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LOWEST ORDER INVARIANTS FOR REAL-ANALYTIC SURFACES IN $\mathbf{C}^2$

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ABSTRACT. Suppose M is a general real-analytic surface in complex euclidean two-space with complex tangent space at a point p. Further suppose M is tangent to order k at p. This paper determines a complete set of kth order local holomorphic invariants for M at p.

1. Introduction. This paper considers the problem of how to locally distinguish one real surface in $\mathbf{C}^2$ from another. The present study is motivated by recent work of Moser and Webster [**2**], who in turn were inspired by an earlier study of Bishop [**1**]. Before stating the main result of Moser and Webster, we must establish some notation and recall some definitions.

We always let M denote a real-analytic real two-dimensional surface in $\mathbf{C}^2$. Because we are concerned only with local properties of M, we always assume the origin O is in M. Moreover, M will also denote $M \cap U$ for any open ball U about O. Likewise, $\mathbf{C}^2$ will denote any (possibly small) open ball about O in $\mathbf{C}^2$. The description "near O" will be used only occasionally to remind the reader of this abuse of notation; however, "near O" is always understood.

DEFINITION 1.1. Two surfaces M and $\tilde{M}$ are (*holomorphically*) *equivalent* (near O), $\tilde{M} \approx M$, provided there exists an invertible holomorphic map $\Gamma: \mathbf{C}^2 \to \mathbf{C}^2$ (near O) so that $\Gamma(O) = O$ and $\Gamma(M) = \tilde{M}$.

DEFINITION 1.2. A property P of a surface M is an (*holomorphic*) *invariant* for M provided any surface which is equivalent to M also has property P.

The purpose of this study is to find some interesting invariants for a general surface M. If M is CR, then it must be either complex and equivalent to $\mathbf{C}$ (near O), or it must be totally real and equivalent to $\mathbf{R}^2$. Thus the only surfaces of interest to this study are those for which O is a CR singularity. That is, the surface is totally real for points arbitrarily near O but is tangent to $\mathbf{C}$ at O. A point on a surface in $\mathbf{C}^2$ with a complex tangent space is called an *exceptional point* by Bishop. Bishop, as well as Moser and Webster, notes that any surface with a CR singularity at O can be assumed to have the form

$$M = \{(z, h(z, \bar{z}))\}, \tag{1.3}$$

where h is a power series in z and $\bar{z}$ with complex coefficients and centered at O; moreover, h vanishes at O to order at least 2. We adopt the notation $h \in \mathcal{O}^2$. If $h \in \mathcal{O}^3$, Bishop calls the origin *exceptionally exceptional.*

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Under the assumption that O is not exceptionally exceptional, Bishop showed that h in the form (1.3) can be written in the form

$$h(z,\bar{z}) = \begin{cases} \gamma z^2 + z\bar{z} + \gamma\bar{z}^2 + \lambda(z,\bar{z}) \\ \text{or} \\ z^2 + \bar{z}^2 + \lambda(z,\bar{z}), \end{cases} \tag{1.4}$$

where $\lambda \in \mathcal{O}^3$. Moreover, γ can be assumed to be a nonnegative real number and, as such, is a holomorphic invariant of M. The main result of Moser and Webster is the following:

THEOREM [MOSER AND WEBSTER]. *Let M be in the form of (1.3) with h in the form of (1.4). Further suppose $0 < \gamma < 1/2$ (the elliptic case). Then there exists a holomorphic coordinate system (z_1, z_2) in which M has the form*

$$\{(z_1, z_1\bar{z}_1 + \Gamma(x_2)\,(z_1^2 + \bar{z}_1^2)\}, \tag{1.5}$$

where $\Gamma(x_2) \doteqdot \gamma + \delta x_2^s$; $\delta = \pm 1$ for s a nonnegative integer, or $\delta = 0$ for $s = \infty$. Moreover, the set $\{\gamma, \delta, s\}$ is a complete set of invariants for M. That is, $\{\gamma, \delta, s\}$ completely characterizes M up to equivalence.

Bishop's paper considers questions about the basic structure of M: Does M bound an analytic subvariety? How can we describe the hull of holomorphy of M? The approach is to first construct a form of M which yields invariants for M, then answer the questions by considering properties of the invariants. Therefore, a normal form (with a complete set of invariants) is most desirable. For a surface M with an elliptic not exceptionally exceptional point at O, the normal form of Moser-Webster (1.5) could not be more elegant. However, the problem of finding a normal form for more general types of surfaces is left open by their results and techniques. In addition, notice that (1.5) is implicit (dependent on z_2), whereas the original form of M (1.3) is explicit as a graph. We might ask if it is possible to find a normal form for M which retains the property of being a graph. It is this question which provides the motivation for our study.

To begin a study of this question it is natural to seek an invariant normal form for the lowest order terms in the power series h in (1.3). Given h in (1.3) with $h \in \mathcal{O}^k$ for $k \geq 2$, we seek an analogue of Bishop's form (1.4). Deriving such an analogue is the point of this paper, which culminates in the proof of our main result.

