_+$ is homeomorphic to the complement in $S^2$ of some closed 0-dimensional subset. The points of $(K \cup D_+) \cap U_0$ can be considered to be (some of the) ends of $U_0$ and the homeomorphism of $U_0 \cap C_+$ into $S^2$ can be extended, in an end-preserving way, to one from $U_0$ onto an open subset of $S^2$.

Suppose now that $K_1 \subset K$ is a closed subset such that $K - K_1$ is totally disconnected (equivalently, 0-dimensional, as $K - K_1$ is locally compact). Using [**DH**, Theorem 2.10], we name a point $b_0 \in K_1$, a neighborhood $U_0$ of $b_0$ as above, and then another neighborhood $W$ of $b_0$ in $U_0$ for which there exists a shape retraction $\underline{r}$: $p^{-1}W \to p^{-1}b_0$ such that the restriction of $\underline{r}$ to $p^{-1}b$ provides a shape equivalence from $p^{-1}b$ to $p^{-1}b_0$ for each $b \in K_1 \cap W$.

PROPOSITION 3.2. *There is a nonempty relatively open subset $\mathcal{O}$ of $K_1 \cap W$ that is totally disconnected* (*equivalently, in this setting*, 0-*dimensional*).

Before embarking on the proof, which occupies most of the remainder of this section, we derive the important

COROLLARY 3.3. *$K$ is totally disconnected.*

PROOF. Specify $K_1$ by requiring $K - K_1$ to be a maximal relatively open subset of $K$ that is 0-dimensional. The separability of $K$ and the Sum Theorem for 0-dimensional sets [**HW**, p. 18] ensure that such a subset exists. Then $K_1 = \varnothing$, for otherwise a nonempty, 0-dimensional, relatively open subset $\mathcal{O}$ of $K_1$ would exist, and $(K - K_1) \cup \mathcal{O}$ would be larger than $K - K_1$. It would be obvious that $(K - K_1) \cup \mathcal{O}$ is relatively open in $K$ and the Sum Theorem of [**HW**] again would reveal that $(K - K_1) \cup \mathcal{O}$ is 0-dimensional. Thus, $K$ itself is 0-dimensional.

LEMMA 3.4. *There is a connected neighborhood $W_1 \subset W$ of $b_0$ and a component $V$ of $W_1 - K_1$ such that every arc $A$ with $\partial A \subset K_1$ and $A - \partial A \subset V \cap C_+$ separates $B$ and one component of $B - A$ is contained in $U_0$.*

PROOF. Choose $W_1 \subset W$ so that $H_{n+1}(p^{-1}W_1) \to H_{n+1}(p^{-1}U_0)$ is trivial. Since $b_0 \in K$, there exists $w \in W_1$ such that $\check{H}_n(p^{-1}w) \to H_n(p^{-1}W_1)$ is trivial. Because $\alpha | K_1 \cap W$ is nonzero, $w \notin K_1$. It follows from the 0-dimensionality of $[(K - K_1) \cup D_+] \cap W_1$ (and from the fact that $\check{H}_n(p^{-1}w') \to H_n(p^{-1}W_1)$ is trivial for all $w'$ in some neighborhood of $w$) that we can choose $w \in W_1 \cap C_+$.

Let $V$ denote the component of $W_1 - K_1$ containing $w$. Observe that $V \cap K \subset K - K_1$ is a relatively closed 0-dimensional subset of $V$ and, hence, $V \cap C_+ = V - K$ is a dense, connected, open subset of $V$.

Let $A$ be an arc with $\partial A \subset K_1$ and $A - \partial A \subset V \cap C_+$. Split $A$ into subarcs $A_1, A_2$ having just an endpoint $e$ in common. Then $e \in V \cap C_+$. An argument like that in Lemma 2.5 verifies that $\check{H}^{n+1}(p^{-1}A; \mathbf{Z}) \cong \mathbf{Z}$. The diagram

$$\begin{array}{ccccccc} H_1(p^{-1}W_1) & \xrightarrow{\gamma} & H_1(p^{-1}W_1, p^{-1}W_1 - p^{-1}A) & \to & \tilde{H}_0(p^{-1}W_1 - p^{-1}A) & \to & 0 \\ & & \downarrow \cong & & & & \\ & & \check{H}^{n+1}(p^{-1}A) \cong \mathbf{Z} & & & & \end{array}$$

indicates that $p^{-1}A$ separates $p^{-1}W_1$ (recall the previous argument in the proof of Lemma 2.3 that $\gamma$ is trivial). Similarly, $p^{-1}A$ separates $p^{-1}U_0$. Hence, $A$ separates both $W_1$ and $U_0$.

Our initial specification that $H_{n+1}(p^{-1}W_1) \to H_{n+1}(p^{-1}U_0)$ be trivial starts the chase through the diagram

$$\begin{array}{ccccc} H_{n+1}(p^{-1}U_0) & \overset{\cong}{\to} & H^1(\mathrm{Cl}\, p^{-1}U_0, \mathrm{Fr}\, p^{-1}U_0) & & \\ & & & \nwarrow j & \\ \uparrow & & & & H^1(\mathrm{Cl}\, p^{-1}U_0, \mathrm{Cl}\, p^{-1}U_0 - p^{-1}W_1) \\ & & & \swarrow \cong & \\ H_{n+1}(p^{-1}W_1) & \overset{\cong}{\to} & H^1(\mathrm{Cl}\, p^{-1}W_1, \mathrm{Fr}\, p^{-1}W_1) & & \end{array}$$

leading to the detection that $j$ is trivial. (The isomorphisms on the left arise from Lefshetz Duality [S, p. 297] while that on the right is an excision.) Since $j$ is trivial, the diagram

