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FUNCTIONS OF UNIFORMLY BOUNDED CHARACTERISTIC ON RIEMANN SURFACES

BY
SHINJI YAMASHITA

ABSTRACT. A characteristic function $T(D, w, f)$ of Shimizu and Ahlfors type for a function f meromorphic in a Riemann surface R is defined, where D is a regular subdomain of R containing a reference point $w \in R$. Next we suppose that R has the Green functions. Letting $T(w, f) = \lim_{D\uparrow R} T(D, w, f)$, we define f to be of uniformly bounded characteristic in R, $f \in \mathrm{UBC}(R)$ in notation, if $\sup_{w\in R} T(w, f) < \infty$. We shall propose, among other results, some criteria for f to be in $\mathrm{UBC}(R)$ in various terms, namely, Green's potentials, harmonic majorants, and counting functions. They reveal that $\mathrm{UBC}(\Delta)$ for the unit disk Δ coincides precisely with that introduced in our former work. Many known facts on $\mathrm{UBC}(\Delta)$ are extended to $\mathrm{UBC}(R)$ by various methods. New proofs even for $R = \Delta$ are found. Some new facts, even for Δ, are added.

0. Introduction. We shall extend the notion of UBC and UBC_0 from the unit disk $\Delta = \{|z| < 1\}$ (see [$\mathbf{Y_1}$ and $\mathbf{Y_2}$]) to hyperbolic Riemann surfaces, prove some results analogous to those in Δ, and add some facts, new, even for Δ. A hyperbolic Riemann surface S is one possessing Green functions; thus, its universal covering surface S^∞ must be conformally equivalent to Δ, so that S^∞ and Δ are identified.

Our study begins with how to define the Shimizu-Ahlfors characteristic function $T(D, w, f)$ on "good" subdomains D containing a point w of a Riemann surface R, hyperbolic or not, on which f is meromorphic. Each point of R is identified with its local-parametric image in the complex plane $\mathbf{C} = \{|z| < \infty\}$. By D we always mean a relatively compact subdomain of R, whose boundary ∂D consists of a finite number of mutually disjoint, analytic, simple and closed curves on R. If we refer to a pair D and $w \in R$ we always assume that $w \in D$. The radius $r = r(D, w) > 0$ of D is defined by

$$\log r = \lim(g_D(z, w) + \log|z - w|)$$

as $z \to w$ within the parametric disk of center w, where $g_D(z, w)$ is the Green function of D with pole at w. We now set

$$T(D, w, f) = \pi^{-1} \int_0^r t^{-1} \left[\int\int_{D_t} f^{\#}(z)^2 \, dx\, dy\right] dt,$$

where $D_t = \{z \in D; g_D(z, w) \geq \log(r/t)\}$, $0 < t < r$, and

$$(0.1) \qquad f^{\#}(z) = \begin{cases} |f'(z)|/(1 + |f(z)|^2), & \text{if } f(z) \neq \infty, \\ (1/f)^{\#}(z), & \text{if } f(z) = \infty, \end{cases}$$

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is not a function on R, yet the second-order differential $f^{\#}(z)^2\,dx\,dy$, $z = x+iy \in R$, is well defined on R. The Green-potential expression

$$T(D,w,f) = \pi^{-1}\int\int_D f^{\#}(z)^2 g_D(z,w)\,dx\,dy, \qquad w \in R, \tag{0.2}$$

will be proved later. The nomenclature of T is justified because for $R = \{|z| < \rho\}$, $D = \{|z| < r\}$ with $0 < r < \rho \leq \infty$, and $w = 0$, we have the usual one because $g_D(z,0) = \log|r/z|$.

Henceforth we always assume that R is hyperbolic and we set

$$T(w,f) = T(R,w,f) = \lim_{D\uparrow R} T(D,w,f) \leq \infty.$$

This means that given $\varepsilon > 0$ we can find a compact set K, $w \in K \subset R$, such that $|T - T(D)| < \varepsilon$ for all $D \supset K$, with the obvious change in case $T = \infty$. Lebesgue's convergence theorem applied to (0.2) yields

$$T(w,f) = \pi^{-1}\int\int_R f^{\#}(z)^2 g(z,w)\,dx\,dy, \tag{0.3}$$

where $g = g_R$ is the Green function on R; (0.2) can be regarded as the case $R = D$.

A meromorphic f on R is said to be of uniformly bounded characteristic, $f \in \mathrm{UBC} \equiv \mathrm{UBC}(R)$ in notation, if the function $T(w,f)$ is bounded on R, while, $f \in \mathrm{UBC}_0 \equiv \mathrm{UBC}_0(R)$ if $\lim_{w\to\partial R} T(w,f) = 0$, that is, for $\varepsilon > 0$ there exists a compact $K \subset R$ such that $T(w,f) < \varepsilon$ in $R\backslash K$.

In §1 we extend our study from the family $M \equiv M(R)$ of meromorphic functions on R to $M_e \equiv M_e(R)$ consisting of multiple-valued meromorphic functions with single-valued moduli on R. We can easily extend the definition of UBC (UBC_0, respectively) for $f \in M$ to $\mathrm{UBC}_e \equiv \mathrm{UBC}_e(R)$ ($\mathrm{UBC}_{e0} \equiv \mathrm{UBC}_{e0}(R)$, resp.) for $f \in M_e$.

In §2, (0.3) for $f \in M_e$ is proved. Thus, criteria are obtained in terms of the Green potentials (Corollary 2.2). The families $\mathrm{BMOA}_e \equiv \mathrm{BMOA}_e(R)$ and $\mathrm{VMOA}_e \equiv \mathrm{VMOA}_e(R)$ are defined for pole-free members of M_e; these are extensions of BMOA and VMOA in the disk. For the definition of $\mathrm{BMOA}(R)$ see [**M**]; note that $\mathrm{BMOA}(R) = M(R) \cap \mathrm{BMOA}_e(R)$. The formulae $\mathrm{BMOA}_e \subset \mathrm{UBC}_e$ and $\mathrm{VMOA}_e \subset \mathrm{UBC}_{e0}$ are now obvious. An expression of T in terms of the limit ($D \uparrow R$) of the mean of $\frac{1}{2}\log(1+|f|^2)$ on ∂D and the limit of

$$N(D,w,f) = \sum_{f(b)=\infty,\, b\in D} g_D(w,b)$$

will be of use to compare T with L. Sario's characteristic function T_S (see [**SN**]). Sario's class $M_eB(R)$ coincides with that of $f \in M_e$ for which $T(w,f)$ is finite for a $w = w(f) \in R$.

§3 is devoted to the study of the least harmonic majorant $\varphi^\wedge$ of

$$\varphi = \tfrac{1}{2}\log(|f_1|^2 + |f_2|^2) \quad \text{for } f \in M_e,$$

where $|f| = |f_1|/|f_2|$, $f_1, f_2 \in M_e$, is an "admissible" decomposition. A new expression $T(w,f) = \varphi^\wedge(w) - \varphi(w)$, $w \in R$, is of use to obtain criteria for $f \in M_e$ to be of UBC_e and of UBC_{e0} in terms of $\varphi^\wedge - \varphi$. Some remarks refer to strong parallels between BMOA_e (VMOA_e, resp.) and UBC_e (UBC_{e0}, resp.).

In §4, criteria for $f \in M$ to be of UBC in terms of the supremum of the function $N(z) = \lim_{D\uparrow R} N(D, w, 1/(f - z))$, $z \in \mathbf{C}^* = \mathbf{C} \cup \{\infty\}$, on the spherical circle of center $f(w)$ are obtained.

The projection π: $\Delta \to R$ is considered in §5, and the identity $T(R, \pi(\delta), f) = T(\Delta, \delta, f \circ \pi)$, $\delta \in \Delta$, for $f \in M(R)$ is proved. As applications we obtain: (1) If $f \in \mathrm{UBC}(S)$ and h: $R \to S$ is an analytic map, then $f \circ h \in \mathrm{UBC}(R)$. (2) If h: $R \to S$ is of type Bl in the sense of M. Heins [**H₁**], and if $f \circ h \in \mathrm{UBC}(R)$, then $f \in \mathrm{UBC}(S)$. Finally, a contribution is made to the classification of Riemann surfaces: $O_{\mathrm{UBC}} \subsetneqq O_{\mathrm{BMOA}}$.