THEOREM 1.6. *Suppose M is a real-analytic two-dimensional surface in $\mathbf{C}^2$ with exceptional point at O. If $M \not\approx \mathbf{C}$, then, near O, M can be written in the form*

$$M = \{(z, P_k(z,\bar{z}) + \bar{z}\mathcal{O}^k)\} \tag{1.7}$$

for a homogeneous polynomial P_k of the form $z^{k-\alpha_0}\bar{z}^{\alpha_0}$ for integers k and α_0 satisfying $k \geq 2$ and $1 \leq \alpha_0 \leq k$, or $z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{k-\alpha_1}\bar{z}^{\alpha_1}$ for integers k, α_0, and α_1 satisfying $k \geq 2$, $1 \leq \alpha_0 \leq \alpha_1 \leq k$, and real number $\gamma > 0$, or

$$z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{k-\alpha_1}\bar{z}^{\alpha_1} + \sum_{j=2}^{\tau} a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j}\bar{z}^{\alpha_j}$$

for integers $k, \tau, \alpha_0, \ldots, \alpha_\tau$ satisfying $k \geq 2$, $2 \leq \tau \leq k-1$, $1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_\tau \leq k$, a real number $\gamma > 0$, and nonzero complex numbers $a_{k-\alpha_j,\alpha_j}$ $(2 \leq j \leq \tau)$

satisfying $0 \leq \operatorname{Arg} a_{k-\alpha_j,\alpha_j} < 2\pi/k_j$, *where the integers* k_j *are defined as follows. For each* $j = 2, 3, \ldots, \tau$ *let* ρ_j *be the integer defined by*

$$0 \leq \rho_j < \alpha_1 - \alpha_0 \quad \textit{and} \quad (\alpha_j - \alpha_0) = m_j(\alpha_1 - \alpha_0) + \rho_j$$

for some integer m_j. *Let* $l_1 \doteqdot \alpha_1 - \alpha_0$. *For given* $l_1, \ldots, l_j$ *let* $l_{j+1} \doteqdot \mathrm{GCD}(l_j, \rho_{j+1})$ *or* $l_{j+1} = l_j$ *if* $\rho_{j+1} = 0$. *For each* $j = 2, \ldots, \tau$ *let* $k_j \doteqdot l_{j-1}/l_j$.

Moreover, P_k *in* (1.7) *is a local holomorphic invariant for* M.

Notice that if $k = 2$ in (1.7) (i.e., O is not exceptionally exceptional) then the form of P_2 is either $z\bar{z}, \bar{z}^2$, or $z\bar{z}+\gamma\bar{z}^2$ for some $\gamma > 0$. Thus it is easy to see, applying Proposition 2.2, that (1.7) is simply another version of Bishop's form (1.4) in the case $k = 2$. Of course, the interesting aspect of Theorem 1.6 is that it produces an invariant normal form for the lowest order term for any real-analytic surface; i.e., for any value of k. For example, if M is tangent to $\mathbf{C}$ to order exactly 3 (it follows from the following discussion that the order of tangency to $\mathbf{C}$ is an invariant of M (Proposition 2.9)) then M has the form $M = \{(z, P_3(z,\bar{z}) + \bar{z}\mathcal{O}^3)\}$, with P_3 in the invariant normal form $z^2\bar{z}, z\bar{z}^2, \bar{z}^3$, or $z^2\bar{z} + \gamma z\bar{z}^2$, $z^2\bar{z} + \gamma\bar{z}^3$, $z\bar{z}^2 + \gamma\bar{z}^3$ for some real $\gamma > 0$, or $z^2\bar{z} + \gamma z\bar{z}^2 + c\bar{z}^3$ for some real $\gamma > 0$ and complex $c \neq 0$. In the latter case $k_2 = 1$ and the condition $0 \leq \operatorname{Arg} c < 2\pi$ is vacuous. The presence of the complex parameter "c" is somewhat surprising in the form of P_3. Notice that if $M = \{(z, z^2\bar{z} + \gamma z\bar{z}^2 + c\bar{z}^3 + \mathcal{O}^4)\}$ and $\tilde{M} = \{(z, z^2\bar{z} + \gamma z\bar{z}^2 + \tilde{c}\bar{z}^3 + \mathcal{O}^4)\}$ with $\gamma > 0$ and $c \neq \tilde{c}$, then $M \not\approx \tilde{M}$. Indeed, Theorem 1.6 yields an interesting collection of holomorphically distinct surfaces parametrized over $\mathbf{R}^+ \times (\mathbf{C}\backslash\{O\})$.

We now begin the proof of Theorem 1.6, leaving other observations and examples until §4.