$$\begin{array}{ccccc} 0 & \to & \tilde{\check{H}}^0(\mathrm{Fr}\, p^{-1}U_0) & \to & \check{H}^1(\mathrm{Cl}\, p^{-1}U_0, \mathrm{Fr}\, p^{-1}U_0) \\ & & \uparrow i & & \uparrow j \\ 0 & \to & \tilde{\check{H}}^0(\mathrm{Cl}\, p^{-1}U_0 - p^{-1}W_1) & \to & H^1(\mathrm{Cl}\, p^{-1}U_0, \mathrm{Cl}\, p^{-1}U_0 - p^{-1}W_1) \end{array}$$

can be used to detect that $i$ is trivial and, consequently, $\mathrm{Fr}\, p^{-1}U_0$ is contained in a single component of $(\mathrm{Cl}\, p^{-1}U_0) - p^{-1}W_1$. Thus, $\mathrm{Fr}\, p^{-1}U_0$ is contained in a single component of $(\mathrm{Cl}\, p^{-1}U_0) - p^{-1}A$. This shows that one component of $(\mathrm{Cl}\, p^{-1}U_0) - p^{-1}A$ is also a component of $M - p^{-1}A$, as well as that one component of $B - A$ is a subset of $U_0$. This completes the proof of Lemma 3.4.

It remains to analyze the set $W_1 \cap \mathrm{Fr}\, V$ with the aim of determining that it is totally disconnected. Thereby we shall discover that $V = W_1$ and that $W_1 \cap K_1$ is a nonempty ($b_0 \in W_1 \cap K_1$), totally disconnected subset of $K_1$.

LEMMA 3.5. *$W_1 \cap \mathrm{Fr}\, V$ contains no arc.*

PROOF. Suppose that $A \subset W_1 \cap \mathrm{Fr}\, V$ is an arc. Recall that the choice of $W$ and $b_0$ was based on the stipulation that the shape retraction $\underline{r}$: $p^{-1}W \to p^{-1}b_0$ restricts to a shape equivalence from $p^{-1}b$ to $p^{-1}b_0$, for each $b \in K_1 \cap W$. It follows that, for each subarc $A'$ of $A$ and $x \in A'$, the inclusion $p^{-1}x \to p^{-1}A'$ is a shape equivalence. Consequently, we can choose connected open sets $P$ and $N$, with $P \subset N \subset W_1$, such that

$$N \cap A = A - \partial A = P \cap A,$$

the inclusion induces an injection of $\underline{\pi}_1(p^{-1}(A - \partial A))$ to $\pi_1(p^{-1}N)$, and

$$\mathrm{Im}\{\underline{\pi}_1(p^{-1}(A - \partial A)) \to \pi_1(p^{-1}N)\} = \mathrm{Im}\{\pi_1(p^{-1}P) \to \pi_1(p^{-1}N)\}.$$

CLAIM. The set $p^{-1}(A - \partial A)$ separates $p^{-1}P$ into precisely two connected pieces. Furthermore, if $A'$ is a subarc of $A - \partial A$ and if $P' \subset N' \subset P$ are connected open sets chosen so that $N' \cap A' = A' - \partial A' = P' \cap A'$, inclusion induces an injection of $\pi_1(p^{-1}(A' - A'))$ to $\pi_1(p^{-1}N')$, and

$$\mathrm{Im}\{\underline{\pi}_1(p^{-1}(A' - \partial A')) \to \pi_1(p^{-1}N')\} = \mathrm{Im}\{\pi_1(p^{-1}P') \to \pi_1(p^{-1}N)\},$$

then distinct components of $p^{-1}P' - p^{-1}(A' - \partial A')$ are contained in distinct components of $p^{-1}P - p^{-1}(A - \partial A)$.

Before proving this, we use it to arrive at the contradiction that establishes Lemma 3.5. Transferring the conclusions via the map $p$, we have that $P - A = Z_1 \cup Z_2$, where $Z_1$ and $Z_2$ are distinct components. An easy consequence of the "furthermore" in the Claim is that every point of $A - \partial A$ is arcwise accessible from each of $Z_1$ and $Z_2$. As one of $Z_1$ or $Z_2$ meets $V$, say $Z_1 \cap V \neq \varnothing$, there is a 4-od $T$ (i.e., $T$ is the cone on a four-point set) with $T \subset (Z_1 \cap C_+) \cup (A - \partial A)$, with the cone point in $V$ and with $T \cap (A - \partial A)$ equal to the four endpoints of $T$. Let $T'$ denote the subset of $T$ that is the "sub-4-od" whose endpoints lie in Fr $V$ but is otherwise contained in $V \cap C_+$. Name the endpoints of $T$ to be $a_1, a_2, a_3, a_4$ and the corresponding endpoints of $T'$ to be $a_1', a_2', a_3', a_4'$. Let $\langle x, y\rangle$ denote the subarc of $T$ connecting any given pair of points $x, y \in T$. Lemma 3.4 implies that for some pair, say $a_1', a_2'$, $p^{-1}(\langle a_1', a_2'\rangle)$ separates $p^{-1}a_3'$ from $p^{-1}a_4'$ and, therefore, that $\langle a_1', a_2'\rangle$ separates $a_3'$ from $a_4'$. This is an impossibility, since $\langle a_3', a_3\rangle \cup Z_2 \cup \langle a_4', a_4\rangle$ is a connected set missing $\langle a_1', a_2'\rangle$.