1. Families UBC_e **and** UBC_{e0}. The functions on R, which we shall actually study, are, for the most part, the "generalized" meromorphic functions on R. Let $M_e \equiv M_e(R)$ be the family of multiple-valued functions $f = \exp(u + iu^*)$ on R, where u is a single-valued function harmonic on R except for countably many logarithmic singularities a_n clustering nowhere in R, such that $u(z) - k_n \log|z - a_n|$ is harmonic in the parameter disk of center a_n with the integral coefficient k_n. The multiple-valuedness of f arises from that of the conjugate function u^* of u on R. It is natural to regard the constant zero as a member of M_e.

The modulus $|f|$ of $f \in M_e$ is single-valued throughout R, and each branch of nonconstant f in the parametric disk of each point $w \in R$ is single-valued there, and has the Laurent expansion

$$c_\lambda(z - w)^\lambda + c_{\lambda+1}(z - w)^{\lambda+1} + \cdots, \tag{1.1}$$

where λ is an integer with $c_\lambda \neq 0$; the branches differ by multiplicative constants of moduli one. Therefore $|c_\lambda|$ is definite and

$$|f(w)| \leq |c_\lambda| \quad \text{if } |f(w)| \neq \infty. \tag{1.2}$$

We call $w \in R$ a zero of f of order λ if $|f(w)| = 0$. Similarly, $w \in R$ is a pole (or, ∞-point) of f of order $-\lambda$ if $|f(w)| = \infty$.

The family $M \equiv M(R)$ of single-valued members of M_e consists of all the meromorphic functions on R. For $a \in \mathbf{C}$ we call $w \in R$ an a-point of order λ of $f \in M$ if w is a zero of order λ of $f - a$.

It is now easy to extend $T(D, w, f)$ and $T(w, f) \equiv T(R, w, f)$ to $f \in M_e$. Actually, $|f'(z)|$ for $|f(z)| \neq \infty$, as well as $f^\#(z)$ defined by (0.1), is definite, so that the differentials $|f'(z)|^2\, dx\, dy$ for pole-free f and $f^\#(z)^2\, dx\, dy$ for arbitrary f are defined on R. The definitions of $\mathrm{UBC}_e \equiv \mathrm{UBC}_e(R)$ and $\mathrm{UBC}_{e0} \equiv \mathrm{UBC}_{e0}(R)$ are thus clear; just extend those of $\mathrm{UBC} \equiv \mathrm{UBC}(R)$ and $\mathrm{UBC}_0 \equiv \mathrm{UBC}_0(R)$ in the introduction to $f \in M_e$.

2. The Shimizu-Ahlfors characteristic function. We begin with the Green-potential expression (0.3) for $f \in M_e$.

THEOREM 2.1. *The identity*

$$T(w, f) = \pi^{-1} \int\!\!\int_R f^\#(z)^2 g(z, w)\, dx\, dy \quad (\leq \infty) \tag{2.1}$$

holds for each $f \in M_e(R)$ *and each* $w \in R$.

PROOF. It suffices to establish

$$T(D, w, f) = \pi^{-1} \int\!\!\int_D f^\#(z)^2 g_D(z, w)\, dx\, dy \tag{2.2}$$

for each pair D, w. Set

$$c_t(z) = \begin{cases} 1, & \text{if } z \in D_t, \\ 0, & \text{if } z \in D \backslash D_t. \end{cases}$$

Then the identity

$$\int_0^r c_t(z)t^{-1}\,dt = g_D(z,w), \qquad z \in D,$$

together with

$$T(D,w,f) = \pi^{-1} \int\int_D \left[\int_0^r c_t(z)t^{-1}\,dt\right] f^{\#}(z)^2\,dx\,dy$$

proves (2.2).

REMARK. Suppose that w is in the parametric disk $U_{w'}$ of center $w' \in R$, and define $r' > 0$ by

$$\log r' = \lim(g_D(z,w) + \log|z-w|),$$

where, this time $z \to w$ within $U_{w'}$. The same proof as above then shows that

$$T(D,w,f) = \pi^{-1} \int_0^{r'} t^{-1} \left[\int\int_{D'_t} f^{\#}(z)^2\,dx\,dy\right] dt,$$

where $D'_t = \{z \in D; g_D(z,w) \geq \log(r'/t)\}$, $0 < t < r'$.

Two corollaries follow from Theorem 2.1.

COROLLARY 2.2. *For $f \in M_e(R)$ the following are valid.*
(I) $f \in \mathrm{UBC}_e(R)$ *if and only if*

$$\sup_{w \in D} \int\int_R f^{\#}(z)^2 g(z,w)\,dx\,dy < \infty. \tag{2.3}$$

(II) $f \in \mathrm{UBC}_{e0}(R)$ *if and only if*

$$\lim_{w \to \partial R} \int\int_R f^{\#}(z)^2 g(z,w)\,dx\,dy = 0. \tag{2.4}$$

This corollary extends [$\mathbf{Y_1}$, Theorem 2.2, p. 352].

A pole-free $f \in M_e(R)$ is said to be of bounded (vanishing, resp.) mean oscillation on R, $f \in \mathrm{BMOA}_e(R)$ ($f \in \mathrm{VMOA}_e(R)$, resp.) in notation, if

$$\sup_{w \in R} \int\int_R |f'(z)|^2 g(z,w)\,dx\,dy < \infty$$

$$\left(\lim_{w \to \partial R} \int\int_R |f'(z)|^2 g(z,w)\,dx\,dy = 0, \text{ resp.}\right).$$

The family $\mathrm{BMOA}(R) = \mathrm{BMOA}_e(R) \cap M(R)$ is introduced by T. A. Metzger [**M**]. An immediate consequence of Corollary 2.2 is the following which extends [$\mathbf{Y_1}$, Theorem 7.1, p. 364].

COROLLARY 2.3. $\mathrm{BMOA}_e(R) \subset \mathrm{UBC}_e(R)$ *and* $\mathrm{VMOA}_e(R) \subset \mathrm{UBC}_{e0}(R)$.

Following Sario we shall define the proximity function $m_S(D,w,f)$, the counting function $N_S(D,w,f)$, and the characteristic function $T_S(D,w,f)$ for $f \in M_e$ and $w \in D$. They are extensions of M. Parreau's [**P**, p. 183 ff.] for $f \in M$. The reader is expected to be familiar with [**SN**, Chapter III] or with the papers [$\mathbf{S_1}$ and $\mathbf{S_2}$].

Let $\log^+ x = \max(\log x, 0)$ for $0 \le x \le \infty$, and set

$$m_S(D, w, f) = -(2\pi)^{-1} \int_{\partial D} \log^+ |f(z)| dg_D^*(z, w),$$

where ∂D is oriented positively with respect to D, and the star means the conjugate. The comparison

$$m_S(D, w, f) \le m(D, w, f) \le \tfrac{1}{2} \log 2 + m_S(D, w, f) \tag{2.5}$$

of $m_S(D, w, f)$ with

$$m(D, w, f) = -(4\pi)^{-1} \int_{\partial D} \log(1 + |f(z)|^2) dg_D^*(z, w)$$

will be of use; (2.5) is a consequence of

$$\log^+ x \le \tfrac{1}{2} \log(1 + x^2) \le \tfrac{1}{2} \log 2 + \log^+ x \quad \text{for } 0 \le x \le \infty. \tag{2.6}$$

For $0 < t < r$, let $n(t, f)$ be the number of the poles of f, counted with the orders, in D_t. We define

$$N_S(D, w, f) = \int_0^r t^{-1}[n(t, f) - n(0, f)]\, dt + n(0, f) \log r,$$

where $n(0, f) = \lim_{t \to +0} n(t, f)$. Set $N(D, w, f) = \sum g_D(w, b)$, where the sum is extended over all the poles b of f in D, each counted with its order. If f is pole-free in D, then $N_S = N = 0$. A routine procedure [**SN**, p. 76] yields that $N_S(D, w, f) = N(D, w, f)$ in case $|f(w)| \ne \infty$. Therefore

$$N(D, w, f) = \int_0^r t^{-1} n(t, f)\, dt$$

for all $w \in D$ because $N(D, w, f) = \infty$ if $|f(w)| = \infty$. The characteristic function for f is now defined by

$$T_S(D, w, f) = m_S(D, w, f) + N_S(D, w, f).$$

For $X = m, N, m_S, N_S$, and T_S, we set $X(w, f) = \lim_{D \uparrow R} X(D, w, f)$.