2. Preliminaries to the proof. Suppose M is a surface of the form

$$M = \{(z, h(z,\bar{z}))\} \tag{2.1}$$

for some $h \in \mathcal{O}^k$, $k \geq 2$. Further suppose $\Gamma\colon \mathbf{C}^2 \to \mathbf{C}^2$ and $f\colon \mathbf{C} \to \mathbf{C}$ are holomorphic coordinate changes which preserve the origin. Let

$$\tilde{M} \doteqdot \{\Gamma(f(z), h(f(z), \bar{f}(\bar{z})))\},$$

where $\bar{f}$ denotes the power series obtained by conjugating the coefficients of the power series f. It follows that $M \approx \tilde{M}$. The idea is to choose Γ and f so that $\tilde{M}$ is of the form (1.7). By way of illustration we obtain

PROPOSITION 2.2. *Any surface of the form* (2.1) *can be written in the form*

$$M = \{(z, h(z,\bar{z}))\} \tag{2.3}$$

for some $h \in \mathcal{O}^k \cap \mathcal{O}(\bar{z})$, $k \geq 2$. *Here we mean* $h \in \mathcal{O}^k$ *and* h *is divisible by* $\bar{z}$; *we will also use the notation* $h \in \bar{z}\mathcal{O}^{k-1}$ *to mean the same thing.*

PROOF. Simply choose $\Gamma\colon \mathbf{C}^2 \to \mathbf{C}^2$ defined by $\Gamma(z_1, z_2) \doteqdot (z_1, z_2 - h(z_1, 0))$. □

Showing that we can obtain holomorphic invariants for M by using Γ and f as above is complicated by the fact that existence of such Γ and f is not a necessary condition for M to be equivalent to $\tilde{M}$. This difficulty will be overcome by use of

LEMMA 2.4. *Let* $M \doteqdot \{(z, h(z,\bar{z}))\}$ *and* $\tilde{M} \doteqdot \{(z, \tilde{h}(z,\bar{z}))\}$ *be two surfaces in the form* (2.3). *If* $M \approx \tilde{M}$ *then there exists a holomorphic coordinate change* $\Gamma\colon \mathbf{C}^2 \to \mathbf{C}^2$ *and real-analytic* $f\colon \mathbf{C} \to \mathbf{C}$ *of the form*

$$f(z,\bar{z}) = bz + \mathcal{O}^2, \qquad b \neq 0, \tag{2.5}$$

such that

$$\Gamma(z, h(z,\bar{z})) = (f(z,\bar{z}), \tilde{h}(f(z,\bar{z}), \bar{f}(\bar{z}, z))). \tag{2.6}$$

PROOF. Suppose $\Gamma\colon \mathbf{C}^2 \to \mathbf{C}^2$ is a holomorphic coordinate change with $\Gamma(O) = 0$ and $\Gamma(M) = \tilde{M}$. Let $\Gamma = (\Gamma_1, \Gamma_2)$ for $\Gamma_1, \Gamma_2\colon \mathbf{C}^2 \to \mathbf{C}$. Define a function $f\colon \mathbf{C} \to \mathbf{C}$ by

$$f(z,\bar{z}) \doteqdot \Gamma_1(z, h(z,\bar{z})). \tag{2.7}$$

It follows that f is real-analytic and invertible. Suppose $\Gamma_1(z_1, z_2) = bz_1 + \mathcal{O}(z_1^2) \cup \mathcal{O}(z_2)$. Then, since $h \in \mathcal{O}^k$,

$$f(z,\bar{z}) = bz + \mathcal{O}(z^2) + \mathcal{O}^k. \tag{2.8}$$

Since f is invertible we have $b \neq 0$, and since $k \geq 2$ we have f in the form (2.5). Equation (2.6) follows because $\Gamma(M) = \tilde{M}$ and $\tilde{M}$ is the graph of $\tilde{h}$. □

The importance of Lemma 2.4 lies in the fact that any $\bar{z}$ dependence in f (2.5) can occur only in the higher-order terms. It is this fact which eventually will allow us to conclude that our form (1.7) yields holomorphic invariants for M.

We now apply Lemma 2.4 to obtain a well-known invariant of M, the order of tangency to $\mathbf{C}$.

PROPOSITION 2.9. *Suppose* $M = \{(z, h(z,\bar{z}))\}$ *is in the form* (2.3). *Then* $k \doteq \operatorname{Ord}_0 h$ *is a holomorphic invariant of* M.

PROOF. Suppose $\tilde{M} = \{(z, \tilde{h}(z,\bar{z}))\}$ is also in the form (2.3) and $M \approx \tilde{M}$. Let P_k, respectively $\tilde{P}_{\tilde{k}}$, denote the leading nonvanishing homogeneous polynomial in the expansion of h, respectively $\tilde{h}$, into a sum of homogeneous polynomials of increasing degree. Thus

$$h(z,\bar{z}) = P_k(z,\bar{z}) + \mathcal{O}^{k+1} \quad \text{and} \quad \tilde{h}(z,\bar{z}) = \tilde{P}_{\tilde{k}}(z,\bar{z}) + \mathcal{O}^{\tilde{k}+1}.$$

Let Γ and f be the mappings guaranteed by Lemma 2.4. Again letting $\Gamma \doteq (\Gamma_1, \Gamma_2)$, it follows from (2.5) and (2.6) that

$$\Gamma_2(z, P_k(z,\bar{z}) + \bar{z}\mathcal{O}^k) = \tilde{P}_{\tilde{k}}(bz, \bar{b}\bar{z}) + \mathcal{O}^{\tilde{k}+1}. \tag{2.10}$$