Proof of the Claim. Since $p^{-1}x$ and $p^{-1}y$ are "connected" by a shape equivalence for $x, y \in A$, an argument like the one given for Lemma 2.2 shows that inclusion induces an isomorphism $\check{H}^n(p^{-1}A') \to \check{H}^n(p^{-1}a)$ and that $\check{H}^{n+1}(p^{-1}A') = 0$ for each subarc $A'$ of $A$ and each $a \in A'$. The exact cohomology sequence for the pair $(p^{-1}A', p^{-1}\partial A')$ and properties of cohomology with compact supports [**S**, p. 321] give

$$\check{H}_c^{n+1}(p^{-1}(A' - \partial A')) \cong \check{H}^{n+1}(p^{-1}A', p^{-1}\partial A') = 0,$$

and a comparison of two such sequences reveals that inclusion induces an isomorphism

$$H_c^{n+1}(p^{-1}(A' - \partial A')) \to H_c^{n+1}(p^{-1}(A'' - \partial A''))$$

for subarcs $A'' \subset A'$ of $A$. The diagram

$$\begin{array}{ccccccc}
H_1(p^{-1}N) & \xrightarrow{\beta} & H_1(p^{-1}N, p^{-1}(N - A)) & & & & \\
\uparrow \phi & & \uparrow \cong & & & & \\
H_1(p^{-1}P) & \xrightarrow{\gamma} & H_1(p^{-1}P, p^{-1}(P - A)) & \to & \tilde{H}_0(p^{-1}(P - A)) & \to & 0 \\
 & & \uparrow \cong & & & & \\
 & & H_c^{n+1}(p^{-1}(A - \partial A)) \cong Z & & & &
\end{array}$$

can be used to establish the Claim, provided $\gamma$ is the trivial homomorphism. [The "furthermore" follows by comparing the diagrams for $A \subset P \subset N$ and $A' \subset P' \subset N'$ and by using the inclusion-induced isomorphism

$$H_c^{n+1}(p^{-1}(A - \partial A)) \to H_c^{n+1}(p^{-1}(A' - \partial A')).]$$

The triviality of $\gamma$ is immediate if $\pi_1(p^{-1}a) = 0$ for some (hence, any) $a \in A$, as the homomorphism $\phi$ is then trivial. An outline of the argument that $\gamma$ is trivial in general will follow; this will be done by showing $\beta(2\phi(e)) = 0$ for each $e \in H_1(p^{-1}P)$ and, hence, as $\beta$ is a homomorphism to $\mathbf{Z}$, $\beta\phi$ is trivial. Of course, it suffices to show that $2\phi(e)$ is an element of

$$\operatorname{Im}\{H_1(p^{-1}(N - A)) \to H_1(p^{-1}N)\}.$$

In fact, we show that in the diagram

$$\begin{array}{ccc} \pi_1(p^{-1}(N-A), x_0) & \xrightarrow{\delta} & \pi_1(p^{-1}N, x_0) \\ & & \downarrow \eta \\ & & \pi_1(p^{-1}P, x_0) \end{array}$$

$2\eta(e) \in \operatorname{Im}\delta$ for each $e \in \pi_1(p^{-1}P, x_0)$, where the basepoint $x_0$ lies in $p^{-1}(P - A)$.

Let $f\colon \tilde{N} \to p^{-1}N$ denote the universal covering space. Choose $y_0 \in f^{-1}(x_0)$. Let $\tilde{P}_0$ denote the component of $f^{-1}(P)$ containing $y_0$ and let

$$\tilde{A}_0 = \tilde{P}_0 \cap f^{-1}p^{-1}(A - \partial A).$$

It follows from the $\pi_1$ conditions stipulated prior to the statement of the Claim that $\tilde{A}_0$ is the only component of $f^{-1}p^{-1}(A - \partial A)$ in $\tilde{P}_0$ and that $\underline{\pi}_1(\tilde{A}_0)$ is trivial. A computation outlined below establishes $H_c^{n+1}(\tilde{A}) \cong \mathbf{Z}$, and then the simply-connected case of this result applies to reveal $\tilde{P}_0 - \tilde{A}_0 = C_1 \cup C_2$, where $C_1$ and $C_2$ are distinct components. For an arbitrary representative $e\colon (I, \{0,1\}) \to (p^{-1}P, x_0)$ of an element (also called) $e$ of $\pi_1(p^{-1}P, x_0)$, if there is a lift $\tilde{e}\colon I \to \tilde{P}_0$ of $e$ with $\tilde{e}(0)$ and $\tilde{e}(1)$ contained in the same $C_i$, then the loop obtained by connecting $\tilde{e}(0)$ and $\tilde{e}(1)$ via a path in that $C_i$ is necessarily null homotopic in $\tilde{N}$ and, consequently, $e$ is homotopic rel$\{0,1\}$ in $p^{-1}N$ to $e'\colon (I, \{0,1\}) \to (f(C_i), x_0)$. As $f(C_i) \cap p^{-1}A = \varnothing$, the element $\eta(e) = \eta(e')$ is in $\operatorname{Im}\delta$. If no lift $\tilde{e}$ to $\tilde{P}_0$ has $\tilde{e}(0)$ and $\tilde{e}(1)$ contained in the same $C_i$, then each lift of $2e$ to $\tilde{P}_0$ must have this property. Thus, in either case, $\eta(2e) \in \operatorname{Im}\delta$.