We shall compare $T_S(w, f)$ with $T(w, f)$ in Corollary 2.5 below.

THEOREM 2.4. *If $|f(w)| \ne \infty$ for $f \in M_e(R)$, then*

$$T(w, f) = m(w, f) + N(w, f) - \tfrac{1}{2} \log(1 + |f(w)|^2). \tag{2.7}$$

If $|f(w)| = \infty$, then

$$T(w, f) = m(w, f) + N_S(w, f) - \log |c_\lambda|, \tag{2.8}$$

where c_λ is defined in (1.1).

REMARK. If $f \in M_e(R)$ is bounded, $|f| \le K$, then $T(w, f) \le \frac{1}{2} \log(1 + K^2)$ by (2.7), so that $f \in \mathrm{UBC}_e(R)$.

COROLLARY 2.5. *If $f \in M_e(R)$ is nonconstant, then*

$$|T(w, f) - T_S(w, f)| \le \tfrac{1}{2} \log 2 + |\log |c_\lambda||. \tag{2.9}$$

Read (2.9), in the specified case, as $T(w, f) = \infty$ if and only if $T_S(w, f) = \infty$.

Fix $w \in R$ and let $\mathrm{BC}_e \equiv \mathrm{BC}_e(R)$ be the family of $f \in M_e$ such that $T_S(w, f) < \infty$. As will be observed later in Remark (a) after Theorem 3.1, BC_e does not depend on w. Note that $\mathrm{BC}_e(R) = M_e B(R)$ in [**SN**, p. 78]; we let $\mathrm{BC}(R) = \mathrm{BC}_e(R) \cap M(R)$. An immediate consequence is the following.

COROLLARY 2.6. $\mathrm{UBC}_e(R) \subset \mathrm{BC}_e(R)$.

Hereafter, mainly in the proofs, we shall frequently use the following abbreviations:

$$X(f) = X(D, w, f) \quad \text{for } X = m, N, T, m_S, N_S, T_S. \tag{2.10}$$

PROOF OF THEOREM 2.4. It suffices to prove

$$T(f) = m(f) + N(f) - \tfrac{1}{2}\log(1 + |f(w)|^2), \quad \text{if } |f(w)| \neq \infty; \tag{2.7$'$}$$

$$T(f) = m(f) + N_S(f) - \log|c_\lambda|, \quad \text{if } |f(w)| = \infty. \tag{2.8$'$}$$

Suppose first that no pole of f lies on ∂D, and let b be all the distinct poles of f with orders $k(b)$ in D. For sufficiently small $\varepsilon > 0$, we let $\gamma_w = \{|z - w| \leq \varepsilon\}$ and $\gamma_b = \{|z - b| \leq \varepsilon\}$. Apply the Green formula to the function $\psi = (1/2)\log(1 + |f|^2)$ on the domain $D_\varepsilon = D \backslash \gamma_w \backslash \bigcup_b \gamma_b$. Since

$$\Delta\psi = (\partial^2/\partial x^2 + \partial^2/\partial y^2)\psi = 2f^{\#2} \quad \text{in } D_\varepsilon,$$

it follows that

$$\begin{aligned}\iint_{D_\varepsilon} & g_D(z, w)\Delta\psi(z)\, dx\, dy \\ &= -\int_{\partial D_\varepsilon} g_D(z, w)\frac{\partial\psi(z)}{\partial\nu}|dz| + \int_{\partial D_\varepsilon} \psi(z)\frac{\partial g_D(z, w)}{\partial\nu}|dz|,\end{aligned} \tag{2.11}$$

where the normal derivatives $\partial/\partial\nu$ are considered in the direction of the inner normal. As to the first integral in the right-hand side of (2.11), that on ∂D is zero, and those on $\partial\gamma_w$ and $\partial\gamma_b$ tend to zero and $2\pi k(b) g_D(b, w)$ as $\varepsilon \to 0$, respectively. Furthermore, as to the second, that on ∂D equals $2\pi m(f)$, and those on $\partial\gamma_w$ and $\partial\gamma_b$ tend to $-2\pi\psi(w)$ and 0 as $\varepsilon \to 0$, respectively. The resulting identity divided by 2π, together with $g_D(b, w) = g_D(w, b)$, yields

$$\pi^{-1}\iint_D g_D(z, w) f^{\#}(z)^2\, dx\, dy = m(f) - \psi(w) + \sum_b k(b) g_D(w, b). \tag{2.12}$$

In view of (2.2) we immediately observe that (2.7′) is true.

Suppose now that ∂D contains at least one pole of f. For t, $0 < t < r$, sufficiently near r, we obtain (2.7′) for $D_t \backslash \partial D_t$ instead of D. Observing that T, m, and hence N, all are continuous in t, one obtains (2.7′) for D by letting $t \uparrow r$.

For the proof of (2.8′) we quote Jensen's formula

$$\log|c_\lambda| = T_S(f) - T_S(1/f), \tag{2.13}$$

valid for f without any assumption on $|f(w)|$ (see [**SN**, (7), p. 77]).

Now, for f with $|f(w)| = \infty$, we first note that (2.7′) for $1/f$ yields

$$N_S(1/f) = N(1/f) = T(1/f) - m(1/f) = T(f) - m(1/f). \tag{2.14}$$

On the other hand, (2.13) for the present f, together with $\log|f| = \log^+|f| - \log^+|1/f|$, yields

$$\log|c_\lambda| = -(2\pi)^{-1}\int_{\partial D} \log|f(z)|\, dg_D^*(z, w) + N_S(f) - N_S(1/f),$$

so that, by (2.14),

$$\begin{aligned}&= -(2\pi)^{-1}\int_{\partial D}\log|f(z)|\,dg_D^*(z,w) + N_S(f) - T(f) + m(1/f)\\ &= N_S(f) - T(f) + m(f),\end{aligned}$$

which completes the proof of (2.8′).

PROOF OF COROLLARY 2.5. We again use (2.10). It suffices to prove

$$|T(f) - T_S(f)| \le \tfrac{1}{2}\log 2 + |\log|c_\lambda||, \tag{2.15}$$

which yields (2.9) on letting $D \uparrow R$. In the case $|f(w)| \neq \infty$, it follows from $N_S(f) = N(f)$, the inequalities (2.5) and (2.6) for $x = |f(w)|$, and (2.7′), that

$$|T(f) - T_S(f) + \log^+|f(w)|| \le \tfrac{1}{2}\log 2. \tag{2.16}$$

As is observed in (1.2), $|f(w)| \le |c_\lambda|$, so that (2.15) now follows from (2.16). In the case $|f(w)| = \infty$, we apply (2.16) to $1/f$ instead of f, to obtain

$$|T(f) - T_S(1/f)| = |T(1/f) - T_S(1/f)| \le \tfrac{1}{2}\log 2.$$

It then follows from (2.13) that

$$|T(f) - T_S(f) + \log|c_\lambda|| \le \tfrac{1}{2}\log 2,$$

whence (2.15).

3. An admissible and the canonical decompositions. For $f \in M_e(R)$ we call

$$|f| = |f_1/f_2| \quad \text{on } R \tag{3.1}$$

an admissible decomposition if $f_k \in M_e$ is pole-free ($k = 1, 2$) and furthermore if both have no common zero, or,

$$|f_1|^2 + |f_2|^2 > 0 \quad \text{on } R. \tag{3.2}$$

This decomposition is not unique.