Suppose Γ_2 has the form

$$\Gamma_2(z_1, z_2) = \sum_{j=1}^{k} a_j z_1^j + cz_2 + \mathcal{O}(z_1^{k+1}) \cup \mathcal{O}(z_2^2). \tag{2.11}$$

By interchanging k and $\tilde{k}$, if necessary, we may assume without loss of generality that $k \leq \tilde{k}$. It then follows from (2.10) and (2.11) that

$$\sum_{j=1}^{k} a_j z^j + cP_k(z,\bar{z}) = \tilde{P}_{\tilde{k}}(bz, \bar{b}\bar{z}) + \mathcal{O}^{k+1}. \tag{2.12}$$

Since $\bar{z}$ divides both $P_k(z,\bar{z})$ and $\tilde{P}_{\tilde{k}}(bz,\bar{b}\bar{z})$, (2.12) implies $a_j = 0$ for $1 \le j \le k$; hence

$$cP_k(z,\bar{z}) = \tilde{P}_{\tilde{k}}(bz,\bar{b}\bar{z}) + \mathcal{O}^{k+1}. \tag{2.13}$$

But, $\Gamma \doteq (\Gamma_1,\Gamma_2)$ invertible with $a_1 = 0$ in (2.11) implies $c \ne 0$ in (2.11). Proposition 2.9 follows immediately from (2.13). □

As indicated above, Proposition 2.9 is a well-known result. We have included this particular proof because it yields the following useful

LEMMA 2.14. *Suppose M and $\tilde{M}$ are two surfaces in the form* (2.3) *with $M = \{(z, P_k(z,\bar{z}) + \mathcal{O}^{k+1})\}$ and $\tilde{M} = \{(z, \tilde{P}_k(z,\bar{z}) + \mathcal{O}^{k+1})\}$ for nonvanishing homogeneous polynomials P and $\tilde{P}$ of degree k. If $M \approx \tilde{M}$ then there are nonzero complex numbers b and c such that*

$$cP_k(z,\bar{z}) = \tilde{P}_k(bz,\bar{b}\bar{z}). \tag{2.15}$$

PROOF. Equation (2.15) follows immediately from (2.13). □

Lemma 2.14 is the key to finding an invariant normal form for the leading homogeneous term. Indeed, because of Lemma 2.14 we need to consider only those coordinate changes of the forms $f(z) = bz$ and $\Gamma(z_1,z_2) = (z_1, cz_2)$ for nonzero complex numbers b and c.

3. The proof of Theorem 1.6.

Suppose M is a surface of the form (2.3) with $h(z,\bar{z}) = P_k(z,\bar{z}) + \mathcal{O}^{k+1}$. Let P_k have the explicit description

$$P_k(z,\bar{z}) \doteq \sum_{\alpha=1}^{k} a_{k-\alpha,\alpha} z^{k-\alpha}\bar{z}^{\alpha}. \tag{3.1}$$

Note that $\alpha \ge 1$ because $\bar{z}$ divides P_k. Let $\alpha_0 \doteq \min\{\alpha\colon a_{k-\alpha,\alpha} \ne 0\}$. It follows from Lemma 2.14 that α_0 is a holomorphic invariant for M. By choosing $\Gamma(z_1,z_2) \doteq (z_1, (1/a_{k-\alpha_0,\alpha_0})z_2)$, we may assume (3.1) has the form

$$P_k(z,\bar{z}) = z^{k-\alpha_0}\bar{z}^{\alpha_0} + \sum_{\alpha>\alpha_0}^{k} a_{k-\alpha,\alpha} z^{k-\alpha}\bar{z}^{\alpha}. \tag{3.2}$$

If $a_{k-\alpha,\alpha} = 0$ for all $\alpha_0 < \alpha \le k$, it follows from Lemma 2.14 that $P_k(z,\bar{z}) = z^{k-\alpha_0}\bar{z}^{\alpha_0}$ is the desired invariant normal form. Otherwise let

$$\alpha_1 \doteq \min\{\alpha > \alpha_0\colon a_{k-\alpha,\alpha} \ne 0\}.$$

It follows, again by Lemma 2.14, that α_1 is an invariant for M. The idea is to find an invariant normal form for the coefficient $a_{k-\alpha_1,\alpha_1}$ while preserving the form (3.2), namely, preserving $a_{k-\alpha_0,\alpha_0} = 1$. To do this we no longer have complete freedom to choose c and b; in particular, Lemma 2.14 implies that c and b must be chosen to satisfy the condition

$$c = b^{k-\alpha_0}\bar{b}^{\alpha_0}. \tag{3.3}$$

If we let $f(z) \doteq bz$ for an arbitrary nonzero b in $\mathbf{C}$ and let $\Gamma(z_1,z_2) \doteq (z_1, z_2/c)$ for c defined by (3.3), we may assume (3.2) has the form

$$\begin{aligned} P_k(z,\bar{z}) = z^{k-\alpha_0}\bar{z}^{\alpha_0} &+ \left[\frac{\bar{b}}{b}\right]^{\alpha_1-\alpha_0} a_{k-\alpha_1,\alpha_1} z^{k-\alpha_1}\bar{z}^{\alpha_1} \\ &+ \sum_{\alpha>\alpha_1} \left[\frac{\bar{b}}{b}\right]^{\alpha-\alpha_0} a_{k-\alpha,\alpha} z^{k-\alpha}\bar{z}^{\alpha}. \end{aligned} \tag{3.4}$$