Finally, to finish off the Claim, we describe the computation of $H_c^{n+1}(\tilde{A}_0)$. Set $\tilde{E}$ equal to that component of $f^{-1}p^{-1}E$ contained in $\tilde{A}_0$, where $E$ is a closed subarc of $A$. Given any point $a_0 \in E$, we specify a shape equivalence $s$ from $p^{-1}a_0$ to a closed orientable $n$-manifold $g_0$ and a shape deformation retraction $\underline{r}'\colon p^{-1}E \to p^{-1}a_0$. Denote by $\tilde{g}_0$ the universal cover of $g_0$ and by $\tilde{s}\colon \tilde{E} \cap f^{-1}p^{-1}a_0 \to \tilde{g}_0$ and $\tilde{\underline{r}}'\colon \tilde{E} \to \tilde{E} \cap f^{-1}p^{-1}a_0$ lifts of $s$ and $\underline{r}'$, respectively. Computing with $\tilde{s}$ and $\tilde{\underline{r}}'$ establishes that $H_c^n(E \cap f^{-1}p^{-1}a_0) \cong \mathbf{Z}$, that $\check{H}_c^{n+1}(\tilde{E}) = 0$, and that inclusion induces an isomorphism

$$\check{H}_c^n(\tilde{E}) \to \check{H}_c^n(\tilde{E} \cap f^{-1}p^{-1}a_0).$$

The exact cohomology sequence (with compact supports) of the pair $(\tilde{E}, \tilde{E} \cap f^{-1}p^{-1}\partial E)$ displays that

$$\check{H}_c^{n+1}(\tilde{E}, \tilde{E} \cap f^{-1}p^{-1}\partial E) \cong \check{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E) \cong \mathbf{Z}.$$

For a pair of closed subarcs $E \subset F$ of $A$, a comparison of the computation just described for each reveals that inclusion induces an isomorphism

$$\check{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E) \cong \check{H}_c^{n+1}(\tilde{F} - f^{-1}p^{-1}\partial F).$$

(Specifically, compare the pair $(\tilde{F}, \tilde{F} \cap f^{-1}p^{-1}\partial F)$ and the pair $(\tilde{F}, \tilde{F} \cap f^{-1}p^{-1}$ $\mathrm{Cl}(F - E))$ to determine that

$$\check{H}_c^{n+1}(\tilde{F}, \tilde{F} \cap f^{-1}p^{-1}\partial F) \cong \check{H}_c^{n+1}(\tilde{F}, \hat{F} \cap f^{-1}p^{-1}\,\mathrm{Cl}(F - E))$$

and, then, use the excision axiom to deduce that the latter group is isomorphic to

$$\check{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E).)$$

Finally, $\check{H}_c^{n+1}(\tilde{A}_0)$ is the direct limit of the groups $\check{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E)$ where $E$ ranges over closed subarcs of $A$ and, hence, $\check{H}_c^{n+1}(A_0) \cong \mathbf{Z}$.

PROOF OF PROPOSITION 3.2. Since $W_1 \cap \mathrm{Fr}\, V$ contains no arc, no compact connected subset of it, other than a singleton, can be locally connected. Consequently, we can show $W_1 \cap \mathrm{Fr}\, V$ to be totally disconnected by proving it contains no sequence $Z_1, Z_2, \ldots$ of pairwise disjoint, compact connected subsets, each with at least two points, that converges in the Hausdorff metric to another compact connected subset $Z$, also with at least two points. The possibility that such a sequence might exist is eliminated by considering the following two cases.

First, $Z$ contains four points $a_1, a_2, a_3, a_4$ which are arcwise accessible from $V$. Then there exists a 4-od $T$ in $(V \cap C_+) \cup Z$, with $T \cap Z = \{a_1, a_2, a_3, a_4\}$. Lemma 3.4 implies that for some pair, say $\{a_1, a_2\}$, the subarc $\langle a_1, a_2 \rangle$ of $T$ separates $a_3$ from $a_4$ in $V$. This is not possible because the connected sets $Z_1, Z_2, \ldots$ miss $T$ but have both $a_3$ and $a_4$ as limit points.

Second, the set of points in $Z$ that are arcwise accessible from $V$ is not dense in $Z$. In this case, there exists a 4-od $T$ in $(V \cap C_+) \cup (W_1 \cap \mathrm{Fr}\, V)$, having its endpoints $a_1, \ldots, a_4$ in $Z$ and containing a "sub-4-od" $T'$, which, in turn, has its endpoints $a_1', \ldots, a_4'$ in $W_1 \cap \mathrm{Fr}\, V$, disjoint from $\{a_1, \ldots, a_4\}$, and satisfying $T' - \{a_1', \ldots, a_4'\} \subset V \cap C_+$. Lemma 3.4 again implies that for some pair, say $\{a_1', a_2'\}$, the subarc $\langle a_1', a_2' \rangle$ of $T'$ separates $a_3'$ from $a_4'$. This is not possible because $\langle a_3', a_3 \rangle \cup Z \cup \langle a_4', a_4 \rangle$ is a connected subset of $B$ missing $\langle a_1', a_2' \rangle$ but containing both $a_3'$ and $a_4'$.

Lemma 3.1 and Corollary 3.3 combine to form our main result.

THEOREM 3.6. *If $M$ is an orientable $(n + 2)$-manifold and $G$ is a usc decomposition of $M$ into continua having the shape of closed, orientable $n$-manifolds, then $B = M/G$ is a 2-manifold (without boundary).*

Orientability of the elements of $G$ is a fundamental hypothesis in Theorem 3.6; orientability of the source manifold $M$ is less crucial, as the next result demonstrates.