LEMMA 3.0. *Each $f \in M_e(R)$ has an admissible decomposition* (3.1), *where one of f_1 and f_2 is a member of $M(R)$.*

PROOF. If f is pole-free, then set $f_1 \equiv f$ and $f_2 \equiv 1$. If f has one pole at least, then by H. Florack's theorem [**F**, Satz II, pp. 3–4], there exists a pole-free $f_2 \in M$ such that $z \in R$ is a zero of order k of f_2 if and only if z is a pole of order k of f. We now obtain (3.1) on setting $f_1 = ff_2$. We note that if f is zero-free, then set $f_1 \equiv 1$ and $f_2 = 1/f$. If f has one zero at least, then there exists a pole-free $h_1 \in M$ such that $z \in R$ is a zero of order k of h_1 if and only if z is a zero of order k of f. We then obtain another admissible decomposition $|f| = |h_1/h_2|$ for $h_2 = h_1/f$.

By a harmonic majorant of a subharmonic function v on R we mean a function u harmonic and $v \le u$ on R. If v has a harmonic majorant on R, then it has the least harmonic majorant $v^\wedge = v_R^\wedge$ in the sense that $v^\wedge$ is a harmonic majorant of v and $v^\wedge \le u$ for each harmonic majorant u of v on R.

For an admissible decomposition (3.1) for $f \in M_e(R)$ we denote

$$\varphi = \tfrac{1}{2}\log(|f_1|^2 + |f_2|^2) \quad \text{on } R. \tag{3.3}$$

This is a finite-valued subharmonic function on R because (3.2) holds and

$$\Delta\varphi(z)\,dx\,dy = 2f^{\#}(z)^2\,dx\,dy \quad \text{on } R. \tag{3.4}$$

Therefore, for $\varphi_1 = \frac{1}{2}\log(|F_1|^2+|F_2|^2)$ for another admissible decomposition $|f| = |F_1/F_2|$, $\varphi - \varphi_1$ is a harmonic function on R; thus φ is unique modulo harmonic functions on R.

THEOREM 3.1. *Suppose that $f \in M_e(R)$. Then $f \in \mathrm{BC}_e(R)$ if and only if φ has a harmonic majorant on R. If $f \in \mathrm{BC}_e(R)$, then*

$$T(w,f) = \varphi_R^{\wedge}(w) - \varphi(w) \quad \textit{for each } w \in R, \tag{3.5}$$

so that the function $\varphi_R^{\wedge} - \varphi$ is independent of a choice of decomposition (3.1).

REMARKS. (a) It follows from Corollary 2.5, together with (3.5) that if $f \in \mathrm{BC}_e(R)$, then $T_S(w,f) < \infty$ for every $w \in R$. Consequently, BC_e does not depend on a choice of $w \in R$.

(b) If $f \in \mathrm{BC}_e(R)$, then (3.5) shows that $T(z,f)$ is a C^∞ function of real variables $x, y, z = x+iy$, together with

$$\Delta T(z,f)\,dx\,dy = -2f^{\#}(z)^2\,dx\,dy$$

by (3.4). This answers the question raised in [$\mathbf{Y_1}$, Remark (b), p. 361]. Note that [$\mathbf{Y_1}$, Theorem 5.1, p. 360] is a specified case of Theorem 3.1 for pole-free $f \in M(\Delta)$.

As an immediate consequence we obtain

COROLLARY 3.2. *Suppose that $f \in M_e(R)$. Then $f \in \mathrm{UBC}_e(R)$ if and only if φ has a harmonic majorant on R and $\varphi_R^{\wedge} - \varphi$ is bounded there. Further, $f \in \mathrm{UBC}_{e0}(R)$ if and only if φ has a harmonic majorant on R and*

$$\lim_{w\to\partial R}[\varphi_R^{\wedge}(w) - \varphi(w)] = 0.$$

The inclusion formula $\mathrm{UBC}_{e0}(R) \subset \mathrm{UBC}_e(R)$ is now obvious. The proof of [$\mathbf{Y_1}$, Lemma 2.1, p. 352] is thus facilitated. In fact, this is an immediate consequence of continuity of $T(w,f)$ observed in Remark (b) above.

For clarity we propose

COROLLARY 3.3. *For $f \in M_e(R)$ to be in $\mathrm{UBC}_e(R)$ it is necessary and sufficient that* $\limsup_{w\to\partial R} T(w,f) < \infty$.

More precisely, there exists a compact set $K \subset R$ such that $T(w,f)$ is bounded in $R\backslash K$.

As the third corollary of Theorem 3.1 we claim that a compact set of capacity zero on R is removable for functions of UBC. A set $E \subset R$ is said to be of capacity zero if the intersection of E with the parameter disk at each point of E is of capacity zero [**T**, p. 55] in **C**. If E is closed further, then E is totally disconnected, so that $R\backslash E$ is connected. The following is never obvious and needs a proof.

COROLLARY 3.4. *Let E be a compact set of capacity zero on R and suppose that $f \in \mathrm{UBC}(R\backslash E)$. Then there exists $F \in \mathrm{UBC}(R)$ such that $F \equiv f$ in $R\backslash E$.*

REMARK. If $f \in \mathrm{UBC}(R\backslash E)$ is pole-free, then, is F pole-free? The answer is "no". Just let $f(z) = 1/z$, $R = \Delta$, and $E = \{0\}$.

PROOF OF THEOREM 3.1. Let w be the point in the definition of $\mathrm{BC}_e(R)$. We use the device (2.10). Since

$$\varphi_D^\wedge(w) = -(2\pi)^{-1}\int_{\partial D}\varphi(z)\,dg_D^*(z,w),$$

it follows from (2.2) and (3.4), together with the Green formula, that

$$2\pi T(f) = \int\int_D (\Delta\varphi(z))g_D(z,w)\,dx\,dy = -\int_{\partial D}\varphi(z)\,dg_D^*(z,w) - 2\pi\varphi(w),$$

so that

$$T(f) = \varphi_D^\wedge(w) - \varphi(w). \tag{3.6}$$

Thus, $\lim_{D\uparrow R}\varphi_D^\wedge(w) < \infty$ if and only if $T_S(w,f) < \infty$, by (2.9). Now, if $\varphi_R^\wedge$ exists, then (3.5) follows because the same calculation shows that (3.6) is true for each $w \in R$.

PROOF OF COROLLARY 3.4. We may assume that f is nonconstant. First of all, $\mathrm{BC}(R)$ coincides with Parreau's class (AM_0) in R [**P**, Definition 1, p. 180]; compare [**P**, Théorème 19, a), p. 181] with [**SN**, Theorem 3B, (33), p. 83]. By the case $\alpha = 0$ in [**P**, Théorème 20, p. 182] there exists $F \in \mathrm{BC}(R)$ ($F = f_1$ in Parreau's proof) with $F \equiv f$ in $R\backslash E$. An admissible decomposition $F = f_1/f_2$ of F in R yields that of f in $R\backslash E$ also. Thus, for φ of (3.3) for F,

$$\varphi_R^\wedge - \varphi = (\varphi_R^\wedge - \varphi_{R\backslash E}^\wedge) + (\varphi_{R\backslash E}^\wedge - \varphi) \quad \text{in } R\backslash E.$$

Since $\varphi_R^\wedge - \varphi$ is continuous on R and since the second term in the right-hand side is bounded in $R\backslash E$, it suffices to show that $\varphi_R^\wedge - \varphi_{R\backslash E}^\wedge$ is bounded in $R\backslash E$. To prove this we remark that there exists a sequence of domains $D = D_E \cup K_E$ in R such that (a) $K_E \supset E$ is compact, (b) ∂K_E consists of piecewise analytic Jordan curves, (c) $D_E = D\backslash K_E \uparrow R\backslash E$ as $D \uparrow R$. Then,

$$\varphi_D^\wedge - \varphi_{D_E}^\wedge = \begin{cases} 0 & \text{on } \partial D; \\ \varphi_D^\wedge - \varphi & \text{on } \partial K_E, \end{cases}$$

so that

$$\varphi_D^\wedge - \varphi_{D_E}^\wedge \le \max_{K_E}(\varphi_R^\wedge - \varphi) \quad \text{in } D_E.$$

Letting $D \uparrow R$ we now obtain

$$\varphi_R^\wedge - \varphi_{R\backslash E}^\wedge \le \max_{K}(\varphi_R^\wedge - \varphi) \quad \text{in } R\backslash E,$$

where K is a compact set whose interior contains E.