Since $a_{k-\alpha_1,\alpha_1} \neq 0$ and b in (3.4) can be any nonzero complex number, we may choose b so that $[\bar{b}/b]^{\alpha_1-\alpha_0} a_{k-\alpha_1,\alpha_1} \doteqdot \gamma$ is a positive real number. Thus we may assume (3.4) has the form

$$P_k(z,\bar{z}) = z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{k-\alpha_1}\bar{z}^{\alpha_1} + \sum_{\alpha>\alpha_1} a_{k-\alpha,\alpha} z^{k-\alpha}\bar{z}^{\alpha} \tag{3.5}$$

for $1 \leq \alpha_0 < \alpha_1 \leq k$ and some real number $\gamma > 0$. Moreover, γ in (3.5) is an invariant for M. To see this suppose $\tilde{M}$ is of the form (2.1) with $\tilde{h}(z,\bar{z}) = \tilde{P}_k(z,\bar{z}) + \mathcal{O}^{k+1}$ and

$$\tilde{P}_k(z,\bar{z}) = z^{k-\alpha_0}\bar{z}^{\alpha_0} + \tilde{\gamma} z^{k-\alpha_1}\bar{z}^{\alpha_1} + \sum_{\alpha>\alpha_1} \tilde{a}_{k-\alpha,\alpha} z^{k-\alpha}\bar{z}^{\alpha}$$

for some positive real number $\tilde{\gamma}$. (Recall that we already know k, α_0, and α_1 are invariants of M.) If $M \approx \tilde{M}$ then applying Lemma 2.14 and condition (3.3) implies the existence of a nonzero complex number b such that

$$\gamma = [\bar{b}/b]^{\alpha_1-\alpha_0}\tilde{\gamma} \quad \text{and} \quad a_{k-\alpha,\alpha} = [\bar{b}/b]^{\alpha-\alpha_0}\tilde{a}_{k-\alpha,\alpha} \quad \text{for all } \alpha_1 < \alpha \leq k. \tag{3.6}$$

Not only does (3.6) yield that $\gamma = \tilde{\gamma}$ (i.e., γ in form (3.5) is an invariant of M), it also yields that $|a_{k-\alpha,\alpha}|$ in form (3.5) is an invariant for M for each $\alpha_1 < \alpha \leq k$. In addition, it follows from (3.6) that in order to preserve the normalization (3.5), we are constrained to require a condition on our future choice of b, namely,

$$[\bar{b}/b]^{\alpha_1-\alpha_0} = 1. \tag{3.7}$$

If $\alpha_1 = k$ or $a_{k-\alpha,\alpha} = 0$ for $\alpha_1 < \alpha \leq k$, then we have obtained the invariant normal form

$$P_k(z,\bar{z}) = z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{k-\alpha_1}\bar{z}^{\alpha_1} \quad \text{for } 1 \leq \alpha_0 < \alpha_1 \leq k$$

and positive real γ. Otherwise, let $\alpha_2, \dots, \alpha_\tau$ denote the integers satisfying $\alpha_1 < \alpha_2 < \cdots < \alpha_\tau \leq k$ for which (3.5) has the form

$$P_k(z,\bar{z}) = z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{k-\alpha_1}\bar{z}^{\alpha_1} + \sum_{j=2}^{\tau} a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j}\bar{z}^{\alpha_j} \tag{3.8}$$

for $\gamma > 0$ and nonzero $a_{k-\alpha_j,\alpha_j}, 2 \leq j \leq \tau$. Choosing any complex number b satisfying (3.7), defining c by (3.3), and applying the coordinate changes $f(z) \doteqdot bz$ and $\Gamma(z_1,z_2) \doteqdot (z_1, z_2/c)$ transforms (3.8) to

$$P_k(z,\bar{z}) = z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{k-\alpha_1}\bar{z}^{\alpha_1} + \sum_{j=2}^{\tau} \left[\frac{\bar{b}}{b}\right]^{\alpha_j-\alpha_0} a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j}\bar{z}^{\alpha_j}. \tag{3.11}$$

We see that our freedom to choose b so as to normalize the coefficient of $z^{k-\alpha_2}\bar{z}^{\alpha_2}$ in (3.9) is completely determined by the set

$$\mathcal{G}(M;2) \doteqdot \{[\bar{b}/b]^{\alpha_2-\alpha_0} : b \in \mathbf{C} \text{ and } [\bar{b}/b]^{\alpha_1-\alpha_0} = 1\}.$$