THEOREM 3.7. *If $G$ is a usc decomposition of an $(n + 2)$-manifold $M$ into continua having the shape of closed, orientable $n$-manifolds, then $B = M/G$ is a 2-manifold possibly with boundary.*

PROOF. Assume $M$ to be nonorientable. Consider the orientable double covering $\theta\colon \tilde{M} \to M$ and usc decomposition $\tilde{G}$ into the components of $\theta^{-1}(g)$, $g \in G$. The nontrivial covering homeomorphism $h$: $\tilde{M} \to \tilde{M}$, which respects elements of $\tilde{G}$, induces an involution $u$ on $\tilde{M}/\tilde{G}$, and the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\theta} & M \\ \tilde{p}\downarrow & & p\downarrow \\ \tilde{M}/\tilde{G} & \xrightarrow{T} & M/G \end{array}$$

induces a map $T$: $\tilde{M}/\tilde{G} \to M/G$ behaving just like the map of $\tilde{M}/\tilde{G}$ to the orbit space of $u$. It is well known **[Wh]** that the orbit space of every involution on a 2-manifold is a 2-manifold with boundary (the boundary may be empty).

Before closing down our analysis of the troublesome spots $D$ and $K$, the discontinuity and degeneracy subsets of $B$, we provide one last analysis.

PROPOSITION 3.8. *Assume $M$ (orientable) satisfies $H_1(M) = 0$ and $K \neq \varnothing$. Then $K \cup D$ consists of exactly one point.*

PROOF. Find an arc $A$ in $B$ with $\partial A \subset K \cup D$, $\partial A \cap K \neq \varnothing$, and $A - \partial A \subset C_+$. If $\partial A \subset K$, then as before $\check{H}^{n+1}(p^{-1}A; \mathbf{Z}) \cong \mathbf{Z}$; if $d \in \partial A \cap D$, then

$$\check{H}^{n+1}(p^{-1}A; \mathbf{Z}/k\mathbf{Z}) \cong \mathbf{Z}/k\mathbf{Z},$$

where $\alpha_d(x) = k$ for all $x \neq d$ sufficiently close to $d$. As a result, with the appropriate coefficients the diagram

$$\begin{array}{c} 0 = H_1(M) \to H_1(M, M - p^{-1}A) \to \tilde{H}_0(M - p^{-1}A) \to \tilde{H}_0(M) = 0 \\ \downarrow \cong \\ \check{H}^{n+1}(p^{-1}A) \end{array}$$

reveals $\tilde{H}_0(M - p^{-1}A)$ to be nontrivial, implying that the arc $A$ separates the 2-manifold $B$, an impossibility.

As applications of Theorem 3.6, we mention some structural limitations upon the manifolds $M$ admitting decompositions into codimension-two submanifolds.

PROPOSITION 3.9. *Suppose $M$ is a connected $(n + 2)$-manifold with at least two ends and with $H_1(M; \mathbf{Z}/2\mathbf{Z}) = 0$. Then $M$ admits no usc decomposition $G$ into continua having the shape of closed, orientable $n$-manifolds.*

PROOF. If it did, the (noncompact) decomposition space $B$ would satisfy $H_1(B; \mathbf{Z}/2\mathbf{Z}) = 0$ **[B]**, indicating that $B$ is the plane. Here the key property is that $B$ has one end. The desired contradiction stems from the fact that proper, monotone maps preserve the number of ends.

PROPOSITION 3.10. *Suppose $M$ is a noncompact, connected $(n + 2)$-manifold such that for each compact subset $C$ of $M$ there exists another compact set $C' \supset C$ such that*

$$H_1(M - C'; \mathbf{Z}/2\mathbf{Z}) \to H_1(M - C; \mathbf{Z}/2\mathbf{Z})$$

*is trivial. Then $M$ admits no usc decomposition $G$ into continua having the shape of closed, orientable $n$-manifolds.*

PROOF. Suppose otherwise. Then $B = M/G$ is a noncompact 2-manifold for which $H_1(B; \mathbf{Z}/2\mathbf{Z})$ is trivial at infinity, by [**B**] and the hypothesis. This is impossible: every noncompact 2-manifold contains loops arbitrarily close to infinity that fail to be null homologous in the complement of a given compactum.

Our chief interest was in determining whether $E^{n+2}$ admits a decomposition ino codimension-two submanifolds, which now is settled.

COROLLARY 3.11. *There is no usc decomposition of $E^{n+2}$ into continua having the shape of closed, orientable n-manifolds.*

REMARK. The upper semicontinuity of the decompositions $G$ has been fundamental to the discussion throughout this paper. Reinforcing this point, a remarkably explicit partition of $E^3$ into (round) circles is given in [**Sz**].

Finally, in the compact case, we have

PROPOSITION 3.12. *Suppose $M^{n+2}$ is a closed, orientable n-manifold for which $H_2(M; \mathbf{Z}/2\mathbf{Z}) = 0$. Then $M$ admits no usc decomposition $G$ into continua having the shape of closed n-manifolds with $\check{H}_1(g; \mathbf{Z}/2\mathbf{Z}) = 0$ for all $g \in G$.*

PROOF. If it did, $B$ would be a closed 2-manifold, and $H_2(B; \mathbf{Z}/2\mathbf{Z}) \neq 0$. Under these hypotheses, the Vietoris-Begle Theorem [**B**] ensures that

$$p_*: H_2(M; \mathbf{Z}/2\mathbf{Z}) \to H_2(B; \mathbf{Z}/2\mathbf{Z})$$

is surjective, a contradiction.