REMARKS. We pose here for some references to $\mathrm{BMOA}_e(R)$ and $\mathrm{VMOA}_e(R)$.

(a) For $f \in M_e(R)$ pole-free, we have

$$\Delta(|f(z)|^2)\,dx\,dy = 4|f'(z)|^2\,dx\,dy, \qquad z \in R.$$

By the Green formula it is now easy to prove the following.

(ai) *f is of Hardy class $H_e^2(R)$, that is, $|f|^2$ has a harmonic majorant on R if and only if for a point $w \in R$,*

$$\int\int_R |f'(z)|^2 g(z,w)\,dx\,dy < \infty.$$

(aii) *$f \in \mathrm{BMOA}_e(R)$ if and only if $f \in H_e^2(R)$ and $(|f|^2)^\wedge_R - |f|^2$ is bounded there, while, $f \in \mathrm{VMOA}_e(R)$ if and only if $f \in H_e^2(R)$ and*

$$\lim_{w \to \partial R} [(|f|^2)^\wedge_R(w) - |f(w)|^2] = 0.$$

The BMOA_e version of Corollary 3.3 is obvious.

(b) The analogue of Corollary 3.4 is valid.

If $E \subset R$ is compact and of capacity zero, and if $f \in \mathrm{BMOA}(R \backslash E)$, then there exists $F \in \mathrm{BMOA}(R)$ with $F \equiv f$ in $R \backslash E$.

The proof is essentially the same as that of Corollary 3.4. The existence of a single-valued $F \in H_e^2(R)$ with $F \equiv f$ in $R \backslash E$ follows from $\alpha = 2$ in [**P**, Théorème 20, p. 182].

Returning to $f \in \mathrm{BC}_e(R)$, $f \not\equiv 0$, we now consider $m_S(w,f)$ and $N(w,f)$; see $X(w,f)$ before Theorem 2.4. Then $m_S(w,f)$ is a nonnegative harmonic function of w on R and $N(w,f) = \sum g(w,b)$, the summation being extended over all the poles b of f in R, each counted with its order, is harmonic in R minus the poles of f. Set

$$F(w,f) = \exp[-m_S(w,f) - im_S^*(w,f)],$$
$$\beta(w,f) = \exp[-N(w,f) - iN^*(w,f)]$$

for $w \in R$; both are bounded by one, and hence are in $\mathrm{UBC}_e(R)$ by the remark after Theorem 2.4. In particular, β is a Blaschke product if $R = \Delta$.

The canonical decomposition, referred to in the title of the present section, is obtained by Sario [**SN**, Theorem 3B, p. 83 and Corollary 7, p. 86], which, for the clarity, we propose in the form of a lemma.

LEMMA 3.5. *For $f \in \mathrm{BC}_e(R)$, $f \not\equiv 0$, we have*

$$|f(w)| = \left| \frac{\beta(w,1/f)F(w,1/f)}{\beta(w,f)F(w,f)} \right| \quad \text{at each } w \in R. \tag{3.7}$$

Note that $1/f \in \mathrm{BC}_e(R)$. The following is an extension of [$\mathbf{Y_1}$, Corollary 4.1, p. 359].

THEOREM 3.6. *If $f \in \mathrm{UBC}_e(R)$, $f \not\equiv 0$, then $F(w,1/f)/F(w,f) \in \mathrm{UBC}_e(R)$.*

It is known that the converse is false for $R = \Delta$; see [$\mathbf{Y_1}$, p. 359]. Further it is known that $\mathrm{UBC}(\Delta)$ is not closed for multiplication and summation [$\mathbf{Y_1}$, Theorem 4.2, p. 359].

PROOF OF THEOREM 3.6. To prove first that

$$\beta(w,f)f(w) \in \mathrm{UBC}_e(R), \tag{3.8}$$

we set

$$\beta(w) = \beta(w,f), \quad h(w) = \beta(w)f(w), \qquad w \in R.$$

Then $\beta \in \mathrm{BC}_e$ and $h \in \mathrm{BC}_e$ is pole-free. First we consider those $w \in R$ for which $|f(w)| \neq \infty$. It follows from (2.7) of Theorem 2.4 that

$$\begin{aligned} m(w,f) &= T(w,f) + \log|\beta(w)| + \tfrac{1}{2}\log(1 + |f(w)|^2) \\ &= T(w,f) + \tfrac{1}{2}\log(|\beta(w)|^2 + |h(w)|^2). \end{aligned}$$

Since

$$m(w,h) \le m(w,\beta) + m(w,f), \quad \text{and}$$
$$A \equiv (|\beta(w)|^2 + |h(w)|^2)/(1+|h(w)|^2) \le 1,$$

it again follows from (2.7) that

$$\begin{aligned} T(w,h) &= m(w,h) - \tfrac{1}{2}\log(1+|h(w)|^2) \\ &\le m(w,\beta) + m(w,f) - \tfrac{1}{2}\log(1+|h(w)|^2) \\ &\le \tfrac{1}{2}\log 2 + T(w,f) + \tfrac{1}{2}\log A \\ &\le \tfrac{1}{2}\log 2 + T(w,f). \end{aligned}$$

Therefore,

$$T(w,h) \le \tfrac{1}{2}\log 2 + \sup_{\varsigma\in R} T(\varsigma, f) \tag{3.9}$$

for $w \in R$ with $|f(w)| \ne \infty$. Since $T(w,h)$ is continuous on R, and since the poles of f are isolated, we observe that $\sup T(w,h)$ for $w \in R$ does not exceed the right-hand side of (3.9), which completes the proof of (3.8).

Now, $1/h \in \mathrm{UBC}_e(R)$ by (3.8). Apply (3.8) to $1/h$ instead of f. Since $|\beta(w,1/f)| = |\beta(w,1/h)|$, we conclude that $F(w,f)/F(w,1/f) \in \mathrm{UBC}_e(R)$ because

$$|F(w,f)/F(w,1/f)| = |\beta(w,1/h)(1/h(w))|.$$

Thus, $F(w,1/f)/F(w,f) \in \mathrm{UBC}_e(R)$.

4. Counting function. For nonconstant $f \in M(R)$, $w \in D$, and $z \in \mathbf{C}^*$ we define

$$N_S(D,w,z,f) = N_S(D,w,1/(f-z)) \quad \text{and} \quad N(D,w,z,f) = N(D,w,1/(f-z)).$$

Then $N(D,w,z,f) = \sum g_D(w,\varsigma)$, where the sum is extended over all z-points ς of f in D, each counted with its order. Further, set

$$N_S(w,z,f) = \lim_{D\uparrow R} N_S(D,w,z,f) \quad \text{and}$$
$$N(w,z,f) = \lim_{D\uparrow R} N(D,w,z,f) = \sum g(w,\varsigma),$$

where the sum is extended over all z-points ς of f in R, each counted with its order. Note that $N(w,z,f) = G(w,z,f)$ in [**SN**, p. 90]. Apparently, $N(w,z,f) = \infty$ if $f(w) = z$. Heins [$\mathbf{H_2}$] called $f \in M(R)$ a Lindelöfian map from R into $\mathbf{C}^*$, or f is Lindelöfian and meromorphic, if $N(w,z,f) < \infty$ for each pair $z \in \mathbf{C}^*$, $w \in R$, with $f(w) \ne z$. It is known that $f \in \mathrm{BC}(R)$ if and only if f is Lindelöfian and meromorphic [**SN**, Theorem 6E, p. 92].

As usual we consider the chordal distance

$$\chi(a,b) = |a-b|/[(1+|a|^2)(1+|b|^2)]^{1/2}, \qquad a,b \in \mathbf{C}^*,$$

with the obvious change for $a = \infty$ or $b = \infty$. The length of a curve on $\mathbf{C}^*$ measured by $d\chi(\varsigma) = |d\varsigma|/(1+|\varsigma|^2)$, $\varsigma \in \mathbf{C}^*$, is the same as its Euclidean length, considered as a curve on the Riemann sphere in the Euclidean space. Set

$$\Gamma(a,\rho) = \{z \in \mathbf{C}^*; \chi(z,a) = \rho\}, \qquad a \in \mathbf{C}^*,\ 0 < \rho < 1,$$

and set, for $w \in R$, $0 < \rho < 1$, and $f \in M(R)$,

$$C(w,\rho,f) = \sup_{z\in\Gamma(f(w),\rho)} N(w,z,f).$$

We begin with criteria for $f \in M(R)$ to be of $\mathrm{BC}(R)$ in terms of C.