Let ρ_2 be the integer defined by $0 \leq \rho_2 < \alpha_1 - \alpha_0$ and $(\alpha_2 - \alpha_0) = m_2(\alpha_1 - \alpha_0) + \rho_2$ for some nonnegative integer m_2. It follows that

$$\mathcal{G}(M;2) = \sqrt[k_2]{1},$$

the group of k_2 roots of unity, where $k_2 \doteq (\alpha_1 - \alpha_0)/\mathrm{GCD}(\alpha_1 - \alpha_0, \rho_2)$. We are using the convention $\mathrm{GCD}(l, 0) \doteq l$ for all $l > 0$. Thus we may choose b (and the appropriate coordinate changes) so that

$$0 \leq \mathrm{Arg}[\bar{b}/b]^{\alpha_2-\alpha_0} a_{k-\alpha_2,\alpha_2} < 2\pi/k_2.$$

That is, the form of $P_k(z, \bar{z})$ in (3.8) can be assumed to satisfy the additional property

$$0 \leq \mathrm{Arg}\, a_{k-\alpha_2,\alpha_2} < 2\pi/k_2. \tag{3.10}$$

It follows from (3.6) that $a_{k-\alpha_2,\alpha_2}$, normalized to satisfy (3.10), is an invariant of M. Furthermore, to preserve the normalization (3.5) (with (3.8)) requires yet another constraint on our future choice of b, namely,

$$[\bar{b}/b]^{\alpha_2-\alpha_0} = 1. \tag{3.11}$$

We proceed in this way to find an invariant normal form for the coefficient of $z^{k-\alpha_3}\bar{z}^{\alpha_3}$, then the coefficient of $z^{k-\alpha_4}\bar{z}^{\alpha_4}$, and so on, until we have obtained the invariant normal form (1.7). □

4. Conclusion. We now present some interesting consequences of Theorem 1.6 and the above proof. To begin, we observe

COROLLARY 4.1. *Let M be any surface of the form*

$$M = \left\{ \left(z, z^{k-\alpha_0}\bar{z}^{\alpha_0} + \sum_{\alpha=\alpha_0+1}^{k} a_{k-\alpha,\alpha} z^{k-\alpha}\bar{z}^{\alpha} + \bar{z}\mathcal{O}^k \right) \right\}.$$

The numbers $|a_{k-\alpha,\alpha}|$, $\alpha = \alpha_0 + 1, \ldots, k$, are invariants for M.

PROOF. Corollary 4.1 follows from the above discussion since $\alpha_0, \ldots, \alpha_\tau$ are invariants for M which pick out the nonzero coefficients, and for any $b \in \mathbf{C}\backslash\{O\}$ and any integer ρ we always have $|(\bar{b}/b)^\rho| = 1$. □

COROLLARY 4.2. *Suppose M is any surface of the form $\{(z, P_k(z, \bar{z}) + \bar{z}\mathcal{O}^k)\}$ with $P_k(z, \bar{z})$ in the invariant normal form of Theorem* 1.6. *There are exactly $K \doteq \prod_{j=2}^{\tau} k_j$ surfaces of the form*

$$\left\{ \left(z, z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{k-\alpha_1}\bar{z}^{\alpha_1} + \sum_{\alpha=\alpha_1+1}^{k} \tilde{a}_{k-\alpha,\alpha} z^{k-\alpha}\bar{z}^{\alpha} + \bar{z}\mathcal{O}^k \right) \right\}$$

which have the same kth order invariants as M. (γ *is the positive real number fixed by Theorem* 1.6.)

PROOF. As before, all $\tilde{a}_{k-\alpha,\alpha}$ must be 0 except for $\tilde{a}_{k-\alpha_j,\alpha_j}$, $2 \leq j \leq \tau$. Corollary 4.2 follows by counting from back to front. That is, fix $\tilde{a}_{k-\alpha_j,\alpha_j} = a_{k-\alpha_j,\alpha_j}$ for all $j = 2, \ldots, \tau - 1$. Clearly, there are k_τ distinct possible choices for $\tilde{a}_{k-\alpha_\tau,\alpha_\tau}$. Similarly, there are $k_{\tau-1}$ choices for $\tilde{a}_{k-\alpha_{\tau-1},\alpha_{\tau-1}}$, etc. □

To emphasize the explicit nature of the proof of Theorem 1.6 we prove the following proposition.

PROPOSITION 4.3. *Let M be any surface as in Corollary* 4.2. *The set of all surfaces of the form $\{(z, \tilde{P}_k(z,\bar{z}) + \mathcal{O}^{k+1})\}$ with the same kth order invariants as M is*

$$\left\{\left\{\left(z, z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma e^{i\theta} z^{k-\alpha_1}\bar{z}^{\alpha_1} + \sum_{j=2}^{\tau}\left\{\exp\left(\frac{\mu_j}{k_j}2\pi i\right)\exp\left(\frac{\alpha_j-\alpha_0}{\alpha_1-\alpha_0}\thei\right) \times \exp\left(\sum_{l=2}^{j}\frac{\mu_{l-1}}{k_{l-1}\rho_{l-1}}\rho_j 2\pi i\right)\right\} \times a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j}\bar{z}^{\alpha_j} + \mathcal{O}^{k+1}\right)\right\} : 0 \le \theta < 2\pi;\ \text{for all } j = 2,\ldots,\tau,\ 0 \le \mu_j < k_j\right\}. \tag{4.4}$$

(*For notational convenience we have employed the convention* $k_1 = \rho_1 = 1$ *and* $\mu_1 = 0$. *All other terms are from Theorem* 1.6.)