**4. Structure of the sheaf $\mathscr{H}^n$.** We shall denote by $\mathscr{H}^n$ the presheaf on $B$ determined by setting $\mathscr{H}^n(U) = H^n(p^{-1}U; \mathbf{Z})$ (and using inclusion induced homomorphisms) as well as the associated sheaf, whose stalk at a point $x \in B$ is $\check{H}^n(p^{-1}x; \mathbf{Z})$. The path leading to the conclusion that $B$ is a 2-manifold also brought us to a determination that $\mathscr{H}^n$ is locally constant except at points of the sparse set $K \cup D_+$. A final refinement is

THEOREM 4.1. *Suppose $M$ is an orientable $(n+2)$-manifold and $G$ is a usc decomposition of $M$ into continua having the shape of closed, orientable n-manifolds. Then the decomposition space $B$ is a 2-manifold and $B$ contains a locally finite (closed) subset $F$ such that $\mathscr{H}^n$ is locally constant at each point of $B - F$. Furthermore, if $\check{H}_1(g; \mathbf{Z}) \cong 0$ for each $g \in G$, then $F = \varnothing$.*

PROOF. Since it is already established that $B$ is a 2-manifold and $\mathscr{H}^n$ is locally constant over $B \setminus K \cup D_+$, we need to show that $K \cup D_+$ is locally finite and, under the additional constraints of the "furthermore", that it is empty.

The local finiteness of $K \cup D_+$ is detected by precluding the existence of three possible types of convergent sequences from $K \cup D_+$.

First, suppose there is a sequence $\{x_i\} \subset K$ with $x_i \to x \in K$. It had been previously observed that fact (1) detects that the local winding function at any point of $K$ is zero at some point of the continuity set $C_+$. As we currently know that $U \cap C_+$ is connected whenever $U$ is an open connected set, each such local winding function is zero at every point of $C_+$ in its domain. Thus, the argument in Lemma

3.4 would show that an arc $A$ connecting $x_i$ and $x_{i+1}$, where $i$ is chosen so that $A$ is contained in an appropriately small neighborhood of $x$ and $A - \partial A \subset C_+$, would separate a connected neighborhood of $x$. This clearly is not possible as $B$ is a 2-manifold (without boundary).

Second, suppose that some point $x \in D_+$ is a limit point of $D_+ - x$. Let $\alpha$ denote the local winding function at $x$ defined on a connected neighborhood $U$ of $x$ and, for $d \in D_+ \cap U$, let $\alpha_d$ denote the local winding function of $d$ defined on a connected neighborhood $V_d \subset U$ of $d$. Since $U \cap C_+$ is connected, $\alpha$ is constant on $U \cap C_+$, say $\alpha(b) = r$. Similarly, $\alpha_d$ is constant on $V_d \cap C_+$, say $\alpha_d(b) = r_d$. According to fact (1), $r = k \cdot r_d$ for $b \in V_d \cap C_+$ and, consequently, the integers $r_d$ are bounded (by $r$). Of course, $r_d > 1$ as $d \in D_+$. Thus, we can specify a sequence $\{x_i\} \subset D_+ - x$ with $x_i \to x$ and with the local winding functions at the $x_i$'s taking on the single value $q > 1$ at points in the continuity set $C_+$. An argument as in Lemma 3.4 using $\mathbf{Z}/q\mathbf{Z}$-coefficients would lead to the contradiction that an arc $A$, chosen as in the preceding paragraph, separates a connected neighborhood of $x$.

Third, suppose that there is a sequence $\{x_i\} \subset D_+$ with $x_i \to x \in K$. As before, we assume that the $x_i$'s are contained in a neighborhood $U$ of $x$ on which the local winding function $\alpha$ at $x$ is defined. Specify connected neighborhoods $V_i \subset U$ of $x_i$ on which the local winding function $\alpha_i$ at $x_i$ is defined and set $r_i$ equal to the constant value of $\alpha_i$ on $V_i \cap C_+$. In contrast with the previous situation, we cannot conclude that the $r_i$'s are bounded, as $\alpha(b) = 0$ for $b \in U_0 \cap C_+$. Consider an arc $A$ connecting $x$ and $x_i$ where $A$ is chosen in a small neighborhood of $x$ containing $x_i$ and where $A - \partial A \subset C$. An argument as in Lemma 3.4 using $\mathbf{Z}/r_i\mathbf{Z}$-coefficients produces the contradiction that $A$ separates a connected neighborhood of $x$ provided the homomorphism $H_1(p^{-1}W_1) \to H_1(p^{-1}W_1, p^{-1}W_1 - p^{-1}A)$ is trivial ($W_1$ being chosen near the start of the proof at Lemma 3.4). In the preceding two paragraphs, the fact that $x \notin A$ facilitates the computation. In the current situation, we must insist that, after choices of neighborhoods $W_1 \subset U_0 \subset U$ of $x$ are made, $i$ is chosen with $p^{-1}A$ so close to $p^{-1}x$ that

$$\operatorname{Im}\{\check{H}_1(p^{-1}x) \to H_1(p^{-1}U)\} = \operatorname{Im}\{H_1(p^{-1}(U_0 - A)) \to H_1(p^{-1}U)\}.$$

This poses no difficulties since generators for $\operatorname{Im}\{\check{H}_1(p^{-1}x) \to H_1(p^{-1}U)\}$ can be found in $H_1(p^{-1}(U_0 - x))$.