THEOREM 4.1. *For $f \in M(R)$ the following are mutually equivalent.*
(I) $f \in \mathrm{BC}(R)$.
(II) *There exists a pair w, ρ, with $w \in R$, $0 < \rho < 1$, such that $C(w, \rho, f) < \infty$.*
(III) *For each pair w, ρ with $w \in R$, $0 < \rho < 1$, we have $C(w, \rho, f) < \infty$.*

A weaker condition than (II) implies (I). Actually, if $N(w, z, f) < \infty$ for a set of $z \in \mathbf{C}^*$ of positive capacity (see §5 for the definition of "capacity" of a set on $\mathbf{C}^*$), then $f \in \mathrm{BC}(R)$ by [**P**, Théorème 22, p. 190]; the set $\Gamma(f(w), \rho)$ is of positive capacity. A reason of proposing (II) is to compare it with (V) in

THEOREM 4.2. *For $f \in M(R)$ the following are mutually equivalent.*
(IV) $f \in \mathrm{UBC}(R)$.
(V) *There exists ρ, $0 < \rho < 1$, such that $\sup_{w \in R} C(w, \rho, f) < \infty$.*
(VI) *For each ρ, $0 < \rho < 1$, $\sup_{w \in R} C(w, \rho, f) < \infty$.*

Postponing the proofs of Theorems 4.1 and 4.2 we first note that the length of $\Gamma(a, \rho)$ is independent of $a \in \mathbf{C}^*$ and is

$$l(\rho) = 2\pi\rho(1 - \rho^2)^{1/2}, \qquad 0 < \rho < 1; \tag{4.1}$$

for example, the length of the equator $\Gamma(0, 1/\sqrt{2}) = \Gamma(\infty, 1/\sqrt{2})$ is $l(1/\sqrt{2}) = \pi$. Set

$$c(\rho) = \max[\rho^{-1}(1 - \rho^2)^{1/2}, \rho(1 - \rho^2)^{-1/2}], \qquad 0 < \rho < 1. \tag{4.2}$$

THEOREM 4.3. *The following estimates hold for $f \in M(R)$, $0 < \rho < 1$, and $w \in R$.*
(VII) $c(\rho)^2 T(w, f) \geq l(\rho)^{-1} \int_{\Gamma(f(w),\rho)} N(w, z, f)\, d\chi(z) - (1/2)\log 2$.
(VIII) $c(\rho)^{-2} T(w, f) \leq l(\rho)^{-1} \int_{\Gamma(f(w),\rho)} N(w, z, f)\, d\chi(z) + (1/2)\log 2$.

The estimates (VII) and (VIII) are motivated by the celebrated Cartan formula [**SN**, (56), p. 89]:

$$T_S(D, w, f) = \log^+ |f(w)| + (2\pi)^{-1} \int_0^{2\pi} N_S(D, w, e^{it}, f)\, dt$$

for $f \in M(R)$, provided that $f(w) \neq \infty$. A merit of (VII) and (VIII) might be that the right-hand sides have no term like $\log^+ |f(w)|$; also no assumption on the value $f(w)$ is posed.

With the aid of (VIII) of Theorem 4.3, (II)$\Rightarrow$(I) of Theorem 4.1 and (V)$\Rightarrow$(IV) of Theorem 4.2 are immediately obtained.

PROOF OF THEOREM 4.3. We may suppose that f is nonconstant. We shall use the following abbreviation like (2.10):

$$N_f(z) = N(D, w, z, f). \tag{4.3}$$

Set

$$\delta = \rho(1 - \rho^2)^{-1/2}, \tag{4.4}$$

so that $l(\rho) = 2\pi\delta(1 + \delta^2)^{-1}$ by (4.1) and $c(\rho) = \max(\delta, \delta^{-1})$ by (4.2). We consider $h \in M(R)$ defined by

$$h = (f - f(w))/(1 + \overline{f(w)} f) \quad (= 1/f \text{ if } f(w) = \infty). \tag{4.5}$$

First we observe that

$$(2\pi)^{-1}\int_0^{2\pi} N_h(\delta e^{it})\,dt = l(\rho)^{-1}\int_{\Gamma(f(w),\rho)} N_f(z)\,d\chi(z). \tag{4.6}$$

For the proof we note that the Möbius transformation

$$\varsigma = (z-f(w))/(1+\overline{f(w)}z)$$

maps the circle $\Gamma(f(w),\rho)$ one-to-one onto the circle $\{|\varsigma|=\delta\}$ and further,

$$|d\varsigma| = (1+\delta^2)\,d\chi(z) \quad \text{for } z\in\Gamma(f(w),\rho).$$

Now, the left-hand side of (4.6), denoted by A, is expressed as

$$A = (2\pi\delta)^{-1}\int_{|\varsigma|=\delta} N_h(\varsigma)|d\varsigma|,$$

which yields (4.6).

Next we claim that

$$|T(h/\delta)-A| \le \tfrac{1}{2}\log 2. \tag{4.7}$$

Since $h(w)=0$, it follows from (2.16) for h/δ instead of f that

$$|T(h/\delta)-T_S(h/\delta)| \le \tfrac{1}{2}\log 2. \tag{4.8}$$

We now remember the identity

$$(2\pi)^{-1}\int_0^{2\pi}\log|b-\delta e^{it}|\,dt = \log\delta + \log^+|b/\delta| \tag{4.9}$$

for each $b\in\mathbf{C}$; see [**N**, p. 178] for the calculation. Jensen's formula [**SN**, (2), p. 76], applied to $h-\delta e^{it}$ with $h(w)-\delta e^{it}=\delta e^{it}$, yields

$$\log\delta = -(2\pi)^{-1}\int_{\partial D}\log|h(z)-\delta e^{it}|\,dg_D^*(z,w) + N(h) - N_h(\delta e^{it}) \tag{4.10}$$

for each $t\in[0,2\pi)$. Calculating the integral means of both sides of (4.10) with respect to dt in $[0,2\pi)$, and observing (4.9), together with $N(h)=N(h/\delta)$, we now obtain

$$\log\delta = \log\delta + m_S(h/\delta) + N(h/\delta) - A,$$

whence $T_S(h/\delta)=A$. The estimate (4.7) follows then from (4.8).

Finally for $c=c(\rho)$, some computations yield $c^{-1}h^{\#} \le (h/\delta)^{\#} \le ch^{\#}$. Since $T(h)=T(f)$, it follows that $c^{-2}T(f)\le T(h/\delta)\le c^2T(f)$.

The estimates (VII) and (VIII) for $R=D$ now follow from the above, together with (4.6) and (4.7). Letting $D\uparrow R$ we arrive at the requested conclusions.

For the remaining proofs of Theorems 4.1 and 4.2 we prove

LEMMA 4.4. *Let $f\in M(R)$, $0<\rho<1$, $w\in R$, and $1<q<\infty$. Then*
(IX) $C(w,\rho,f)\le qc(\rho)^2T(w,f)+(q/2)\log 2+\log[(q+1)/(q-1)]$.

On setting $q=2$, say, we see that (I)⇒(III) of Theorem 4.1 and (IV)⇒(VI) of Theorem 4.2 follow from (IX). Since (III)⇒(II) and (VI)⇒(V) are trivial, this completes the proofs of Theorems 4.1 and 4.2.