PROOF. For any fixed θ in (4.4) there are $\prod_{j=2}^{\tau} k_j$ surfaces in the set (4.4). Thus by Corollary 4.2 it suffices to show that each surface in (4.4) reduces to the form of M. To this end, fix θ and choose b such that $(\bar{b}/b)^{\alpha_1-\alpha_0} = e^{-i\theta}$. Observe that for each $j = 2,\ldots,\tau$,

$$\left(\frac{\bar{b}}{b}\right)^{\alpha_j-\alpha_0} = \exp\left(-\frac{\alpha_j-\alpha_0}{\alpha_1-\alpha_0}\theta i\right).$$

The surface in (4.4) for this θ reduces to

$$\left\{\left(z, z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{k-\alpha_1}\bar{z}^{\alpha_1} + \exp\left(\frac{\mu_2}{k_2}2\pi i\right) a_{k-\alpha_2,\alpha_2} z^{k-\alpha_2}\bar{z}^{\alpha_2} + \sum_{j=3}^{\tau}\left\{\exp\left(\frac{\mu_j}{k_j}2\pi i\right)\exp\left(\sum_{l=3}^{j}\frac{\mu_{l-1}}{k_{l-1}\rho_{l-1}}\rho_j 2\pi i\right)\right\} \times a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j}\bar{z}^{\alpha_j} + \mathcal{O}^{k+1}\right)\right\}. \tag{4.5}$$

Choose b such that $(\bar{b}/b)^{\alpha_1-\alpha_2} = 1$ and recall the definitions of $\rho_2,\ldots,\rho_\tau$ to see that (4.5) reduces to

$$
\begin{aligned}
(4.6)\qquad &\Bigg\{\Bigg(z, z^{k-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{k-\alpha_1}\bar{z}^{\alpha_1} \\
&\quad + \exp\left(\frac{\mu_2}{k_2}2\pi i\right)\left(\frac{\bar{b}}{b}\right)^{\rho_2} a_{k-\alpha_2,\alpha_2} z^{k-\alpha_2}\bar{z}^{\alpha_2} \\
&\quad + \sum_{j=3}^{\tau}\left\{\exp\left(\frac{\mu_j}{k_j}2\pi i\right)\exp\left(\sum_{l=3}^{j}\frac{\mu_{l-1}}{k_{l-1}\rho_{l-1}}\rho_j 2\pi i\right)\left(\frac{\bar{b}}{b}\right)^{\rho_j}\right\} \\
&\qquad\qquad \times a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j}\bar{z}^{\alpha_j} + O^{k+1}\Bigg)\Bigg\}.
\end{aligned}
$$

Recalling the definition of k_2 and the fact that $0 \leq \mu_2 < k_2$, we can choose b such that

$$\left(\frac{\bar{b}}{b}\right)^{\rho_2} = \exp\left(-\frac{\mu_2}{k_2}2\pi i\right).$$

Observe that for each $j = 3, \ldots, \tau$,

$$\left(\frac{\bar{b}}{b}\right)^{\rho_j} = \exp\left(-\frac{\mu_2}{k_2\rho_2}\rho_j 2\pi i\right).$$

It follows that (4.6) reduces to

$$
\begin{aligned}
(4.7)\qquad &\Bigg\{\Bigg(z, \ldots a_{k-\alpha_2,\alpha_2} z^{k-\alpha_2}\bar{z}^{\alpha_2} + \exp\left(\frac{\mu_3}{k_3}2\pi i\right) a_{k-\alpha_3,\alpha_3} z^{k-\alpha_3}\bar{z}^{\alpha_3} \\
&\quad + \sum_{j=4}^{\tau}\left\{\exp\left(\frac{\mu_j}{k_j}2\pi i\right)\exp\left(\sum_{l=4}^{j}\frac{\mu_{l-1}}{k_{l-1}\rho_{l-1}}\rho_j 2\pi i\right)\right\} \\
&\qquad\qquad \times a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j}\bar{z}^{\alpha_j}\Bigg)\Bigg\}.
\end{aligned}
$$

We can now choose b so that

$$\left(\frac{\bar{b}}{b}\right)^{\alpha_1-\alpha_0} = \left(\frac{\bar{b}}{b}\right)^{\rho_2} = 1 \quad \text{and} \quad \left(\frac{\bar{b}}{b}\right)^{\rho_3} = \exp\left(-\frac{\mu_3}{k_3}2\pi i\right).$$

Observe that

$$\left(\frac{\bar{b}}{b}\right)^{\rho_j} = \exp\left(-\frac{\mu_3}{k_3\rho_3}\rho_j 2\pi i\right) \quad \text{for all } j = 4, \ldots, \tau.$$

The argument proceeds and the proof of Proposition 4.3 is clear. □

Perhaps the most interesting feature of the invariant normal form of P_k in Theorem 1.6 is the presence of conditions on the principal arguments of the invariants $a_{k-\alpha_j,\alpha_j}$ for $2 \leq j \leq \tau$. These conditions are determined by various groups of roots of unity. These groups, in turn, are determined by the invariants $\alpha_0, \alpha_1, \ldots, \alpha_\tau$ and, therefore, are themselves invariants of the surface. We conclude this study with a brief look at these invariant groups.