We now assume that $\check{H}_1(p^{-1}x; \mathbf{Z}) \cong 0$ for each $x \in B$. Suppose, contrary to the Claim of the "furthermore", that $b \in K \cup D_+$. Specify an open 2-cell $\Delta \subset B$ with $\Delta \cap (K \cup D_+) = \{b\}$ so that there is a shape strong deformation retraction $\underline{r}$: $p^{-1}\Delta \to p^{-1}b$, the deformation occurring in a neighborhood of $p^{-1}b$. Set $\Delta^* = \Delta - \{b\}$ and choose any $c \in \Delta^*$. Consider the diagram

$$\begin{array}{ccccc} H^n(p^{-1}\Delta) & \xrightarrow{\alpha} & H^n(p^{-1}\Delta^*) & \to & H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \\ \downarrow j^* & & \downarrow i^* & & \\ \check{H}^n(p^{-1}b) & \xrightarrow{\phi} & \check{H}^n(p^{-1}c) & & \end{array}$$

where the top row is part of the exact sequence of the pair $(p^{-1}\Delta, p^{-1}\Delta^*)$, $\phi$ is induced by the restriction of $\underline{r}$, and the remaining homomorphisms are induced by inclusions. In particular, $\phi j^* = i^*\alpha$. The existence of $\underline{r}$ assures that $j^*$ is into. The assumption that $\check{H}_1(p^{-1}b) \cong 0$ yields, by duality [**S**, p. 296] that $H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \cong 0$. Consequently, the contradiction that $\phi$ is onto (and hence an isomorphism) will be reached once $i^*$ is shown to be onto.

The universal coefficient theorem [**S**, p. 243] yields the short exact sequence

$$0 \to \operatorname{Ext}(H_n(p^{-1}\Delta, p^{-1}\Delta^*), \mathbf{Z}) \to H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*)$$
$$\to \operatorname{Hom}(H_{n+1}(p^{-1}\Delta, p^{-1}\Delta^*), \mathbf{Z}) \to 0$$

and duality [**S**, p. 296] produces

$$H_n(p^{-1}\Delta, p^{-1}\Delta^*) \cong \check{H}^2(p^{-1}b) \quad \text{and} \quad H_{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \cong \check{H}^1(p^{-1}b).$$

The same exact sequence applied to $\check{H}^1(p^{-1}b)$ reveals that $\check{H}^1(p^{-1}b) \cong 0$ and, then, to $\check{H}^2(p^{-1}b)$ reveals that $\check{H}^2(p^{-1}b)$ is free. It follows that $H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \cong 0$.

We begin the computation of $i^*$'s surjectiveness by expressing $\Delta - b = \Delta^*$ as the union of two open disks $R_1$ and $R_2$ whose intersection is a pair of open disks $V_1$ and $V_2$ with $c \in V_1$. Further, we specify a point $d \in V_2$ and arcs $A_1 \subset R_1$ and $A_2 \subset R_2$ connecting $e$ and $d$. The strategy is to decompose $i^*$ as

$$H^n(p^{-1}\Delta^*) \underset{\delta}{\to} H^n(p^{-1}(A_1 \cup A_2)) \underset{\beta}{\to} \check{H}^n(p^{-1}c),$$

and then to determine that both inclusion induced homomorphisms $\delta$ and $\beta$ are surjective.

The Mayer-Vietoris sequence

$$\begin{array}{ccc}
\check{H}^n(p^{-1}(A_1 \cap A_2)) & \to & \check{H}^{n+1}(p^{-1}(A_1 \cup A_2)) \to 0 \\
\downarrow \cong & & \downarrow \cong \\
\check{H}^n(p^{-1}c) \oplus \check{H}^n(p^{-1}d) & & \mathbf{Z} \\
\downarrow \cong & & \\
\mathbf{Z} \oplus \mathbf{Z} & &
\end{array}$$

contains adequate information for discovering that $\beta$ is surjective. The computation of $\check{H}^{n+1}(p^{-1}(A_1 \cup A_2))$ is essentially done in Lemma 2.3 while the zero at the right is a consequence of Lemma 2.2. A further consequence of the latter is that, for $i = 1, 2$, there is an element $a_i \in \check{H}^n(p^{-1}A_i)$ that restricts to a generator in both $\check{H}^n(p^{-1}c)$ and $\check{H}^n(p^{-1}d)$. Hence, essentially either $\gamma(a_1, a_2) = (0, 0)$ or $\gamma(a_1, a_2) = (2, 0)$. The latter would force $\check{H}^{n+1}(p^{-1}(A_1 \cup A_2))$ to be a torsion group, which it is not, while the former readily shows that $\check{H}^n(p^{-1}(A_1 \cup A_1)) \to \check{H}^n(p^{-1}A_1)$ is surjective, and, hence, $\beta$ is surjective.

Detecting the surjectivity of

$$\check{H}^n(p^{-1}\Delta^*) \overset{\delta}{\to} \check{H}^n(p^{-1}(A_1 \cup A_2))$$

is accomplished by comparing the spectral sequences of $p|: p^{-1}\Delta^* \to \Delta^*$ and $p|: p^{-1}(A_1 \cup A_2) \to A_1 \cup A_2$. (The sequence is derived in [**G**] and is described adequately for our purpose in [**DW2**].) We only need to extract the commuting diagram

$$\begin{array}{ccccc}
\check{H}^n(p^{-1}\Delta^*; \mathbf{Z}) & \to & H^0(\Delta^*; \mathscr{H}^n) & \to & 0 \\
\downarrow \delta & & \downarrow & & \\
\check{H}^n(p^{-1}A_1 \cup A_2; \mathbf{Z}) & \underset{\eta}{\to} & H^0(A_1 \cup A_2; \mathscr{H}^n) & \to & 0
\end{array}$$

where the right-hand vertical homomorphism is inclusion induced. Furthermore, the homomorphism is an isomorphism since the early computation that

$$\check{H}^{n+1}\left(p^{-1}(A_1 \cup A_2); \mathbf{Z}\right) \cong \mathbf{Z}$$

detects that the sheaf $\mathscr{H}^n$ is constant on $\Delta^*$. The surjectiviy of $\delta$ follows once we determine that the homomorphism $\eta$ is an isomorphism. From the spectral sequence of

$$p|: p^{-1}(A_1 \cup A_2) \to A_1 \cup A_2$$

we determine that kernel $\eta \cong H^1(A_1 \cup A_2; \mathscr{H}^{n-1})$ but $\mathscr{H}^{n-1}$ is the zero-sheaf as each stalk $\check{H}^{n-1}(p^{-1}x; \mathbf{Z})$ is dually isomorphic to $\check{H}_1(p^{-1}x; \mathbf{Z}) \cong 0$ (by assumption).