PROOF OF LEMMA 4.4. Fix D, $w \in D$. As is noted by O. Lehto [**L**] the function $N_f(z)$ (see (4.3)) of $z \in \mathbf{C}^*$ is subharmonic in $\mathbf{C}^* \setminus \{f(w)\}$ and N_f has the logarithmic singularity at $f(w)$ in the sense that

$$N_f(z) + \log|z - f(w)| \qquad (N_f(z) - \log|z| \text{ if } f(w) = \infty)$$

is subharmonic in $\mathbf{C}^* \setminus \{-1/\overline{f(w)}\}$. Thus,

$$A' \equiv \sup_{\chi(z,f(w)) \geq \rho} N_f(z) = \sup_{z \in \Gamma(f(w),\rho)} N_f(z). \tag{4.11}$$

Our task is therefore to show that

$$A' \leq qc(\rho)^2 T(f) + (q/2)\log 2 + \log[(q+1)/(q-1)]. \tag{4.12}$$

Since $T(w, f) \geq T(f)$ and since $N_f \uparrow N(w, f)$ as $D \uparrow R$, (IX) follows from (4.12).

We consider again h of (4.5) for which $h(w) = 0$. Then $u(z) \equiv N_h(1/z) = N_{1/h}(z)$ is subharmonic in $\mathbf{C}$ and $u(z) - \log|z|$ is subharmonic in $\mathbf{C}^* \setminus \{0\}$. By (4.6) and (VII) of Theorem 4.3 we obtain

$$\begin{aligned} &(2\pi)^{-1} \int_0^{2\pi} u(\delta^{-1} e^{it})\, dt = (2\pi)^{-1} \int_0^{2\pi} u(\delta^{-1} e^{-it})\, dt \\ &\quad = (2\pi)^{-1} \int_0^{2\pi} N_h(\delta e^{it})\, dt \leq c(\rho)^2 T(f) + \tfrac{1}{2}\log 2 \equiv \alpha, \end{aligned} \tag{4.13}$$

where δ is defined in (4.4).

Our aim is to prove that

$$|z| \leq \delta^{-1} \Rightarrow u(z) \leq q\alpha + \log\lambda, \qquad \lambda = (q+1)/(q-1). \tag{4.14}$$

Then, the identities

$$A' = \sup_{\delta \leq |z| \leq \infty} N_h(z) = \sup_{|z| \leq \delta^{-1}} u(z),$$

together with (4.14), show (4.12).

Let $u_\delta^\wedge$ be the least harmonic majorant of u in the disk $|z| < \delta^{-1}$, which can be expressed by the Poisson formula,

$$u_\delta^\wedge(z) = (2\pi)^{-1} \int_0^{2\pi} \frac{\delta^{-2} - |z|^2}{|\delta^{-1} e^{it} - z|^2} u(\delta^{-1} e^{it})\, dt, \qquad |z| < \delta^{-1}.$$

Set $\Lambda = \lambda^{-1}\delta^{-1}$. Then, on the circle $|z| = \Lambda$ we have the estimate

$$u_\delta^\wedge(z) \leq \frac{\Lambda^{-1} + \delta}{\Lambda^{-1} - \delta} (2\pi)^{-1} \int_0^{2\pi} u(\delta^{-1} e^{it})\, dt,$$

whence, by (4.13), we obtain

$$\max_{|z|=\Lambda} u_\delta^\wedge(z) \leq q\alpha. \tag{4.15}$$

The maximum principle now yields

$$|z| \leq \Lambda \Rightarrow u(z) \leq u_\delta^\wedge(z) \leq q\alpha. \tag{4.16}$$

On the other hand, by the maximum principle for $u(z) - \log|z|$, with (4.15), we have

$$\begin{aligned} \Lambda < |z| < \infty \Rightarrow u(z) - \log|z| &\leq \sup_{|\varsigma|=\Lambda} (u(\varsigma) - \log|\varsigma|) \\ &\leq q\alpha - \log\Lambda. \end{aligned}$$

Therefore,

$$\Lambda < |z| \leq \delta^{-1} \Rightarrow u(z) \leq q\alpha - \log \Lambda + \log \delta^{-1} \\ = q\alpha + \log \lambda. \tag{4.17}$$

The estimate (4.14) now follows from (4.16) and (4.17).

5. Uniformization. For a projection map π: $\Delta \to R$, there exists the group $\mathcal{G}$ of conformal homeomorphisms from Δ onto Δ, called the covering transformation group, such that $\pi \circ \gamma = \pi$ for each $\gamma \in \mathcal{G}$. Thus, for $f \in M(R)$, the function $f \circ \pi$ is $\mathcal{G}$-automorphic in the sense that $(f \circ \pi) \circ \gamma = f \circ \pi$ for all $\gamma \in \mathcal{G}$. We begin with

THEOREM 5.1. *For each $f \in M(R)$ and each $\delta \in \Delta$, we have*

$$T(R, \pi(\delta), f) = T(\Delta, \delta, f \circ \pi). \tag{5.1}$$

Set

$$\|f\|_{\mathrm{UBC}(R)} = \sup_{w \in R} T(R, w, f)$$

for $f \in M(R)$. Besides the obvious consequence of Theorem 5.1 that $f \in \mathrm{BC}(R)$ if and only if $f \circ \pi \in \mathrm{BC}(\Delta)$, we have

COROLLARY 5.2. *For $f \in M(R)$, the equality*

$$\|f\|_{\mathrm{UBC}(R)} = \|f \circ \pi\|_{\mathrm{UBC}(\Delta)} \tag{5.2}$$

holds. Thus, $f \in \mathrm{UBC}(R)$ if and only if $f \circ \pi \in \mathrm{UBC}(\Delta)$.

A subset A of $\mathbf{C}^*$ is said to be of positive capacity if A contains a closed subset of $\mathbf{C}^*$ of positive elliptic capacity [**T**, p. 90], or equivalently, $A \backslash \{\infty\}$ contains a bounded (in $\mathbf{C}$) and closed set of positive capacity [**T**, p. 55].

COROLLARY 5.3. *Suppose that for $f \in M(R)$, the exceptional set $\mathbf{C}^* \backslash f(R)$ is of positive capacity. Then $f \in \mathrm{UBC}(R)$.*

Applying the known theorem [$\mathbf{Y_2}$, Theorem 1] to $f \circ \pi$ we observe that $f \circ \pi \in \mathrm{UBC}(\Delta)$, whence $f \in \mathrm{UBC}(R)$.

COROLLARY 5.4. *Suppose that a subdomain G of $\mathbf{C}^*$ has the projection π: $\Delta \to G$. Then, $\pi \in \mathrm{UBC}(\Delta)$ if and only if $\mathbf{C}^* \backslash G$ is of positive capacity.*

The "only if" part is a consequence of [**N**, Satz 1, p. 213] because $\pi \in \mathrm{BC}(\Delta)$. The "if" part is a consequence of Corollary 5.3.

PROOF OF THEOREM 5.1. We note that, by the construction of f_1 and f_2 in Lemma 3.0, both are members of M. Thus,

$$f \circ \pi = F_1/F_2, \quad F_k = f_k \circ \pi, \qquad k = 1, 2,$$

yields a canonical decomposition of $f \circ \pi$ in Δ. For φ of (3.3) we set

$$\Phi = \varphi \circ \pi = \tfrac{1}{2} \log(|F_1|^2 + |F_2|^2). \tag{5.3}$$

We shall show that

$$\Phi_\Delta^\wedge - \Phi = (\varphi_R^\wedge - \varphi) \circ \pi; \tag{5.4}$$

that $\Phi_\Delta^\wedge$ exists if and only if $\varphi_R^\wedge$ exists is clear in the following context. The identity (5.1) now follows from (3.5) with (5.4).

For the proof of (5.4) it suffices to observe that $\Phi_{\Delta}^{\wedge} = \Phi_R^{\wedge} \circ \pi$. Since $\varphi \leq \varphi_R^{\wedge}$, it follows that $\Phi \leq \varphi_R^{\wedge} \circ \pi$, whence $\Phi_{\Delta}^{\wedge} \leq \varphi_R^{\wedge} \circ \pi$. To prove the inverse we remark that $\Phi_{\Delta}^{\wedge}$ is $\mathcal{G}$-automorphic by (5.3). Then the function ψ on R well-defined by

$$\psi(z) = \Phi_{\Delta}^{\wedge}(\varsigma) \quad \text{for } z \in R \text{ with } \pi(\varsigma) = z,\ \varsigma \in \Delta,$$

is harmonic on R. Therefore, it follows from $\Phi_{\Delta}^{\wedge} \geq \Phi = \varphi \circ \pi$ that $\psi \geq \varphi$, whence $\psi \geq \varphi_R^{\wedge}$, so that $\Phi_{\Delta}^{\wedge} \geq \varphi_R^{\wedge} \circ \pi$ in Δ.