Suppose $P_k(z,\bar{z})$ is in the invariant normal form of Theorem 1.6. Let $\alpha_0, \alpha_1, \ldots, \alpha_\tau$ be the invariants given in Theorem 1.6. The groups in question are described as follows: For notational consistency let $\mathcal{G}(M;O)$ denote $\mathbf{C}\backslash\{O\}$ and $\mathcal{G}(M;1)$ denote the unit circle. For each $2 \le j \le \tau$ let

$$\mathcal{G}(M;j) \doteqdot \{r^{\alpha_j-\alpha_0} : r^{\alpha_l-\alpha_0} = 1 \text{ for } 1 \le l \le j-1\}.$$

The numbers $k_2, \ldots, k_\tau$ in Theorem 1.6 are the orders of $\mathcal{G}(M;2), \ldots, \mathcal{G}(M;\tau)$, respectively; indeed, for each $2 \le j \le \tau$ it follows that $\mathcal{G}(M;j)$ is the group of k_j roots of unity. Notice that $\mathcal{G}(M;j)$ is the group containing all the numbers which can be used to reduce (by multiplication) the coefficient of $z^{k-\alpha_j}\bar{z}^{\alpha_j}$ to the invariant form of Theorem 1.6. The groups $\mathcal{G}(M;O)$ and $\mathcal{G}(M;1)$ are the same for all surfaces (with trivial exception being those for which $P_k(z,\bar{z}) = z^{k-\alpha_0}\bar{z}^{\alpha_0}$) and yield all possible information if $\tau \le 1$. Notice this includes the case $k = 2$, since the construction requires $\tau \le k-1$. Hence, the invariant groups are interesting only in the case $k \ge 3$ and $\tau \ge 2$, that is, in the case $P_k(z,\bar{z})$ has at least three nonvanishing terms.

In the case $k = 3$ and $\tau = 2$ we see that $\mathcal{G}(M;2) = \{\mathrm{id}\}$, thus accounting for the fact that the coefficient of $\bar{z}^3$ is invariant without any condition on its argument. Indeed, if $\alpha_1 - \alpha_0 = 1$ then $\mathcal{G}(M;j) = \{\mathrm{id}\}$ for all $2 \le j \le \tau$, and we obtain no restrictive condition on any of the remaining nonzero invariant coefficients of $P_k(z,\bar{z})$. Hence, the first occurrence of a restrictive condition on the argument of an invariant coefficient is for $k = 4$ and $\tau = 2$. Thus, we consider the following example.

Suppose M is tangent to $\mathbf{C}$ to order exactly 4. Then M has the form $M = \{(z, P_4(z,\bar{z}) + \bar{z}\mathcal{O}^4)\}$ with P_4 in the invariant form $z^{4-\alpha_0}\bar{z}^{\alpha_0}$ for some $1 \le \alpha_0 \le 4$; or $z^{4-\alpha_0}\bar{z}^{\alpha_0} + \gamma z^{4-\alpha_1}\bar{z}^{\alpha_1}$ for some $1 \le \alpha_0 < \alpha_1 \le 4$ and $\gamma > 0$; or $z^3\bar{z} + \gamma z^2\bar{z}^2 + a_{1,3}z\bar{z}^3 + a_{0,4}\bar{z}^4$ for some $\gamma > 0$ and complex $a_{1,3}$ and $a_{0,4}$ not both equal to 0; or $z^2\bar{z}^2 +_\gamma z\bar{z}^3 + c\bar{z}^4$ for some $\gamma > 0$ and complex $c \ne 0$; or $z^3\bar{z} + \gamma z\bar{z}^3 + c\bar{z}^4$ for some $\gamma > 0$ and nonzero c satisfying $0 \le \operatorname{Arg} c < \pi$.

For a given $k \ge 3$ and $\tau \ge 2$, suppose we wish to give an example for which the invariant form of $P_k(z,\bar{z})$ places the most stringent condition possible on the argument of some coefficient of $P_k(z,\bar{z})$. It is evident that we must choose the example so that $\mathcal{G}(M;2)$ has greatest possible order. This is accomplished by choosing $\alpha_0 = 1$, $\alpha_1 = k-1$, and $\alpha_2 = k$, thus producing the invariant form

$$P_k(z,\bar{z}) = z^{k-1}\bar{z} + \gamma z\bar{z}^{k-1} + c\bar{z}^k$$

for some $\gamma > 0$ and nonzero complex number c satisfying $0 \le \operatorname{Arg} c < 2\pi/(k-2)$.

We close with the observation that we can apply elementary number-theoretic arguments to the invariants $\alpha_0, \alpha_1, \ldots, \alpha_\tau$ and produce many observations concerning the restrictive conditions on the invariant coefficients of $P_k(z,\bar{z})$. Although some of these observations are amusing, we have overcome the temptation to present them here.

References

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