**5. Examples.** As a summary, we want to compare what is known about usc decompositions $G$ of orientable $(n+k)$-manifolds $M$ into connected, orientable $n$-manifolds in the two low-codimensional cases where $k = 1$ and $k = 2$. In either case $B = M/G$ is now known to be a boundaryless $k$-manifold.

Sometimes the decomposition map $p$: $M \to B$ can be employed to study the structure of $M$. This investigative tool is quite effective when $p$ is an approximate fibration. Unfortunately, that is not always the prevailing situation. Example 3 of [**D**] provides a decomposition $G$ of an $(n+1)$-manifold $M$ into homology $n$-spheres, some simply connected and others not; the decomposition map cannot be an approximate fibration because the decomposition elements have different shapes (different homotopy types). There is a similar decomposition of the $(n+2)$-manifold $M \times E^1$ into homology $n$-spheres.

When $k = 1$ the structure of $M$ is exposed, to some extent, by one unifying feature: the elements of $G$ are all pairwise homologically equivalent [**D**, Corollary 6.3]. Consequently, the shape (neighborhood) retractions $V \to g$ induce homology isomorphisms $H_j(g') \to H_j(g)$ for all $g'$ in $V$ sufficiently close to $g$. The first fact may appear to stem from the property that the continuity set $C_+$ of $B$ coincides with $B$ itself. However, when $k = 2$ the elements of $G$ need not all be homologically equivalent, not even if $C_+ = B$.

EXAMPLE. A usc decomposition $G$ of a connected, orientable $(n+2)$-manifold $M$ $(n \geqslant 2)$ into orientable $n$-manifolds such that (1) in the resulting decomposition space $B$, each point belongs to the continuity set $C_+$, and (2) $G$ contains a pair of homologically inequivalent elements.

The crux of the matter is manifested when $n = 2$, so focus on that case. Start with a knot $K$ in $S^3$, like the trefoil, whose closed complement fibers over $S^1$ with fiber $T$, where $T$ is a disk having a positive number of handles. Attach a (4-dimensional) 2-handle $H$ to $B^4$ along $U(K)$, a tubular neighborhood of $K$ in $S^3 = \partial B^4$, via the framing determined by the longitude $\partial T$ of the knot space $\mathrm{Cl}(S^3 - U(K))$. The interior $M$ of the resulting 4-manifold with boundary represents the desired manifold.

The exceptional element $g_0$ of the decomposition $G$ yet to be specified is the 2-sphere in $M$ formed by the cone over $K$ from the center of $B^4$ and the core of the attached 2-handle, $h$. Topologically the other, more standard elements of $G$ all are the union $\hat{T}$ of $T$ and a 2-cell capping off $\partial T$. Each lies in $M$ as a copy of $\hat{T}$ in

$$r \cdot (S^3 - U(K)) \subset B^4 \quad (\text{where } 0 < r < 1)$$

concentric with one of the given fibers of $S^3 - U(K)$, a copy of $B^2$ in

$$B^2 \times r \cdot \partial D^2 \subset B^2 \times D^2 = h,$$

and an annulus connecting the boundaries of the first two in the cone over $U(K)$ in $B^4$, as suggested by Figure 1.

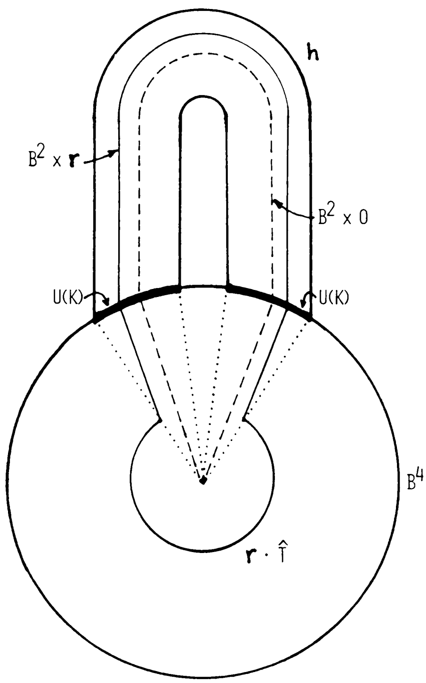


FIGURE 1

Away from the exceptional element $g_0$, the decomposition map behaves like the projection of $\hat{T} \times (E^2 - 0)$ to $E^2 - 0$. One can see that the winding function $\alpha_b$ is locally constant at $b = p(g_0)$ by observing the existence of a retraction $r_0$: $M \to g_0$ such that $r_0(M \cap B^4) \subset g_0 \cap B^4$ and $r_0 | h \cap M$ acts like the projection of $B^2 \times \operatorname{Int} D^2$ to its core $B^2 \times 0 \subset g_0$. For geometric reasons, $r_0 | g'$: $g' \to g_0$ is a degree-one map, for each $g' \in G[E]$. Obviously, for each $g \neq g_0$, $H_1(g') = H_1(\hat{T}) \not\cong 0$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37996-1300