We now extend [$\mathbf{Y_2}$, Theorem 3]. Let S be a hyperbolic Riemann surface with the Green functions $g_S(z,w)$. An analytic mapping h: $R \to S$ is said to be of type Bl (see [$\mathbf{H_1}$]) if for each $w \in S$, the function $g_S(h(z),w) - \sum g_R(z,\varsigma)$ of $z \in R$ is singular, that is, it does not dominate any strictly positive and bounded harmonic function on R, where the summation is taken over all roots $\varsigma \in R$ of the equation $h = w$, counted with their multiplicities.

THEOREM 5.5. *The following hold for $f \in M(S)$.*

(I) *If $f \in \mathrm{UBC}(S)$, then for each analytic map h: $R \to S$, we have*

$$\|f \circ h\|_{\mathrm{UBC}(R)} \leq \|f\|_{\mathrm{UBC}(S)}, \tag{5.5}$$

so that $f \circ h \in \mathrm{UBC}(R)$.

(II) *If h: $R \to S$ is of type Bl, and if $f \circ h \in \mathrm{UBC}(R)$, then*

$$\|f \circ h\|_{\mathrm{UBC}(R)} = \|f\|_{\mathrm{UBC}(S)}, \tag{5.6}$$

so that $f \in \mathrm{UBC}(S)$.

PROOF. Let π_R: $\Delta \to R$ and π_S: $\Delta \to S$ be the projection maps. Then a branch of $\pi_S^{-1} \circ h \circ \pi_R$ is single-valued in Δ, which we denote simply, $H = \pi_S^{-1} \circ h \circ \pi_R$. Set $F = f \circ \pi_S$, so that $f \circ h \circ \pi_R = F \circ H$. If $f \in \mathrm{UBC}(S)$, then $F \in \mathrm{UBC}(\Delta)$ with

$$\|F\|_{\mathrm{UBC}(\Delta)} = \|f\|_{\mathrm{UBC}(S)} \tag{5.7}$$

by Corollary 5.2. On the other hand, it follows from [$\mathbf{Y_2}$, Theorem 3, (I)] that

$$\|F \circ H\|_{\mathrm{UBC}(\Delta)} \leq \|F\|_{\mathrm{UBC}(\Delta)}. \tag{5.8}$$

Corollary 5.2 again shows that

$$\|F \circ H\|_{\mathrm{UBC}(\Delta)} = \|f \circ h\|_{\mathrm{UBC}(R)}, \tag{5.9}$$

so that (5.5) follows from (5.7), (5.8), and (5.9).

Now, if h is of type Bl, then H is of type Bl by [$\mathbf{H_1}$, Corollary, p. 472] because S is hyperbolic. Thus, if $f \circ h \in \mathrm{UBC}(R)$, then by (5.9), $F \circ H \in \mathrm{UBC}(\Delta)$, so that, by [$\mathbf{Y_2}$, Theorem 3, (II)] we have $\|F \circ H\|_{\mathrm{UBC}(\Delta)} = \|F\|_{\mathrm{UBC}(\Delta)}$, which, together with (5.7) and (5.9), yields (5.6).

REMARKS. (a) Is it true that $f \in \mathrm{UBC}_0(R)$ if and only if $f \circ \pi \in \mathrm{UBC}_0(\Delta)$? This is open, as far as the author knows. The similar question for VMOA is also open.

(b) Let $N(R)$ denote the family of normal meromorphic functions on R; namely, $f \in N(R)$ if and only if there is a constant $c > 0$ such that $f^{\#}(z) \leq c\sigma_R(z)$, $z \in R$, where $\sigma_R(z)|dz|$ is the hyperbolic metric on R; note that $\sigma_\Delta(z) = (1 - |z|^2)^{-1}$. It is easy to observe that $f \in N(R)$ if and only if $f \circ \pi \in N(\Delta)$. It then follows from the known inclusion formula $\mathrm{UBC}(\Delta) \subset N(\Delta)$ [$\mathbf{Y_1}$, Theorem 3.1] that $\mathrm{UBC}(R) \subset N(R)$.

(c) By the remarks after the proof of Corollary 3.4, we can prove the analogue of (5.1) for pole-free $f \in M(R)$ by the similar method. Namely,

$$\int\int_R |f'(z)|^2 g(z, \pi(\delta))\,dx\,dy = \int\int_\Delta |(f\circ\pi)'(z)|^2 g_\Delta(z,\delta)\,dx\,dy.$$

Metzger [**M**] obtained this by analyzing the Green functions.

(d) Define the BMOA norm for pole-free $f \in M(R)$ by

$$\|f\|_{\mathrm{BMOA}(R)} = \sup_{w\in R}\int\int_R |f'(z)|^2 g(z,w)\,dx\,dy.$$

The BMOA analogue of (5.2) is then $\|f\|_{\mathrm{BMOA}(R)} = \|f\circ\pi\|_{\mathrm{BMOA}(\Delta)}$.

(e) A BMOA analogue of Theorem 5.5 is valid. Instead of [$\mathbf{Y_2}$, Theorem 3] we use the corresponding one found by K. Stephenson [**St**, Theorem 3, p. 572, in particular]. The results are:

If $f \in \mathrm{BMOA}(S)$, *then*

$$\|f\circ h\|_{\mathrm{BMOA}(R)} \le \|f\|_{\mathrm{BMOA}(S)}$$

for each analytic map $h\colon R \to S$.

If $h\colon R\to S$ *is of type Bl, and if* $f\circ h \in \mathrm{BMOA}(R)$, *then* $\|f\circ h\|_{\mathrm{BMOA}(R)} = \|f\|_{\mathrm{BMOA}(S)}$, *so that* $f \in \mathrm{BMOA}(S)$.

(f) Metzger [**M**] introduced the family O_{BMOA} of Riemann surfaces on which BMOA consists only of constants. By an obvious reason we include Riemann surfaces of O_G in O_{BMOA}. Let O_{UBC} be the family of Riemann surfaces which are either of O_G or admit no nonconstant UBC functions. We shall prove the strict inclusion formula

$$O_{\mathrm{UBC}} \subsetneqq O_{\mathrm{BMOA}}.$$

Our work should be finding $R \in O_{\mathrm{BMOA}}\backslash O_{\mathrm{UBC}}$. Let E be a compact set of linear measure zero, yet of positive capacity lying on the real axis. Then $R = \mathbf{C}^*\backslash E$ is the desired. Since the function z is of $\mathrm{UBC}(R)$ by Corollary 5.3, it follows that $R \notin O_{\mathrm{UBC}}$. On the other hand, each $f \in \mathrm{BMOA}(R)$ is of class $H^2(R)$, that is, $|f|^2$ has a harmonic majorant on R. It is familiar (see, for example, [$\mathbf{Y_3}$, p. 334]) that f can be extended holomorphically to $\mathbf{C}^*$, so that, f must be a constant. Thus $R \in O_{\mathrm{BMOA}}$.

(g) Let $\mathrm{UBCA}(\Delta)$ be the set of all pole-free members of $\mathrm{UBC}(\Delta)$. It is apparent that $\mathrm{BMOA}(\Delta) \subset \mathrm{UBCA}(\Delta)$. On observing [**B**, Corollary 2, p. 15], one might suspect that $\mathrm{BMOA}(\Delta) = \mathrm{UBCA}(\Delta)$. This is not the case. Let

$$f(z) = (1+z)/(1-z).$$

By Corollary 5.3, $f \in \mathrm{UBCA}(\Delta)$, yet $f \notin \mathrm{BMOA}(\Delta)$ because f is not Bloch in the sense that $(1-|z|^2)|f'(z)|$ is unbounded in Δ.

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, FUKASAWA, SETAGAYA, TOKYO 158, JAPAN